Simulations of Relativistic Solitons and their Radiation Emission

Daniel Seipt

Helmholtz Institute Jena

HIJ Scientific Council
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“Wave of Translation” on Union Canal (1834)



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- Lagrange’s theory (leading order h/H, $v^2 = gH$) — dispersion-free waves:

$$0 = \frac{\partial^2 h}{\partial t^2} - v^2 \frac{\partial^2 h}{\partial x^2} = \left(\frac{\partial}{\partial t} - v \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) h(x + vt, x - vt)$$

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- Korteweg-de-Vries equation (1895) — nonlinearity and dispersion:

$$0 = \frac{\partial h}{\partial t} + h \frac{\partial h}{\partial x} + \frac{\partial^3 h}{\partial x^3}$$

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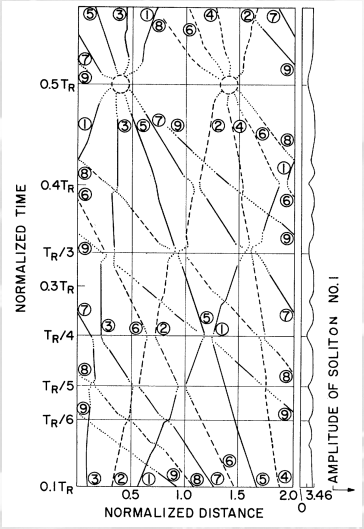
$$0 = \frac{\partial h}{\partial t} + h \frac{\partial h}{\partial x} + \frac{\partial^3 h}{\partial x^3}$$

- Solitonic waves: nonlinearity and dispersion compensate. Stable propagating wavepacket.

$$h(x, t) = 3v \operatorname{sech}^2 \left(\sqrt{v} \frac{x - vt}{2} \right)$$

- The wave amplitude correlates with wave velocity v

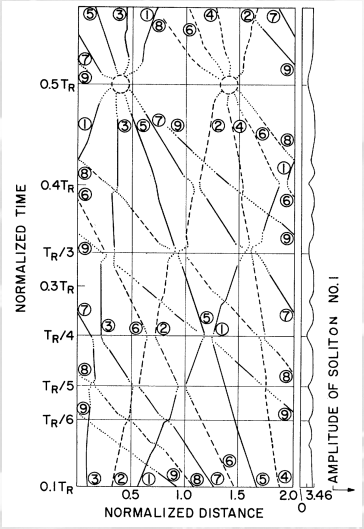
Revival of Solitons — Advent of Computer Simulations (1950s-60s)



- Interaction of "solitons" in a collisionless plasma and the recurrence of initial states
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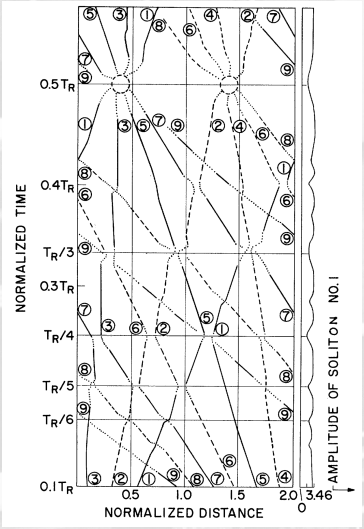
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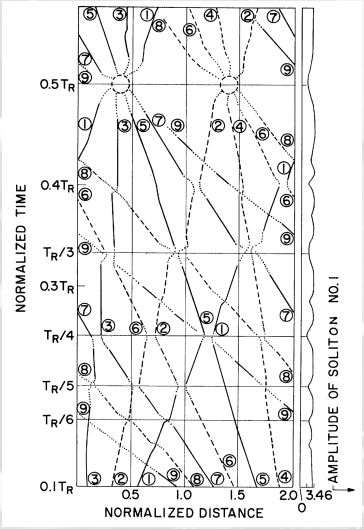
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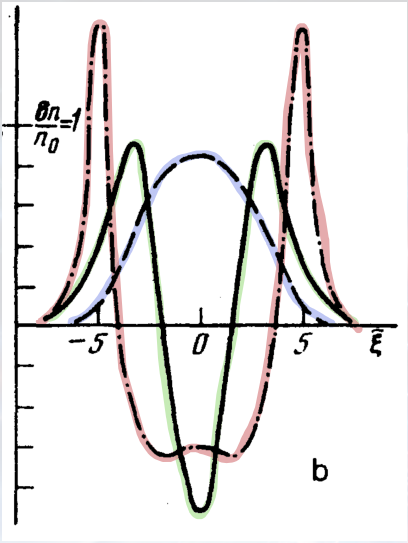
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- ... but computer simulations are more and useful
- "Fermi–Pasta–Ulam, solitons and the fabric of nonlinear and computational science ..." — Zabusky 2005

Relativistic LPI *are* Highly Non-linear (1979)

- Prediction of the "Laser Electron Accelerator" using computer simulations

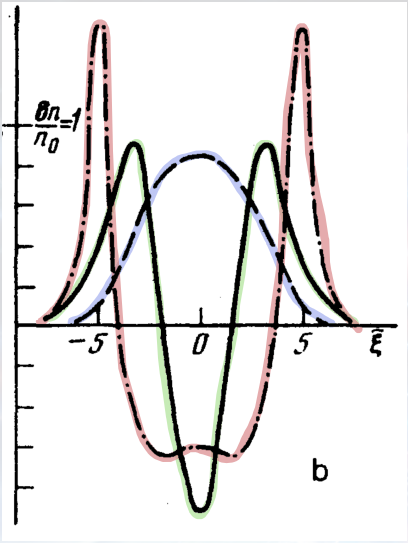
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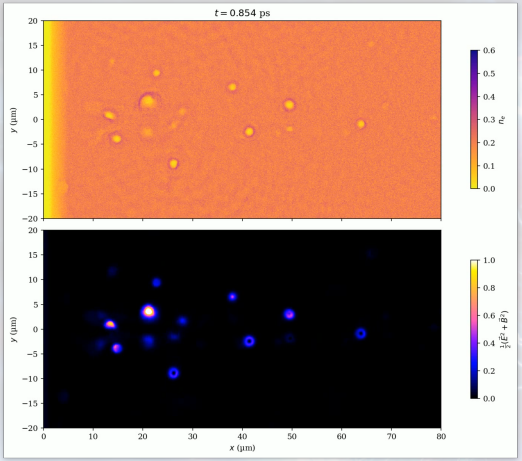
... resonant absorption Valeo and Kruer PRL 1974

... plasma turbulence Mima et al, PoP 2001

... self-induced transparency Tushentsov PRL 2001

Kinetic Simulations of Relativistic Solitons in Wake of Laser Pulses (2000)

Soliton Formation



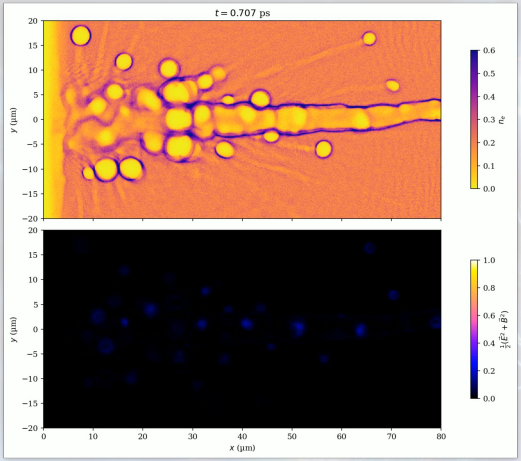
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Key features seen in sims:

- Trapping of electromagnetic energy from the driver
- Non-linear frequency down-conversion of laser light
- Associated to the break-up of the laser pulse and filamentation
- Stationary or slowly propagating

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Solitons $\rightarrow$ Post-solitons



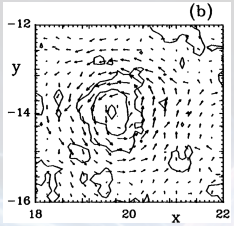
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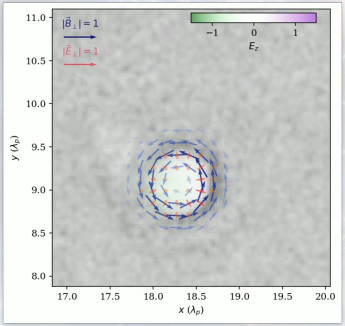
Key features seen in sims:

- Trapping of electromagnetic energy from the driver
- Non-linear frequency down-conversion of laser light
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- Stationary or slowly propagating
- Solitons in strict sense only on the electron time-scale. On ion time-scale Coulomb explosion into post-solitons

Naumova et al, PRL 87, 185004 (2001).

Detailed Look at Relativistic Soliton Properties

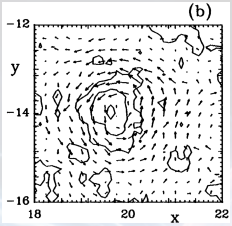


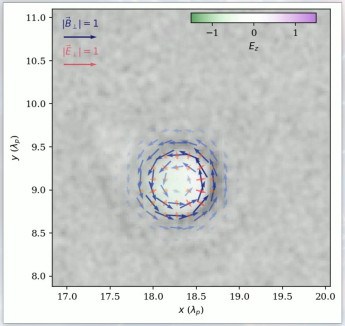


Properties:

- Size $\sim \lambda_p$
- Density depression with wall
- Trapped em. field $a_s \gtrsim 1$
- Radial electric field
- Azimuthal magnetic field
- Debate: Do solitons emit radiation?

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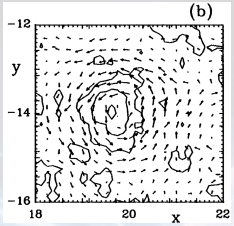


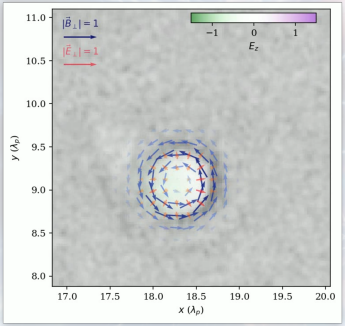
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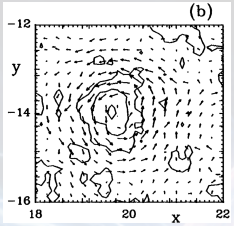
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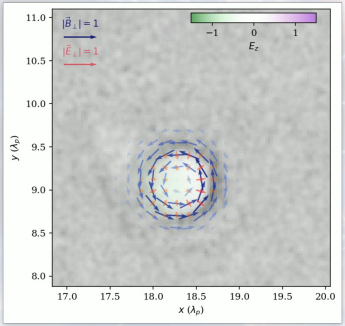
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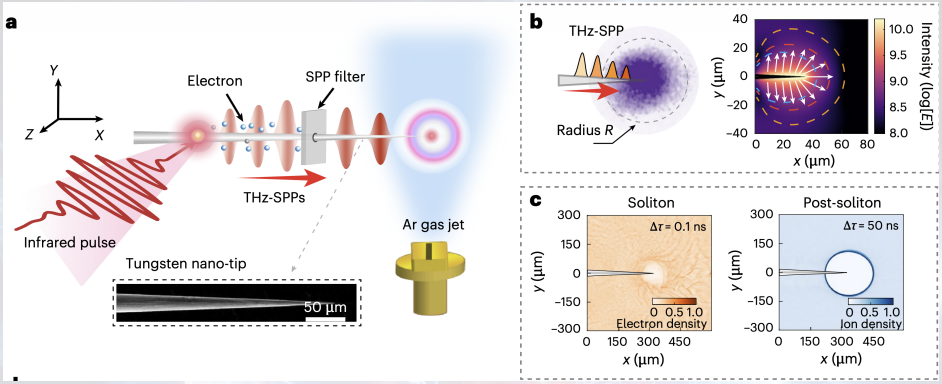
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Yes! Currents in the wall Esirkepov PRL 2002 (LP), 2004 (CP).

THz Solitons and Ball Lightning



[. . .] unambiguous characterization is impeded by the micrometre- and picosecond-level spatiotemporal scale associated ***with the optical EM soliton****.*

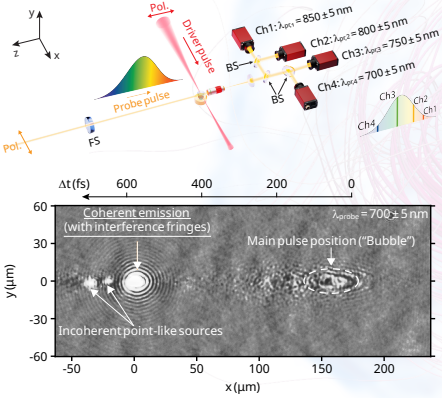
C. Zhou et al., Ball-lightning-like relativistic terahertz solitons, Nature Photonics, **20**, 727, June 2026

S. V. Bulanov, *ibid.* p. 611.


HI Jena, 2026

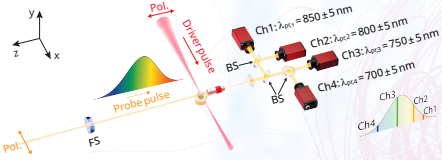
Long-lived, stationary, coherent, broadband optical emission in laser wake

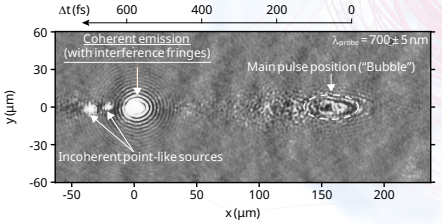
M. Kaluza



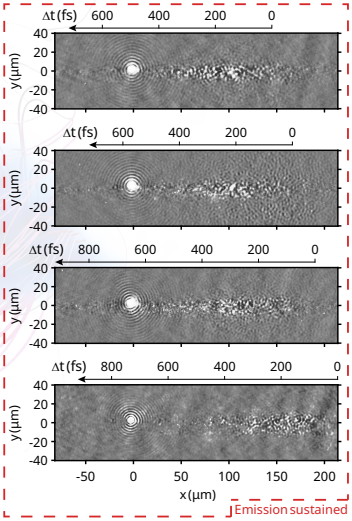
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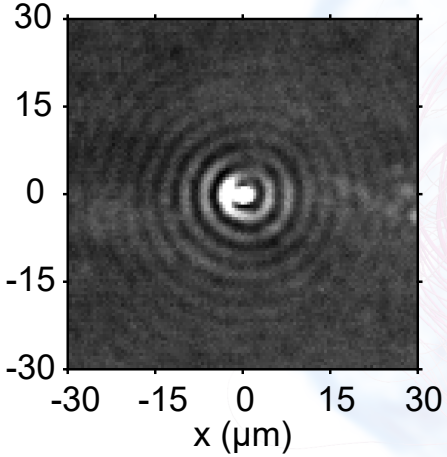


Looks like emission from solitons, need simulations to confirm.



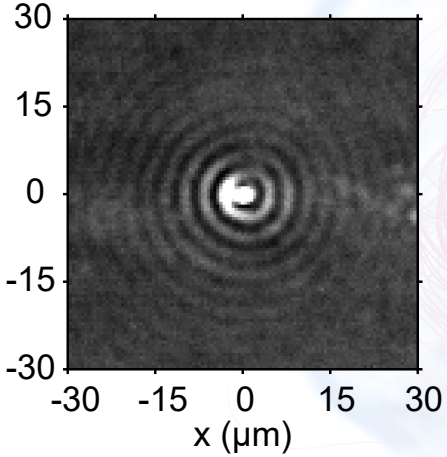
Micron-sized Coherent Light Emission

Example of measured probe image

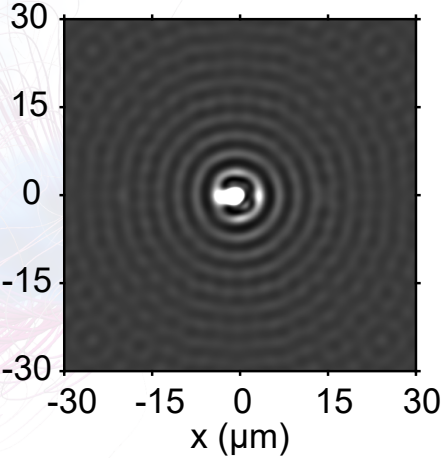


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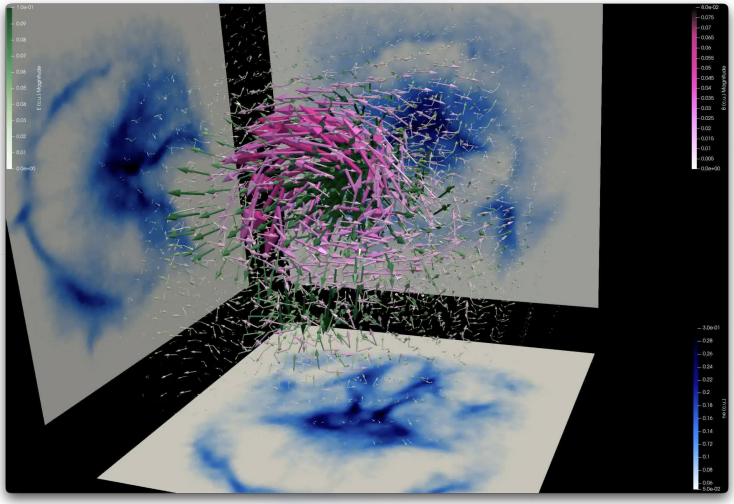


Calculated synthetic probe image



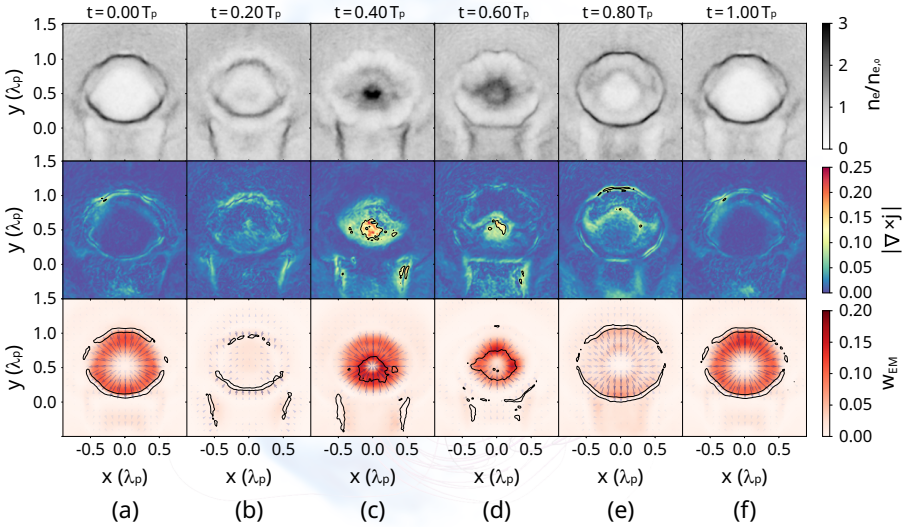
Remainder of talk: Methodology for generating synthetic probe images

Results of Smilei) PIC Simulation



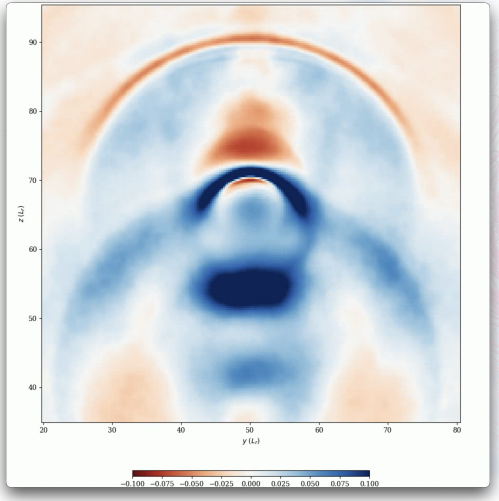
Simulations performed at $\approx 20\times$ higher density, scaled focal spot and pulse duration

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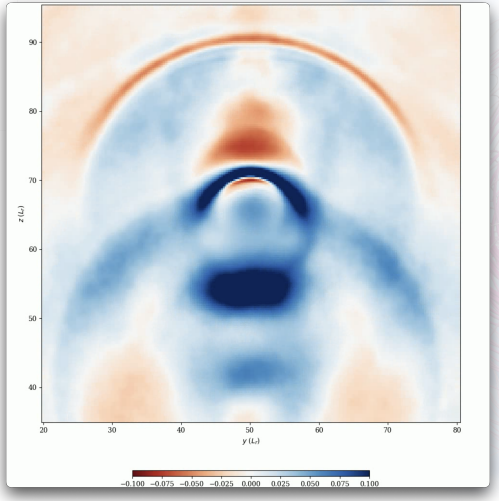
Simulated object has characteristics of a soliton

Radiation Emission: PIC and Beyond

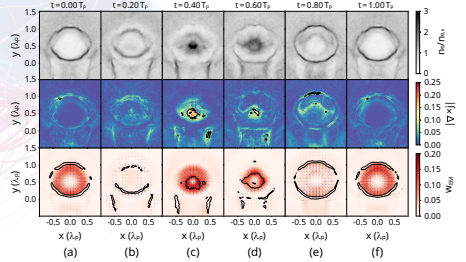


- PIC results show outgoing em. waves
- ? But how to calculate probe images?
- ! Need to post-process PIC data

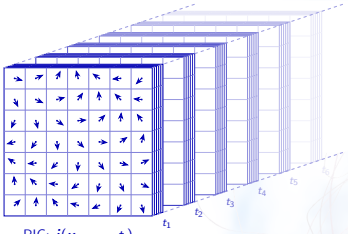
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- Radition emission: $\Box B = \nabla \times j$
- Source size $\ll L_s$

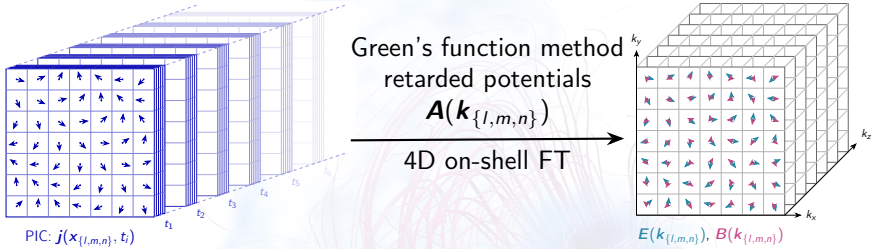


Synthetic Images: Postprocessing Code `plasmaglow`

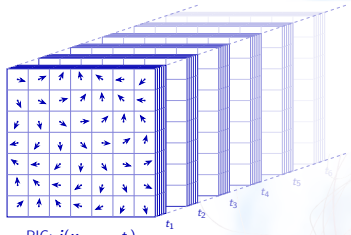


PIC: $\boldsymbol{j}(\boldsymbol{x}_{\{l,m,n\}}, t_i)$

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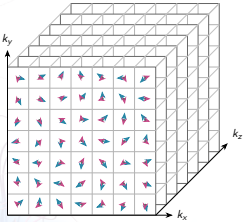
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Green's function method
retarded potentials
$\boldsymbol{A}(\boldsymbol{k}_{\{l,m,n\}})$

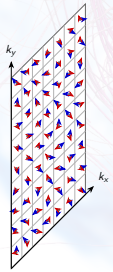
4D on-shell FT



$\boldsymbol{E}(\boldsymbol{k}_{\{l,m,n\}})$, $\boldsymbol{B}(\boldsymbol{k}_{\{l,m,n\}})$

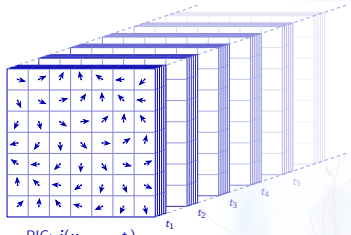
$\longrightarrow$

backpropagation
interpolation
(probe addition)
$\mathcal{U}_{\mathrm{obj}}(\boldsymbol{k}_\perp, \omega_{\mathrm{pr}})$



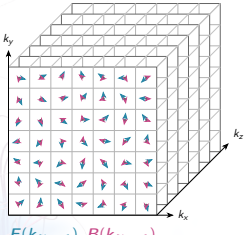
$(\boldsymbol{E}(\boldsymbol{k}_\perp, \omega), \boldsymbol{B}(\boldsymbol{k}_\perp, \omega))$

Synthetic Images: Postprocessing Code `plasmaglow`



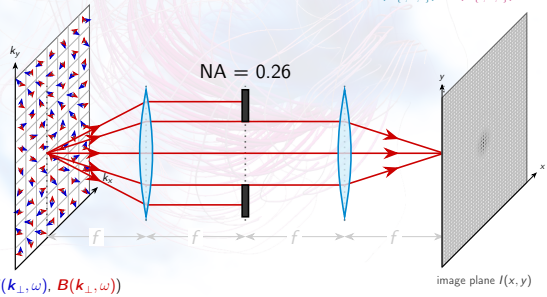
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$\boldsymbol{A}(\boldsymbol{k}_{\{l,m,n\}})$

4D on-shell FT



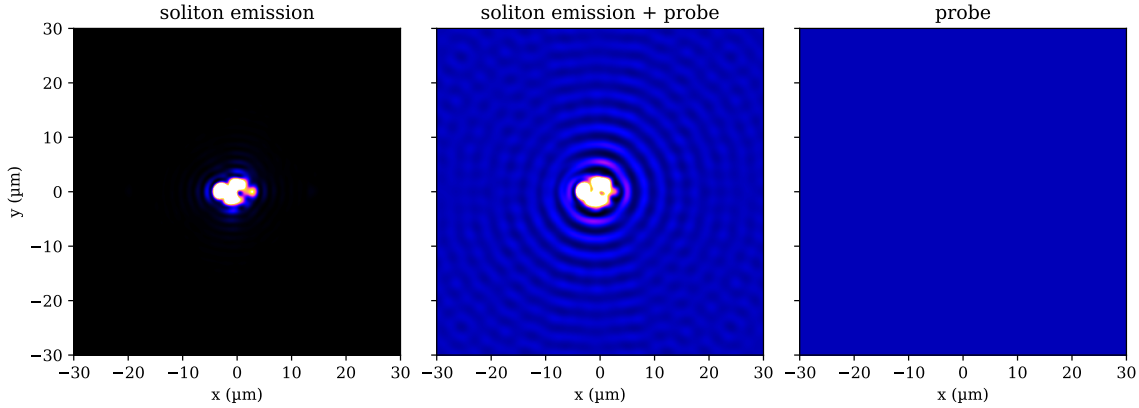
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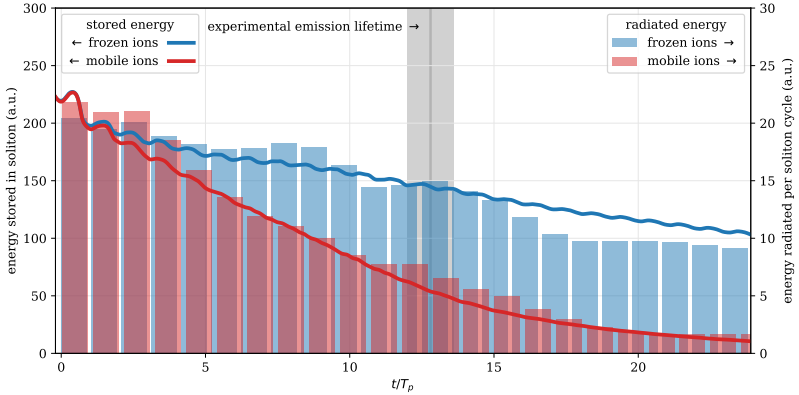


$\mathcal{U}_{\text{img}}(\boldsymbol{x}_\perp, \omega_{\text{pr}})$
$\downarrow$
$I(x,y)$

Probe Images Results



Energy Stored and Radiation Intensity



$$I \propto \mathcal{E}_{stored} = \frac{m}{q} \int d^3x\, \rho(x) \left(\sqrt{\frac{\rho^2}{\rho^2 - \boldsymbol{j}^2}} - 1 \right) + \int d^3x\, \frac{\boldsymbol{E}^2 + \boldsymbol{B}^2}{2}$$

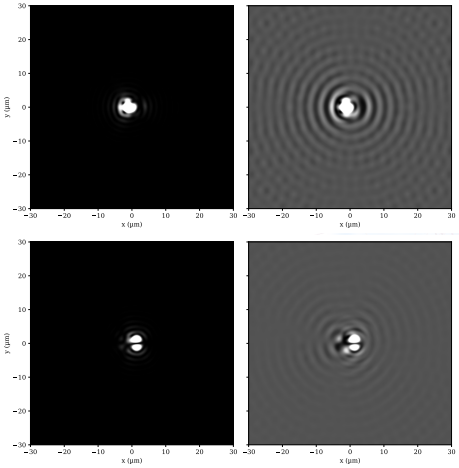
What's Next?

How to improve experimental method?

- More frames & longer delay window?

$\Rightarrow$ Cover complete lifetime in single shot

What's Next?



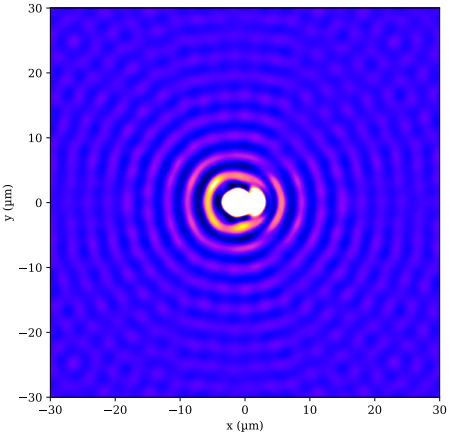
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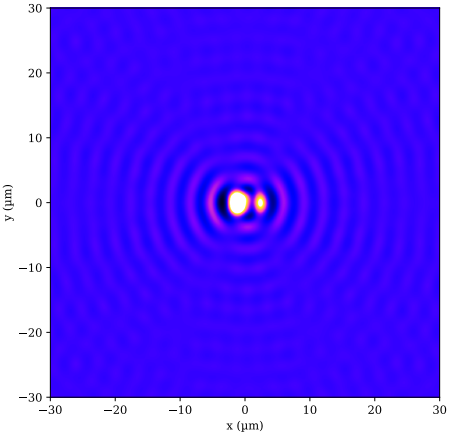
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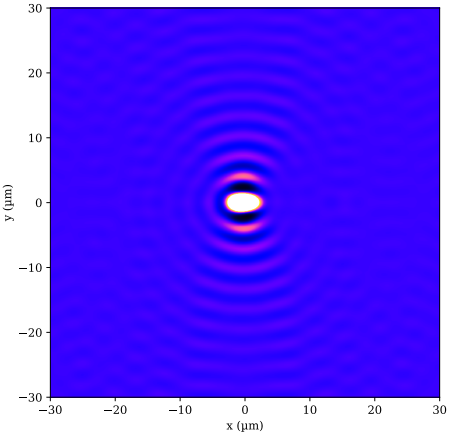
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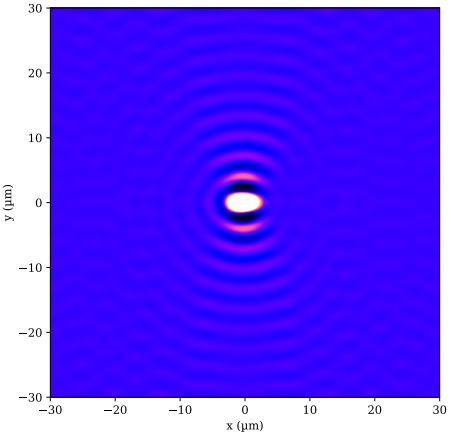
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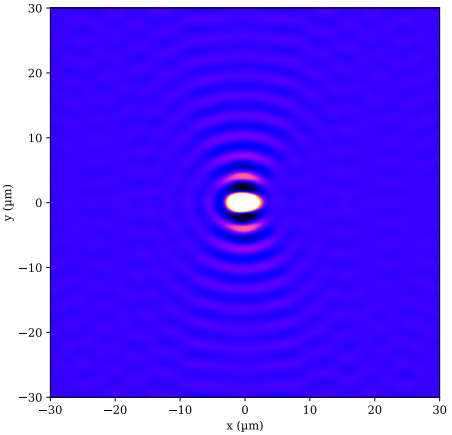
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Howard HPLSE 2023

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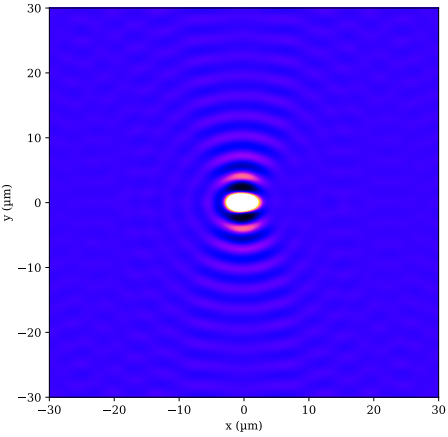
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- Online probe diagnostic?

⇒ Self-consistent impact of plasma on probe light

What's Next?



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... and theory?

- Better understanding of formation mechanisms?
- ...

Conclusions and Outlook

- PIC simulations of relativistic solitons
- Synthetic probe images with `plasmaglow` reproduce experimental results
- Great agreement with JETi200 experiment
- **Synergy between theory and experiment in LPA and HFS**

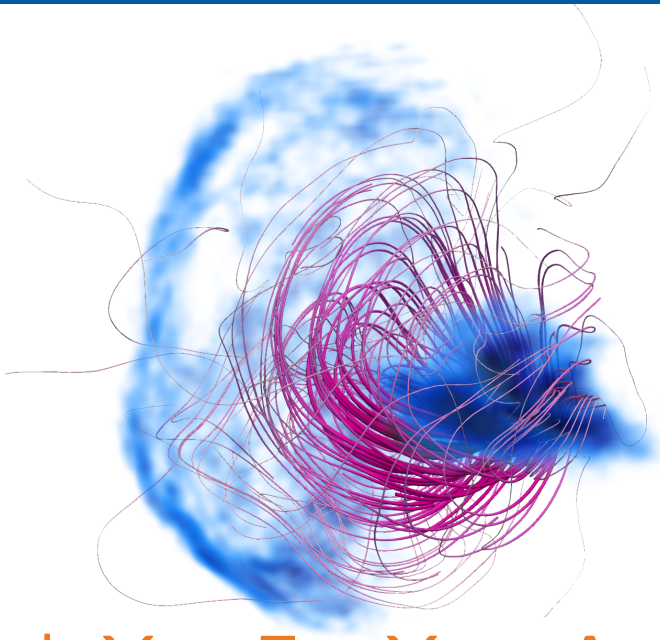
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Applications of multi-frame soliton imaging?

- Diagnostic for LPA? Can it help monitor stable LWFA operation?
- Soliton formation as signal for nonlinear bubble evolution?
- Non-perfectly matched laser focal spot $\rightarrow$ filamentation?
- Self-truncated ionization injection?
- Efficient staging: drive laser pulses near depletion limit

$\rightarrow$ Solitons near back end of guiding channel?



Thank You For Your Attention!