Worldline approach to lattice QCD at finite β

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Lattice QCD with staggered fermions

$$\mathcal{Z}_{QCD} = \int \mathcal{D}\chi \mathcal{D}\overline{\chi} \mathcal{D}U \; e^{S_G + S_F}$$

$$S_G = \frac{\beta}{N} \sum_x \sum_{\mu<\nu} \mathrm{ReTr}\left(U_{\mu,x} U_{\nu,x+\hat{\mu}} U^\dagger_{\mu,x+\hat{\nu}} U^\dagger_{\nu,x} \right)$$

$$S_F = \sum_{x,\mu,f} \eta_{\mu,x} \gamma^{\delta_{\mu 4}} \left(e^{+\mu_f a_\tau \delta_{\mu 4}} \overline{\chi}^f_x U_{\mu,x} \chi^f_{x+\hat{\mu}} - e^{-\mu_f a_\tau \delta_{\mu 4}} \overline{\chi}^f_{x+\hat{\mu}} U^\dagger_{\mu,x} \chi^f_x \right) + \sum_{x,f} 2am_f \overline{\chi}^f_x \chi^f_x$$

$$\eta_{\mu,x} = (-1)^{\sum_{\nu<\mu} x_\nu}$$

Straightforward approach:

- Integrate out lattice fermions $\Rightarrow \det(\not{D}(\mu_f, m_f)) = \overline{\det(\not{D}(-\overline{\mu_f}, m_f))}$
- Sample over gauge fields $\Rightarrow$ **sign problem**
- Approach justified by the Grassmann nature of fermion components

But:

- Physical states are color singlets

Worldline approach at strong coupling

$$\mathcal{Z}_{QCD}(\beta=0) = \int \mathcal{D}\chi \mathcal{D}\overline{\chi} \mathcal{D}U \; e^{\not{S}_G} e^{S_F}$$

$$\mathcal{Z}_{QCD}(\beta=0) = \int \prod_x d\chi_x d\overline{\chi}_x \; e^{2am\overline{\chi}_x \chi_x} \prod_\mu \underbrace{\int dU \; e^{\mathrm{Tr}\left(K^\dagger_{\mu,x}(+\mu)U + K_{\mu,x}(-\mu)U^\dagger\right)}}_{\mathcal{I}_{N_c}(K^\dagger_{\mu,x}(+\mu), K_{\mu,x}(-\mu))}$$

$$K^{ij}_{\mu,x}(\pm\mu) = \eta_{\mu,x}\gamma^{\delta_{\mu 4}} e^{\pm\mu a_\tau \delta_{\mu 4}} \; \chi^i_x \overline{\chi}^j_{x+\hat{\mu}}$$

Alternative approach: change order of integration! Rossi & Wolff '84

- Integrate out lattice gauge fields first $\Rightarrow$ color singlets
- Integrate out lattice fermions $\Rightarrow$ worldlines of color singlets

But:

- Integrals over gauge fields are only known at $\beta = 0$: **one-link integrals**

Worldline approach at strong coupling

$$\mathcal{Z}_{QCD}(\beta=0) = \int \mathcal{D}\chi\mathcal{D}\overline{\chi}\mathcal{D}U \; e^{\not{S}_G} e^{S_F}$$

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One-link integral with fermionic sources: Rossi & Wolff '84

$$\mathcal{I}_{N_c}(K^\dagger_{\mu,x}(+\mu), K_{\mu,x}(-\mu)) = \sum_{k=0}^{N_c} \frac{(N_c-k)!}{N_c!k!}(\gamma^{2\delta_{\mu 4}} M_x M_{x+\hat{\mu}})^k + \kappa\gamma^{N_c}\left(e^{+\mu a_\tau N_c}\overline{B}_x B_{x+\hat{\mu}} + (-1)^{N_c} e^{-\mu a_\tau N_c}\overline{B}_{x+\hat{\mu}} B_x\right)$$

$M_x = \overline{\chi}_x\chi_x$, $\quad B_x = \frac{1}{N!}\varepsilon_{i_1\dots i_{N_c}}\chi^{i_1}_x \dots \chi^{i_{N_c}}_x$ $\qquad \kappa = 0, 1$ for $U(N_c)$ or $SU(N_c)$ (resp.)

MDP model at strong coupling

Integrating out the lattice fermions *after* the gauge fields results in a combinatorial partition function for color singlets:

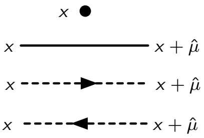
$$\mathcal{Z}_{QCD}(\beta = 0) = \sum_{\{n,k,\ell\}} w_M(n)\ w_D(k)\ w_B(\ell)$$

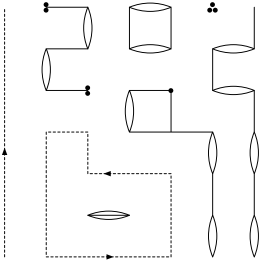
$$w_M(n) = \prod_x \frac{N_c!}{n_x!}(2ma)^{n_x}, \qquad w_D(k) = \prod_l \frac{(N_c - k_l)!}{N_c! k_l!}\gamma^{2k_l \delta_{\mu 4}},$$

$$w_B(\ell) = \sigma(\ell)\left(\prod_{x\in\ell} N_c!\right)^{-1} \gamma^{N_c \Delta N_{\hat{4}}} e^{N_c N_\tau r_\ell a_\tau \mu}, \qquad \sigma(\ell) = (-1)^{r_\ell + N_-(\ell) + \lfloor N_c/2 \rfloor} \prod_{l \in \ell} \eta_l$$

DOF:

- **Monomers:** n_x
- **Dimers:** $k_{\mu,x}$
- **Baryons:** $b^+_{\mu,x}$, $b^-_{\mu,x}$





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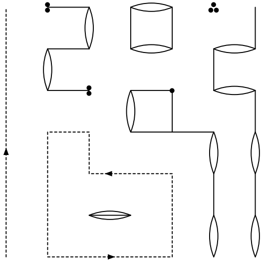
Constraints:

$\bar{\chi}$: $n_x+\sum_\mu(k_{\mu,x}+k_{-\mu,x}+N_c b^+_{\mu,x})=N_c$

χ: $n_x+\sum_\mu(k_{\mu,x}+k_{-\mu,x}+N_c b^-_{\mu,x})=N_c$

$\Longrightarrow \sum_\mu b^+_{\mu,x}=\sum_\mu b^-_{\mu,x}$

Self-avoiding baryon loops



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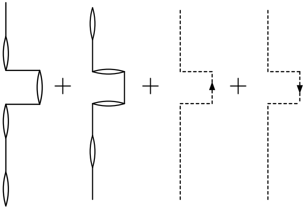
Karsch-Mütter resummation:

Some non-contractible baryon loops wrapping the Euclidean time have *negative weight*
$\Longrightarrow$ sign problem even at $\mu = 0$!

Resummation of configurations makes the sign problem milder. Karsch & Mütter '89

$$w_P(\ell) \propto 1 + \sigma(\ell)\cosh(N_c N_\tau r_\ell a_\tau \mu)$$

New DOF: **self-avoiding polymers**.



Hubbard-Stratonovich transformations

In order to go beyond $\beta = 0$, we multiply the partition function by "1":

$$\mathcal{Z} = \int du\ e^{b|w(u)|^2} \underbrace{\int dP_a(z)}_{1}, \qquad dP_a(z) = \frac{a}{2\pi} dz d\bar{z}\ e^{-\frac{a}{2}|z|^2}$$

and use **Hubbard-Stratonovich transformations** to simplify the action:

$$z \mapsto \left(\frac{b}{a}\right)^{1/2} (z - w(u))$$

$$\textbf{action}: \quad -\frac{a}{2}|z|^2 \quad \mapsto \quad -\frac{b}{2}|z|^2 + b\mathrm{Re}\left(\overline{w}(u) z\right) - \frac{b}{2}|w(u)|^2$$

$$\textbf{measure}: \quad \frac{a}{2\pi} dz d\bar{z} \quad \mapsto \quad \frac{b}{2\pi} dz d\bar{z}$$

$$\mathcal{Z} = \int du\ e^{b|w(u)|^2} \int dP_a(z) = \int dP_b(z)\ du\ e^{b\mathrm{Re}(\overline{w}(u) z)}$$

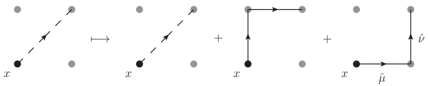
2-link action

Multiply the partition function by "1":

$$\mathcal{Z}_G = \int \mathcal{D}U \; e^{-S_{G,4}(U)} \times \underbrace{\int dP(\widetilde{Q})}_{1}$$

and HS-transform the new auxiliary variable $\widetilde{Q}$

$$\widetilde{Q}_{\mu\nu,x} = \left(\frac{\beta}{N_c}\right)^{1/2} \left(Q_{\mu\nu,x} - U_{\mu,x} U_{\nu,x+\hat{\mu}} - U_{\nu,x} U_{\mu,x+\hat{\nu}}\right)$$



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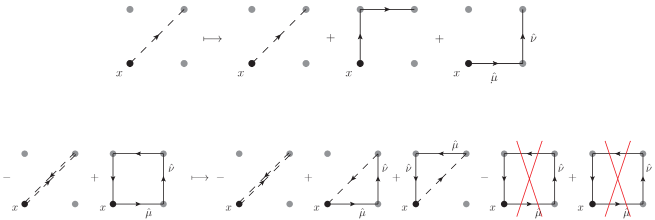

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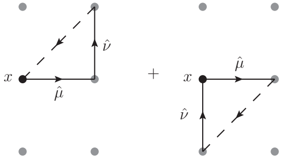
The partition function becomes

$$\int dP(Q) \; \mathcal{D}U \; e^{-S_{G,2}(Q,U)}$$

where $S_{G,2}$ is the **2-link action**:

$$S_{G,2}(Q,U) = -\tfrac{\beta}{N_c} \sum_{x,\mu\neq\nu} \mathrm{ReTr}\left((Q_{\mu\nu,x}U^{\dagger}_{\nu,x+\hat{\mu}})^{\dagger} U_{\mu,x}\right)$$

Fabricius & Haan '84



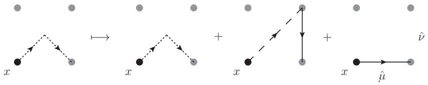
1-link action

Multiply the partition function again by "1":

$$\mathcal{Z}_G = \int dP(Q)\, \mathcal{D}U\, e^{-S_{G,2}(Q,U)} \times \underbrace{\int dP(\widetilde{R})}_{1}$$

and HS-transform the new auxiliary variable $\widetilde{R}$:

$$\widetilde{R}_{\mu\nu,x} = \left(\frac{\beta}{N_c}\right)^{\frac{1}{2}} \left(R_{\mu\nu,x} - Q_{\mu\nu,x} U^{\dagger}_{\nu,x+\hat{\mu}} - U_{\mu,x}\right)$$



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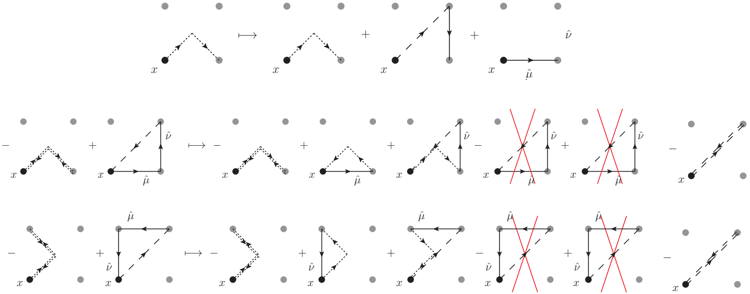

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and HS-transform the new auxiliary variable $\widetilde{R}$:

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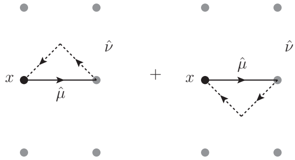
The partition function becomes

$$\int dP(Q)\, dP(R)\, \mathcal{D}U\, e^{-S_{G,1}(Q,R,U)}$$

where $S_{G,1}$ is the **1-link action**:

$$S_{G,1}(Q,R,U) = -\frac{\beta}{N_c} \sum_{x,\mu} \mathrm{ReTr}\left(J^{\dagger}_{\mu,x}(Q,R) U_{\mu,x}\right),$$

$$J_{\mu,x}(Q,R) = \sum_{\nu \neq \mu} \left(R^{\dagger}_{\nu\mu,x-\hat{\nu}} Q_{\mu\nu,x-\hat{\nu}} + R_{\mu\nu,x}\right)$$



n-link actions: numerical results

$$\mathcal{Z}_G = \int \mathcal{D}U \; e^{-S_{G,4}(U)} = \int dP(Q)\; \mathcal{D}U \; e^{-S_{G,2}(Q,U)} = \int dP(Q,R)\; \mathcal{D}U \; e^{-S_{G,1}(Q,R,U)}$$

Algorithm:

1. Gaussian heatbath update of $\widetilde{Q}$ $\rightarrow$ Construct $Q \equiv \widetilde{Q} + UU + UU$
2. Gaussian heatbath update of $\widetilde{R}$ $\rightarrow$ Construct $R \equiv \widetilde{R} + QU^\dagger + U$
3. Construct each $J_l \equiv \sum (R^\dagger Q + R)$
4. (Pseudo) heatbath update of U_l

	$SU(2)$		$SU(3)$	
	β	plaquette	β	plaquette
$S_{G,4}$	2.35	0.6170(1)	5.75	0.5591(2)
$S_{G,2}$	2.35	0.6170(2)	5.75	0.5588(2)
$S_{G,1}$	2.35	0.6170(2)	5.75	0.5592(3)

0-link action: pure gauge

The integrand of the partition function with action $S_{1,G}$ factorizes and becomes a **product of one-link integrals**, which can be integrated exactly:

$$\mathcal{Z}_G = \int dP(Q,R) \prod_l \underbrace{\int dU \; e^{\frac{\beta}{N_c} \mathrm{Re}\mathrm{Tr}(J_l^\dagger U)}}_{\mathscr{I}_{N_c}\left(\frac{\beta}{2N_c} J_l^\dagger, \frac{\beta}{2N_c} J_l\right)}$$

Examples:

$U(1)$: $\mathscr{I}_1(A,B) = I_0(\sqrt{AB}),$

$SU(2)$: $\mathscr{I}_2(A,B) = \dfrac{2 I_1(\|A,B\|)}{\|A,B\|}, \quad \|A,B\|^2 = \dfrac{1}{4}\left(\mathrm{Tr}(AB) + \det A + \det B\right)$

$SU(3)$: $\mathscr{I}_3(A^\dagger, A) = \displaystyle\sum_{j,k,l,n=0}^{\infty} \frac{2\, \mathrm{Tr}(A^\dagger A)^j \, \mathrm{Tr}_{AS}(A^\dagger A)^l \; \det(A^\dagger A)^k \; (\det A^\dagger + \det A)^n}{j!k!l!n!(j+2k+3l+n+2)!(k+2l+n+1)!}$

Eriksson, Svartholm & Skagerstam '81

where $I_\nu(z)$ are modified Bessel functions.

This is an exact rewriting of the partition function of pure Yang-Mills with all the link variables integrated out.

0-link action: full theory

The generalization to $N_f > 0$ is formally obtained by solving the group integrals for generic Grassmann-even-valued sources:

$$\mathcal{Z}_{QCD} = \int \mathcal{D}\chi\mathcal{D}\overline{\chi}\; e^{2a\Sigma_f m_f \overline{\chi}^f \chi^f} \int dP(Q,R) \prod_l \underbrace{\int dU\; e^{\frac{\beta}{2N_c}\mathrm{Tr}((J_l+\Sigma_f a_f K_l^f)^\dagger U + (J_l+\Sigma_f b_f K_l^f) U^\dagger)}}_{\mathscr{I}_{N_c}\left(\frac{\beta}{2N_c} J_l^\dagger + \Sigma_f a_f K_l^{f\dagger},\, \frac{\beta}{2N_c} J_l + \Sigma_f b_f K_l^f\right)}$$

Use the Wright function:

$$\phi_\nu(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!(k+\nu)!} = z^{-\nu/2} I_\nu(2\sqrt{z}), \qquad \phi'_\nu(z) = \phi_{\nu+1}(z)$$

$U(1)$, $N_f = 1$:

$$\begin{aligned}
\mathscr{I}_1(\overline{J} + a\overline{K}, J + bK) &= \phi_0((\overline{J} + a\overline{K})(J + bK)) \\
&= \phi_0(\overline{J}J)\left(1 + \frac{\phi_1(\overline{J}J)}{\phi_0(\overline{J}J)}\left(aJ\overline{K} + b\overline{J}K + ab\overline{K}K\right) + \frac{\phi_2(\overline{J}J)}{\phi_0(\overline{J}J)} ab\overline{J}J\overline{K}K\right)
\end{aligned}$$

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$U(1)$, arbitrary N_f:

$$\mathscr{I}_1\left(\overline{J} + a\cdot\overline{K}, J + b\cdot K\right) = \phi_0(\overline{J}J) \sum_{k_{ff'}=0}^{1}{}' \sum_{q_f^{\pm}=0}^{1}{}' \frac{\phi_m(\overline{J}J)}{\phi_0(\overline{J}J)} \prod_{f,f'=1}^{N_f} \frac{(a_f b_{f'} \overline{K}_f K_{f'})^{k_{ff'}}}{k_{ff'}!} \frac{(a_f J\overline{K}_f)^{q_f^+}}{q_f^+!} \frac{(b_{f'}\overline{J}K_{f'})^{q_{f'}^-}}{q_{f'}^-!}$$

$$0 \le k = \sum_{f,f'} k_{ff'} \le 2N_f - 1 \qquad q_f^+ + \sum_{f'} k_{ff'} \le 1 \qquad 0 \le m = k + q^+ + q^- \le 2N_f$$

$$0 \le q^{\pm} = \sum_f q_f^{\pm} \le 2N_f \qquad q_f^- + \sum_{f'} k_{f'f} \le 1$$

0-link action: full theory

The generalization to $N_f > 0$ is formally obtained by solving the group integrals for generic Grassmann-even-valued sources:

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$SU(2), N_f = 1$:

$$\mathscr{I}_2 = \phi_1(\|J\|)\left(1 + \sum_{k=1}^{2}(k+1)!\frac{\phi_{k+1}(\|J\|)}{\phi_1(\|J\|)}\frac{(2-k)!}{2!k!}(ab)^k \mathrm{Tr}(K^\dagger K)^k + \frac{\phi_2(\|J\|)}{\phi_1(\|J\|)}\left(a^2 \det K^\dagger + b^2 \det K\right)\right.$$
$$\left. + \sum_{k=0}^{2}{}' \sum_{b^\pm=0}^{1}{}' \sum_{q^\pm=0}^{2}{}' \frac{\phi_{m+1}(\|J\|)}{k!\phi_1(\|J\|)}\frac{a^{p^+}}{q^+!}\frac{b^{p^-}}{q^-!}\mathrm{Tr}(K^\dagger K)^k \det(K^\dagger)^{b^+} \det(K)^{b^-} \mathrm{Tr}(\widetilde{J}K^\dagger)^{q^+}\mathrm{Tr}(\widetilde{J}^\dagger K)^{q^-}\right)$$

$0 \le k + b^+ + b^- \le 3$
$1 \le q^+ + q^- \le 4$
$1 \le m = k + b^+ + b^- + q^+ + q^- \le 4$

$\widetilde{J} = J - J^\dagger + \mathrm{Tr}(J^\dagger) I_2$
$\|J\| = \mathrm{Tr}(J^\dagger J) + \det J^\dagger + \det J$

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$SU(2)$, arbitrary N_f:

$$\mathscr{I}_2 = \sum_{k_{ff'}=0}^{2}{}' \sum_{b^\pm=0}^{1}{}' \sum_{q^\pm=0}^{2}{}' \sum_{d^\pm_{ff'}=0}^{2}{}' \phi_{m+1}(\|J\|) \prod_{f,f'=1}^{N_f} \frac{\mathrm{Tr}(a_f b_{f'} K^{f\dagger} K^{f'})^{k_{ff'}}}{k_{ff'}!} \frac{\mathrm{Tr}(a\widetilde{J} K^{f\dagger})^{q_f^+}}{q_f^+!} \frac{\mathrm{Tr}(b\widetilde{J}^\dagger K^{f'})^{q_{f'}^-}}{q_{f'}^+!}$$

$$\times \prod_{f,f'=1}^{N_f} \frac{\det(K^{f\dagger})^{b_f^+}}{b_f^+!} \frac{\det(K^{f'})^{b_{f'}^-}}{b_{f'}^+!} \prod_{f<f'} \frac{\left(a_f a_{f'} \Phi(K^{f\dagger}, K^{f'\dagger})\right)^{d^+_{ff'}}}{d^+_{ff'}!} \frac{\left(b_f b_{f'} \Phi(K^f, K^{f'})\right)^{d^-_{ff'}}}{d^-_{ff'}!}$$

$$0 \le k \le N_f \qquad 0 \le c = k + b^+ + b^- + d^+ + d^- \le 4N_f - 1 \qquad q_f^+ + \sum_{f'} k_{ff'} + \sum_{f'} d^+_{ff'} + b_f^+ \le 2$$

$$0 \le b^\pm \le N_f \qquad 0 \le q = q^+ + q^- \le 4N_f \qquad q_f^- + \sum_{f'} k_{f'f} + \sum_{f'} d^-_{ff'} + b_f^- \le 2$$

$$0 \le q^\pm \le N_f \qquad 0 \le m = c + q \le 4N_f \qquad \Phi(A,B) = \mathrm{Tr}(A)\mathrm{Tr}(B) - \mathrm{Tr}(AB)$$

$$0 \le d^\pm_{ff'} \le \lfloor N_f/2 \rfloor \qquad \|J\| = \mathrm{Tr}(J^\dagger J) + \det J^\dagger + \det J$$

MDP model for $U(1)$, $N_f = 1$

Integrating out the lattice fermions *after* the gauge fields results in a combinatorial partition function for color singlets:

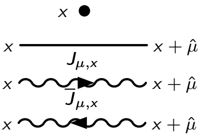
$$\mathcal{Z}_{QED}(\beta) = \int dP(Q,R) \prod_l \phi_0(\beta|J_l|) \sum_{\{n,k,\ell\}} w_G(n,\ell,J)\; w_M(n)\; w_D(k)\; w_F(\ell,J)$$

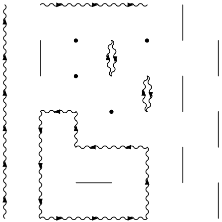
$$w_G(k,q,J) = \prod_l \frac{\phi_{m_l}(\beta|J_l|)}{\phi_0(\beta|J_l|)}, \quad w_M(n) = \prod_x (2ma)^{n_x}, \quad w_D(k) = \prod_l \gamma^{2k_l \delta_{\mu 4}}, \quad m_l = k_l + q_l^+ + q_l^-$$

$$w_F(\ell,J) = \sigma(\ell)\gamma^{(N_{+\hat{4}}+N_{-\hat{4}})} e^{r_\ell N_\tau a_\tau \mu} \prod_{l\in\ell} \bar{J}_l^{q_l^-} J_l^{q_l^+} \in \mathbb{C}, \qquad \sigma(\ell) = (-1)^{r_\ell + N_-(\ell)} \prod_{l\in\ell} \eta_l$$

DOF:

- **Monomers:** n_x
- **Dimers:** $k_{\mu,x}$
- **Electrons:** $q^+_{\mu,x}$, $q^-_{\mu,x}$





MDP model for $U(1)$, $N_f = 1$

Integrating out the lattice fermions *after* the gauge fields results in a combinatorial partition function for color singlets:

$$\mathcal{Z}_{QED}(\beta) = \int dP(Q,R) \prod_l \phi_0(\beta|J_l|) \sum_{\{n,k,\ell\}} w_G(n,\ell,J)\; w_M(n)\; w_D(k)\; w_F(\ell,J)$$

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$$w_F(\ell,J) = \sigma(\ell)\gamma^{(N_{+\hat{4}}+N_{-\hat{4}})} e^{r_\ell N_\tau a_\tau \mu} \prod_{l\in\ell} \bar{J}_l^{q_l^-} J_l^{q_l^+} \in \mathbb{C}, \qquad \sigma(\ell) = (-1)^{r_\ell + N_-(\ell)} \prod_{l\in\ell} \eta_l$$

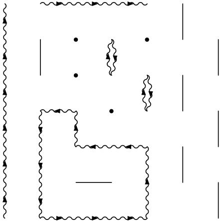
Constraints:

$\bar{\chi}$: $n_x + \sum_\mu (k_{\mu,x} + k_{-\mu,x}) + \sum_\mu q^+_{\mu,x} = 1$

χ: $n_x + \sum_\mu (k_{\mu,x} + k_{-\mu,x}) + \sum_\mu q^-_{\mu,x} = 1$

$\Longrightarrow \sum_\mu q^+_{\mu,x} = \sum_\mu q^-_{\mu,x}$

Self-avoiding electron loops



MDP model for $SU(2)$, $N_f = 1$

Integrating out the lattice fermions *after* the gauge fields results in a combinatorial partition function for color singlets:

$$\mathcal{Z}_{QCD}(\beta) = \int dP(Q,R) \prod_l \phi_1(\beta \| J_l \|) \sum_{\{n,k,\ell,\ell'\}} w_G(k,\ell',J)\; w_M(n)\; w_D(k)\; w_B(\ell)\; w_F(\ell',J)$$

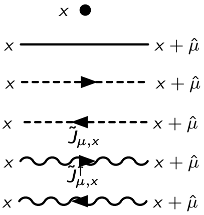
$$w_G(k,\ell',J) = \prod_l \frac{\phi_{m_l}(\beta|J_l|)}{\phi_0(\beta|J_l|)}, \qquad w_M(n) = \prod_x \frac{2!}{n_x!}(2ma)^{n_x}, \qquad w_D(k) = \prod_l \gamma^{2k_l\delta_{\mu 4}} \frac{(2-k_l)!}{2!k_l!},$$

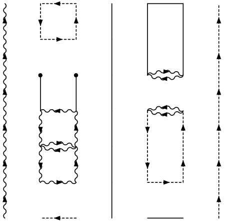
$$w_B(\ell) = \sigma(\ell)\gamma^{2\Delta N_{\hat{4}}} e^{r_\ell N_\tau a_\tau \mu}, \qquad w_F(\ell',J) = \sigma(\ell')\gamma^{\Delta N_{\hat{4}}} e^{r_{\ell'} N_\tau a_\tau \mu} \mathrm{Tr}\left(\mathcal{P}\prod_{l\in\ell'} J_l^{\dagger q_l^-} J_l^{q_l^+}\right) \in \mathbb{C}$$

$$m_l = k_l + q_l^+ + q_l^-$$

DOF:

- **Monomers:** n_x
- **Dimers:** $k_{\mu,x}$
- **Baryons:** $b^+_{\mu,x}$, $b^-_{\mu,x}$
- **Quarks:** $q^+_{\mu,x}$, $q^-_{\mu,x}$





MDP model for $SU(2)$, $N_f = 1$

Integrating out the lattice fermions *after* the gauge fields results in a combinatorial partition function for color singlets:

$$\mathcal{Z}_{QCD}(\beta) = \int dP(Q,R) \prod_l \phi_1(\beta \| J_l \|) \sum_{\{n,k,\ell,\ell'\}} w_G(k,\ell',J)\ w_M(n)\ w_D(k)\ w_B(\ell)\ w_F(\ell',J)$$

$$w_G(k,\ell',J) = \prod_l \frac{\phi_{m_l}(\beta|J_l|)}{\phi_0(\beta|J_l|)}, \qquad w_M(n) = \prod_x \frac{2!}{n_x!}(2ma)^{n_x}, \qquad w_D(k) = \prod_l \gamma^{2k_l\delta_{\mu 4}} \frac{(2-k_l)!}{2!k_l!},$$

$$w_B(\ell) = \sigma(\ell)\gamma^{2\Delta N_{\hat{4}}} e^{r_\ell N_\tau a_\tau \mu}, \qquad w_F(\ell',J) = \sigma(\ell')\gamma^{\Delta N_{\hat{4}}} e^{r_{\ell'} N_\tau a_\tau \mu} \mathrm{Tr}\left(\mathcal{P}\prod_{l\in\ell'} J_l^{\dagger q_l^-} J_l^{q_l^+}\right) \in \mathbb{C}$$

$$m_l = k_l + q_l^+ + q_l^-$$

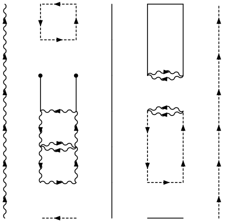
Constraints:

$$\bar{\chi}:\ n_x + \sum_\mu \left(k_{\mu,x} + k_{-\mu,x} + q^+_{\mu,x} + 2b^+_{\mu,x}\right) = 2$$

$$\chi:\ n_x + \sum_\mu \left(k_{\mu,x} + k_{-\mu,x} + q^-_{\mu,x} + 2b^-_{\mu,x}\right) = 2$$

$$\Longrightarrow \sum_\mu (q^+_{\mu,x} + 2b^+_{\mu,x}) = \sum_\mu (q^-_{\mu,x} + 2b^-_{\mu,x})$$

Non-self-avoiding quark loops



Sign problem

Quark worldlines have with complex weights:

$$U(1):\ \ w_F(\ell') = \sigma(\ell')\gamma^{\Delta N_{\hat{4}}} e^{r_{\ell'} N_\tau a_\tau \mu} \prod_{l\in\ell'} \bar{J}_l^{q_l^-} J_l^{q_l^+} \in \mathbb{C}$$

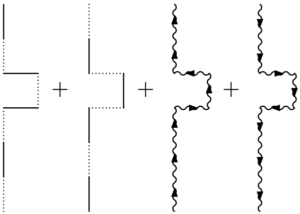
$$SU(2):\ \ w_F(\ell') = \sigma(\ell')\gamma^{\Delta N_{\hat{4}}} e^{r_{\ell'} N_\tau a_\tau \mu}\, \mathrm{Tr}\left(\mathcal{P} \prod_{l\in\ell'} J_l^{\dagger q_l^-} J_l^{q_l^+}\right) \in \mathbb{C}$$

A naïve **Karch-Mütter resummation** of quark loops does not help, e.g. in $U(1)$:

$$w_F(\ell')+w_F(-\ell') \propto \cosh(r_{\ell'} N_c N_\tau a_\tau \mu + i \arg(J_{\ell'})) \in \mathbb{C}$$

But due to the $\{Q,R\} \to \{\overline{Q},\overline{R}\}$ symmetry of the 0-link pure gauge action and measure, the complex phase is resummed away:

$$\begin{aligned} &w_F(\ell,Q,R) + w_F(-\ell,Q,R) \\ +&w_F(\ell,\overline{Q},\overline{R}) + w_F(-\ell,\overline{Q},\overline{R}) \\ &\propto \cosh(r_\ell N_c N_\tau a_\tau \mu)\ \mathrm{Re}(J_\ell(Q,R)) \in \mathbb{R} \end{aligned}$$



Monte Carlo

$$
\begin{aligned}
\mathcal{Z}_{QCD} &= \int dP_1(Q,R) \prod_l \mathcal{I}_{N_c}(J_l^\dagger, J_l) \sum_{\{n,k,\ell\}} w_G(k,\ell,J)\; w_M(n)\; w_D(k,J)\; w_P(\ell,J) \\
&\equiv \mathcal{Z}_{N_f=0} \left\langle \sum_{\{n,k,\ell\}} w_G(k,\ell,J)\; w_M(n)\; w_D(k,J)\; w_P(\ell,J) \right\rangle_{N_f=0}
\end{aligned}
$$

Algorithm:

1. Gaussian heatbath update of the auxiliary fields Q, R
2. Construct $J_l \equiv J_l(Q,R)$ on each link
3. Accept J_l with probability $\min(1, \prod_l \mathcal{I}_{N_c}(J_l^\dagger, J_l))$
4. Propose a MDP configuration $\{n,k,\ell\}$.
5. Accept the MDP configuration with probability $\min(1, w_G w_M w_D w_P)$.

Conclusions

- MDP models are simple only at $\beta = 0$; $O(\beta^n)$ corrections are cumbersome.
- Auxiliary variables allow **full and exact integration of gauge fields at arbitrary β** $\Rightarrow$ MDP model of full lattice QCD.
- Karsch-Mütter resummation can be applied to quark loops to remove complex phases from their weights.

Next:

- Lattice simulations of the full MDP models towards weak coupling.