QCD with one Flavor and the Sign Problem at Fixed θ Angle

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Acknowledgments

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Tilo Wettig (Regensburg)

Relevant Papers

L. Ravagli and J.J.M. Verbaarschot, QCD in One Dimension at Nonzero Chemical Potential, Phys. Rev. D76 (2007) 05406 [arXiv:0704.1111]

J.C. Osborn K. Splittorff and J. J. M. Verbaarschot, Chiral Symmetry Breaking and the Dirac Spectrum at Nonzero Chemical Potential, Phys. Rev. Lett. 94 (2005) 202001 [arXiv[hep-th/0501210]].

J.J.M Verbaarschot and T. Wettig, The Spectrum of the Dirac Operator for QCD with on flavor at fixed θ angle, (2014), in preparation.

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I.One Flavor QCD

Microscopic Spectral Density

Chiral Condensate at $\theta = 0$

Sign Problem

QCD for $N_f = 1$

Partition Function

$$Z = e^{mV\Sigma}.$$

The chiral condensate is constant as a function of m

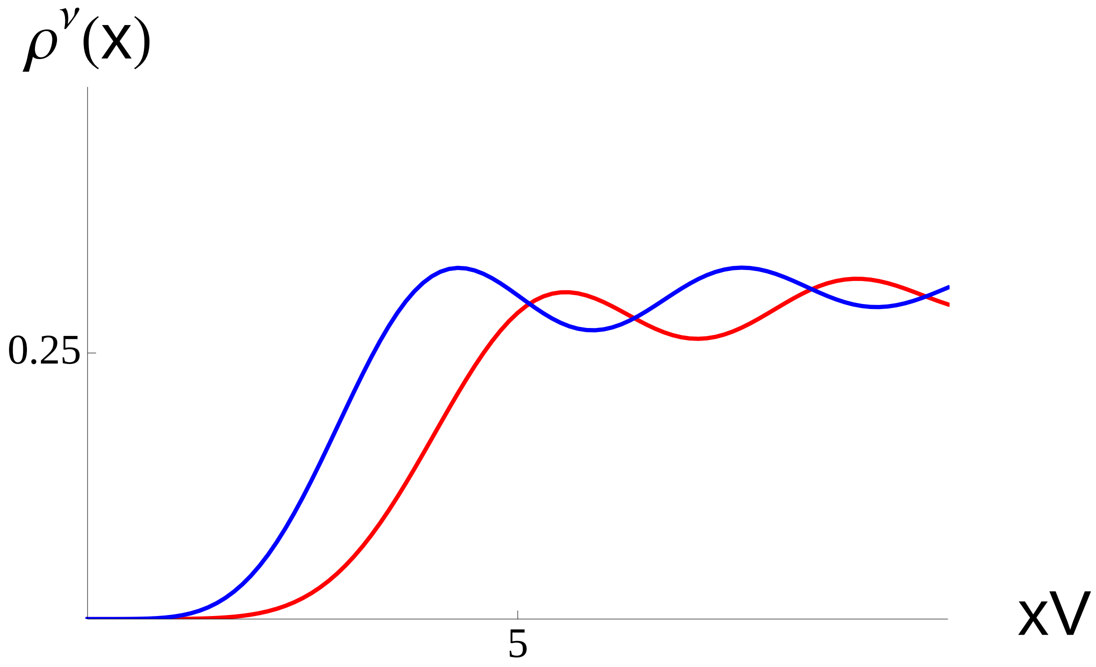
$$\langle \bar{q}q \rangle = \frac{1}{V}\frac{d}{dm}\log Z = \Sigma.$$

The spectral density in the ϵ domain is given by

$$\begin{aligned} \rho_\nu(x) &= \frac{x}{2}(J_\nu^2(x) - J_{\nu+1}(x)J_{\nu-1}(x)) + |\nu|\delta(x) \\ . &+ \frac{m}{m^2+x^2}\left[mJ_\nu(x)J_{\nu+1}(x) - x\frac{I_{\nu+1}(m)}{I_\nu(m)}J_\nu(x)\right]. \end{aligned}$$

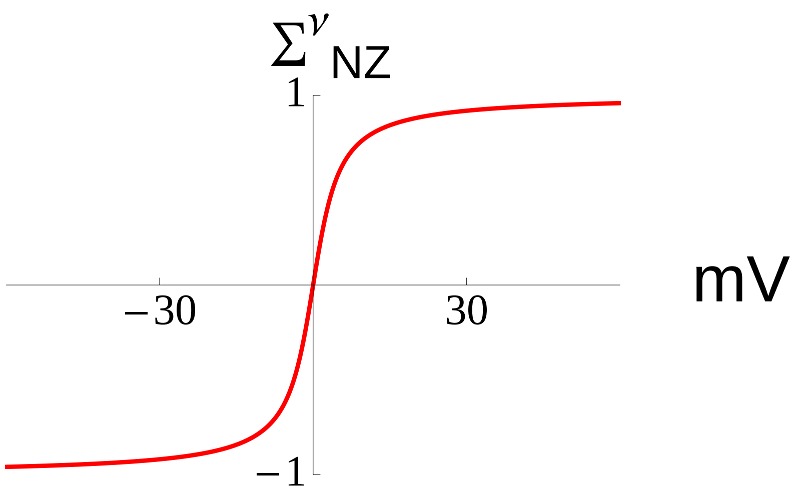
Damgaard-Osborn-Toublan-JV-1999

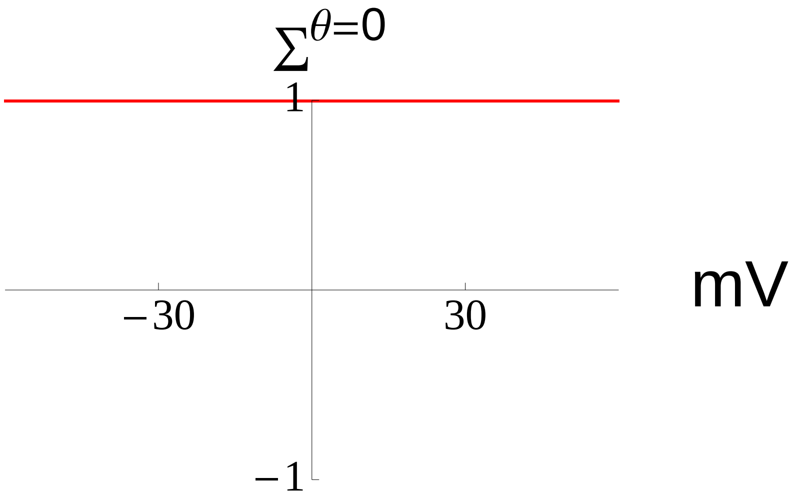
Microscopic Spectral Density at fixed ν



The one microscopic spectral density for $\nu = 2$ and $mV = 1$ (red) compared to the quenched result for $\nu = 2$ (blue).

Chiral Condensate for $N_f = 1$





Chiral condensate due to the nonzero modes for $\nu = 2$

$$\Sigma^{\nu}_{NZ}(m) = \Sigma^{\nu}(m) - \frac{|\nu|}{mV}.$$

Leutwyler-Smilga-1992

Chiral condensate at $\theta = 0$

$$\Sigma(m, \theta = 0) = \frac{\sum_{\nu=-\infty}^{\infty} Z_\nu(m) \Sigma^\nu(m)}{\sum_{\nu=-\infty}^{\infty} Z_\nu(m)}.$$

This condensate follows from the spectral density at $\theta = 0$

$$\rho(m, \theta = 0) = \frac{\sum_{\nu=-\infty}^{\infty} Z_\nu(m) \rho^\nu(m)}{\sum_{\nu=-\infty}^{\infty} Z_\nu(m)}.$$

Can be evaluated numerically in the ϵ domain of QCD.

Damgaard-1999, Kanazawa-Wettig-2011

For $m < 0$, the negative values of $\Sigma^\nu(m)$ should average to a positive number. This is only possible if the weight $Z_\nu(m)$ is not positive definite.

Sign Problem for QCD with $N_f = 1$

Because

$$\det(D+m) = m^\nu \prod_k (\lambda_k^2 + m^2).$$

QCD at $\theta = 0$ has a sign problem for $m < 0$.

Magnitude of the Sign Problem for QCD with $N_f = 1$

Partition function at $\theta = 0$

$$Z_{QCD}(m) = \sum_{\nu=-\infty}^{\infty} I_\nu(mV\Sigma) = e^{mV\Sigma}.$$

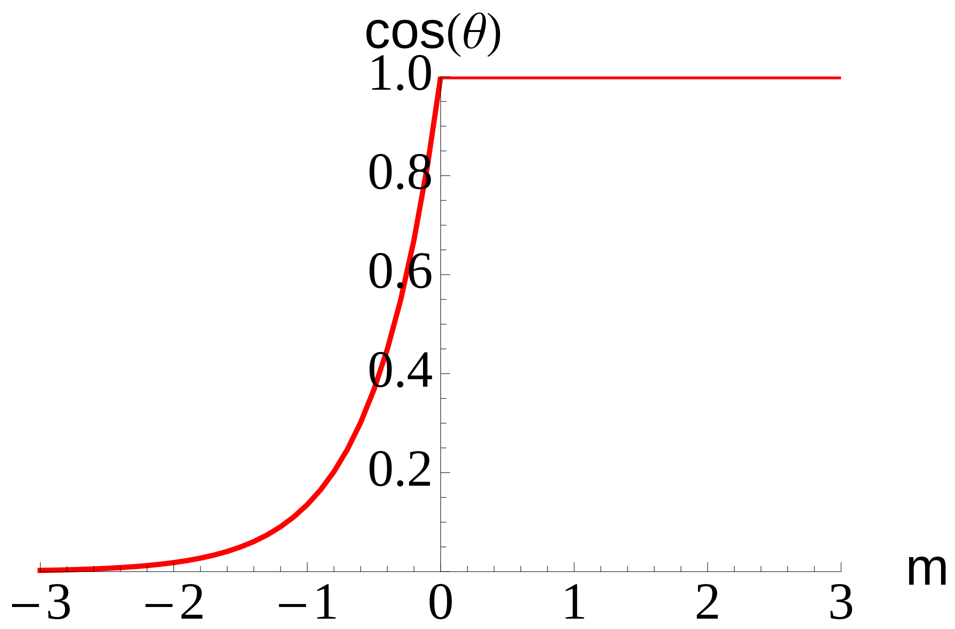
Phase quenched partition function

$$Z_{|QCD|}(m) = \sum_{\nu=-\infty}^{\infty} |I_\nu(mV\Sigma)| = e^{|m|V\Sigma}.$$

Average sign

$$\cos\theta = \frac{Z_{QCD}(m)}{Z_{|QCD|}(m)} = e^{(m-|m|)V\Sigma}.$$

Sign Problem for QCD with $N_f = 1$



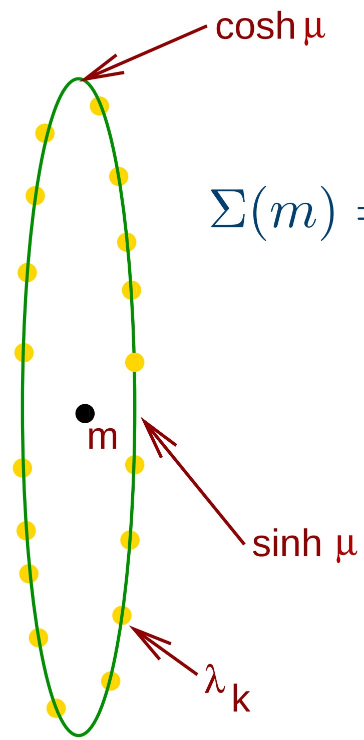
$$\cos\theta = \frac{Z_{QCD}(m)}{Z_{|QCD|}(m)} \approx \frac{Z_{\rm rooted}(m)}{Z_{\rm unrooted}(m)}.$$

II. One dimensional QCD at Nonzero Chemical Potential

Chiral Condensate

Sign Problem

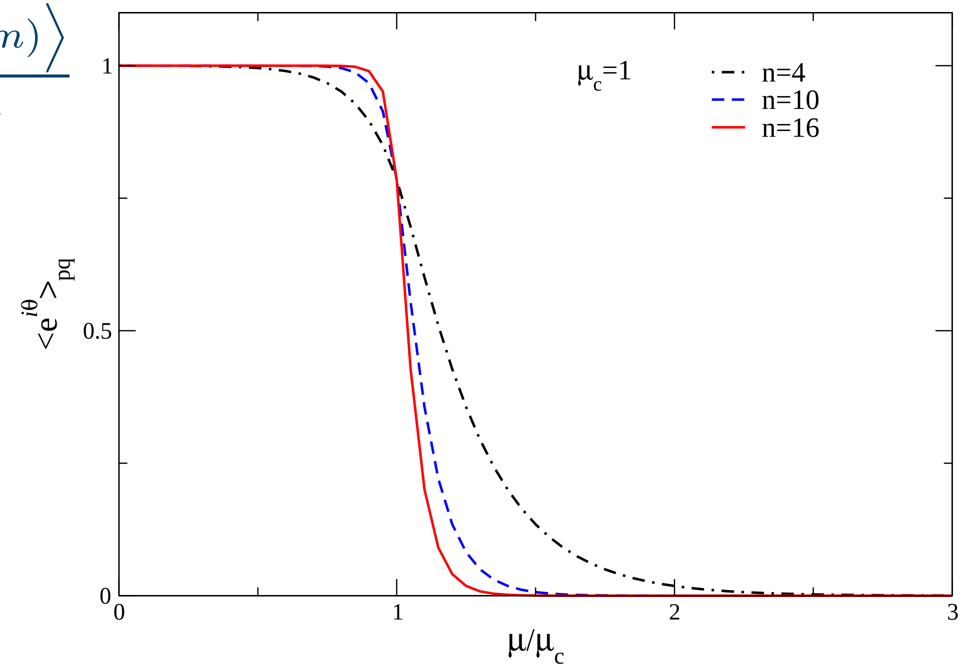
Chiral Condensate $U(1)$ QCD in 1d



Dirac spectrum of 1d QCD

$$\Sigma(m) = \frac{\left\langle \sum_k \frac{1}{\lambda_k + m} \prod_k (\lambda_k + m) \right\rangle}{\left\langle \prod_k (\lambda_k + m) \right\rangle}$$

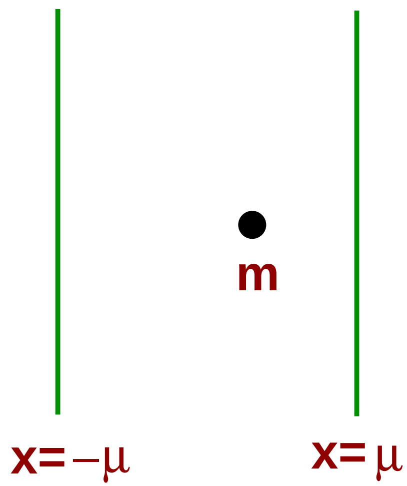
determinant with
a complex phase

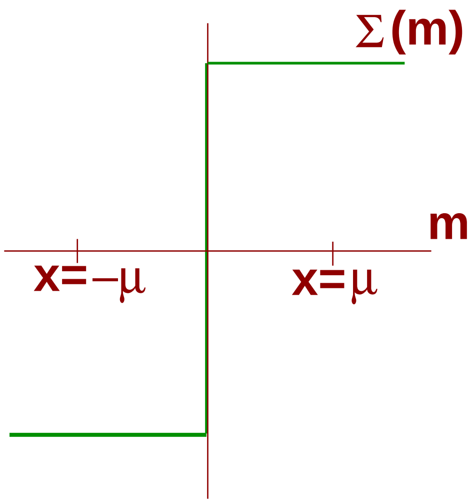


Ravagli-JV-2007, Aarts-Splittorff-2010

Eigenvalues are equally spaced on an ellipse with a random overall phase.

Chiral Condensate for $U(1)$ QCD in 1d





The chiral condensate is continuous across the ellipse where the eigenvalues are located.

In the limit of a dense spectrum, $\Sigma(m)$ is discontinuous across the imaginary axis despite the fact that there are no eigenvalues for $\mu \neq 0$.

III. Silver Blaze Problem

Silver Blaze Problem

Spectral Density

Solution of Silver Blaze Problem

Silver Blaze Problem

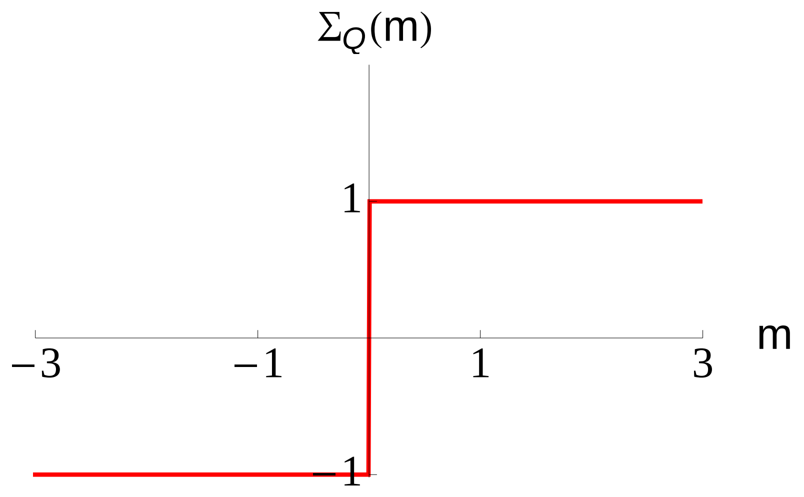
- The spectrum of the Dirac operator for one flavor QCD at fixed topological charge is as if chiral symmetry is broken spontaneously.
- In particular, in the thermodynamic limit, the chiral condensate has a discontinuity when the mass crosses the line of eigenvalues.
- General arguments show that the chiral condensate at $\theta = 0$ does not have a discontinuity.
- One flavor QCD has a sign problem for $m < 0$.
- What is the solution of the “Silver Blaze Problem”?

The OSV Mechanism

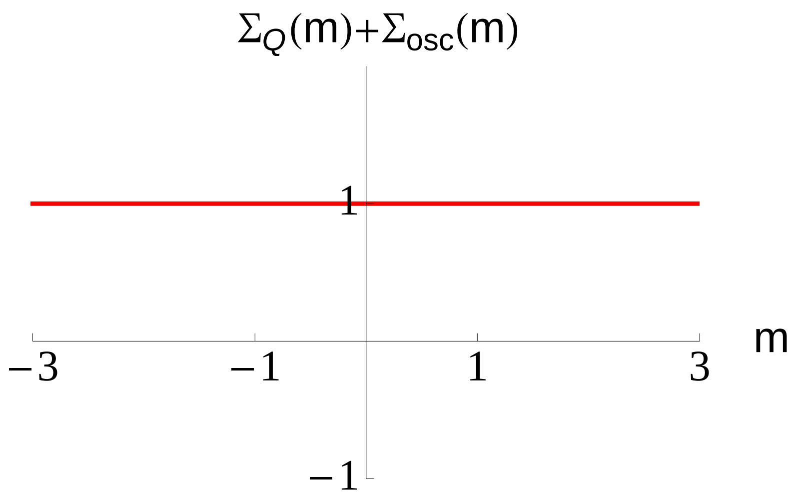
- This mechanism makes it possible to obtain a chiral condensate that does not change when the mass crosses a line or area of eigenvalues.
- For a positive definite eigenvalue density this is not possible according to the Banks-Casher formula.
- When the eigenvalue density is not positive definite (due to the fermion determinant), the OSV mechanism replaces the Banks-Casher formula.

Let us see how it can work.

How to get a Constant Chiral Condensate?



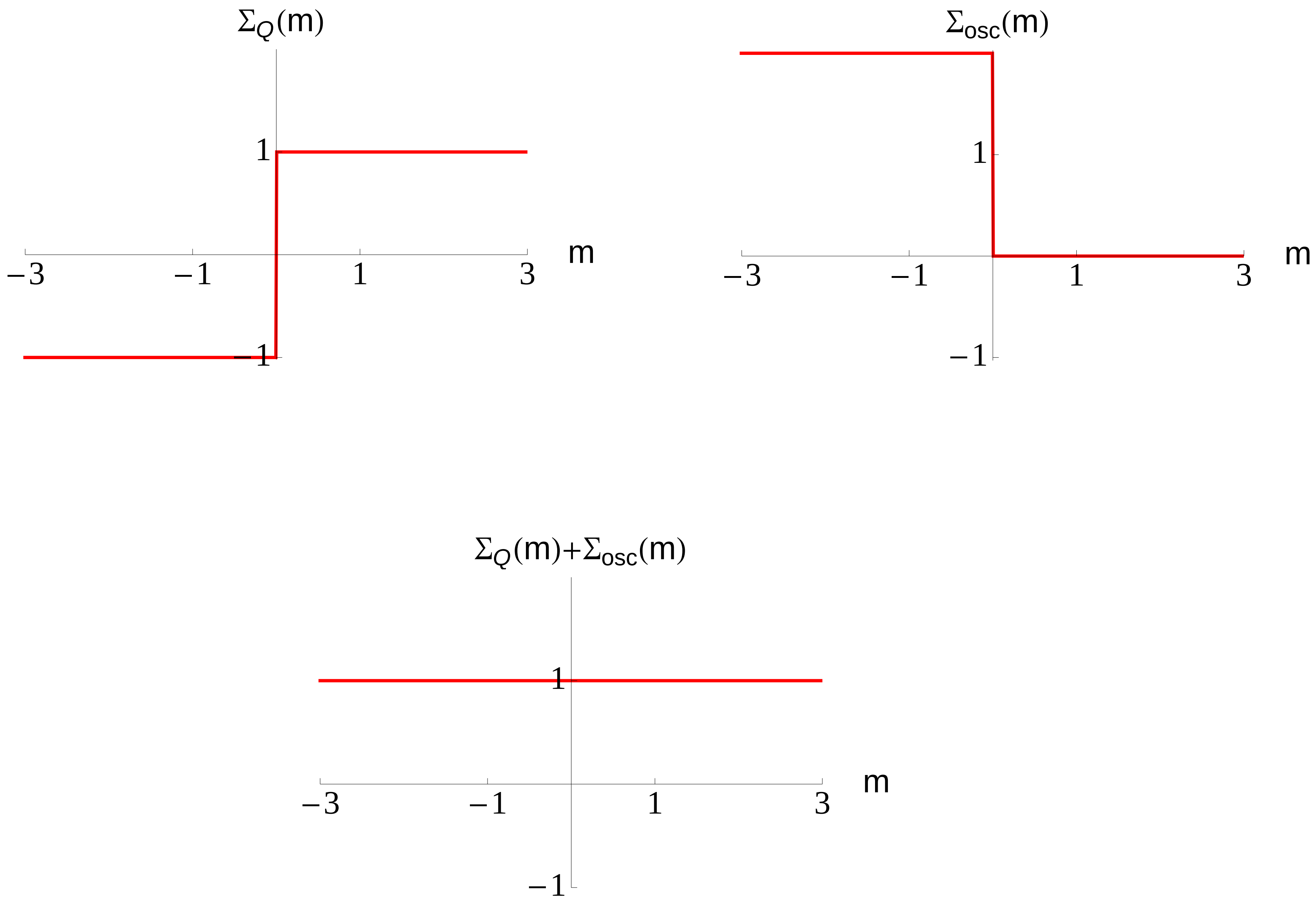
Behavior of the quenched chiral condensate due to a line of eigenvalues.



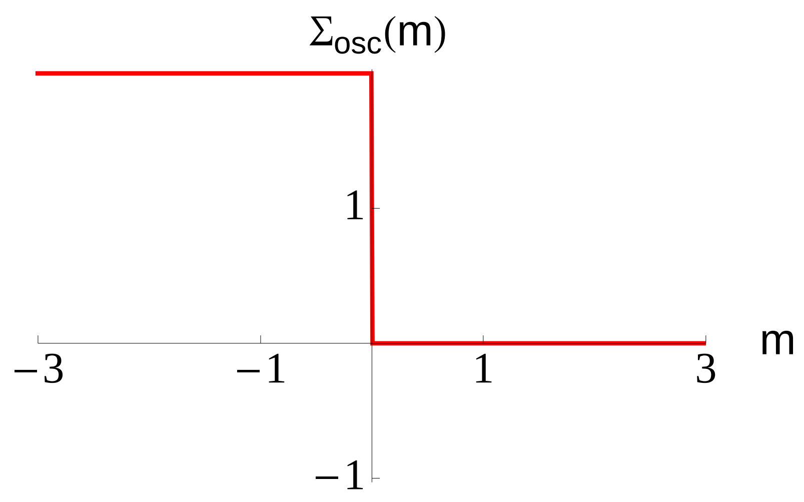
Behavior of the chiral condensate due to a line of eigenvalue when the not positive definite determinantal measure is included.

This implies that the not positive definite measure should give a correction to the spectral density that results in a mass dependence of the chiral condensate given by $\Sigma_{\rm osc}(m) = 2\theta(-m)$

OSV Mechanism in Pictures



How can this be Generated by a Spectral Density?



$$2\theta(-m) = \int d\lambda \frac{\rho_{osc}(\lambda, m)}{i\lambda - m}.$$

What is $\rho_{\rm osc}(\lambda, m)$?

Hint,

$$\theta(m) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\tau \frac{e^{im\tau + m\epsilon}}{\tau - i\epsilon}.$$

Solution satisfying $\rho(\lambda) = \rho(-\lambda)$

$$\rho_{\rm osc}(\lambda, m) = \frac{1}{\pi}(e^{iV\lambda - Vm} + e^{-iV\lambda - Vm})$$

$$\int_{-\infty}^{\infty} d\lambda \frac{1}{i\lambda - m}\frac{1}{\pi}(e^{V(i\lambda - m)} + e^{V(i\lambda - m)}) = 2\theta(-m) - 2\theta(m)e^{-2Vm}$$

We conclude that in the thermodynamic limit, if the condensate does not change over a line with average eigenvalue spacing $\sim \pi/V$, the spectral density satisfying $\rho(\lambda) = \rho(-\lambda)$ is given by

$$\rho(\lambda, m) = \frac{1}{\pi}(1 - e^{V(i\lambda - m)} - e^{-V(i\lambda + m)}).$$

This is a completely general result that does not rely on approximations.

What is the Most General Class of Spectra?

What we need is a spectral density that gives rise to a chiral condensate that behaves as

$$\Sigma(m) \to \begin{cases} 0 & \text{for} \quad m \to \infty \\ 2 & \text{for} \quad m \to -\infty \end{cases}$$

You can check that

$$\rho(x,m) = -\frac{4}{\pi}\frac{x^2}{x^2+m^2}\int_0^1 \frac{tdt}{\sqrt{1-t^2}} e^{-2mVt^2} J_1(2xVt)$$

gives rise to a chiral condensate with this property.

IV. One Dimensional QCD at Nonzero Chemical Potential

Spectral Density

Chiral Condensate

Spectral Density for 1d QCD

For large V and small μ the eigenvalues of the Dirac operator are located on two parallel lines $x \pm \mu$ resulting in the spectral density and the chiral condensate

$$\begin{aligned}\Sigma(m) &= \int \frac{dxdy}{2\pi} \frac{1}{m-x-iy} \underbrace{\delta(|x|-\mu)\left[1-\frac{(e^{V(x+iy)}+e^{-V(x+iy)})}{e^{Vm}+e^{-Vm}}\right]} \\ &= \tanh(Vm). \qquad\qquad \rho(x,y) \quad \text{for} \quad N_f = 1\end{aligned}$$

In the thermodynamic limit $(V \to \infty)$ this results in a discontinuity across $m = 0$, but not at $m \pm \mu$. Osborn-Splittorff-JV-2005, Ravagli-JV-2008

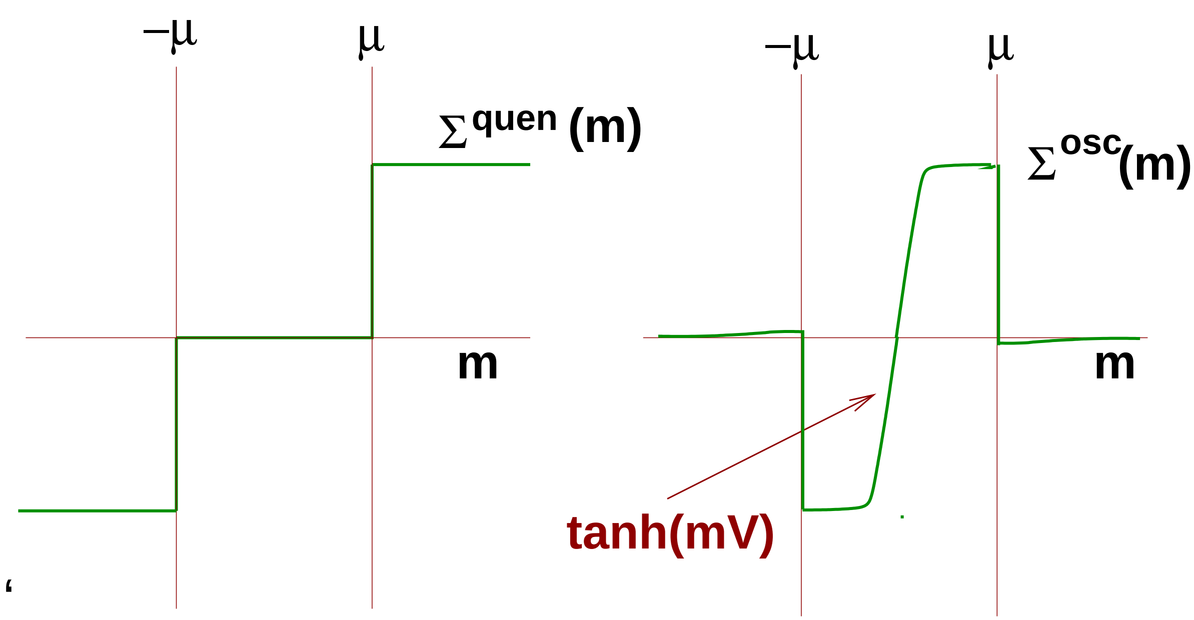
Chiral Condensate in 1d

The first term ($\sim \delta(|x| - \mu)$) gives the quenched contribution

$$\Sigma^{\rm quenched}(m) = {\rm sign}(m - \mu) + {\rm sign}(-m + \mu).$$

This follows from electrostatic arguments with eigenvalues as charges.
The second term is evaluated as

$$\Sigma^{\rm osc}(m) = \tanh(mV) - {\rm sign}(m - \mu) - {\rm sign}(-m + \mu).$$



The chiral condensate becomes discontinuous in the continuum limit.

Ravagli-JV-2007

V. One Flavor QCD

Spectral Density

Zero Modes

Solution of Silver Blaze Problem

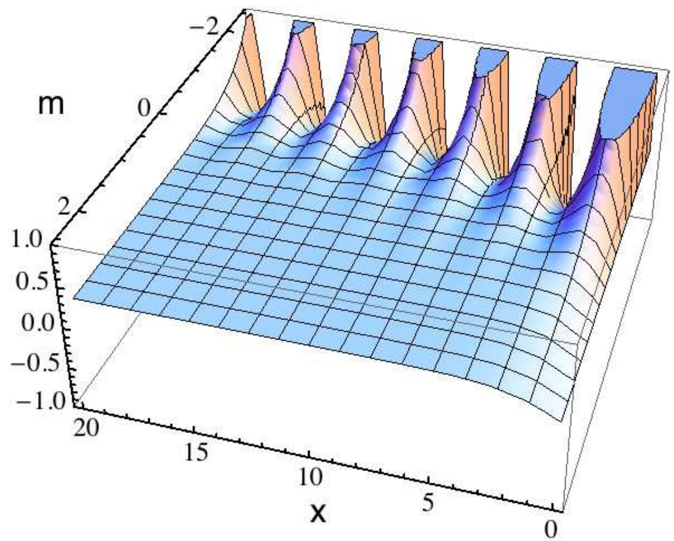
Another Possibility for $N_f = 1$ QCD

- It has been argued that the spectral density for $N_f = 1$ at $\theta = 0$ vanishes around zero so the chiral constant remains continuous in the thermodynamic limit. Creutz-2007
- To confirm if the OSV mechanism holds we have to calculate the spectral density of the Dirac operator for $N_f = 1$ QCD.
- Actually this can be done analytically in the ϵ domain of QCD. The result can be expressed as a simple one dimensional integral.

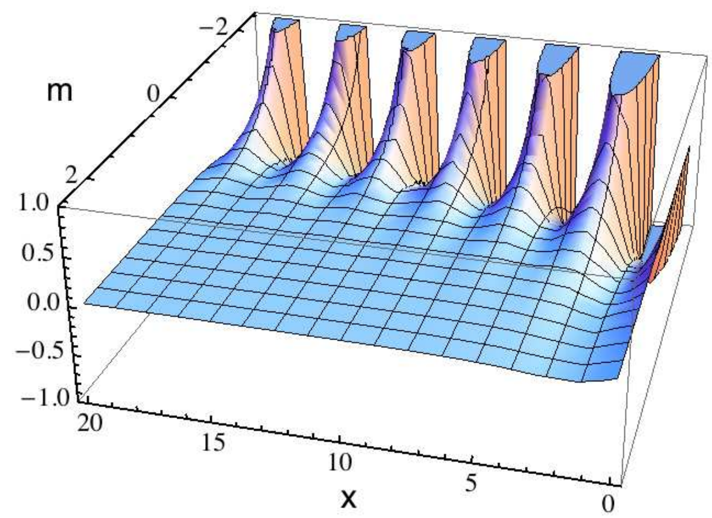
We decompose

$$\begin{aligned} \rho^\nu(x,m) &= \rho_q^\nu(x) + \rho_d^\nu(x,m), \\ \rho(x,m) &= \rho_q(x,m) + \rho_d(x,m). \end{aligned}$$

The Dirac Spectrum for $N_f = 1$ at $\theta = 0$



The quenched part of the spectral density at $\theta = 0$, $\rho_q(x, m)$.



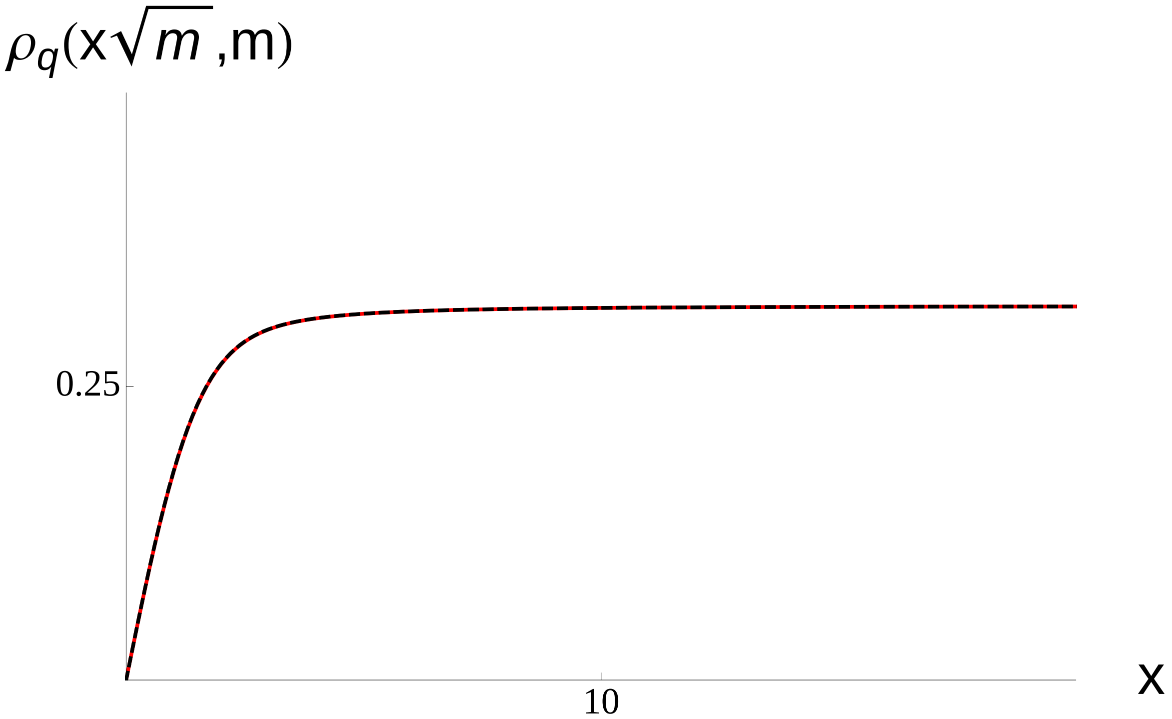
The dynamical part of the spectral density at $\theta = 0$, $\rho_d(x, m)$.

Analytical result

$$\rho_q(x, m) \quad = \quad \frac{1}{\pi} \int_0^1 \frac{dt}{t\sqrt{1-t^2}} e^{-2mVt^2} J_1(2xVt).$$

and similar expressions for the contribution of the dynamical quarks.

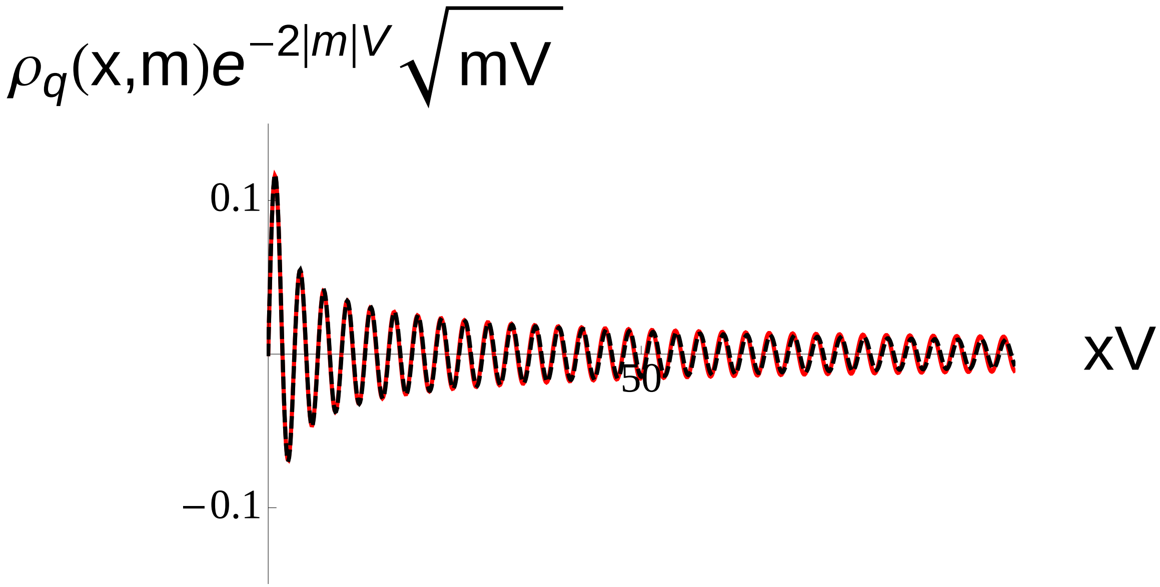
Asymptotic Scaling for $m > 0$



$$\rho_q(x\sqrt{m}, m > 0) \sim \frac{xV}{\sqrt{2\pi m V}} e^{-Vx^2/4m}[I_0(Vx^2/4m) + I_1(Vx^2/4m)]$$

Wettig-JV-2014

Asymptotic Scaling for $m < 0$



$$\rho_q(x, m < 0) \sim \frac{e^{2|m|V}}{\sqrt{8\pi|m|V}} J_1(2xV).$$

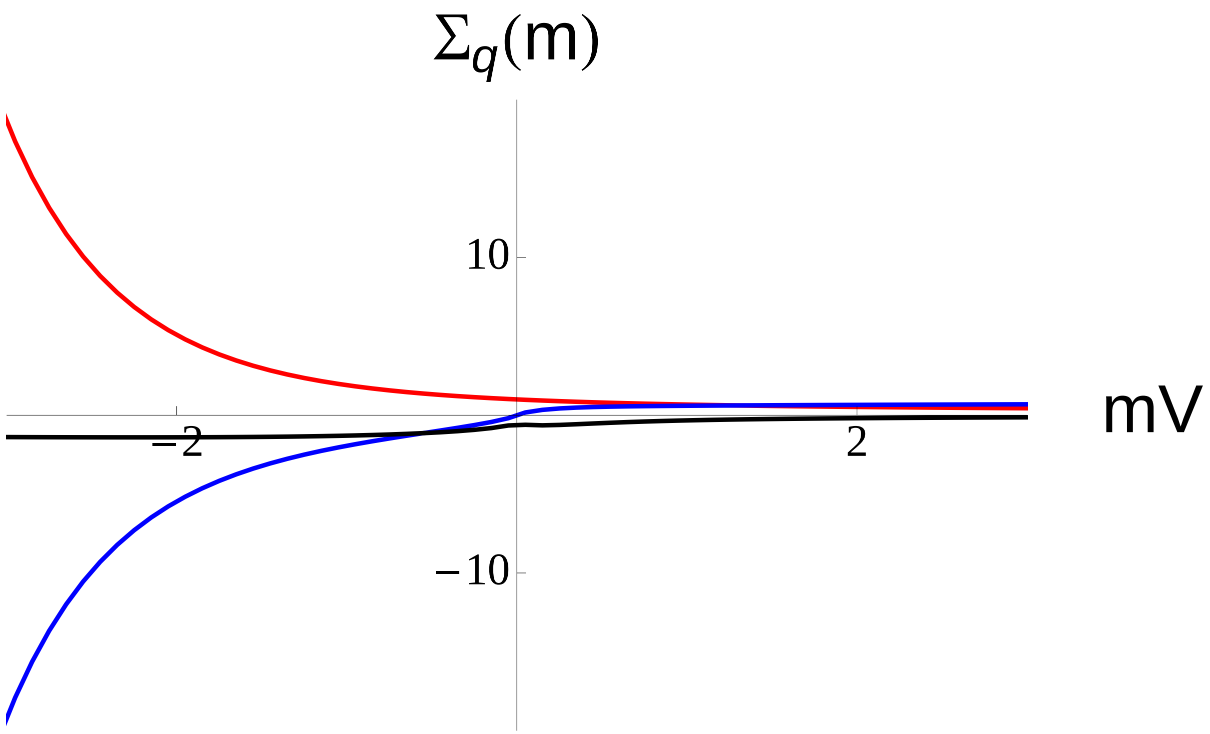
Wettig-JV-2014

Contribution from zero modes

Leutwyler-Smilga-1992

$$\rho_{ZM}(x,m) \quad = \quad e^{|m|V}\sum_{\nu}|\nu|I_\nu(mV)\delta(x) = e^{-mV}(I_0(mV)+I_1(mV))\delta(x)$$

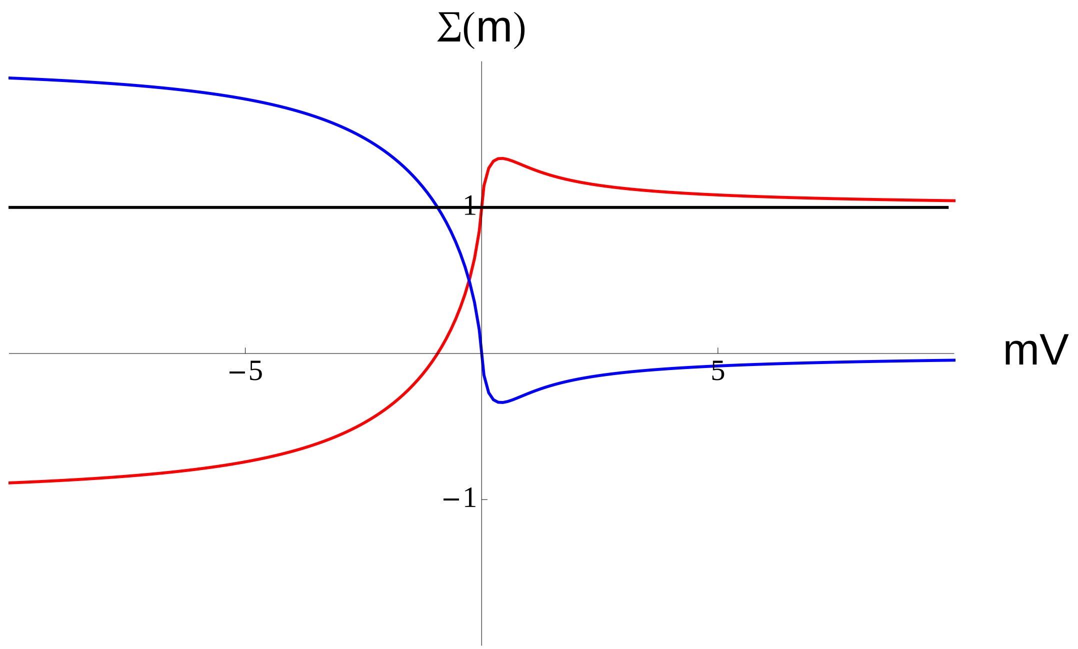
Contribution of Zero Modes



The exponentially large contribution of the zero modes is canceled by the same contribution from the quenched part of the spectral density.

Kanazawa-Wettig-2011

Solution of the Silver Blaze Problem



Chiral condensate as a function of the quark mass m . The red curve shows the chiral condensate due to the quenched part of the spectral density, while the blue curve represents the condensate due to the oscillating part. The black curve is the sum of the two.

V. Conclusions

- Solution of the Silver Blaze problem requires an oscillating spectral density with period $\sim 1/V$ and an amplitude that grows exponentially with the volume.

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XQCD at Stony Brook, June 19-21

Stony Brook, 19-21 June 2014
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XQCD 2014

Stony Brook University

Aims & Scope

eXtreme QCD (XQCD) is a series of international workshop-style conferences, held annually, which aims to cover recent advances in the theory and phenomenology of QCD under extreme conditions of temperature and/or baryon density, together with related topics. This year's meeting is co-organised by Stony Brook University and Brookhaven National Laboratory.

Topics

Further Information

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