# Complex Saddle Points in QCD

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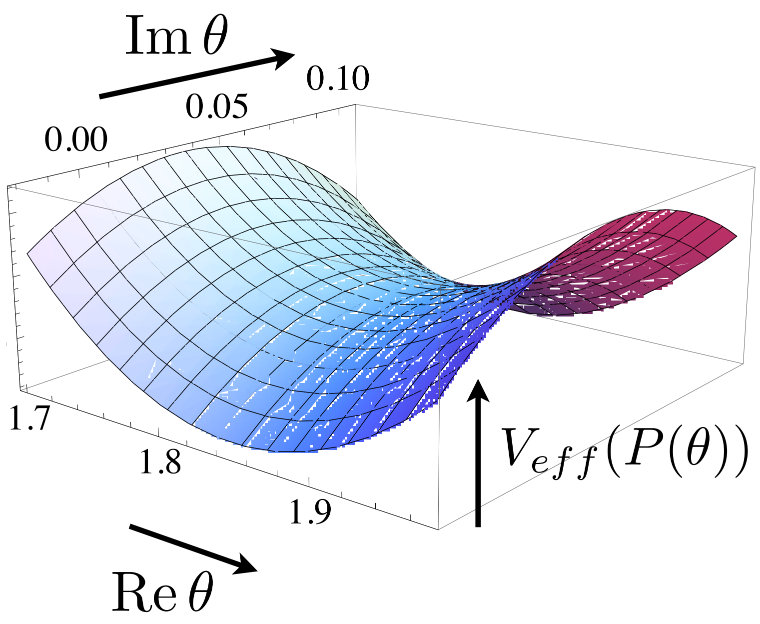
## Overview




Look for saddle points in the effective action of the Polyakov loop in finite-density QCD

Three consequences

- Conjugate Loop > Polyakov Loop
- Color Neutrality
- Complex mass matrix

# Outline

- Introduction

  - Polyakov loop

  - Problems with fermion determinant in mean-field theory

- Formalism

  - Complex saddle points from *CK* symmetry

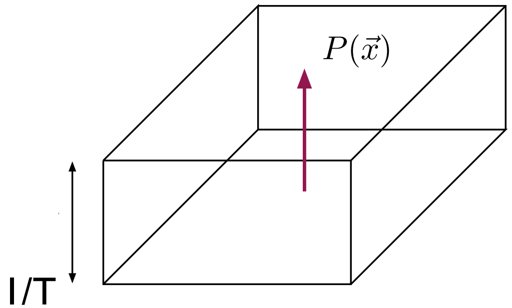
  - Polyakov loop and mass matrix at saddle points

- Application: $N_f = 2$ light fermions

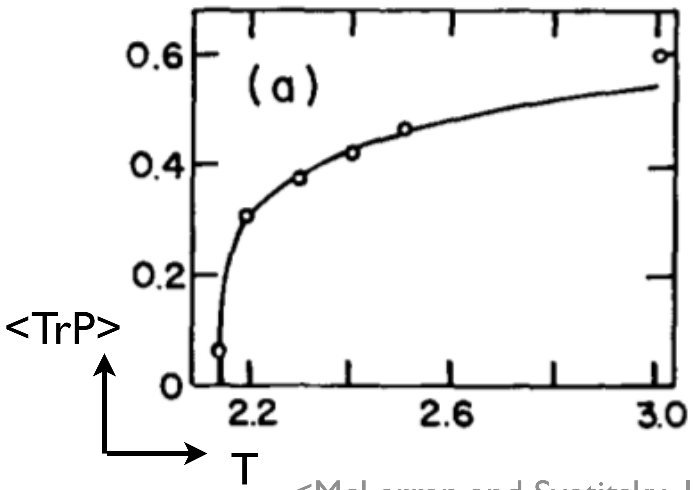
- Conclusions

# Introduction

# The Polyakov Loop




Pure SU(2)




<McLerran and Svetitsky, 1980>

- The order parameter for the (de-)confined phase

$$P(\vec{x}) = \mathcal{P}e^{i\int_0^{1/T} dx_4 A_4(x)}$$

- Free energy of a heavy quark, $F_q$

$$\langle \mathrm{tr} P(\vec{x}) \rangle = e^{-F_q/T}$$

- Free energy of a heavy antiquark, $F_{\bar{q}}$

$$\langle \mathrm{tr} P^\dagger(\vec{x}) \rangle = e^{-F_{\bar{q}}/T}$$

- Low temperature: unbroken center symmetry

$$\langle \mathrm{tr} P(\vec{x}) \rangle = 0 \quad \rightarrow \quad F_q = \infty$$

Confined phase

- High temperature: broken center symmetry

$$\langle \mathrm{tr} P(\vec{x}) \rangle \neq 0 \quad \rightarrow \quad F_q = \mathrm{Finite}$$

Deconfined phase

# “Sign problem” in mean-field theory

<Dumitru, Pisarski and Zschiesche, 2005>

<Fukushima and Hidaka, 2007>

- The fermion determinant is complex.

$$\det M\left(-\mu\right) = \left[\det M\left(\mu\right)\right]^*$$

$\longrightarrow$ complex effective potential

- Constant Polyakov loop background

$$V_{1L} = -2T\,\mathrm{tr}\int\frac{d^3k}{(2\pi)^3}\log\left[1+Pe^{-(\omega-\mu)/T}\right]\left[1+P^\dagger e^{-(\omega+\mu)/T}\right]$$

1. $\langle \mathrm{tr}P \rangle$ and $\langle \mathrm{tr}P^\dagger \rangle$ not real
2. $\langle \mathrm{tr}P \rangle = \langle \mathrm{tr}P^\dagger \rangle$ at $\mu \neq 0$
3. Not color neutral: $\langle n_{red} \rangle = \langle n_{green} \rangle \neq \langle n_{blue} \rangle$

This talk: How going to the complex plane can address these issues.

# Formalism

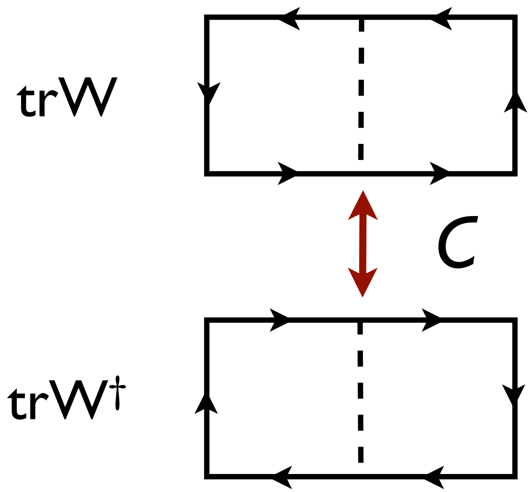
# Global Symmetry: μ = 0

- Charge conjugation (*C*)

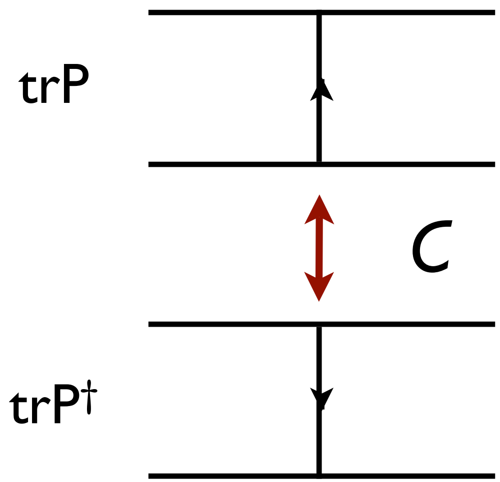
$$Z = \int DU e^{-S_{YM}} \det M(\mu)$$

$$\mathcal{C} : A_\mu \to -A_\mu^t \quad \Rightarrow \quad \det M(0) = [\det M(0)]^*$$

- Expansion of the fermion determinant (Hopping parameter expansion)




$$W = \mathcal{P} e^{ig \oint dx_\mu A_\mu}$$




$$P = \mathcal{P} e^{ig \int_0^{1/T} dx_4 A_4}$$

## Global Symmetry: μ ≠ 0

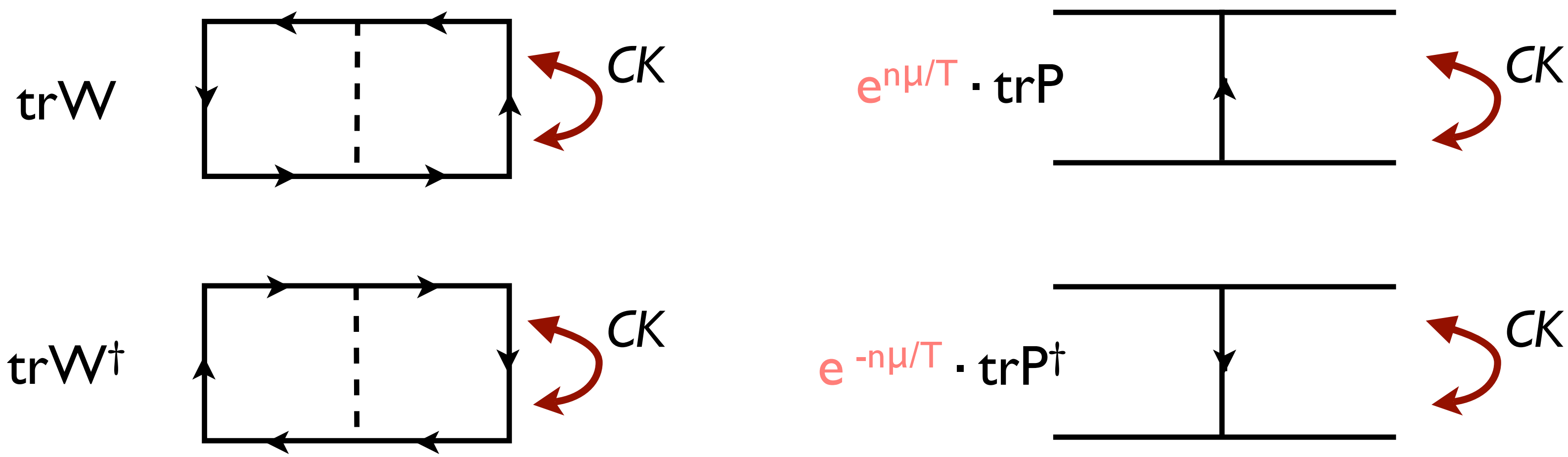
- Charge + Complex Conjugations (*CK*)

$$\mathcal{C} : A_\mu \rightarrow -A_\mu^t$$

$$\mathcal{K} : A_\mu \rightarrow A_\mu^*, \quad i \rightarrow -i$$

$$\det M(\mu) = \underbrace{[\det M(-\mu)]^*}_{CK}$$

- Each loop is invariant under *CK*




Any observables should respect *CK* symmetry at μ≠0.

# The Polyakov Loop

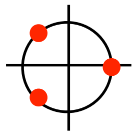
- Choose a constant background in the Polyakov gauge.

SU(3): $$P = \begin{pmatrix} e^{i\theta_1} & & \\ & e^{i\theta_2} & \\ & & e^{i\theta_3} \end{pmatrix}$$

- μ = 0

P is real and detP = 1 $\longrightarrow$ $\theta_1 + \theta_2 + \theta_3 = 0$ and $\theta_1 + \theta_2 = 0$

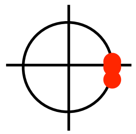
Confined



$\theta = 2\pi/3$

$\langle \mathrm{tr} P \rangle = 0$

Deconfined



$\theta = 0$

$\langle \mathrm{tr} P \rangle \neq 0$

With this choice, we always have $\langle \mathrm{tr} P \rangle = \langle \mathrm{tr} P^\dagger \rangle$.

- μ ≠ 0: complexify the eigenvalues

$$\theta_j \to z_j = \theta_j + i\psi_j$$

## The Polyakov Loop: SU(N)

- The Polyakov loop is invariant under CK

$$P = e^{i\int dx_4 A_4} = \mathrm{diag}\left(e^{iz_1}, \cdots e^{iz_j} \cdots, e^{iz_N}\right)$$

$$\mathcal{C} : A_\mu \to -A_\mu^t \qquad \mathcal{K} : \begin{array}{ccc} i & \to & -i \\ A_\mu & \to & A_\mu^* \end{array}$$

- The eigenvalues of Polyakov loop comes with a conjugate pair:

If $P\,|z_j\rangle = e^{iz_j}\,|z_j\rangle$ then

$$\begin{aligned} P\,\mathcal{CK}\,|z_j\rangle &= \mathcal{CK}\,P\,|z_j\rangle \\ &= e^{-iz_j^*}\mathcal{CK}\,|z_j\rangle \end{aligned}$$

Thus $z_j$ and $-z_j^*$ are the eigenvalues.

# The Polyakov Loop: SU(N)

- The eigenvalues can be uniquely specified:

For an eigenvalue $z_j = \theta_j + i\psi_j$, we have $\sum_{j=1}^{N} z_j = 0$ & $z_k = -z_j^*$

For SU(N), there are [N/2] real parts, $\theta_j$, and [(N-1)/2] imaginary parts, $\psi_j$.

- For SU(3): $(z_1, z_2, z_3) = (\theta - i\psi, -\theta - i\psi, +2i\psi)$

$$P = \begin{pmatrix} e^{iz_1} & & \\ & e^{iz_2} & \\ & & e^{iz_3} \end{pmatrix} \rightarrow \begin{pmatrix} e^{i\theta+\psi} & & \\ & e^{-i\theta+\psi} & \\ & & e^{-2\psi} \end{pmatrix}$$

$$\begin{aligned} \mathrm{tr}P &= 2e^{\psi}\cos\theta + e^{-2\psi} \\ \mathrm{tr}P^\dagger &= 2e^{-\psi}\cos\theta + e^{2\psi} \end{aligned}$$

It is real and trP ≠ trP$^\dagger$ if $\psi \neq 0$.

# Perturbative calculation

- One-loop effective potential for $N_f$ massless fermions at finite μ:

$$
\begin{aligned}
V_f &= -2N_f T\,\mathrm{tr}\int \frac{d^3k}{(2\pi)^3}\log\left[1+Pe^{-(\omega-\mu)/T}\right] \\
&= -\frac{4N_f T^4}{\pi^2}\sum_{j=1}^{N_c}\left(\eta(4)-\frac{\eta(2)}{2}(z_j-i\frac{\mu}{T})^2+\frac{\eta(0)}{4!}(z_j-i\frac{\mu}{T})^4\right)
\end{aligned}
$$

<Korthals Altes, Pisarski and Sinkovics 2000>

in general $V_f$ is complex when $z_j$ is complex.

- CK-symmetric background: $(z_1, z_2, z_3) = (\theta - i\psi, -\theta - i\psi, +2i\psi)$

$$
\begin{aligned}
\frac{V_f(\theta,\psi,\mathrm{T},\mu)}{N_f} &= -\frac{\mu^4}{2\pi^2}+T^2\left(-\mu^2+\frac{2\theta^2\mu^2}{\pi^2}-\frac{6\mu^2\psi^2}{\pi^2}\right)+\frac{4T^3\left(\theta^2\mu\psi+\mu\psi^3\right)}{\pi^2} \\
&+\frac{T^4\left(-7\pi^4+20\pi^2\theta^2-10\theta^4-60\pi^2\psi^2+60\theta^2\psi^2-90\psi^4\right)}{30\pi^2}
\end{aligned}
$$

Real

## Saddle Point: 1st derivative

- Effective potential: $V_{eff} = V_{eff}(A_4)$

$$\frac{A_4}{T} = \theta \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} - i\psi \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

Parameters of the Cartan algebra

- First derivative

$$\frac{\partial V_{eff}}{\partial \theta} = 0 \longrightarrow \langle n_{red} \rangle - \langle n_{green} \rangle = 0$$

$$\frac{\partial V_{eff}}{\partial \psi} = 0 \longrightarrow \langle n_{red} \rangle + \langle n_{green} \rangle - 2 \langle n_{blue} \rangle = 0$$

$$\therefore \quad \langle n_{red} \rangle = \langle n_{green} \rangle = \langle n_{blue} \rangle$$

Color neutral

# Saddle Point: 2nd derivative

- Mass Matrix

$$M_{ab} = \frac{\partial^2 V_{eff}}{\partial A_4^a \partial A_4^b} = \frac{g^2}{T^2} \begin{pmatrix} \frac{1}{4}\frac{\partial^2 V_{eff}}{\partial \theta^2} & \frac{-i}{4\sqrt{3}}\frac{\partial^2 V_{eff}}{\partial \theta \partial \psi} \\ \frac{-i}{4\sqrt{3}}\frac{\partial^2 V_{eff}}{\partial \theta \partial \psi} & \frac{-1}{12}\frac{\partial^2 V_{eff}}{\partial \psi^2} \end{pmatrix}$$

- Saddle

$$\frac{\partial^2 V_{eff}}{\partial \theta^2} > 0, \quad \frac{\partial^2 V_{eff}}{\partial \psi^2} < 0 \quad \longleftrightarrow$$

Stability conditions for color chemical potential

- Eigenvalues of the mass matrix

$$\begin{aligned} m_{ev}^2 &= \frac{g^2 T^2}{12\pi^2}\left[A\left(\theta,\psi,T,\mu\right) \pm 2\sqrt{B\left(\theta,\psi,T,\mu\right)}\right] \\ (\theta,\psi \to 0) \;\to\; & \frac{g^2}{3}\left[N_c T^2 + \frac{N_f}{2}\left(T^2 + \frac{3}{\pi^2}\mu^2\right)\right] \end{aligned}$$

- When B<0, the mass eigenvalues become complex → More on this later.

# Application

(Light Fermions)

# Phenomenological model of QCD: “Model A”

<P. Meisinger, T. Miller, and M. Ogilvie, 2002>

- Gluonic potential

$$V_g = 2\mathrm{Tr}[\int \frac{d^3k}{(2\pi)^3} \log(1 - Pe^{-\omega_k/T})]$$

Introduce mass parameter, $M_A$: $\omega_k = \sqrt{k^2 + M_A^2}$

$$V_A = V_g(P) + V_d(P, M_A)$$

where

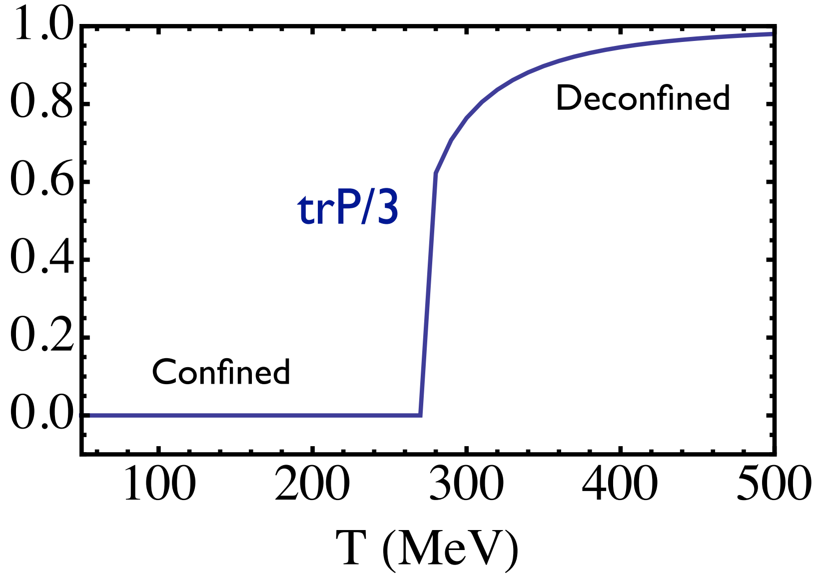
$$V_d = \frac{M_A^2 T^2}{2\pi^2} \sum_{n=1}^{\infty} \frac{|\mathrm{tr} P^n|^2}{n^2}$$

“confining term”

$$V_g = -\frac{2T^4}{\pi^2} \sum_{n=1}^{\infty} \frac{|\mathrm{tr} P^n|^2}{n^4} \longrightarrow -\frac{(N_c^2 - 1)}{45}\pi^2 T^4$$

Stefan-Boltzmann

$M_A$ = 596 MeV

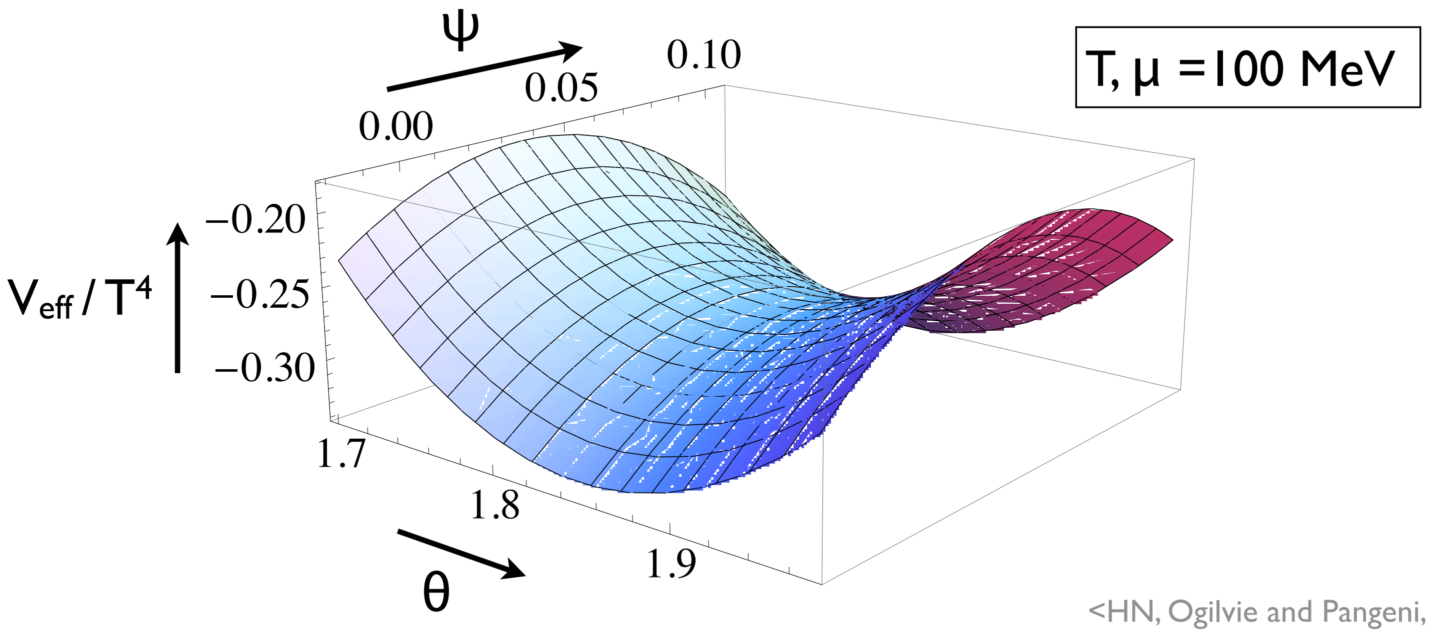



- Complexify: $\theta_i \rightarrow z_i$

# Phenomenological model of QCD: “Model A”

- Total effective potential: find the saddle points in the θ-ψ plane.

$$V_{eff}(\theta, \psi, T, \mu, M_A) = V_g + V_f + V_d$$




<HN, Ogilvie and Pangeni, 2014>

# Polyakov Loop

$$\mathrm{tr}P = 2e^{\psi}\cos\theta + e^{-2\psi}$$
$$\mathrm{tr}P^{\dagger} = 2e^{-\psi}\cos\theta + e^{2\psi}$$

- Crossover

- $\mathrm{tr}P^{\dagger} > \mathrm{tr}P$

  - It is easier to add antiquark for $\mu > 0$

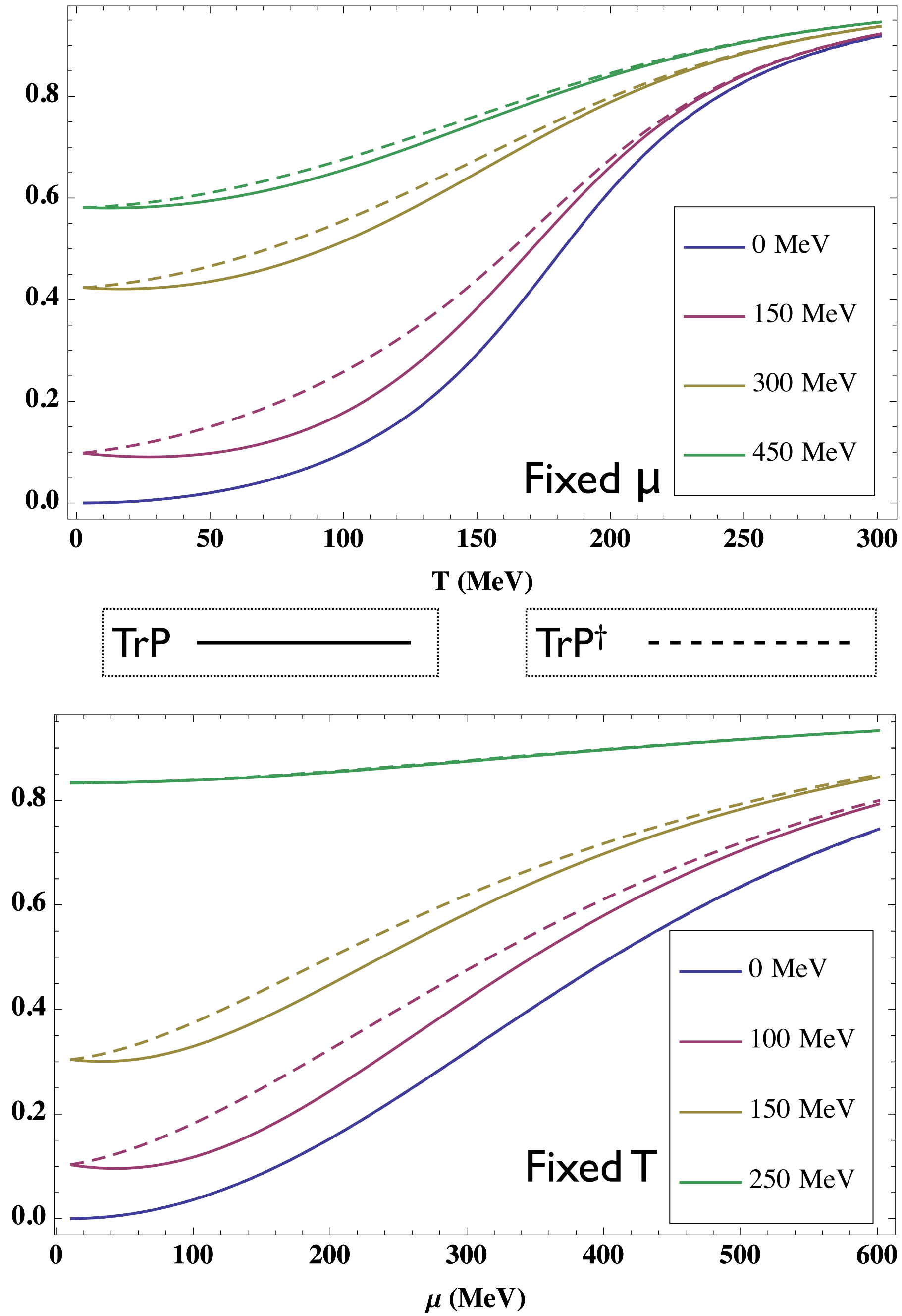
  - Similar results:

    → Weiss mean-field approximation

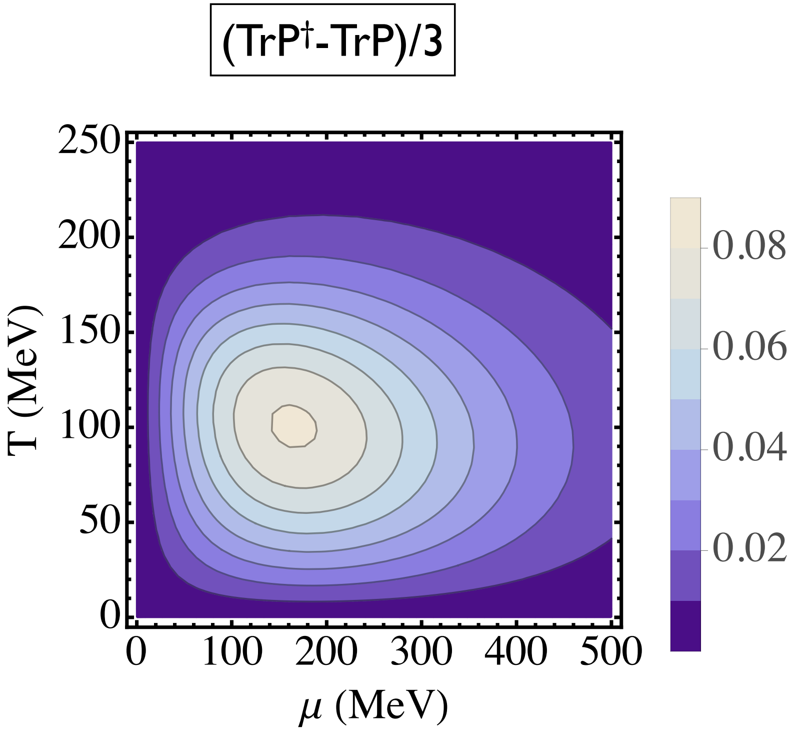
    <Abuki and Fukushima, 2009>

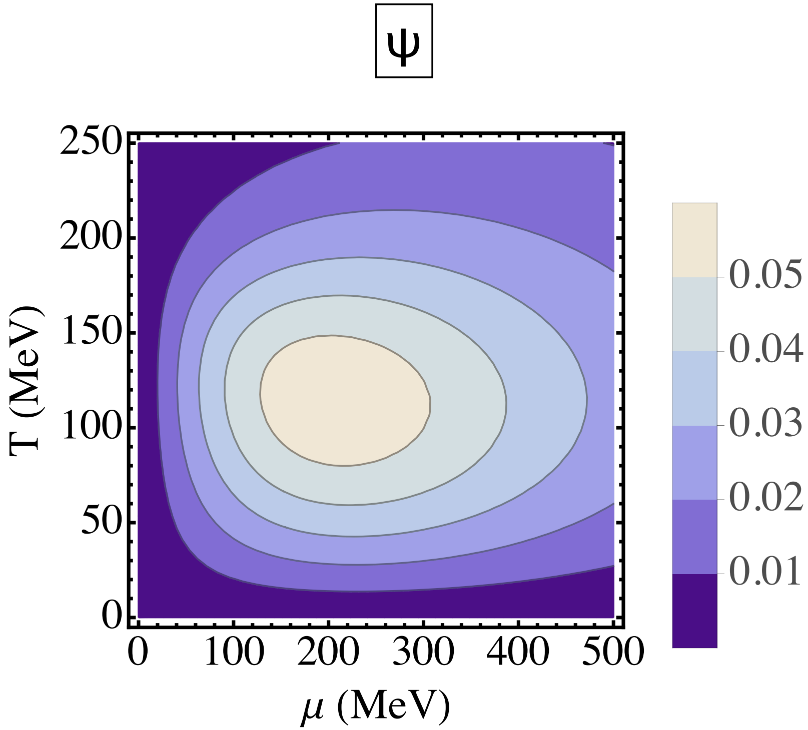
    → $S^3 \times S^1$ at large N

    <Hands, Hollowood and Myers, 2010>



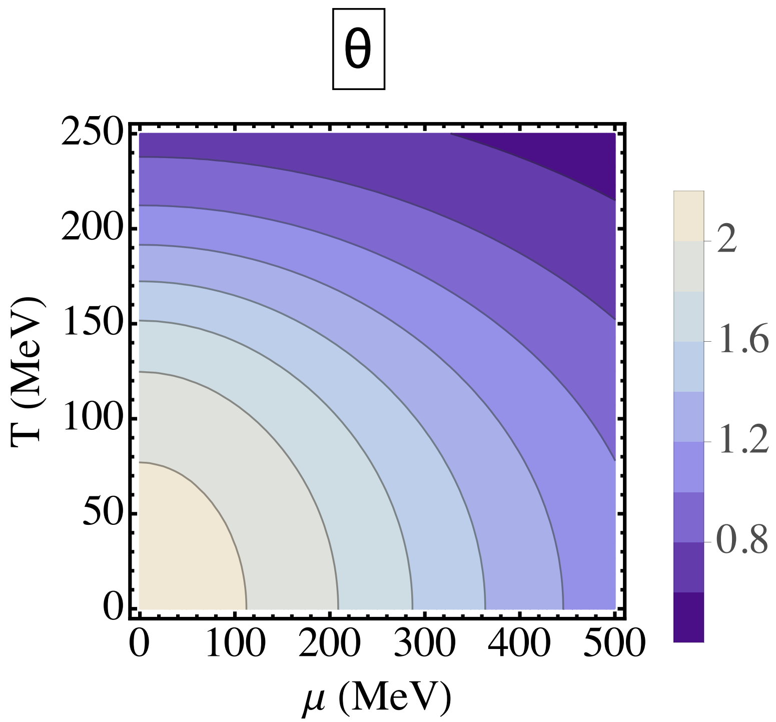
# Polyakov Loop

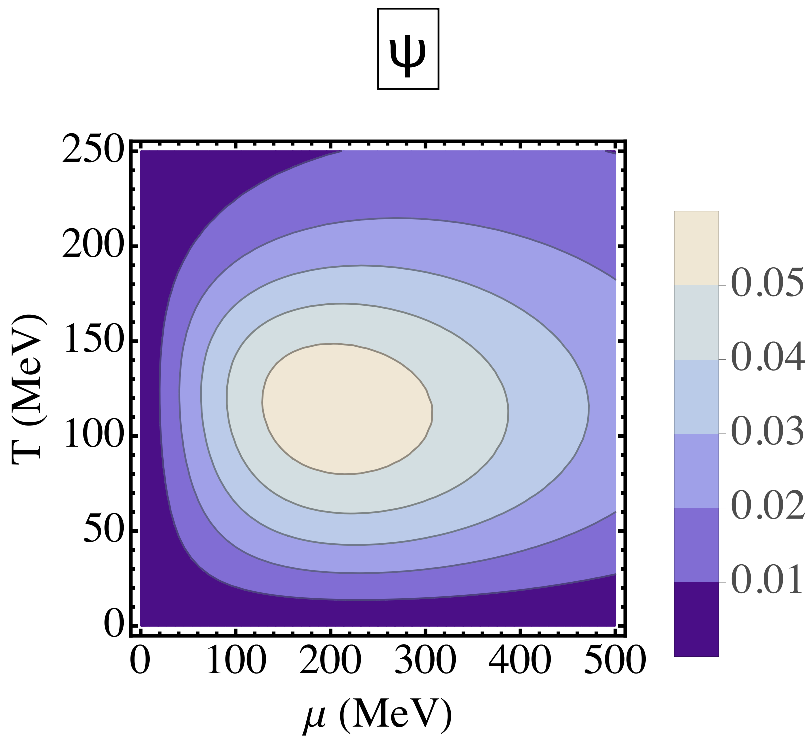



$\langle \mathrm{tr} P \rangle \neq \langle \mathrm{tr} P^{\dagger} \rangle$ only when $\psi \neq 0$

## Polyakov Loop




<u>At high T-μ</u> (T, μ >>$M_A$):

$$\theta \sim \frac{3M_A^2\pi}{8\pi^2T^2+6\mu^2}$$

$$\psi \sim \frac{9M_A^4\pi^2T\mu}{4\left(4\pi^2T^2+3\mu^2\right)^3}$$

## Mass Matrix

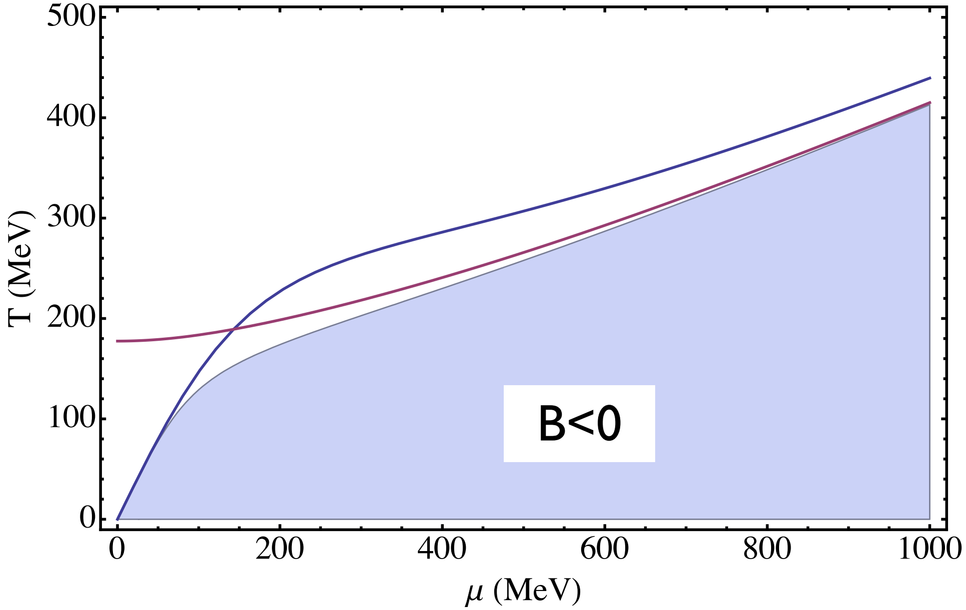
- Disorder line

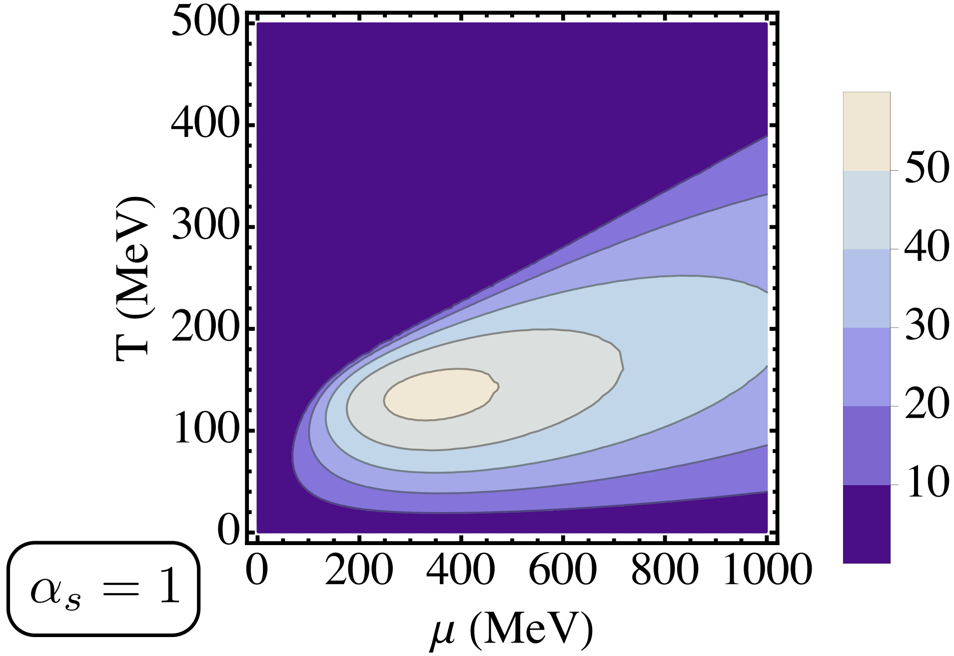
$$M_{ab} = \frac{\partial^2 V_{eff}}{\partial A_4^a \partial A_4^b}$$

$$m_{ev}^2 = \frac{g^2 T^2}{12\pi^2}\left[A \pm 2\sqrt{B}\right]$$

- Complex mass: Im[$m_{ev}$]

The magnitude is not negligible.



# Mass Matrix

- The complex conjugate pair comes from *CK* symmetry

$$M_{ab} \quad = \quad \frac{\partial^2 V_{eff}}{\partial A_4^a \partial A_4^b}$$

- An effective potential for a boson with mass, $m^2 = a \pm i\, b$:

$$\frac{1}{2}\int \frac{d^dk}{(2\pi)^d} \log\left[\left(k^2 + a + ib\right)\left(k^2 + a - ib\right)\right]$$

$$= \frac{1}{2}\int \frac{d^dk}{(2\pi)^d} \log\left[(k^2 + a)^2 + b^2\right] > 0$$

Because a > 0 ,
no instability

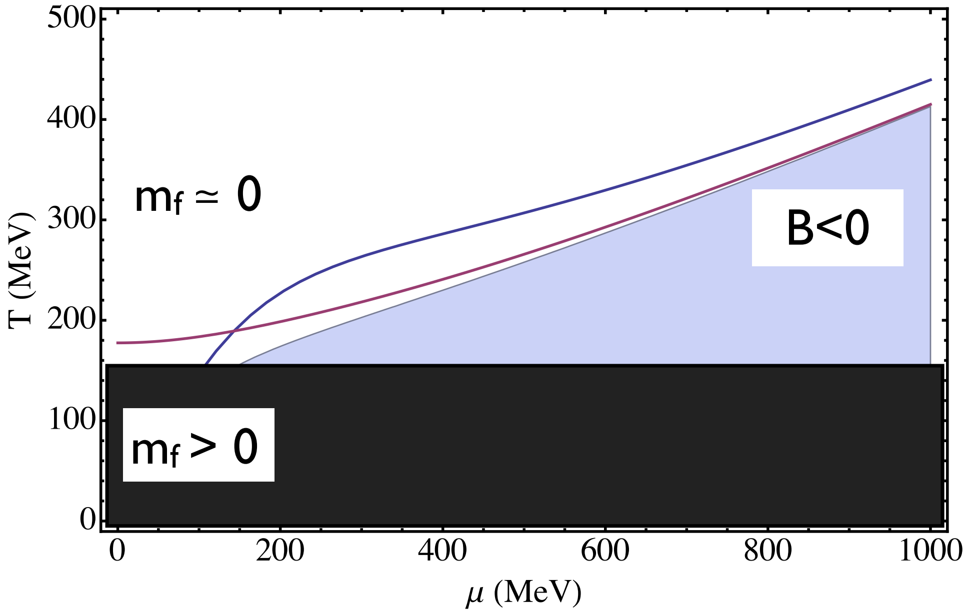
- Color density oscillation if b ≠ 0

$$\langle A_4(r) A_4(0)\rangle \sim \frac{\mathrm{Exp}\left[-r\left(\mathrm{Re}\, m\right)\right]}{r} \cos\left[r\left(\mathrm{Im}\, m\right)\right]$$

Similar behavior for baryon number correlators
was observed in the flux-tube model by Patel.

<Patel, 2012>

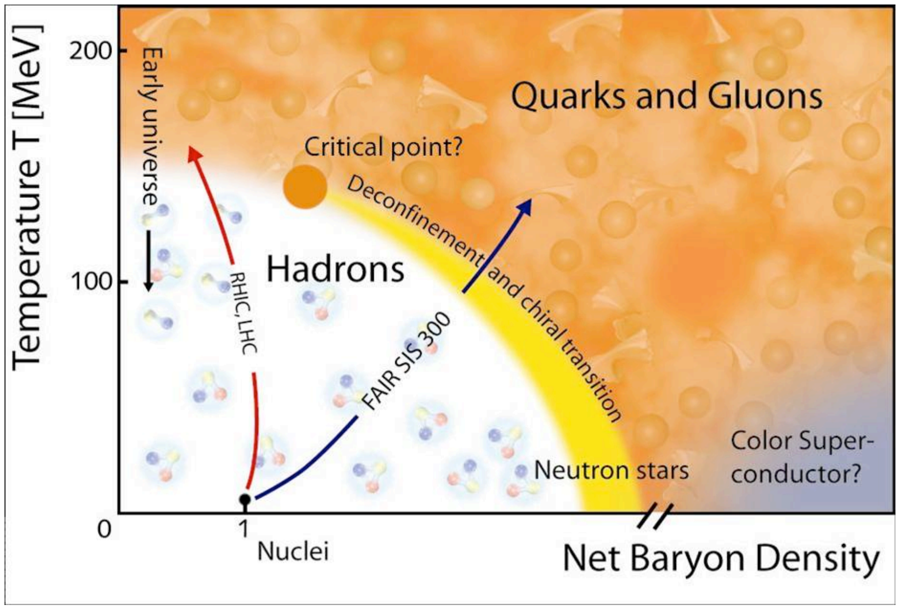
# Mass Matrix




Relevant for GSI/FAIR...

but need more work:

- Check for the model-dependence.
- Need to add chiral symmetry breaking.
- Look for useful experimental signature.




Still in working progress.

# Conclusions

(and future directions)

# Conclusions

- Complex saddle points from *CK* symmetry can fix the fermion sign problems in mean-field theory.

- $\mathrm{tr}P \neq \mathrm{tr}P^\dagger$ in the crossover region at finite $\mu$

- Color neutrality is naturally achieved at the saddle point.

- Complex mass eigenvalues: Oscillatory behavior in color charge density.

# Future direction

## In mean-field theory:

- Check the model dependence (in progress)
- Add chiral symmetry breaking via PNJL (in progress)
- Add diquark condensate → Explore phase diagram of QCD.

## In LQCD:

- Imaginary chemical potential?
- Relevance for Lefschetz thimble?