Complex Langevin dynamics for SU(3) gauge theory with a θ term

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Pure Gauge Lagrangian of $SU(3)$:

$$\mathcal{L}_{PG} \; = \; -\frac{1}{2} F^a_{\mu\nu} F^a_{\mu\nu} \; - \; i \, \frac{\theta}{32\pi^2} \; F^a_{\mu\nu} \tilde{F}^a_{\mu\nu}$$

$$\tilde{F}^a_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^a_{\rho\sigma} \; ; \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g \; f^{abc} A^b_\mu A^c_\nu$$

where :

$$\int d^4x \, \frac{\theta}{32\pi^2} \; F^a_{\mu\nu} \tilde{F}^a_{\mu\nu} \; = \; Q_{top}$$

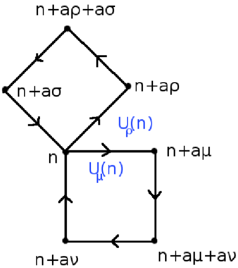
is the *topological charge*

Discretization on the Lattice

Topological density and charge on lattice :

$$q_L(n) = -\frac{1}{2^4 \times 32\pi^2} \sum_{\mu\nu\rho\sigma=\pm 1}^{\pm 4} \tilde{\epsilon}_{\mu\nu\rho\sigma} Tr[\Pi_{\mu\nu}(n)\Pi_{\rho\sigma}(n)]$$

$$Q_L = \sum_n q_L(n)$$



Renormalization

The topological charge density must be corrected by a renormalization factor introduced by the lattice cut-off at the quantum level

$$q_L(n) \to a^4 Z_L(g^2) q(x) + O(a^6)$$

Various methods to take care of Z_L :

- Cooling
- Smearing
- Wilson Flow
- etc...

Wilson Flow

The Wilson Flow equation :

$$\dot{V}_\mu(x,\tau) = -g^2 \, [\partial_{x,\mu} S(V(\tau))] \, V_\mu(x,\tau)$$
$$V_\mu(x,0) = U_\mu(x)$$

It has some advantages for our purpose :

- Its process can be accurately controlled since associated to a differential equation,
- it can, in principle, be extended to any gauge group

Implementation of Wilson Flow on the Lattice :

$$V_\mu(x, \tau + \epsilon) = R_\mu(x, \epsilon)\ V_\mu(x, \tau)$$

$$R(x, \epsilon) \ = \ e^{-\ \epsilon\ \sum_a\ Tr\left[\lambda_a\ (\Gamma_\mu(x) - \Gamma_\mu^{-1}(x))\right]\ \lambda_a}$$

$$\Gamma_\mu(x) \ = \ \sum_{\nu \neq \mu} \Pi_{\mu\nu}(x)$$

NOTE

This is just as the Langevin evolution without noise

Sign Problem

Since

$$\mathcal{S}_\theta \;=\; i\theta Q_{top}$$

is purely imaginary $\Rightarrow$ SIGN PROBLEM .

A number of progress have been made, on the Lattice, in studying θ - T plane of the theory using :

- analytical continuation from imaginary θ $(\theta = \theta_R + i\theta_I)$
- Reweightening , Taylor expansion
- large N expansion

The first two, however, are limited by the small value of θ , the last is affected from the corrections for N=3

Complex Langevin

In principle Complex Langevin Dynamics is a method to access the whole θ - T plane (provided $\beta > \beta_{min}$)

Very careful with the proofs of correctness

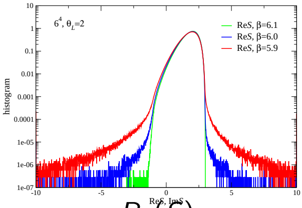
- Compactness of the distribution in the complex plane
- agreement of CL with MC methods for θ_I
- smoothness of $\langle O \rangle$ going from θ_I to θ_R

Compactness

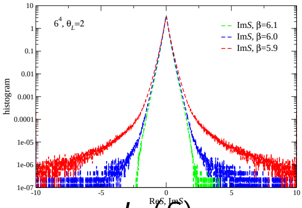
Dynamics :

- 1 Complex Langevin update + several Gauge Cooling steps
- GC. is a gauge transformation that locally minimize the Unitarity Norm $\mathrm{UN}(n) = \sum_\mu \mathrm{Tr}(U_\mu(n) U_\mu^\dagger(n))$, $U \in SL(3, C)$
- We use GC. to keep the distribution compact, as close as possible to the SU(N) manifold

Histogram of the distribution of S for $\theta_L = 2$:

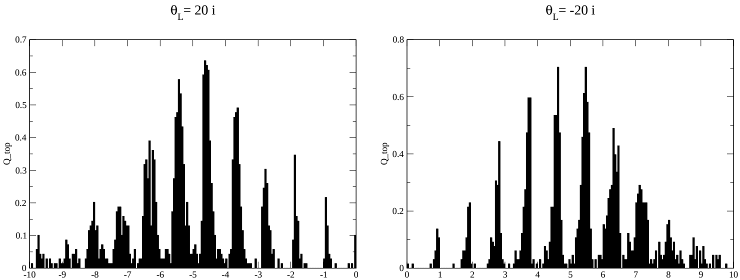


$Re(S)$



$Im(S)$

Test dynamics choosing $\theta = i\ \theta_I$



$\langle Q \rangle_{\theta_L = 20i} = -4.93(5)$ $\quad$ $\langle Q \rangle_{\theta_L = -20i} = +4.95(4)$

- Use Complex Langevin evolution
- NO unitarization
- Wilson Flow for studying the topology of the configuration
- Gauge cooling to stabilize dynamics
- without gc. : explores $SL(3, C)$, and eventually breaks down

$\Longrightarrow$ Test of approach

Exploring Real θ

Preliminary Results for $N = 6^4$

So far :

- bare lattice parameter θ_L , i.e. not renormalization
- the lattice version of $F\tilde{F}$ contributes to the eq of motion
- no renormalization of the topological operators

Expectation value for the topological charge :

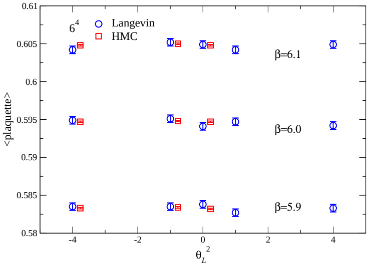
$$\langle Q \rangle = \frac{\int DU\ e^{-S_{YM}+i\theta Q} Q}{Z}$$

$$= \frac{\int DU\ e^{-S_{YM}} (\cos(\theta Q) + i \sin(\theta Q))\, Q}{Z}$$

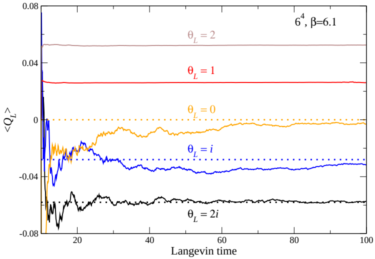
- θ imaginary $\Longrightarrow$ $\langle Q \rangle$ real
- θ real $\Longrightarrow$ $\langle Q \rangle$ imaginary

Smoothness

We look at the behaviour of the plaquette and the topological charge going from θ_I to θ_R



$\langle Plaq \rangle$



$\langle Q_{top} \rangle, \ (\beta = 6.1)$

Smooth behaviour of both observables with θ

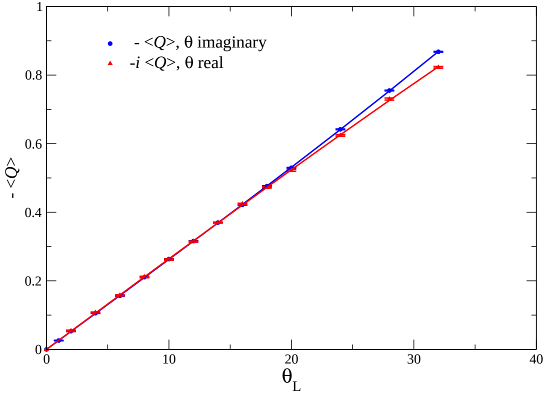
$$Z(\theta) \;=\; \int D[A]\; e^{-S_{YM}}\; e^{i\theta Q_{top}} \;=\; \exp[-VF(\theta)] \;;$$

$$F(\theta) \;=\; \sum_k \frac{1}{(2k)!}\; F^{2k}(0)\; \theta^{2k} \;;$$

The distribution of $\langle Q_{top} \rangle$ with θ is thus is expected to have the form :

- $\langle Q \rangle_{\theta_I} = -\; V \dfrac{d}{d\theta_I} F(\theta_I) = -\; V \chi_L\; \theta_I (1 \;-\; 2b_2\; \theta_I^2 + 3b_4\; \theta_I^4 - \ldots)$
- $\langle Q \rangle_{\theta_R} = i\; V \dfrac{d}{d\theta_R} F(\theta_R) = i\; V \chi_L\; \theta_R (1 \;+\; 2b_2\; \theta_R^2 + 3b_4\; \theta_R^4 + \ldots)$

Deviation from linear behaviour of $\langle Q \rangle_\theta$ at large θ :



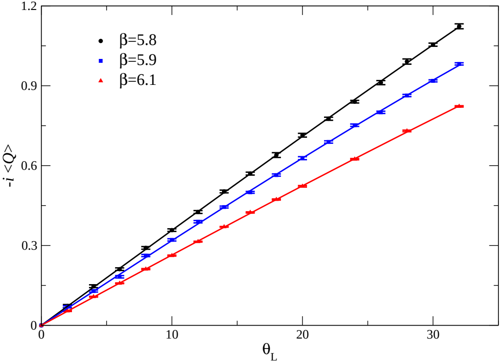
$\beta = 6.1$

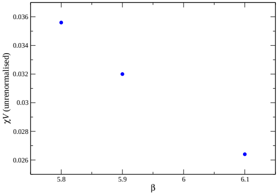
lines are fits

$y = b_0\theta_L(1+2b_2\theta_L^2)$

with
$b_0 = 0.026$
for both and
$b_2 \sim \pm 1 \cdot 10^{-5}$

Drop of the lattice topological susceptibility χ_L for increasing values of β





The effect will be enhanced including the renormalization factor $Z(\beta)$

Conclusions and Outlook

- We have good control of the CL dynamics for real θ at values of β high enough ($\beta \gtrsim 5.8$), i.e. satisfaction of the criteria for correctness
- We showed agreement of $\langle Q \rangle_\theta$ calculated independently at θ_R and at θ_I
- We showed the expected behaviour of the χ_{top} with β

Outlook :

- Explore the phase diagram more widely
- Possibly find a way to analyze the topological content of the configurations at θ real