Complex Langevin for chiral Random Matrix Theory at nonzero chemical potential

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GSI, Germany, Feb 20 2014

What Complex Langevin and the Dirac spectrum

$$(iD_\eta\gamma_\eta + \mu\gamma_0)\psi_j = z_j\psi_j$$

Why Understand complex Langevin with fermionic sign problem

... in a case with the $\mu = m_\pi/2$ barrier

... and where exact analytic results are known

How Analytically and explicit simulations

Find To take-home

Special requirement for Dirac eigenvalues in complex Langevin

Perfect convergence even for $\mu > m_\pi/2$

First a reminder: statistical physics @ $\mu = 0$

$\mu = 0$ $\langle\bar{\psi}\psi\rangle(m)$ from the Dirac eigenvalues

Eigenvalue equation

$$(iD_\eta\gamma_\eta)\psi_j = i\lambda_j\psi_j$$

$$\langle\bar{\psi}\psi\rangle(m) = \langle\sum_j \frac{2m}{\lambda_j^2 + m^2}\rangle$$

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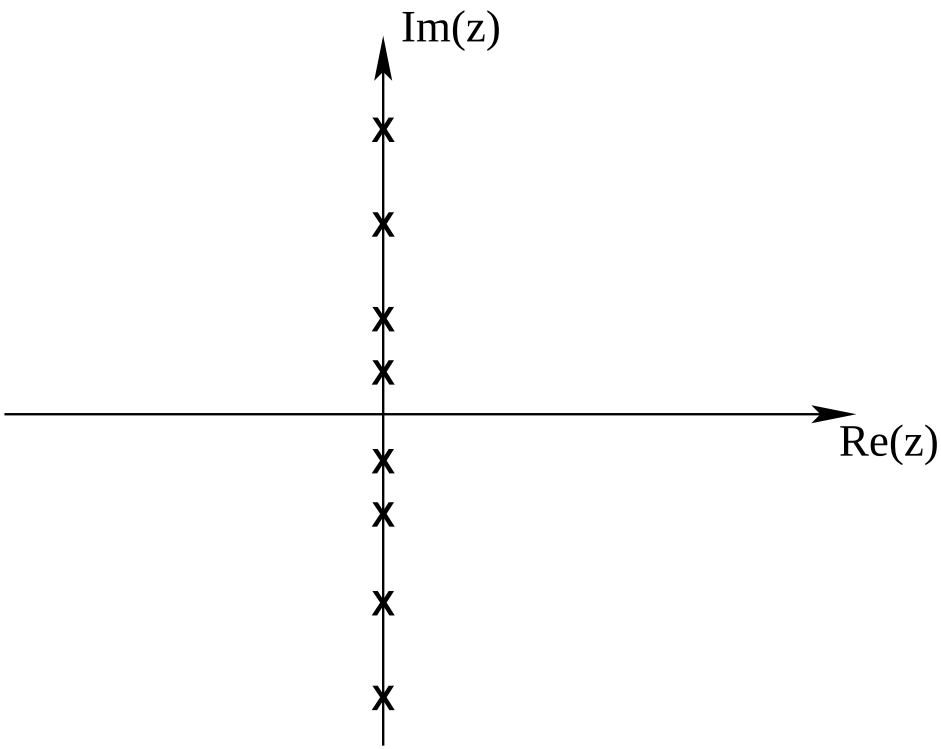
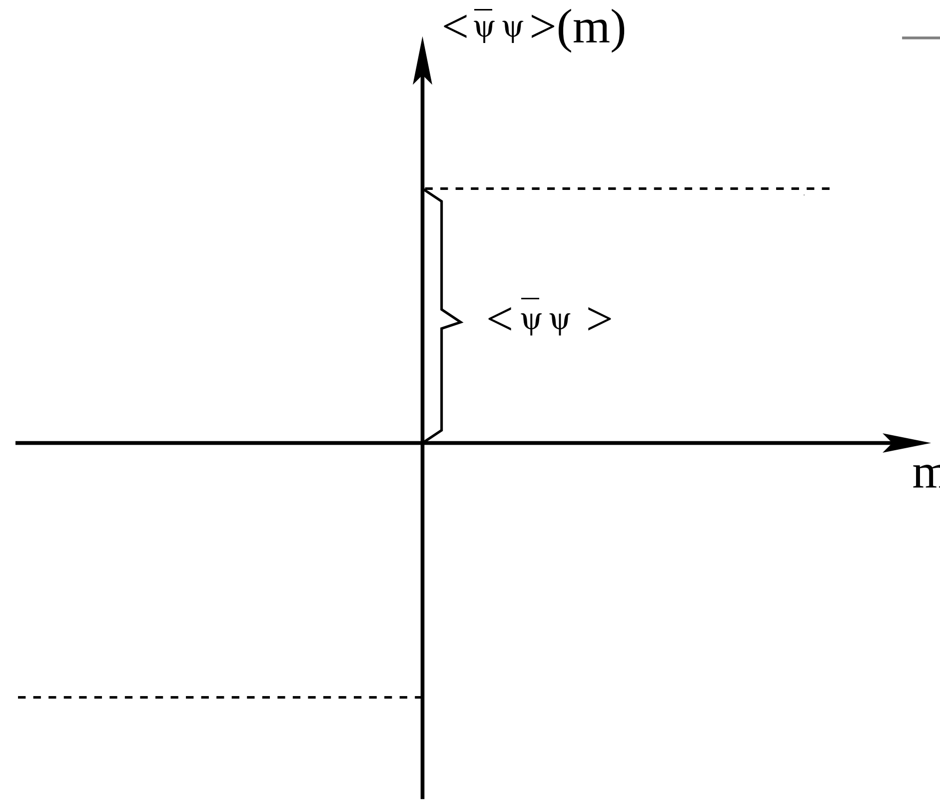
Contributions add up statistically

$\mu = 0$

Banks Casher





$$\langle\bar{\psi}\psi\rangle = \frac{\pi}{V}\rho(0)$$

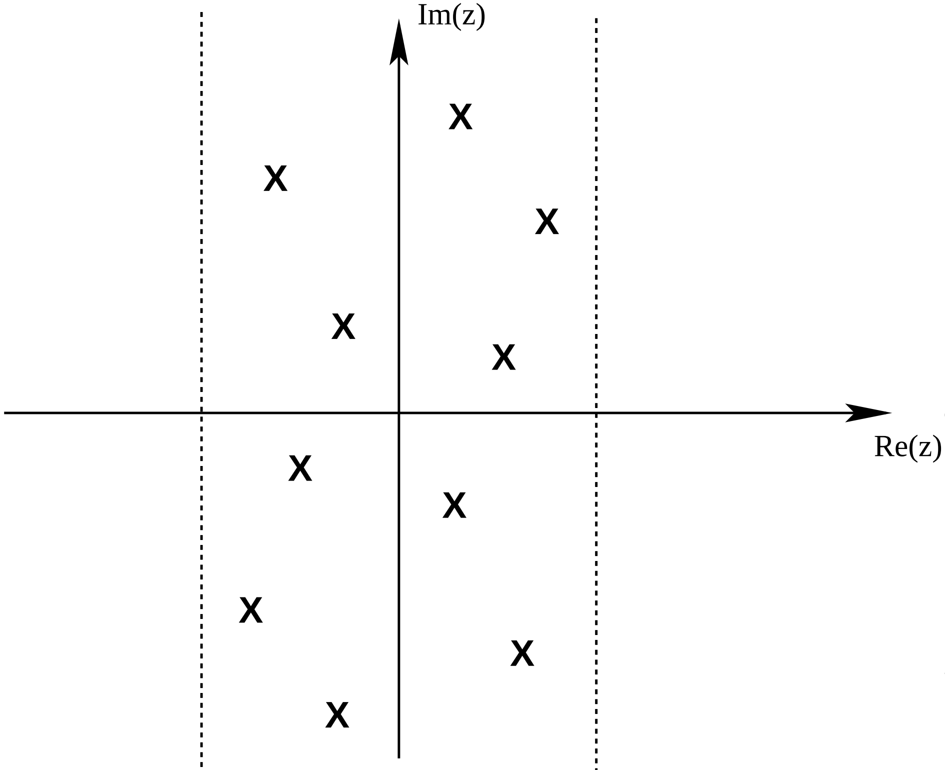
Banks Casher NPB 169 (1980) 103

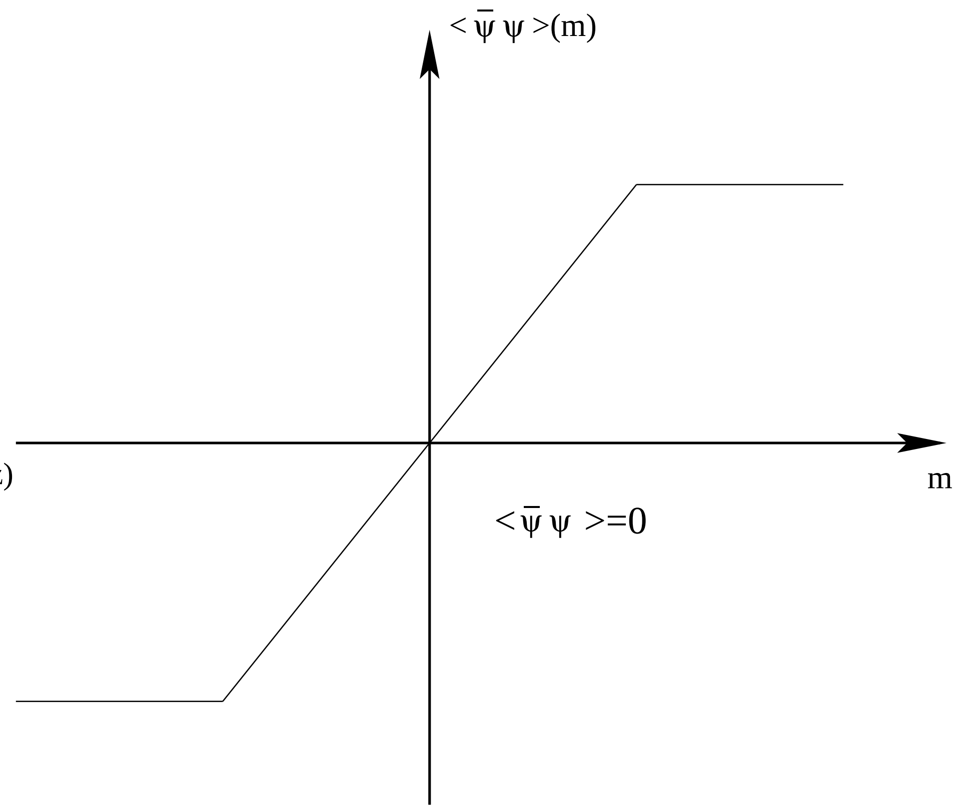
Now: non-statistical physics $\mu \neq 0$

Eigenvalue equation @ $\mu \neq 0$

$$(iD_\eta \gamma_\eta + \mu\gamma_0)\psi_j = z_j \psi_j$$

Electrostatic analogy suggests

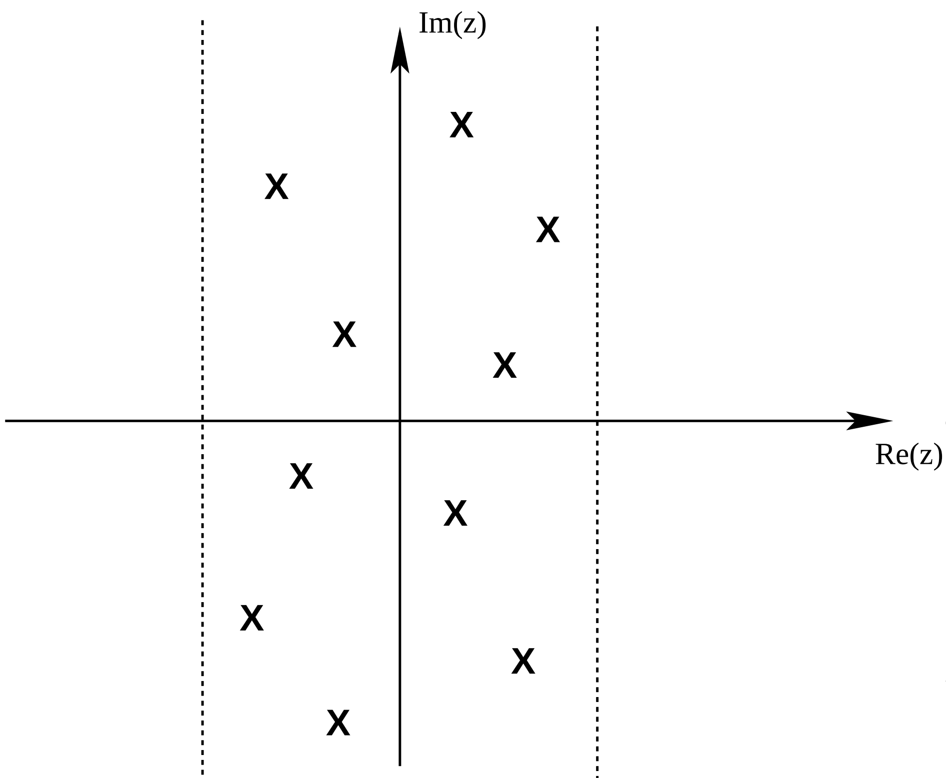




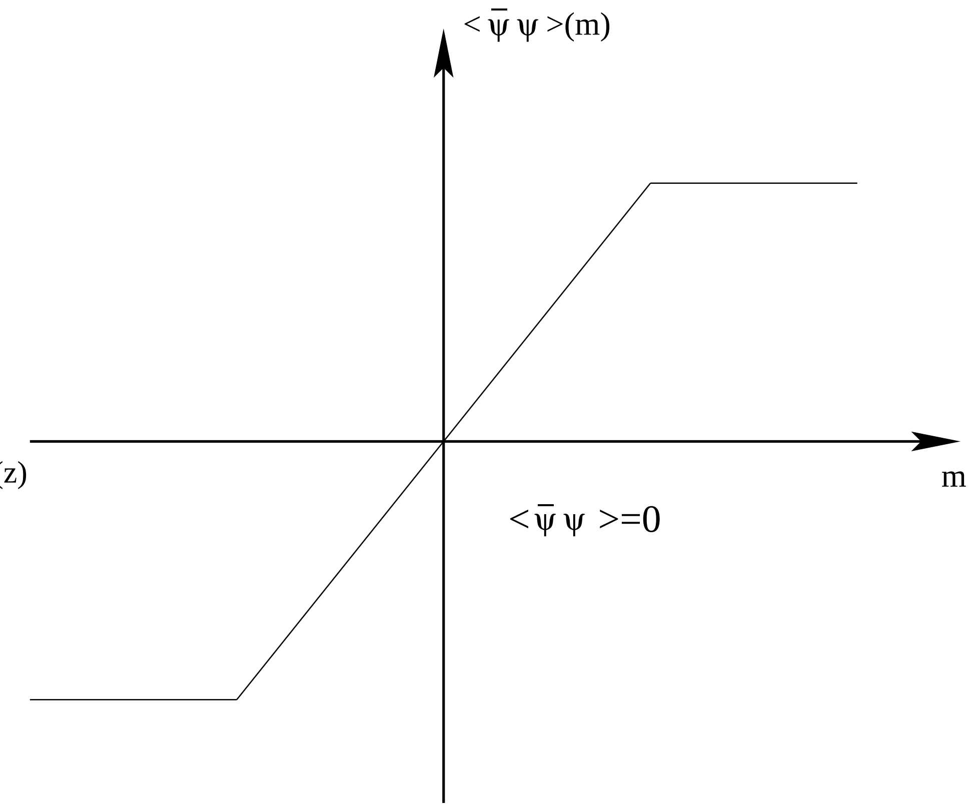
Quark mass hits the strip of the eigenvalues at $\mu = m_\pi/2$

$\Leftrightarrow$

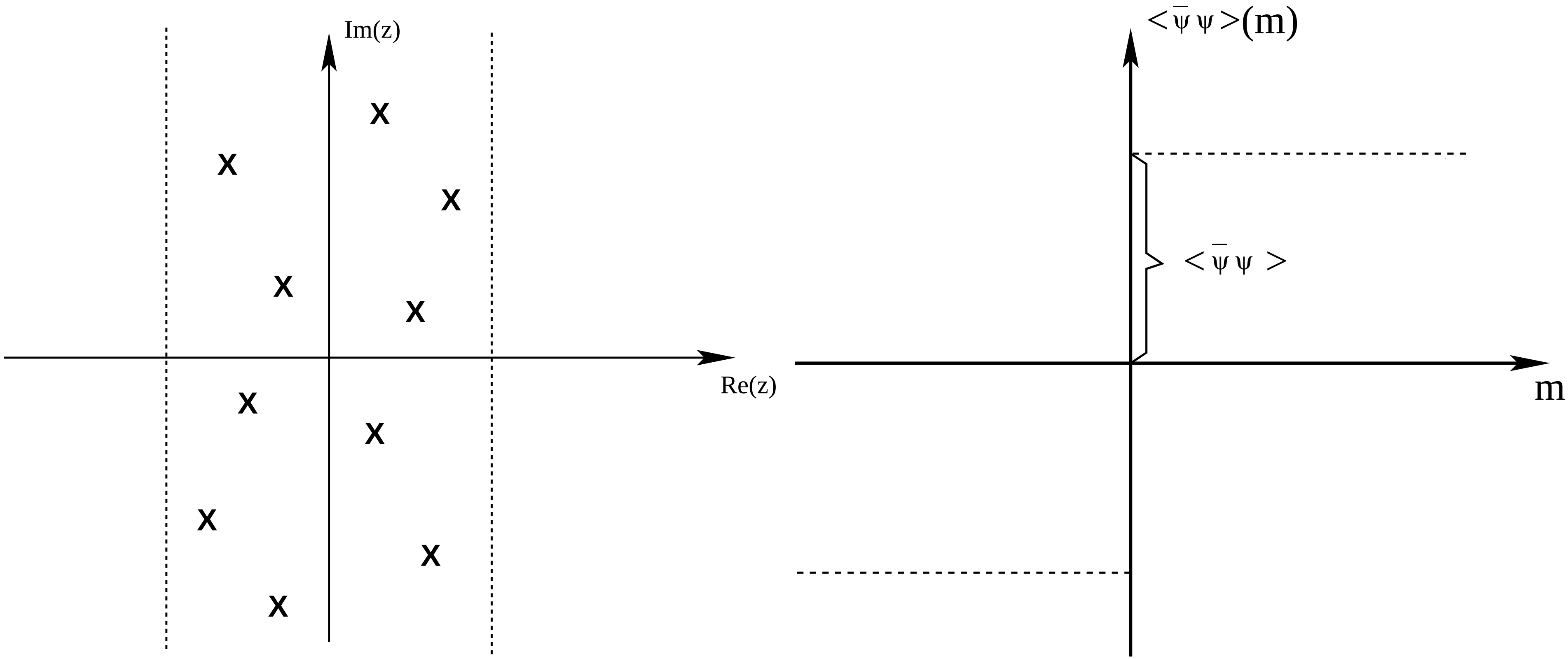
$$\det(iD_\eta\gamma_\eta + \mu\gamma_0 + m)^* = \det(iD_\eta\gamma_\eta - \mu\gamma_0 + m)$$



The phase quenched theory



$\mu \neq 0$ *The silver blaze problem*



Eigenvalues move into the complex plane

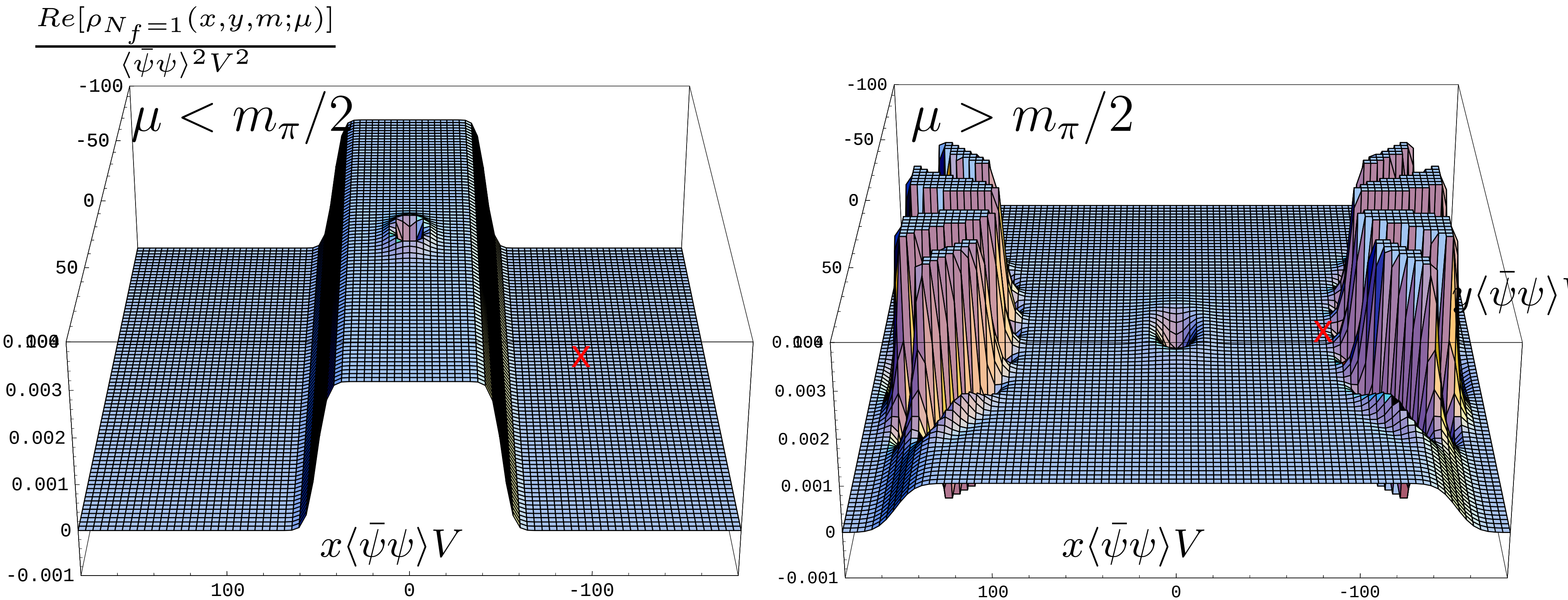
the discontinuity of the chiral condensate remains

Barbour et al. NPB 275 (1986) 296
Gibbs PLB 182 (1986) 369

Cohen PRL 91 (2003) 222001

The unquenched eigenvalue density

$m\langle\bar{\psi}\psi\rangle V = 100$ increasing $2\mu^2 F_\pi^2 V$



For $\mu > m_\pi/2$ the density is complex and oscillates

Osborn PRL 93 (2004) 222001

Akemann Osborn Splittorff Verbaarschot NPB 712 (2005) 287

Definition of the eigenvalue density

Eigenvalue equation

$$(iD_\eta \gamma_\eta + \mu\gamma_0)\psi_j = z_j \psi_j$$

Eigenvalue density

$$\rho^{N_f}(z, z^*, m; \mu) \equiv \left\langle \sum_j \delta^2(z - z_j) \right\rangle_{\rm QCD}$$

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$$\rho^{N_f}(z, z^*, m; \mu) \quad \equiv \quad \left\langle \sum_j \delta^2(z - z_j) \right\rangle_{\mathrm{QCD}}$$

$$\langle \mathcal{O} \rangle_{\mathrm{QCD}} \equiv \frac{\int \mathrm{d}A\, \mathcal{O} \det(iD_\eta\gamma_\eta + \mu\gamma_0 + m)^{N_f}\ \mathsf{e}^{-S_{\mathrm{YM}}(A)}}{\int \mathrm{d}A \det(iD_\eta\gamma_\eta + \mu\gamma_0 + m_f)^{N_f}\ \mathsf{e}^{-S_{\mathrm{YM}}(A)}}$$

The chiral condensate from the eigenvalue density

$$
\begin{aligned}
\langle \bar{\psi}\psi \rangle(m) &= \frac{1}{V}\partial_m \log Z(m;\mu) \\
&= \frac{1}{V}\int d^2z\, \rho(z) \frac{2m}{-z^2+m^2}
\end{aligned}
$$

The oscillations of the density are
responsible for chiral symmetry breaking

Osborn Splittorff Verbaarschot PRL 94 (2005) 202001

Q: Can complex Langevin do this?

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A: No

Complex Langevin - basics

Previous talk by Denes Sexty

$S(\mathbf{x})$ complex valued on the real fields $\mathbf{x} = (x^{(1)}, \ldots, x^{(N)})$

Complexify all fields $x^{(k)} \to x^{(k)} + iy^{(k)}$

Drift

$$\begin{aligned} x_{t+1}^{(k)} &= x_t^{(k)} - \mathrm{Re}\left[\frac{dS}{dx^{(k)}}\right]_{\mathbf{x}=\mathbf{x_t}+\mathbf{iy_t}} dt + \eta\sqrt{dt} \\ y_{t+1}^{(k)} &= y_t^{(k)} - \mathrm{Im}\left[\frac{dS}{dx^{(k)}}\right]_{\mathbf{x}=\mathbf{x_t}+\mathbf{iy_t}} dt \end{aligned}$$

$t = 0, \ldots, T$ is the Langevin time

Expectation values

$$\langle \mathcal{O} \rangle = \frac{1}{T} \sum_{t=0}^{T} \mathcal{O}(x_t^{(k)} + iy_t^{(k)})$$

Parisi PLB **131**, 393 (1983)

Klauder Acta Phys. Austriaca **25**, 251 (1983)

Complex Langevin - Dirac spectrum

Eigenvalue equation

$$(iD_\eta \gamma_\eta + \mu \gamma_0)\psi_j = z_j \psi_j$$

$$\langle \bar{\psi}\psi \rangle_{CL}(m) = \frac{1}{T} \sum_t \sum_j \frac{2m}{-(z_j^{(t)})^2 + m^2}$$

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Equivalently

$$\langle\bar{\psi}\psi\rangle_{CL}(m) = \int d^2 z \frac{2m\rho_{N_f}^{(CL)}(z, m; \mu)}{-z^2 + m^2}$$

$$\rho_{N_f}^{(CL)}(z, m; \mu) = \frac{1}{T}\sum_t \sum_j \delta^{(2)}(z - z_j^{(t)})$$

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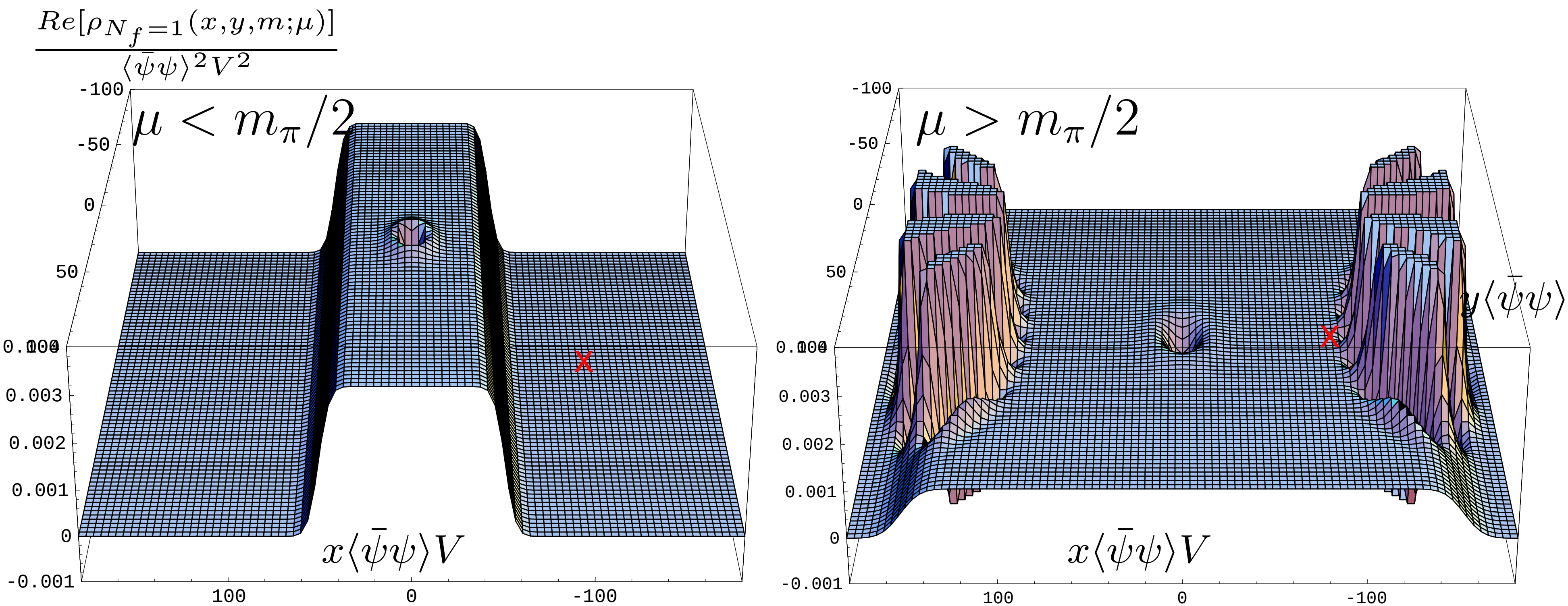
Key Point: $\rho_{N_f}^{(CL)}(z,m;\mu)$ always real and positive

Q: Can complex Langevin do this?

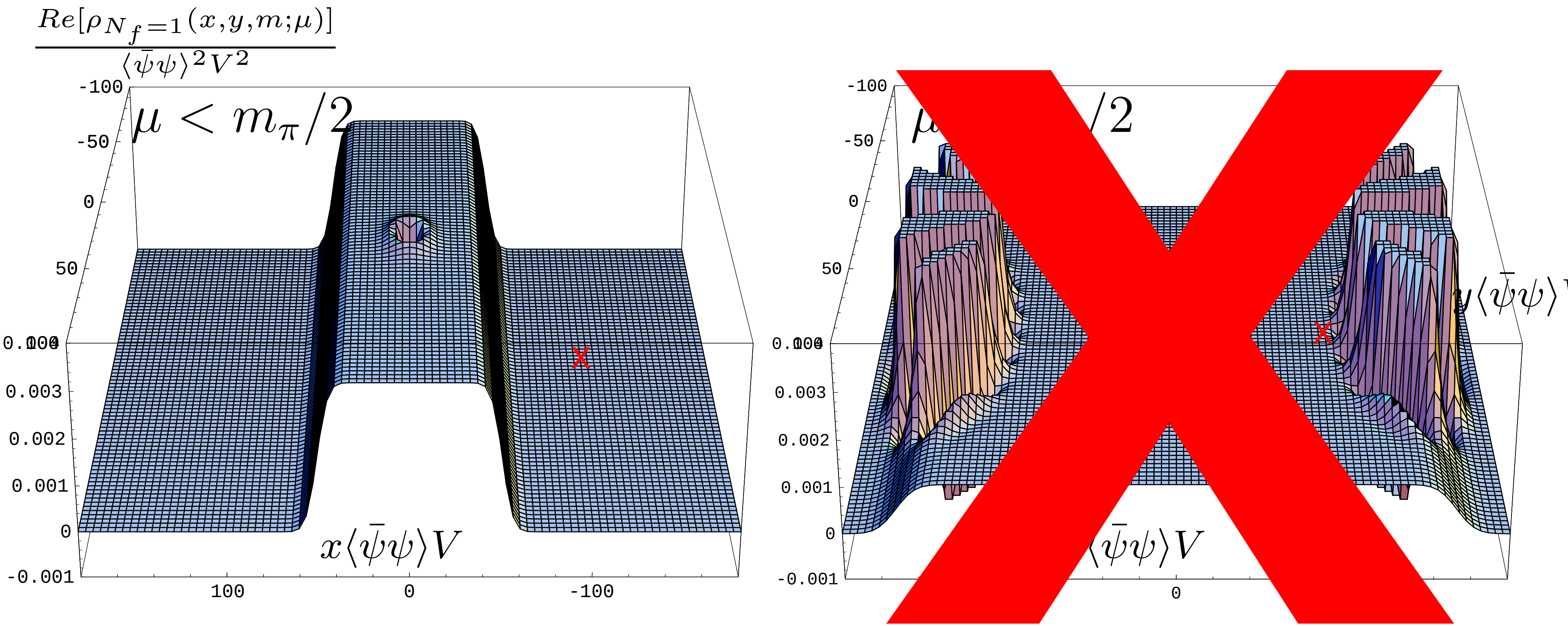


For $\mu > m_\pi/2$ the density is complex and oscillates

Osborn PRL 93 (2004) 222001

Akemann Osborn Splittorff Verbaarschot NPB 712 (2005) 287

A: No



The eigenvalue density from complex Langevin is real and positive

Splittorff, ... *to appear*

Q: So complex Langevin fails for $\mu > m_\pi/2$?

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A: No, not necessarily!

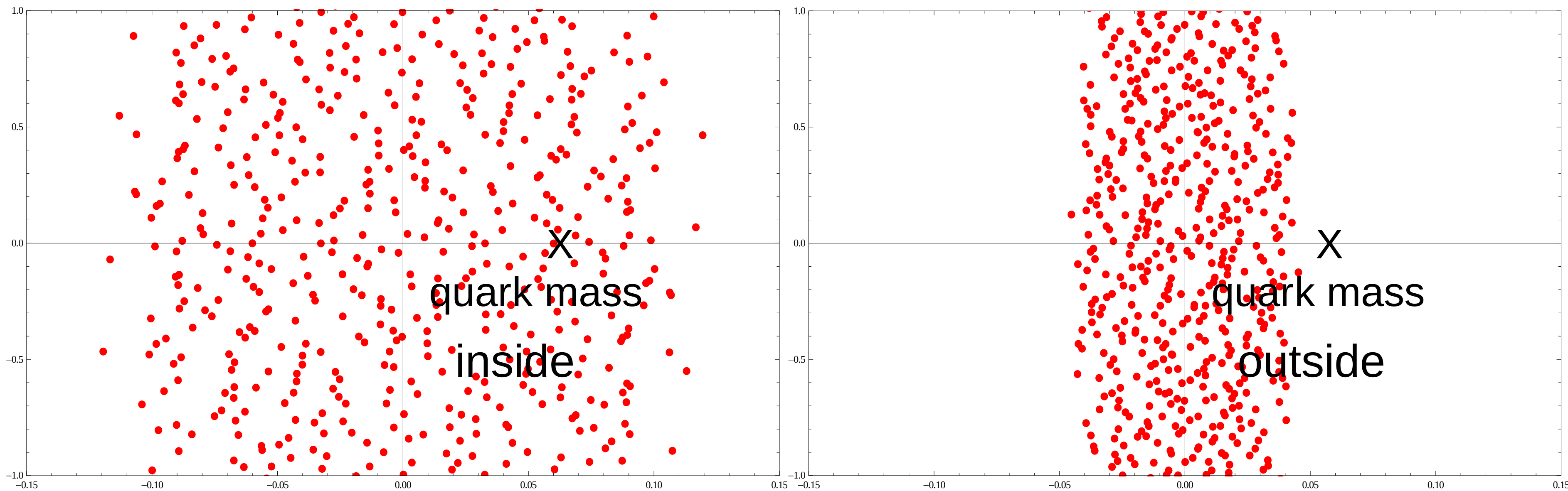
Q: So complex Langevin fails for $\mu > m_\pi/2$?

A: No, not if the Dirac spectrum flows inside the quark mass!

Illustration: *Quark mass* & *the eigenvalue distribution*

Initially in the flow | After sufficient Langevin time

$\mu > m_\pi/2, t = 1$ | $\mu > m_\pi/2, t >> 1$



Q: But what about

Quark mass hits the strip of the eigenvalues at $\mu = m_\pi/2$

$$\Leftrightarrow$$

$$\det(iD_\eta\gamma_\eta + \mu\gamma_0 + m)^* = \det(iD_\eta\gamma_\eta - \mu\gamma_0 + m)$$

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A:

$$\det(iD_\eta\gamma_\eta + \mu\gamma_0 + m)^* \neq \det(iD_\eta\gamma_\eta - \mu\gamma_0 + m)$$

Requirement for Dirac eigenvalues in complex Langevin

In the chiral limit:

$\sim V$ *eigenvalues must end up on the imaginary axis close to the origin in order to make* $\langle \bar{\psi}\psi \rangle \neq 0$

Test this in Random matrix Theory

Complex Langevin for Chiral Random Matrix Theory

$$Z = \int_{\mathbb{C}^{n\times n}} d\Phi_1 d\Phi_2 \; e^{-n\mathrm{Tr}\Phi_1^\dagger\Phi_1 - n\mathrm{Tr}\Phi_2^\dagger\Phi_2}$$

$$\times \det{}^{N_f}\begin{pmatrix} m & e^{\mu}\Phi_1 - e^{-\mu}\Phi_2^\dagger \\ -e^{-\mu}\Phi_1^\dagger + e^{\mu}\Phi_2 & m \end{pmatrix}$$

Same flavor symmetries and explicit flavor symmetry breaking as QCD

Same Hermiticity properties as QCD

Complex Langevin for Chiral Random Matrix Theory

$$
\begin{aligned}
Z &= \int_{\mathbb{C}^{n\times n}} d\Phi_1 d\Phi_2 \; e^{-n\mathrm{Tr}\Phi_1^\dagger\Phi_1 - n\mathrm{Tr}\Phi_2^\dagger\Phi_2} \\
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\end{aligned}
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Same flavor symmetries and explicit flavor symmetry breaking as QCD

Same Hermiticity properties as QCD

Same microscopic low energy theory (Chiral Perturbation Theory)

Complex Langevin for Chiral Random Matrix Theory

$$Z = \int_{\mathbb{C}^{n\times n}} d\Phi_1 d\Phi_2 \, e^{-n\mathrm{Tr}\Phi_1^\dagger\Phi_1 - n\mathrm{Tr}\Phi_2^\dagger\Phi_2} \times {\det}^{N_f} \begin{pmatrix} m & e^{\mu}\Phi_1 - e^{-\mu}\Phi_2^\dagger \\ -e^{-\mu}\Phi_1^\dagger + e^{\mu}\Phi_2 & m \end{pmatrix}$$

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Same Hermiticity properties as QCD

Same microscopic low energy theory (Chiral Perturbation Theory)

Same $\langle \exp(i\theta) \rangle$

Shuryak and Verbaarschot NPA **560**, 306 (1993)
Stephanov PRL **76**, 4472 (1996)
Osborn PRL **93**, 222001 (2004)
Bloch Bruckmann Kieburg Splittorff Verbaarschot PRD **87**, 034510 (2013)
Osborn Splittorff Verbaarschot PRD **78**, 065029 (2008)

G. Akemann, Int. J. Mod. Phys. A **22**, 1077 (2007)

Complex Langevin for Chiral Random Matrix Theory

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Advantage - we know the analytic solution for any n

Shuryak and Verbaarschot NPA **560**, 306 (1993)
Stephanov PRL **76**, 4472 (1996)
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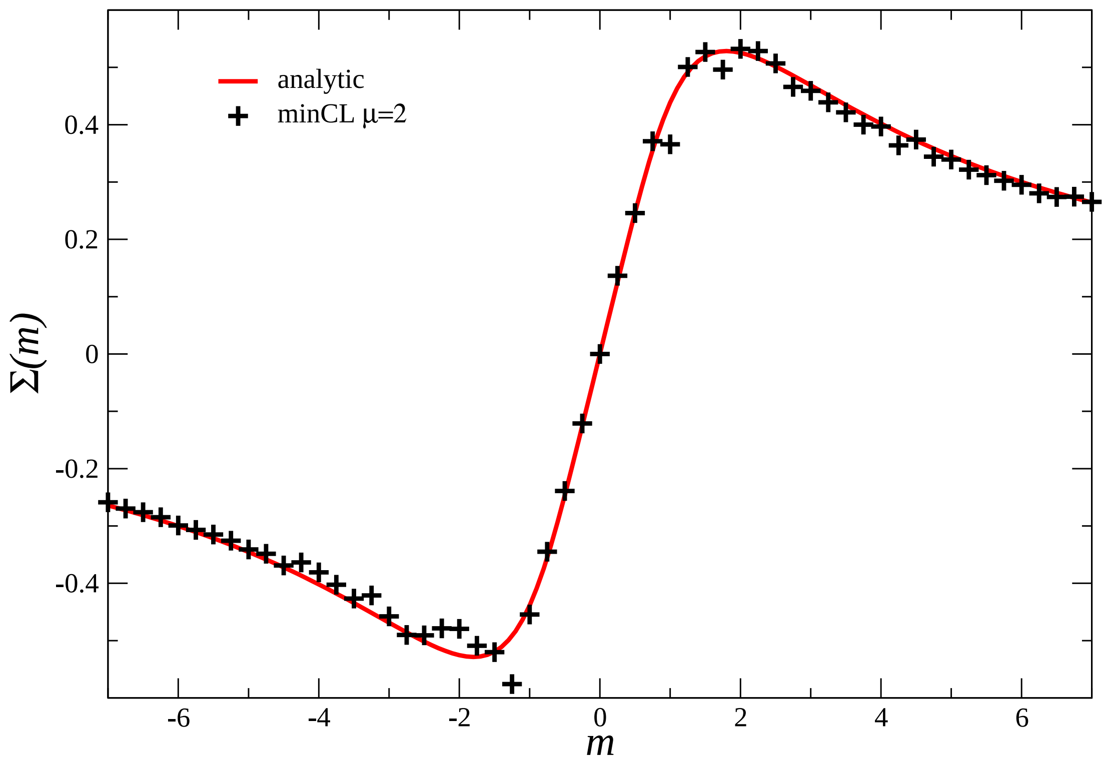
G. Akemann, Int. J. Mod. Phys. A **22**, 1077 (2007)

Chiral condensate from Complex Langevin for Chiral Random Matrix Theory

$N_f = 2$

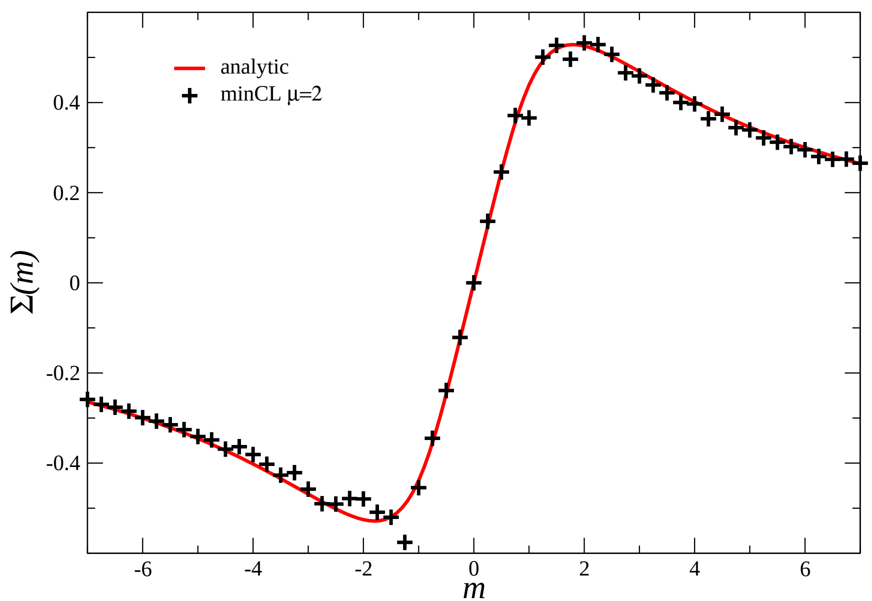


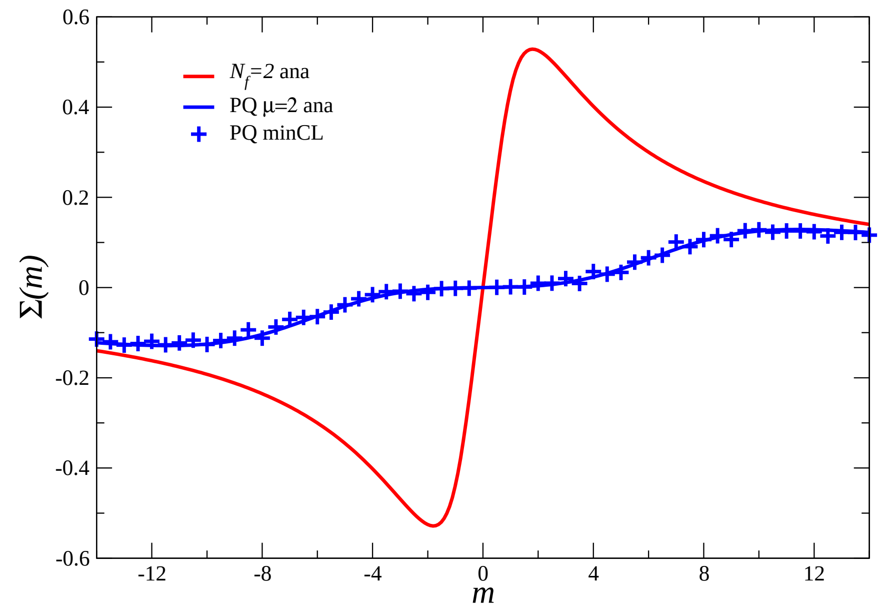
Splittorff, ... *to appear*

Chiral condensate from Complex Langevin for Chiral Random Matrix Theory

$N_f = 2$

Phase quenched

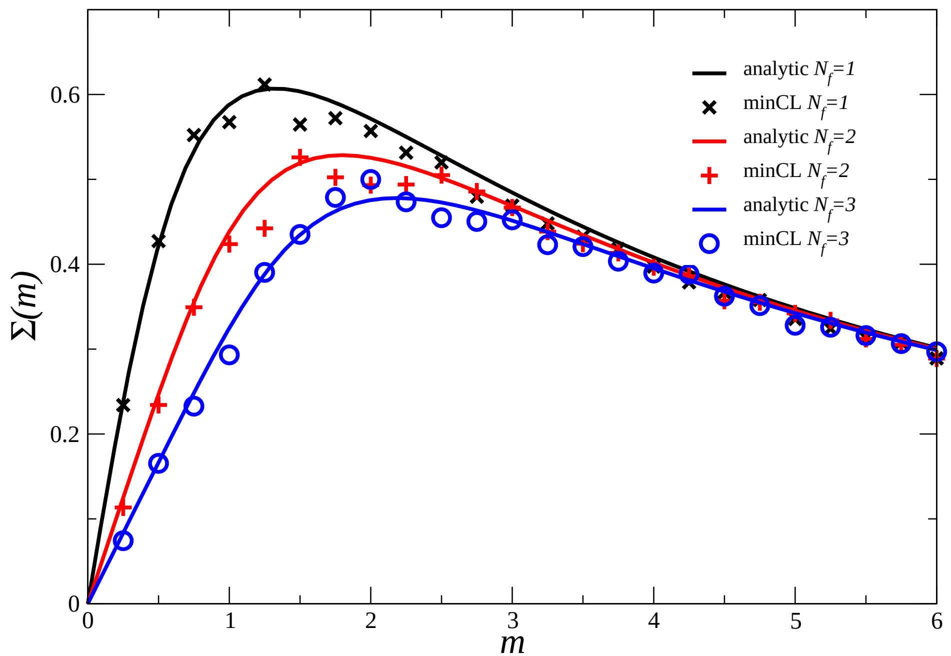




Splittorff, ... *to appear*

Chiral condensate from Complex Langevin for Chiral Random Matrix Theory

$N_f = 1, 2$ and 3 with $\mu = 2$



Splittorff, ... *to appear*

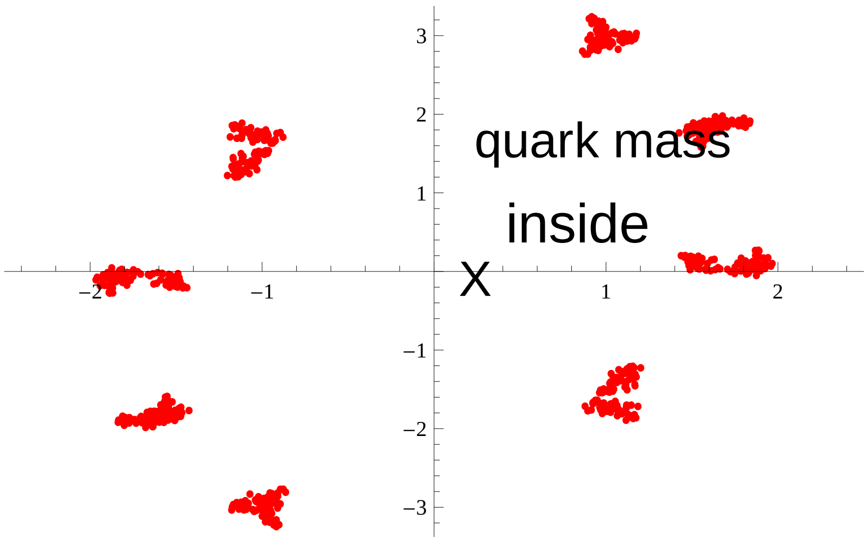
Q: So the eigenvalues flow inside the quark mass?

A: Yes!

Simulation CL of chRMT: *Quark mass & the eigenvalue distribution*

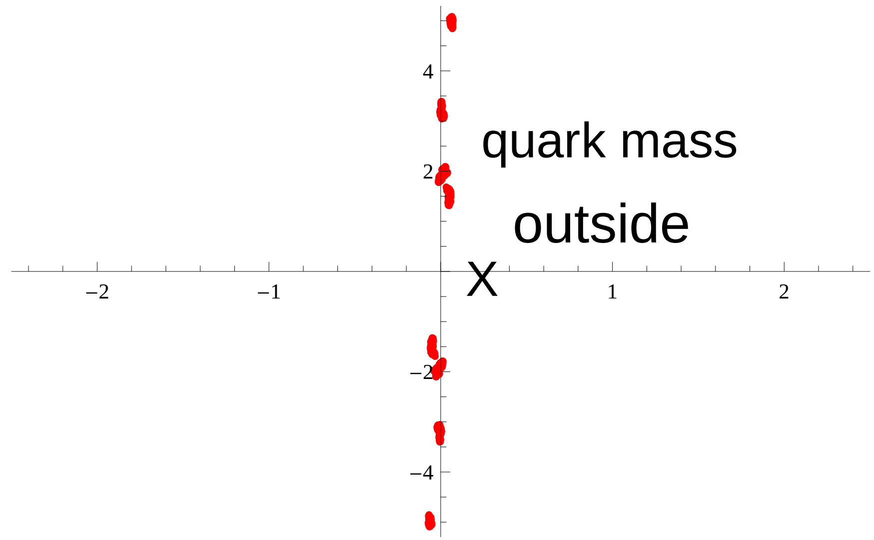
Initially in the flow

$$\mu > m_\pi/2, t = 1, .., 100$$



After sufficient Langevin time

$$\mu > m_\pi/2, t = T - 100, ..., T$$



But wait ... how about the log?

With a complex fermion determinant

Should we use the multivalued $\log$

$$S = S_g - \log\left(\det D\right)$$

or the principal part of the log

$$S = S_g - \mathrm{Log}\left(\det D\right)$$

In both cases

$$Z = \int \mathcal{D}U \, e^{-S} = \int \mathcal{D}U \, \det(D) \, e^{-S_g}$$

The Langevin force

$$-\frac{dS(x)}{dx}\Big|_{x\to x+iy}$$

For the $\log$ we have

$$d_x \log(\det(D(x))) = \frac{\det'(D(x))}{\det(D(x))}$$

but for the principal part we have

$$d_x \mathrm{Log}(\det(D(x))) = \frac{\det'(D(x))}{\det(D(x))}$$

only if $\det(D(x)) \in \mathbb{C}/\mathbb{R}_-$

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only if $\det(D(x)) \in \mathbb{C}/\mathbb{R}_-$

The drift is affected by our choice of log if the determinant circles the origin!

Observation: Failure of complex Langevin correlated with cut of the log

Criterion: *measurements can only be trusted if the flow of the determinant does not frequently trace out a path surrounding the origin*

Møllgaard, Splittorff PRD 88, 116007 (2013)

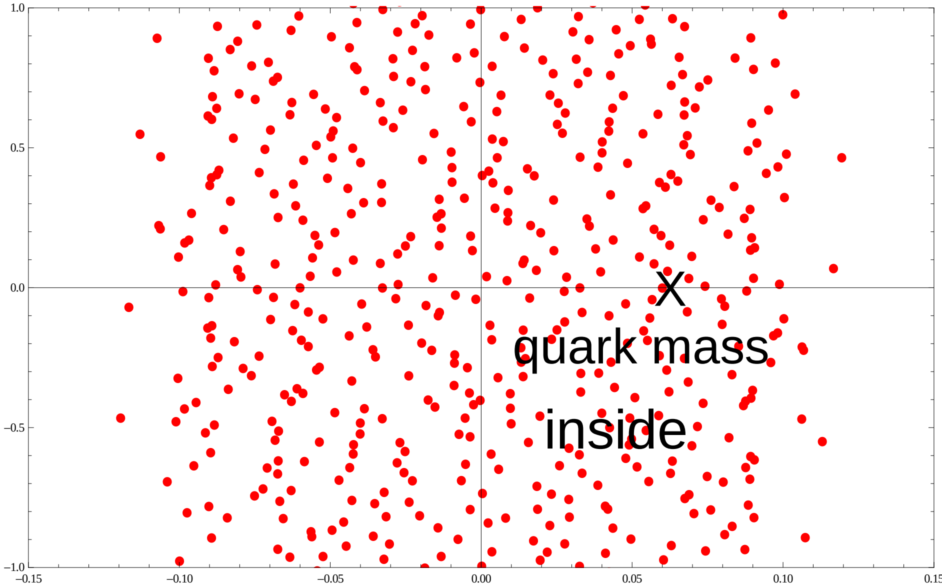
Next talk by Erhard Seiler

Q: Can complex Langevin flow the eigenvalues inside the quark mass without the determinant circles the origin?

Illustration: *Quark mass & the eigenvalue distribution*

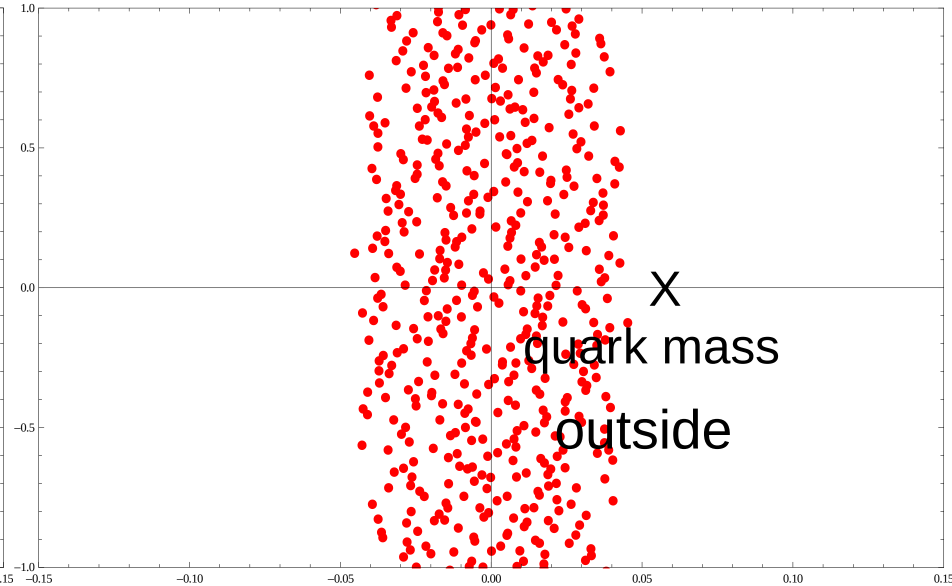
Initially in the flow

$\mu > m_\pi/2, t = 1$



After sufficient Langevin time

$\mu > m_\pi/2, t >> 1$



Q: Can complex Langevin flow the eigenvalues inside the quark mass without the determinant circles the origin?

A1 (observation): yes, in the RMT simulation with sufficiently small dt

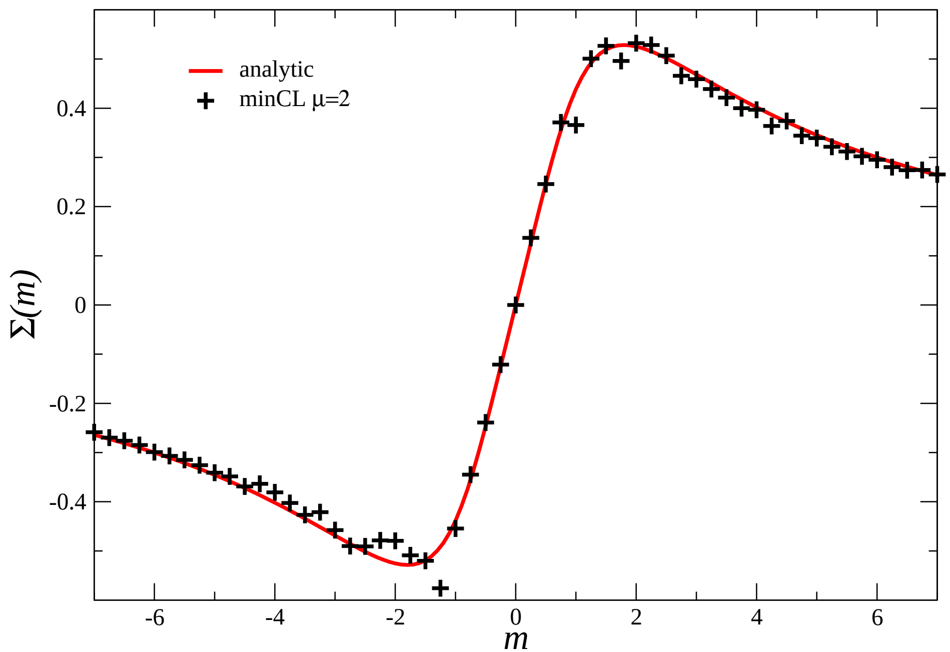
A2 (partial): complex Langevin stays close to Lefschetz thimbles and on these the phase of the fermion determinant is constant

Aarts, PRD **88**, 094501 (2013) + private communications

Splittorff, ... *to appear*

Chiral condensate from Complex Langevin for Chiral Random Matrix Theory

$N_f = 2$ with $\mu = 2$



$\leftarrow m_\pi/2 < \mu$

Splittorff, ... *to appear*

Conclusion

Complex Langevin and the Dirac spectrum

New requirement for Dirac eigenvalues in complex Langevin

Tested in chiral Random Matrix Theory

Successful simulation even for $\mu > m_\pi/2$

Thanks to

Anders Møllgaard

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Nucu Stamatescu

Erhard Seiler

Denes Sexty

Philippe de Forcrand

Francesco Di Renzo

Luigi Scorzato

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