Θ dependence of 4D SU(N) gauge theories at finite temperature

Ettore Vicari

University of Pisa and INFN

ABSTRACT: The dependence of 4D SU(N) gauge theories on the topological θ term is addressed at zero and finite temperature, and in particular in the large-N limit. General arguments and numerical analyses exploiting the lattice formulation show that it drastically changes across the deconfinement transition. The low-T phase is characterized by a large-N scaling with θ/N as relevant variable, while in the high-T phase the scaling variable is just θ and the free energy is essentially determined by the instanton-gas approximation.

SIGN 2014

4D $SU(N)$ gauge theories have a nontrivial θ dependence

$$\mathcal{L}_{\theta,\text{Euclidean}} = \frac{1}{4} F^a_{\mu\nu}(x) F^a_{\mu\nu}(x) - i\theta \frac{g^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x),$$

$q(x) = \frac{g^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x)$ is the topological charge density.

The topological θ term violates parity and time reversal

$|\theta| \lesssim 10^{-9}$ from experimental bounds on the neutron electric dipole moment: $|d_n| < 3 \times 10^{-26}$ e cm, and $d_n \sim \theta e m_\pi^2 / m_n^3 \approx 10^{-16} \theta \, e \, \text{cm}$.

Nevertheless θ dependence remains an interesting issue,

for example, $U(1)_A$ problem $\rightarrow$ the axial $U(1)_A$ symmetry is not realized in the QCD spectrum, neither explicitly nor as a Goldstone mechanism ($m_{\eta'} > \sqrt{3} m_\pi$), being violated at quantum level

θ **dependence** vanishes in perturbation theory.

In the semiclassical picture, contributions from classical instanton solutions with nontrivial topology, $\int d^4x \, q[A_I(x)] = Q$, give rise to tunneling between n-vacua, leading to θ vacua: $|\theta\rangle = \sum_n e^{in\theta}|n\rangle$

The U(1)$_A$ charge is not conserved due to the chiral anomaly $\partial_\mu j_5^\mu(x) = i2N_f q(x)$.

A robust numerical evidence of a nontrivial θ dependence from MC simulations of the lattice formulation of the theory.

At finite T: this issue is related to the expected softening of the U(1)$_A$ breaking, to understand the main features of its T dependence,

effective U(1)$_A$ symmetry restoration, T-dep of η' mass, nature of the hadron-to-quarkgluon transition, spectrum of the excitations, etc

possible evidences from heavy-ion collisions, e.g. claims of a softening of the η' mass from Au+Au collisions at RHIC (Csorgo *etal*, PRL 1010)

Plan of the talk:

- General scenario for the θ dep from $T = 0$ to high-T
- θ dependence within the large-N framework $\longrightarrow$ large-N scaling expected to hold at $T = 0$
- analytic and periodic θ dependence from instanton-gas approximations $\longrightarrow$ expected to be effective at $T \gg T_c$
- sharp change of θ dependence across the deconfinement transition
- Monte Carlo simulations in the presence of θ are affected by the sign problem: expansion around $\theta = 0$ and simulations at imaginary θ
- Overview of lattice results for the θ expansion at $T = 0$ and finite T, across the deconfinement transition.
- Conclusions and a few remarks on θ dependence in full QCD

θ dependence of the ground-state and free energy

$T = 0$ ground-state energy:

$$E(\theta) = -\frac{1}{V_4}\ln\int[dA]\exp\left(-\int d^4x\mathcal{L}_\theta\right)$$

$$\mathcal{L}_\theta = \frac{1}{4}F^a_{\mu\nu}(x)F^a_{\mu\nu}(x) - i\theta q(x), \qquad q(x) = \frac{g^2}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}F^a_{\mu\nu}(x)F^a_{\rho\sigma}(x)$$

The free energy at finite temperature (Gross, Pisarski, Yaffe, RMP 1981)

$$F(\theta,T) = -\frac{1}{\mathcal{V}_4}\ln \mathrm{Tr}e^{-H/T} = -\frac{1}{\mathcal{V}_4}\ln\int[dA]\exp\left(-\int_0^{1/T}dt\int d^3x\,\mathcal{L}_\theta\right),$$

$$\mathcal{V}_4 \equiv T/V_3, \qquad A_\mu(1/T,\mathbf{x}) = A_\mu(0,\mathbf{x}), \qquad E(\theta) = F(\theta,0)$$

In the pure gauge theory θ is a dimensionless RG invariant parameter, i.e. it does not renormalize in appropriate RG schemes, such as the $\overline{\mathrm{MS}}$ scheme

The ground-state/free energy can be parametrized as

$$\mathcal{F}(\theta,T) \equiv F(\theta,T) - F(0,T) = \frac{1}{2}\chi(T)\theta^2 s(\theta,T)$$

$\chi(T) = \int d^4x \langle q(x)q(0)\rangle_{\theta=0} = \langle Q^2\rangle_{\theta=0}/\mathcal{V}_4$ is the topological susceptibility, $s(\theta,T)$ is a dimensionless even function of θ such that $s(0,T)=1$.

Analyticity at $\theta = 0$ (CP is not broken at $\theta=0$, Vafa, Witten, PRL 1984) $\longrightarrow$
$s(\theta,T) = 1 + b_2(T)\theta^2 + b_4(T)\theta^4 + \cdots$, (V, Panagopoulos, PhysRep 2009)

b_i are dimensionless RG invariant quantities,

related to the zero-momentum n-point correlation functions of $q(x)$, e.g. $b_2 = -\chi_4/(12\chi)$ and $\chi_4 = \int d^4x_1 d^4x_2 d^4x_3 \langle q(0)q(x_1)q(x_2)q(x_3)\rangle_c|_{\theta=0}$, and the **cumulants** of $P(Q)$.

If $b_{2n} = 0$ then the distribution is Gaussian $P(Q) = \frac{1}{\sqrt{2\pi\langle Q^2\rangle}} \exp\left(-\frac{Q^2}{2\langle Q^2\rangle}\right)$

Analogously, θ dependence of the spectrum

$\sigma(\theta) = \sigma\left(1 + s_2\theta^2 + ...\right)$, where σ is the string tension at $\theta = 0$.

Similarly for the lowest glueball state:

$M(\theta) = M\left(1 + g_2\theta^2 + ...\right)$, where M is the 0^{++} glueball mass at $\theta = 0$. At $\theta \neq 0$, the lightest glueball state does not have a definite parity anymore, but it becomes a mixed state of 0^{++} and 0^{-+} glueballs.

The coefficients of the above expansions can be computed from appropriate correlators at $\theta = 0$, involving the particle sources and the topological charge density.

(Del Debbio, Manca, Panagopoulos, Skouroupathis, V., 2006)

Within the large-N framework ($N \to \infty$, $g^2 N$ fixed) the U(1)$_A$ problem is explained by a θ dependence at the leading $1/N$ order

WV relations: $\chi = \frac{f_s^2 m_s^2}{4N_f}$ or $\frac{4N_f}{f_\pi^2}\chi = m_{\eta'}^2 + m_\eta^2 - 2m_K^2$ (Witten, Veneziano, 1979)

Large-N scaling to $\mathcal{L}_\theta = \frac{1}{4} F_{\mu\nu}^a(x) F_{\mu\nu}^a(x) - i\theta \frac{g^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}^a(x) F_{\rho\sigma}^a(x)$

$\longrightarrow$ the relevant scaling variable is $\bar{\theta} \equiv \theta/N$

$$f(\theta) \equiv \frac{F(\theta) - F(0)}{\sigma^2} = \frac{1}{2} C\theta^2 (1 + b_2\theta^2 + b_4\theta^4 + ...) = N^2 \bar{f}(\bar{\theta})$$

$\bar{f}(\bar{\theta})$ has a nontrivial large-N limit: $\frac{1}{2} C_\infty \bar{\theta}^2 (1 + \bar{b}_2 \bar{\theta}^2 + \bar{b}_4 \bar{\theta}^4 + \cdots)$,

where $C \equiv \chi/\sigma^2 = C_\infty + c_2/N^2 + ...$, and $b_{2j} = \bar{b}_{2j}/N^{2j} + ...$.

A multibranched $F(\theta)$, $F(\theta) - F(0) = \mathcal{A}\,\mathrm{Min}_k\,(\theta + 2\pi k)^2 + O\,(1/N)$ (Witten, AP 1980, PRL 1998), avoids the apparent incompatibility with periodicity in θ.

This scenario is analytically verified within the theoretical laboratory of the **2D** CP^{N-1} **models** with a N-component complex field z ($\bar{z}z = 1$)

$$\mathcal{L}_\theta = \frac{N}{2g}\overline{D_\mu z}\, D_\mu z - i\theta q(x), \quad D_\mu = \partial_\mu + iA_\mu, \;\; A_\mu = i\bar{z}\partial_\mu z,$$

$$q(x) = \frac{1}{2\pi}\,\epsilon_{\mu\nu}\,\partial_\mu A_\nu, \qquad F(\theta) = -\frac{1}{V}\ln\int [dA]\exp\left(-\int d^2x \mathcal{L}_\theta\right)$$

They share several features with QCD: asymptotic freedom, topology, θ vacua.

Unlike 4D $SU(N)$ gauge theories, **systematic** $1/N$ **expansion**, keeping g fixed, around the large-N saddle-point solution.

Large-N scaling analogous to 4D SU(N) gauge theories at $T = 0$:

$$f(\theta) \equiv \xi^2[F(\theta) - F(0)] = \frac{1}{2}C\theta^2\left(1 + \sum_{n=1} b_{2n}\theta^{2n}\right), \quad C \equiv \chi\xi^2, \;\; \chi = \int d^2x \langle q(0)q(x)\rangle,$$

$$f(\theta) \approx N\bar{f}(\bar{\theta} \equiv \theta/N), \quad \bar{f}(\bar{\theta}) = \frac{1}{2}\widehat{C}\bar{\theta}^2(1 + \sum_{n=1}\bar{b}_{2n}\bar{\theta}^{2n}), \quad \widehat{C} = NC, \;\; \bar{b}_{2n} = N^{2n}b_{2n}$$

$1/N$ **calculations confirm the large-N scaling** see e.g. V, Panagopoulos, PR 2009

Semiclassically θ dependence arises from instantons.

The one-instanton contribution $e^{-8\pi^2/g^2} e^{i\theta} = \left(e^{-8\pi^2/(g^2 N)} e^{i\theta/N}\right)^N$ suggests an exponentially small θ dep. **This conclusion is incorrect**: the instanton gas approximation fails due to infrared divergences.

At finite temperature, periodic instantons in $1/T$ with integer Q.

T **provides the infrared cutoff** to the instanton-size distribution, $n_I(\rho) \sim e^{-S(A_I)} \sim e^{-[8\pi^2/g^2 + 2N(\pi\rho T)^2]}$. (Gross,Pisarski,Yaffe, RMP 1981)

Dilute instanton-gas (DIG) approximation at finite T summing over n_+ instantons and n_- antiinstantons:

$$
\begin{aligned}
Z_\theta &= \mathrm{Tr}\, e^{-H_\theta/T} \approx \sum \frac{1}{n_+! n_-!} (\mathcal{V}_4 D)^{n_+ + n_-} e^{-\frac{8\pi^2(n_+ + n_-)}{g^2} + i\theta(n_+ - n_-)} \\
&= \exp\left[\cos\theta \times 2\mathcal{V}_4 D \times e^{-8\pi^2/g^2}\right]
\end{aligned}
$$

therefore $\mathcal{F}(\theta, T) \equiv F(\theta, T) - F(0, T) \approx \chi(T)\,(1 - \cos\theta)$

At high T ... dilute instanton-gas (DIG) approximation

At one loop $\partial F/\partial\theta = \sin\theta \int_0^\infty d\rho n_{_I}(\rho) \sim \sin\theta \times T^4 e^{-8\pi^2/g^2(T)}$

$$\mathcal{F}(\theta,T) \approx \chi(T)\,(1-\cos\theta)\,, \qquad \chi(T) \approx T^4 \exp[-8\pi^2/g^2(T)] \sim T^{-\frac{11}{3}N+4},$$

using $8\pi^2/g^2(T) \approx (11/3)N\ln(T/\Lambda) + O(\ln\ln T/\ln^2 T)$

DIG is a **good approximation when the overlap between instantons becomes negligible**, thus at large T where $\chi(T)$ is suppressed

The high-T DIG θ dependence qualitatively differs from that at $T=0$:

(•) analytic and periodic θ dependence

(•) The large-N scaling is not realized by the DIG approximation: the relevant variable for the instanton gas is θ, and **not** θ/N

(•) $\chi(T)$ gets exponentially suppressed in the large-N regime, suggesting a rapid decrease of the topological activity with increasing N at high T

• The low-T and high-T phases are separated by a **1$^{\rm st}$-order deconfinement transition**, at $T_c/\sqrt{\sigma} \approx 0.545(2) + 0.46(2)/N^2$ (Lucini, etal, 2004,2012) getting stronger with increasing N, $L_h \sim N^2$

• for $T \ll T_c \rightarrow$ large-N scaling with θ/N **as scaling variable** $\rightarrow$ $\chi/\sigma^2 \approx C_\infty + c/N^2$ and $b_k \approx \bar{b}_k/N^k$.

Does it extend up to T_c^-?

• for $T \gg T_c \rightarrow$ analytic θ dependence by DIG approximation: $\mathcal{F} \approx \chi(T)(1 - \cos\theta)$ with $\chi(T) \sim T^{-\frac{11}{3}N+4}$.

Does it extend down to T_c^+?

The change between the low-T and high-T θ dependence occurs around the deconfinement transition.

Some hints also from models like ADS-CFT, holographic models, etc... (Witten, PRL 1998; Parnashev, Zhitnisky PRD 1998; Unsal PRD 2012, etc)

A quantitative study of θ dependence requires a nonpertubative approach: **Wilson lattice formulation of QCD**, from the critical continuum limit of a statistical 4D lattice model: $Z = \int [dU] \exp(-S_L)$

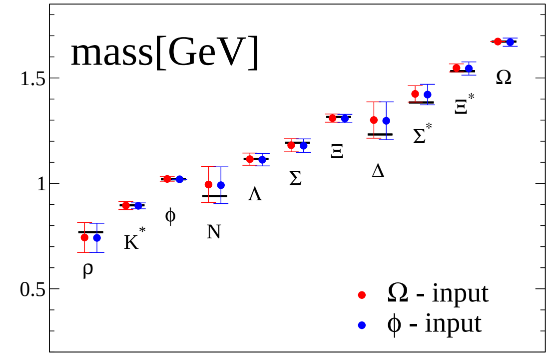
$$S_L = -\frac{2a^4}{g_0^2} \sum \mathrm{Re}\mathrm{Tr}\,[U_\mu(x) U_\nu(x+a\hat{\mu}) U_\mu^\dagger(x+a\hat{\nu}) U_\nu^\dagger(x)], \quad U_\mu \in SU(N)$$

Formally, in the $a \to 0$ limit, one recovers $S = \int d^4x \frac{1}{2g_0^2} \mathrm{Tr}\, F_{\mu\nu} F_{\mu\nu}$.

The statistical theory develops a mass gap, and therefore a length scale ξ.
The continuum theory is defined in the critical limit $g_0^2 \to 0$, when $\xi \to \infty$:
$\xi = a \exp \int^{g_0} dg/\beta_L(g) \sim a(b_0 g_0^2)^{-b_1/2b_0^2} \exp[1/(2b_0 g_0^2)]$, where
$\beta_L = a dg_0/da = -b_0 g_0^3 - b_1 g_0^5 + \ldots$

The lattice formulation lends itself to statistical-physics techniques, such as **MC simulations**

Impressive agreement ...



Topology from the lattice is a nontrivial issue

The lattice regularization makes the topology strictly trivial, because its configuration space is simply connected. The physical topological properties are expected to be recovered in the continuum limit.

From a QFT point view, problems are related to the peculiar singular behavior of the two-point function $\langle q(0)q(x)\rangle$ when $x \to 0$

Various methods to estimate Q:

Smoothing methods read Q after smoothing the configurations. Several variants, such as cooling, smearing, gradient flow, ..., which become equivalent for not-so-large β, $\beta \gtrsim 6$ for SU(3), see e.g. Bonati, D'Elia, arXiv:1401.2441

The **fermionic definition** through the index of the overlap Dirac operator provides a well–defined estimator for Q, but at a much higher computational cost. Other methods: Geometrical, Off-equilibrium, etc...

Several numerical checks and comparisons have shown that they provide accurate and reliable results.

The complex nature of the θ term $i\theta q(x)$ in the Euclidean QCD Lagrangian prohibits a direct MC simulation at $\theta \neq 0$.

Information on the θ dependence of physically relevant quantities can be obtained by computing their expansion around $\theta = 0$:

Expansion around $\theta = 0$: $\quad F(\theta) - F(0)) = \frac{1}{2}\chi\theta^2(1 + b_2\theta^2 + b_4\theta^4 + ...)$

χ and b_{2n} from correlation functions $\langle q(x_1)q(x_2)...q(x_{2n})\rangle$ at $\theta = 0$,

$$b_2 = -\frac{\chi_4}{12\chi}, \qquad \chi_4 = \frac{1}{V}\left[\langle Q^4\rangle - 3\langle Q^2\rangle^2\right]_{\theta=0}, \qquad Q = \sum_x q(x)$$

$$b_4 = \frac{\chi_6}{360\chi}, \qquad \chi_6 = \frac{1}{V}\left[\langle Q^6\rangle - 15\langle Q^2\rangle\langle Q^4\rangle + 30\langle Q^2\rangle^3\right]_{\theta=0}$$

In the continuum limit, $b_{2k,L} \approx b_{2k} + a^2\sigma^2$ for $a \to 0$

$b_{2n} \to$ deviations from a Gaussian $P(Q) = \frac{1}{\sqrt{2\pi\langle Q^2\rangle}} \exp\left(-\frac{Q^2}{2\langle Q^2\rangle}\right)$

b_k requires large statistics, due to the cancellation of volume factors

Alternatively, **imaginary θ term** $\theta_i = -i\theta$ (Panagopoulos, V, JHEP 2011)

$$Z_L = \int [dA] \exp\left(-S_L + \Theta_L Q_L\right), \quad S_L = -\frac{2}{g_0^2} \sum_{x,\mu>\nu} \mathrm{Re}\,\mathrm{Tr}\,\Pi_{\mu\nu}(x),$$

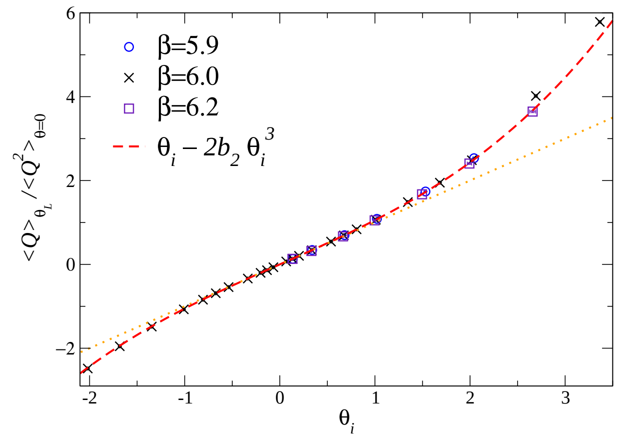
$Q_L \equiv \sum_x q_L(x)$ is a discretization of $q(x)$, $\theta_i = Z_\theta \Theta_L$ where $z_\theta = \langle Q Q_L \rangle / \langle Q^2 \rangle |_{\theta=0}$.

Replacing $\theta \equiv -i\theta_i$ in the free energy, to be eventually extended to real θ

$$\Phi(\theta_i) \equiv \mathcal{F}(-i\theta_i) = -\frac{1}{2}\chi\theta_i^2 s(-i\theta_i) = -\frac{1}{2}\chi\theta_i^2 \left(1 - b_2\theta_i^2 + b_4\theta_i^4 + \cdots\right)$$

Good evidence of scaling for $|\theta_i| \lesssim \pi$ for $N = 3$, thus of the existence of a nontrivial continuum limit for any value of θ_i.

- The θ_i dep is well described by the first few terms of the expansion around $\theta = 0$.
- θ-dep of T_c (D'Elia, Negro, PRL 2012)



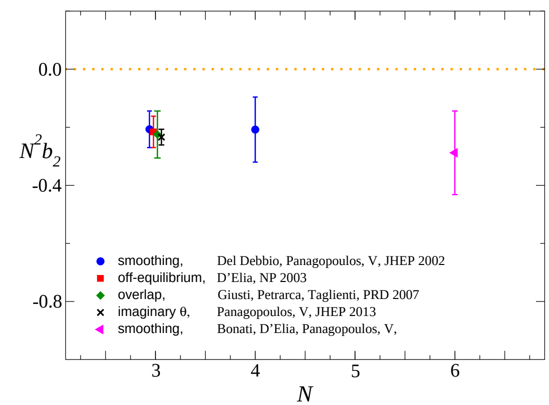
θ dependence at $T = 0$

- $\chi \equiv \partial^2 F(\theta)/\partial\theta^2|_{\theta=0} \neq 0$ for SU(3): $\chi/\sigma^2 = 0.028(2)$ by various methods

- **Nonzero large-N limit**: $\chi/\sigma^2 = 0.022(2)$, from MC simulations for $N > 3$ (by Cundy, Del Debbio, Lucini, Panagopoulos, Teper, V., Wenger, ... They support the expected large-N behavior: $\chi/\sigma^2 = C_\infty + c_2/N^2$

- **Nonzero higher-order terms** of the expansion around $\theta = 0$,

$F(\theta) - F(0) = \frac{1}{2}\chi\theta^2(1 + b_2\theta^2 + b_4\theta^4 + \ldots)$,
SU(3) estimates: $b_2 = -0.026(3)$ and $|b_4| \lesssim 0.001$
(using various methods to determine Q)



- **Vanishing large-N limit** of $b_k = O(N^{-k})$,

results consistent with $b_2 \approx \bar{b}_2/N^2$, $\bar{b}_2 \approx -0.2$,
see the plot of $N^2 b_2$ vs N

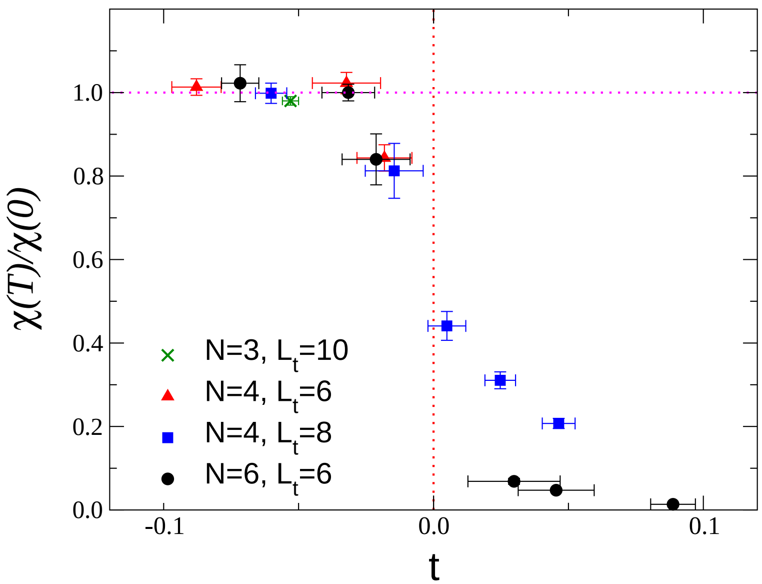
Deviations from a simple Gaussian behavior are already small at $N = 3$.

b_k requires large statistics, due to the cancellation of volume factors

• χ **at finite** T

Several MC results (Alles, Bonati, Del Debbio, D'Elia, Di Giacomo, Lucini, Panagopoulos, Teper, V., Wenger, ...)

$\chi(T)/\chi(T=0)$ vs $t \equiv T/T_c - 1$ across the transition $\longrightarrow$



• χ remains substantially unchanged in the low-T confined phase.

• A sharp change across the first-order transition, likely discontinuous

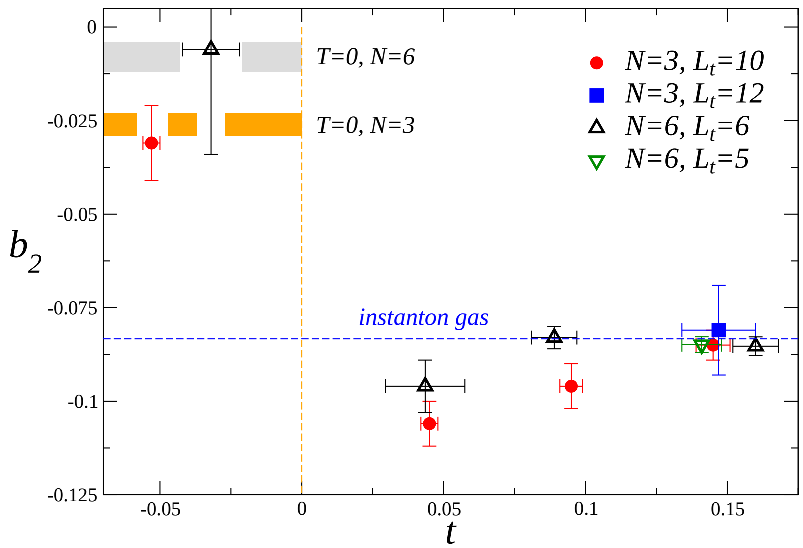
• In the high-T phase χ shows a clear suppression, which becomes stronger with increasing N, in qualitative agreement with one-loop DIG $\chi(T) \sim T^{-\frac{11}{3}N+4}$ for $T \gg T_c$, but larger T are necessary for a quantitative check of the one-loop DIG approximation of $\chi(T)$

• Higher-order terms of $F(\theta,T) = \frac{1}{2}\chi\theta^2(1 + b_2\theta^2 + b_4\theta^4 + \cdots)$ provide a more significant probe of DIG regimes, avoiding the problem of the logarithmic corrections of the prefactor

High-stat MC for $N = 3, 6$ to check large N (smoothing techniques for Q)

(Bonati, D'Elia, Panagopoulos, V, PRL 2013)

b_{2k} are compared with $T = 0$ results and DIG approx ($b_2 = -1/12$) $\longrightarrow$



• Sharp change across the deconfinement transition, likely discontinuous

• For $T > T_c$, **rapid approach to DIG θ dependence**, with deviations visible only for $t \approx 0.05$. The approach appears faster with increasing N.

• $b_4 = 0.0024(4)$ for $N = 6$ and $t = 0.09$ to be compared with $b_4 = 1/360$.

Deviations from dilute instanton gas at $t \equiv (T - T_c)/T_c \lesssim 0.1$

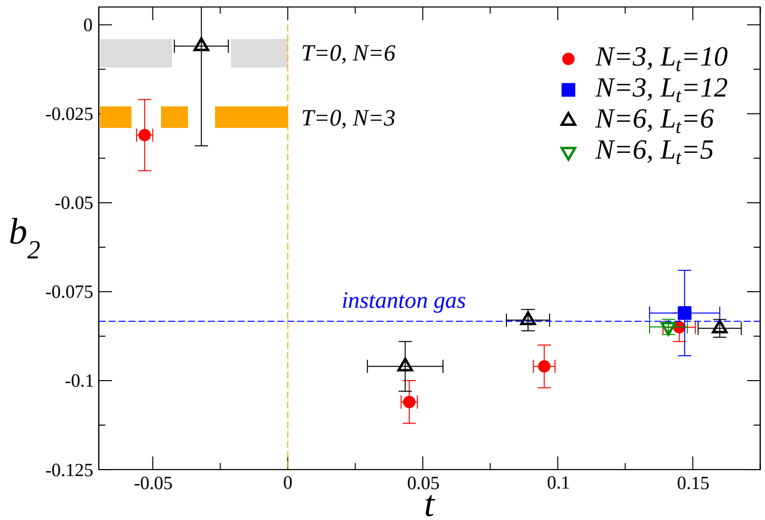
The approach to the DIG regime can be parametrized by a virial-like expansion: the asymptotic formula is corrected by a term proportional to the square of the instanton density

Since $\chi(T) \sim \rho_{\rm inst}$,

$\mathcal{F}(\theta, T) \approx \chi(1 - \cos\theta) + \chi^2 \kappa(\theta) + O(\chi^3)$

$\kappa(\theta) = \sum_{k=2} c_{2k} \sin(\theta/2)^{2k}$

Thus, $b_2 = -\frac{1}{12} + \frac{1}{8} c_4 \chi + O(\chi^2)$.



The behavior $\chi \sim T^{-11N/3+4}$ implies a rapid approach to the asymptotic DIG value, which becomes faster with increasing N.

The hard-core approximation of instanton interactions give a negative correction, i.e. $c_4 < 0$, explaining the approach from below to the DIG value $b_2 = -1/12$.

Summary of the θ dependence in 4D SU(N) gauge theories

$$\mathcal{F}(\theta,T) \equiv F(\theta,T) - F(0,T) = \frac{1}{2}\chi(T)\theta^2 \left(1 + b_2(T)\theta^2 + b_4(T)\theta^4 + \cdots\right)$$

• **Low-T phase** characterized by a **large-N scaling** with θ/N as relevant variable: $\chi/\sigma^2 \approx C_\infty + c/N^2$ and $b_k \approx \bar{b}_k/N^k$

• **Sharp change across the deconfinement transition**, likely discontinuous.

• **High-T phase**: large-N scaling is lost, the topological activity is much reduced. **The dilute instanton-gas regime sets in just above T_c**, giving an analytic dependence $F(\theta) - F(0) \approx \chi(T)(1 - \cos\theta)$.

• MC simulations nicely support the above scenario.

• The crossover around the transition becomes sharper with increasing N, suggesting that the DIG regime sets in just above T_c at large N.

Full QCD: $\frac{1}{4}F^a_{\mu\nu}(x)F^a_{\mu\nu}(x) + \sum_f \bar\psi_f(\mathrm{Re}\, m_f + i\mathrm{Im}\, m_f\,\gamma_5)\psi_f - i\theta q(x)$

θ-$\mathrm{Im}\, m_f$ are related by chiral transformations $\psi \to e^{i\alpha\gamma_5}\psi$

• θ is not RG invariant, indeed in the quark massless limit

$$\begin{pmatrix} i2N_f\, q(x) \\ \partial_\mu j^5_\mu(x) \end{pmatrix}_R = \begin{pmatrix} 1 & z-1 \\ 0 & z \end{pmatrix} \begin{pmatrix} i2N_f\, q(x) \\ \partial_\mu j^5_\mu(x) \end{pmatrix}_B$$

where $z = 1 + \frac{g^4}{16\pi^4}\frac{3c_F}{8}N_f\frac{1}{\epsilon} + O\left(g^6\right)$, and $\epsilon = 2 - d/2$, $c_F = (N^2-1)/(2N)$.

• However, the residual θ parameter when $\mathrm{Im}\, m_f = 0$ does not renormalize, essentially due to the nonrenormalizability of the anomaly equation $\partial_\mu j^5_\mu(x) = i2p(x) + i2N_f q(x)$ V, Panagopoulos, PR 2009, last arXiv.

The massless limit appears singular in this respect, just because the θ term can be completely eliminated by a chiral transformation without any physical effect.

The correct continuum limit $a \to 0$ of lattice QCD, with real fermion masses and θ term, is obtained by keeping the parameter θ fixed (apart from a finite lattice renormalization, like pure SU(N) gauge theories)

Analogous suppression of θ-dep is expected in full QCD at high-T (DIG), with a suppression of the $\mathrm{U}(1)_A$ breaking in the quark-gluon plasma.

The DIG approximation suggests that the $\mathrm{U}(1)_A$ symmetry is not exactly recovered at finite T. Residual small instanton effects in the chiral limit. In $N_f = 2$ QCD, although $\chi \sim m^2\, T^{-11N/3+16/3}$, the Dirac zero modes induce a residual contribution to the $\mathrm{U}(1)_A$ symmetry breaking:

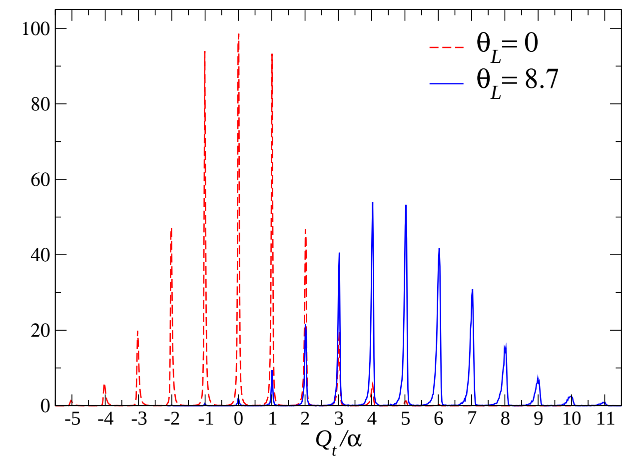
$$\chi_\pi - \chi_\delta \sim T^{-11N/3+16/3}$$

θ dependence and $\mathrm{U}(1)_A$ breaking is relevant for the nature of the finite-T transition in the chiral limit: in the case of a continuous transition its suppression leads to $[\,U(2)_L \otimes U(2)_R\,]\,/\,U(2)_V$ universality class, different from $\mathrm{O}(4)/\mathrm{O}(3)$ (Pelissetto, V, PRD 2013, Pisarski, Wilczek, PRD 1984)

- Numerical studies show the suppression (but apparently not complete) of the $U(1)_A$-breaking effects at and above the finite-T transition: Hints for an early onset of DIG by recent MC simulations of full QCD, by looking at the behavior of the relevant susceptibilities (Bazavov, etal, PRD 2012; Buchoff etal, 2013, Cossu et al, 2013).

Some results from the smoothing method.

Distribution of Q, for $N = 3$, $\beta = 6.2$ at $\theta = 0$ and $\theta_i \approx 1.5$



Stability of the lattice computation of b_2 in the high-T phase, for $N = 3$ and $N = 6$