# Dual variables approach to finite density lattice field theories

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# Abelian gauge Higgs systems at finite density

## *Dual representation for the charged scalar field*

- Lattice action: $\quad ( \, \phi_x \in \mathbb{C} \, , \quad M^2 = 8 + m^2 \, )$

$$S = \sum_x \Big[ \, M^2 \, |\phi_x|^2 \; + \lambda \, |\phi_x|^4 \, \Big] - \sum_{x,\nu} \Big[ e^{-\mu \, \delta_{\nu 4}} \, \phi_x^\star \, \phi_{x+\widehat{\nu}} \; + \; e^{\mu \, \delta_{\nu 4}} \, \phi_x \, \phi_{x+\widehat{\nu}}^\star \Big]$$

- Expand the nearest neighbor terms of $e^{-S}$:

$$\prod_{x,\nu} \exp \left( e^{-\mu \, \delta_{\nu 4}} \, \phi_x^\star \, \phi_{x+\widehat{\nu}} \right) \; = \; \prod_{x,\nu} \sum_{j_{x,\nu}=0}^{\infty} \frac{\left(e^{-\mu \, \delta_{\nu 4}}\right)^{j_{x,\nu}}}{j_{x,\nu} \, !} \; \left(\phi_x^\star\right)^{j_{x,\nu}} \; \left(\phi_{x+\widehat{\nu}}\right)^{j_{x,\nu}}$$

- The $j_{x,\nu}$ are the new, "dual" degrees of freedom.

## *Dual representation - integrating out the fields*

- Integral over $\phi_x$ at site $x$: $\quad$ ($\Sigma_j, \overline{\Sigma}_j$ are sums of $j_{y,\nu}$ connected to $x$)

$$\int_{\mathbb{C}} d\,\phi_x \;\; e^{-M^2|\phi_x|^2 - \lambda|\phi_x|^4} \, (\phi_x)^{\Sigma_j} \, (\phi_x^\star)^{\overline{\Sigma}_j} \;\; =$$

- Polar coordinates $\phi_x = r e^{i\theta}$ to separate radial and U(1) parts (symmetry):

$$\int_0^\infty dr \; r^{\,\Sigma_j + \overline{\Sigma}_j + 1} \; e^{-M^2 r^2 - \lambda r^4} \int_{-\pi}^{\pi} d\theta \; e^{\,i\theta\,(\Sigma_j - \overline{\Sigma}_j)} \;\; = \;\; \mathcal{I}(\Sigma_j + \overline{\Sigma}_j) \; \delta(\Sigma_j - \overline{\Sigma}_j)$$

- At every site there is a weight factor $\mathcal{I}(\Sigma_j + \overline{\Sigma}_j)$ and a constraint.

- The constraint $\delta(\Sigma_j - \overline{\Sigma}_j)$ enforces vanishing $j$-flux at each site.

## *Dual representation – final form*

- The original partition function is mapped exactly to a sum over configurations of the dual variables $k_{x,\nu} \in \mathbb{Z}$ and $l_{x,\nu} \in \mathbb{N}_0$. $k_{x,\nu}$ and $l_{x,\nu}$ are linear combinations of the original $j$:

$$Z = \sum_{\{k,l\}} \mathcal{W}(k,l)\, \mathcal{C}(k)$$

- Weight factor from radial d.o.f. and combinatorics:

$$\mathcal{W}(k,l) = \prod_{x,\nu} \frac{e^{-\mu\, k_{x,4}\, \delta_{\nu,4}}}{(|k_{x,\nu}| + l_{x,\nu})!} \prod_x \mathcal{I}\Big( \sum_\nu \Big[ |k_{x,\nu}| + |k_{x-\widehat{\nu},\nu}| + 2(l_{x,\nu} + l_{x-\widehat{\nu},\nu}) \Big] \Big)$$

- Constraint from integrating over the symmetry group:

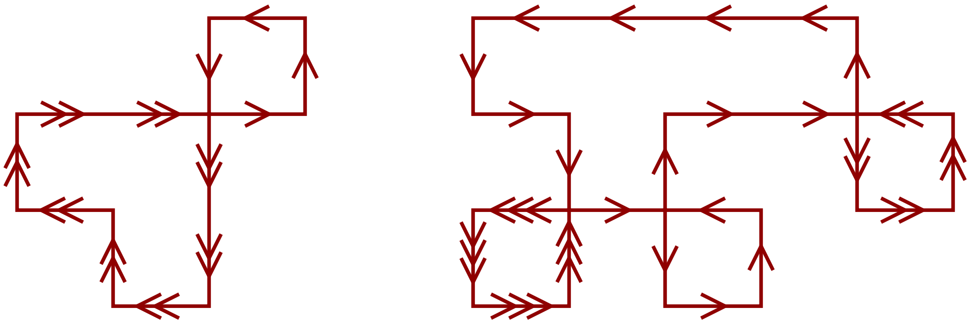
$$\mathcal{C}(k) = \prod_x \delta\Big( \sum_\nu \left[ k_{x,\nu} - k_{x-\widehat{\nu},\nu} \right] \Big)$$

## *Admissible configurations are loops:*

- Constraint from integrating over the symmetry group:

$$\forall\, x \;:\quad \sum_{\nu} \left[\, k_{x,\nu} - k_{x-\widehat{\nu},\nu} \,\right] \;=\; 0 \qquad\qquad (\, \vec{\nabla}\vec{k} \;=\; 0\,)$$

- Admissible configurations of dual variables are oriented loops of flux:



- Finite $\mu$: Different weight for forward and backward temporal flux.

## *Adding gauge fields: U(1) gauge Higgs model*

- Nearest neighbor terms with link variables $U_{x,\nu} \in$ U(1):

$$\exp\left(\phi_x^\star \, U_{x,\nu} \, \phi_{x+\widehat{\nu}}\right) \;=\; \sum_{j_{x,\nu}=0}^{\infty} \frac{\left( U_{x,\nu} \right)^{j_{x,\nu}}}{j_{x,\nu}\,!} \; (\phi_x)^{j_{x,\nu}} \; (\phi_{x+\widehat{\nu}}^\star)^{j_{x,\nu}}$$

$\Rightarrow$ Matter loops are dressed with gauge transporters.

- Expanding the plaquette terms ...

$$e^{\beta \, U_{x,\rho} \, U_{x+\hat{\rho},\sigma} \, U^\star_{x+\hat{\sigma},\rho} \, U^\star_{x,\sigma}} \;=\; \sum_{p_{x,\rho\sigma}} \frac{\beta^{\, p_{x,\rho\sigma}}}{p_{x,\rho\sigma}\,!} \left[ U_{x,\rho} \, U_{x+\hat{\rho},\sigma} \, U^\star_{x+\hat{\sigma},\rho} \, U^\star_{x,\sigma} \right]^{p_{x,\rho\sigma}}$$

... new integer valued dual variables $p_{x,\rho\sigma}$ on the plaquettes.

- Integrating out the link variables gives rise to new constraints that connect matter flux and the dual plaquette variables.

## *Dual form of the partition function for the gauge Higgs model*

The partition sum is mapped exactly to a sum over loops and surfaces:

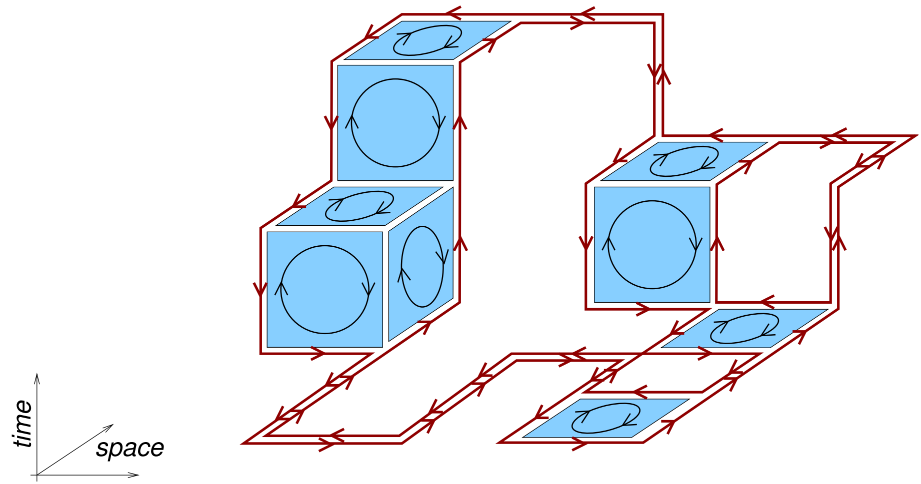
$$Z \;=\; \sum_{\{p,k,l\}} \mathcal{W}(p,k,l)\;\; \mathcal{C}(p,k)$$

- $\mathcal{W}$ positive weight factors.
- $\mathcal{C}$ constraints that turn the sum over configurations of dual variables into summing over surfaces and loops.

T. Sterling, J. Greensite, A. Patel, T. DeGrand, C. DeTar, M. Panero,
V. Azcoiti, E. Follana, A. Vaquero, G. Di Carlo, T. Korzec, U. Wolff ...

M. Endres, PRD 75, 2007
C. Gattringer, A. Schmidt, PRD 86, 2012
T. Korzec and U. Wolff, NPB 871, 2013
Y. Delgado Mercado, C. Gattringer, A. Schmidt, Comp. Phys. Comm. 184, 2013
P.N. Meisinger, M. Ogilvie, arXiv:1306.1495
Y. Delgado Mercado, C. Gattringer, A. Schmidt, PRL 111, 2013

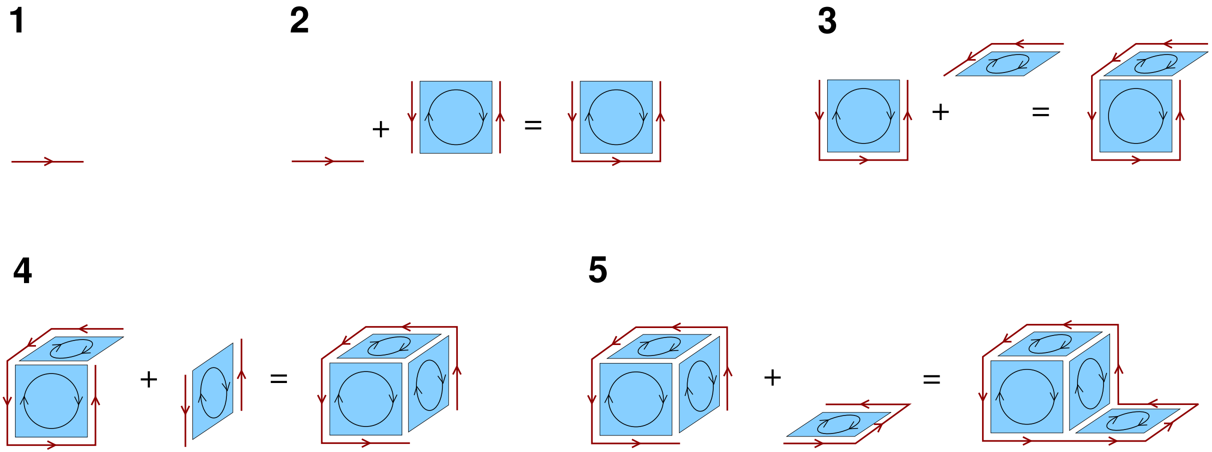
## *An admissible configuration for dual U(1) gauge Higgs theory:*




Chemical potential favors flux forward in time. (2 flavors)

## *Generalized worm algorithm for gauge Higgs systems:*

Worm starts by inserting a unit of matter flux. Adding segments transports the defect across the lattice until the defect is healed in a final step.




Y. Delgado Mercado, C. Gattringer, A. Schmidt, Comp. Phys. Comm. 184, 2013

## *Bulk observables*

- Bulk observables are obtained as derivatives of the free energy with respect to the parameters.

- Example: Observables related to the particle number:

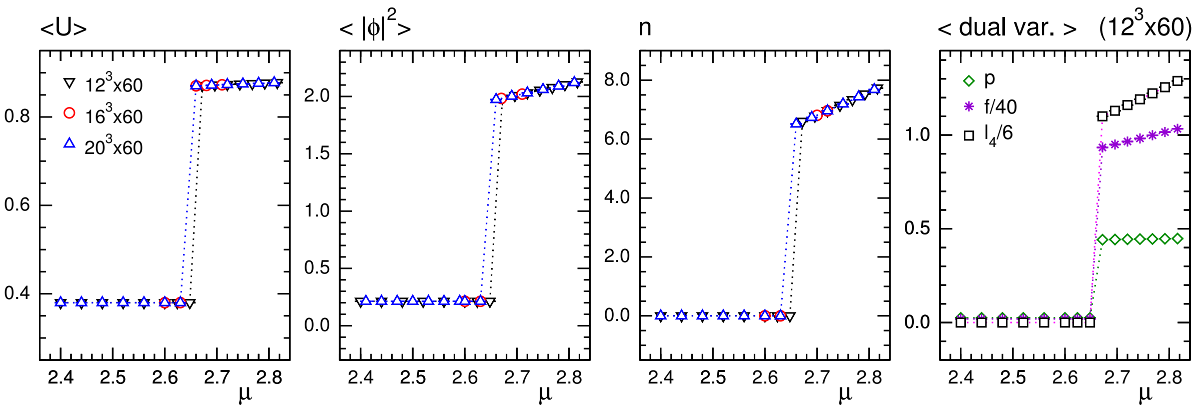
$$n \;=\; \frac{1}{N_s^3\, N_t} \frac{\partial \, \ln Z}{\partial \mu}$$

- Dual form: Particle number = temporal winding number of $k$-flux.

$$n \;=\; \frac{1}{N_s^3\, N_t} \left\langle \sum_x k_{x,4} \right\rangle$$

- Dual bulk observables are related to moments of the dual variables.

# *Observables in the confining phase at low T*

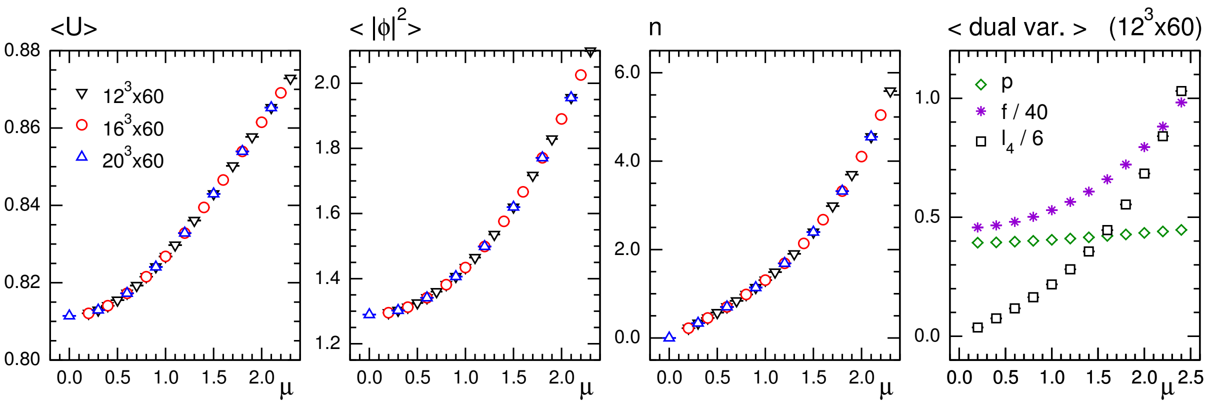



In the confining phase the dependence on the chemical potential $\mu$ sets in only when $\mu$ reaches the mass of the lowest excitation. "Silver Blaze behaviour"

The corresponding BEC is accompanied by a condensation of dual variables.

Y. Delgado Mercado, C. Gattringer, A. Schmidt, PRL 111, 2013

## *Observables in the Higgs phase at low T*




In the Higgs phase there is no mass gap and the non-trivial $\mu$-dependence starts at $\mu = 0$.

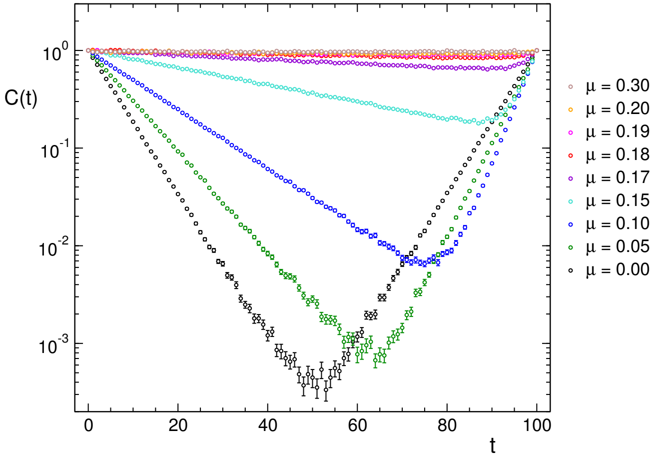
No condensation of dual variables.

Y. Delgado Mercado, C. Gattringer, A. Schmidt, PRL 111, 2013

## *Spectroscopy at finite density ⇒ Dual spectroscopy*

- Propagator: Use generalized ensemble with an additional open string.
- The 2-point function is obtained by sampling the positions of head and tail of the open string.

T. Korzec, I. Vierhaus, U. Wolff,
Comp. Phys. Comm. 182, 2011




Asymmetric propagation for $\mu < \mu_c \simeq 0.17$.
Slopes understood quantitatively.

C. Gattringer, T. Kloiber, PLB 720, 2013

# A different complex action problem: Gauge theory with a $\theta$-angle

In cooperation with Michael Müller-Preussker

## $\theta$-angle

Partition function with a $\theta$-angle:

$$Z(\theta) \; = \; \int D[U] D[\phi] \; e^{- S_g[U] \, - \, S_m[U,\phi] \, - \, i \, \theta \, Q[U]}$$

$Q[U]$ .... topological charge with $Q[U] \in \mathbb{Z}$

Complex action problem for $\theta \neq 0$

Here we consider scalar electrodynamics in 2 dimensions.
In that case the topological charge reads:

$$Q \; = \; \frac{1}{4\pi} \int d^2 x \; \varepsilon_{\mu\nu} \, F_{\mu\nu}(x)$$

## *Scalar $QED_2$ with a $\theta$-angle*

Partition function with a $\theta$-angle:

$$Z(\theta) \;=\; \int D[U] D[\phi] \; e^{-S_g[U] \,-\, S_m[U,\phi] \,-\, i\,\theta\, Q[U]}$$

$$S_g[U] = -\frac{\beta}{2} \sum_x \left[U_p(x) + U_p(x)^*\right] \quad , \quad U_p(x) \;=\; U_{x,1}\, U_{x+\hat{1},2}\, U^*_{x+\hat{2},1}\, U^*_{x,2}$$

$$S_m[U,\phi] = \sum_x \left[\, M^2\, |\phi_x|^2 \;+ \lambda\, |\phi_x|^4 \,\right] - \sum_{x,\nu} \left[\phi^\star_x\, U_{x,\nu}\, \phi_{x+\widehat{\nu}} \;+\; \phi_x\, U^*_{x,\nu}\, \phi^\star_{x+\widehat{\nu}}\right]$$

$$i\, Q[U] = \frac{1}{4\pi} \sum_x \left[U_p(x) - U_p(x)^*\right]$$

## *Dual representation*

- Partition function is a sum over $k_{x,\nu} \in \mathbb{Z}$, $l_{x,\nu} \in \mathbb{N}_0$ and $p_x \in \mathbb{Z}$.

$$Z = \sum_{\{k,l,p\}} \mathcal{C}(p,k)\, \mathcal{W}_m(k,l)\, \mathcal{W}_g(p)$$

- Constraints from integrating over the symmetry group:

$$\mathcal{C}(p,k) = \prod_x \delta\Big(\sum_\nu [\, k_{x,\nu} - k_{x-\widehat{\nu},\nu}\,]\Big)\, \delta\Big(p_x - p_{x-\hat{2}} + l_{x,1}\Big)\, \delta\Big(p_{x-\hat{1}} - p_x + l_{x,2}\Big)$$

- Weight factors:

$$\mathcal{W}_m(k,l) = \prod_{x,\nu} \frac{1}{(|k_{x,\nu}| + l_{x,\nu})!} \prod_x \mathcal{I}\Big(\sum_\nu \Big[\, |k_{x,\nu}| + |k_{x-\widehat{\nu},\nu}| + 2(l_{x,\nu} + l_{x-\widehat{\nu},\nu})\Big]\Big)$$

$$\mathcal{W}_g(p) = \prod_x I_{|p_x|}(\rho) \left(\frac{2\pi\beta - \theta}{2\pi\beta + \theta}\right)^{p_x/2} \quad , \quad \rho = 2\sqrt{\left(\frac{\beta}{2}\right)^2 - \left(\frac{\theta}{4\pi}\right)^2}$$

$\theta$-angle breaks the symmetry $p_x \leftrightarrow -p_x$

## *Summary*

- Complex action problems for scalar field theories and abelian gauge-Higgs models are resolved by mapping them to positive dual representations.

- Dual degrees of freedom are surfaces for gauge fields and loops for matter. Update with generalized worm algorithms.

- Bulk observables, phase diagram, 2-point functions ...

- Models with dual representation serve as test cases for other techniques. $\Rightarrow$ Talk by Ydalia Delgado Mercado

- Also the complex action problem caused by a $\theta$-angle is tractable.