Subset method for the sign problem in one-dimensional QCD

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Outline

- introduction: random matrix model and a simple example
- subset method in random matrices – the first — Bloch '11
  - why it works — Bloch, FB, Kieburg, Splittorff, Verbaarschot '12
- subset method in gauge theories
  - 1d QCD – the obvious and not-so-obvious — Bloch, FB, Wettig '13
  - $\geq$ 2d QCD – the hard — `talk by J. Bloch`

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similar approaches — Barbour, Davies, Sabeur ’88; Hasenfratz, Toussaint ’92

Aarts, Kaczmarek, Karsch, Stamatescu ’02; Bringoltz ’10

on clusters in Potts model — Alford, Chandrasekharan, Cox, Wiese ’01

Introduction

- grandcanonical partition function of QCD (one flavor):

$$Z(\mu) = \int DA\, e^{-S_g[A]} \det(\not{D}[A] + m\,\mathbb{1} + \mu\gamma_0)$$

- and of a random matrix model: Osborn '04

$$Z(\mu) = \int d\Phi_{1,2}\, e^{-N\,\mathrm{Tr}\,(\Phi_1\Phi_1^\dagger + \Phi_2\Phi_2^\dagger)} \det \begin{pmatrix} m & e^{\mu}\Phi_1 - e^{-\mu}\Phi_2^\dagger \\ -e^{-\mu}\Phi_1^\dagger + e^{\mu}\Phi_2 & m \end{pmatrix}$$

– 4d path integral $\rightsquigarrow$ ordinary matrix integral

– gauge fields/lattice links $\rightsquigarrow$ complex $N \times N$ matrices

– gauge action $\rightsquigarrow$ Gaussian weight

both: massless Dirac operator is chiral

– chemical potential: $e^{a\mu}U_0$, $e^{-a\mu}U_0^\dagger$ (lattice) $\rightsquigarrow$ $e^{\pm\mu}\Phi_{1,2}$

both: massless Dirac operator is not anti-hermitian anymore

$\Rightarrow$ sign problem for $\mu \neq 0$

- simplest case: 1×1 matrices, $m = 0$ (problem worst)

  new variables $x = \Phi_1 \Phi_2,\; y = \Phi_1/\Phi_2$, specialize to $y = 1$:

$$Z(\mu) \sim \int_{\mathbb{C}} \mathrm{d}x\, e^{-2|x|} \Big(\underbrace{|x| - \mathrm{Re}\, x \cosh 2\mu}_{\not\geq 0} - \underbrace{\mathrm{i}\, \mathrm{Im}\, x \sinh 2\mu}_{\text{imag.}} \Big)$$

- the term causing a negative integrand (and the imag. part) is odd in x

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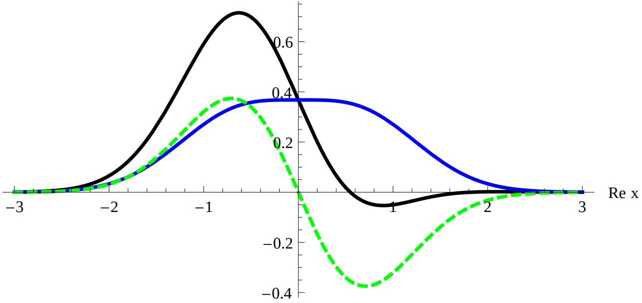
add (average) integrand at x and $-x \Rightarrow$ positive integrand



original integrand

odd term

new integrand

$\mu = 1.8$

$(\mathrm{Im}\,x = 0,\ y = 1)$

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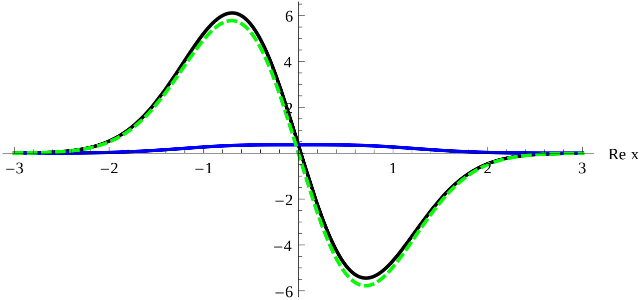

original integrand

odd term

new integrand

$\mu = 1.8$

($\mathrm{Im}\, x = 0$, $y = 1$)

- note that we have removed the μ-dependence altogether:

$$Z(\mu) \sim \int_{\mathbb{C}} \mathrm{d}x\, e^{-2|x|} \Big(|x| - \underbrace{\mathrm{Re}\, x \cosh 2\mu - \mathrm{i}\, \mathrm{Im} x \sinh 2\mu}_{\text{gone}} \Big)$$

  'Silver blaze' forever

  not so in QCD, model is of limited use

- for the generalization note that $x \to -x$ means $\Phi_{1,2} \to i \cdot \Phi_{1,2}$

Subsets for the random matrix model

- use N_sth roots of unity: $e^{i\theta_k}$ with $\theta_k = 2\pi \frac{k}{N_s}$, $k = 0, .., N_s - 1$

average the weights over subsets $\{e^{i\theta_k}\,\Phi_{1,2}\}$

(0) integration variables $\Phi_{1,2}$ remain in the integration range
"fields remain in the configuration space"

(1) transformation is invertible
angles yield an exact N_s-fold covering of configuration space

(2) Gaussian weight is invariant
det's add up to positive for $N_s > 2N$ ✓

(3) number of configurations in the subset:
linear $N_s = \mathcal{O}(N)$, not $\mathcal{O}(e^N)$ ✓

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Why does it work?

Bloch, FB, Kieburg, Splittorff, Verbaarschot ’12

- reinterprete phases as imaginary chem. potential:

$$e^{\mu}\Phi_{1,2} \to e^{\mu}e^{i\theta_k}\Phi_{1,2} = e^{\mu+i\theta_k}\Phi_{1,2} \qquad \text{same for } e^{-\mu}\Phi^\dagger_{1,2}$$

- fugacity expansion: part. function and det.s are polynomials in e^{μ}

$$\det D(\mu) = \sum_{q=-2N}^{2N} e^{q\mu} D_q \qquad \ldots\text{canonical determinants}$$

subset weights:

$$\sigma(\mu) = \sum_{q=-2N}^{2N} e^{q\mu} \underbrace{\left(\frac{1}{N_s}\sum_k e^{2\pi i \frac{k}{N_s} q}\right)}_{\neq 0 \text{ iff } q \mod N_s = 0} D_q = D_0 + e^{\pm N_s\mu} D_{\pm N_s} + \ldots$$

$$\overset{\text{large}}{=} D_0 \geq 0 \quad \ldots\text{integrand at } \mu = 0, \text{ ‘no mu, no cry’}$$

$\Rightarrow$ μ-independence (Silver Blaze) at the level of the weight

$D_1, D_2, \ldots$ removed, they integrate to zero can. part. functions anyhow

Subset method in gauge theories

Bloch, FB, Wettig '13

- first idea:

phases on temporal links: $U_0 \longrightarrow e^{i\theta_k}\, U_0 \quad \theta_k = 2\pi\frac{k}{N_s},\ k = 0, .., N_s - 1$

(0) links remain in configuration space: $e^{i\theta_k} U_0 \in SU(N_c)$?
iff restricted to center transformations (det must be 1), $N_s = N_c$

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NB: in $U(N_c)$ all phases are allowed, N_s is arbitrary
μ-dependence removed $\rightleftharpoons$ no baryons in $U(N_c)$

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(1) transformation is invertible, i.e. exact three-fold covering ✓

(2) gauge action (plaquettes) invariant ✓
det's add up to positive??

(3) 3 configs. in subset
effort = constant, not even $\mathcal{O}(V)$

- interpretation as imag. chemical potential $\mu/T + i\theta_k$ ✓

  projects out baryon sectors: $Z(\mu) = \sum_{q \bmod 3=0} e^{q\mu/T} Z_q$ ✓

  μ-dependence kept ✓

- cancellations in part. function:

$$\int \mathrm{d}U\, e^{-S_g} \Big(\underbrace{\det{}_0(U) + \det{}_{\pm 3}(U) e^{\pm 3\mu/T} + \ldots}_{\substack{\text{give the nonvanishing } Z\text{'s} \\ \text{`essential'} \\ \text{already positive??}}} + \underbrace{\det{}_{\pm 1}(U) e^{\pm \mu/T} + \ldots}_{\substack{\text{integrate to vanishing } Z\text{'s} \\ \text{`artificial'}^* \\ \text{removed by subsets}}} \Big)$$

*: these terms contribute to Polyakov loops ('winding observables')

Subset method in 0+1d QCD

Bloch, FB, Wettig '13

- temporal links U_0 through $\prod_{x_0} U_0 = P \ldots$ Polyakov loop $\in SU(3)$
- Dirac operator: $\det D \sim \det_{3\times3}\left(A\,1_3 + e^{\mu/T}P + e^{-\mu/T}P^\dagger\right)$ Bilic, Demeterfi '88

  with mass parameter $A = 2\cosh(\frac{\text{arsinh}(am)}{aT}) \geq 2$

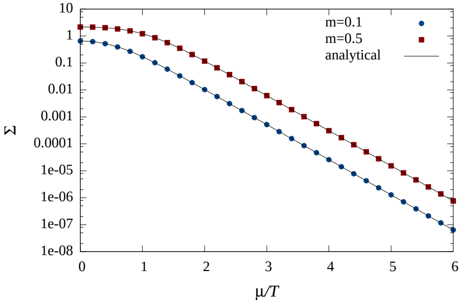
  and chemical potential through $\prod_{x_0} e^{a\mu} = e^{\mu/T}$
- no plaquettes
- analytically solvable — Bilic, Demeterfi '88; Ravagli, Verbaarschot '07
- subsets: $\{P, e^{2\pi i/3}P, e^{-2\pi i/3}P\}$

  part. function for $N_f = 1$:

$$Z(\mu) = \int d_{\text{Haar}}P\,\Big(\underbrace{A^3 - 3A + A|\text{tr}P|^2}_{\text{positive}\ \checkmark} + \underbrace{\ldots e^{\pm\mu/T} + \ldots e^{\pm2\mu/T}}_{\text{removed by subsets}} + \underbrace{1\cdot e^{\pm3mu/T}}_{\text{positive}\ \checkmark}\Big)$$

$\Rightarrow$ sign problem solved by center subsets ✓

- measure observables, e.g. chiral condensate:



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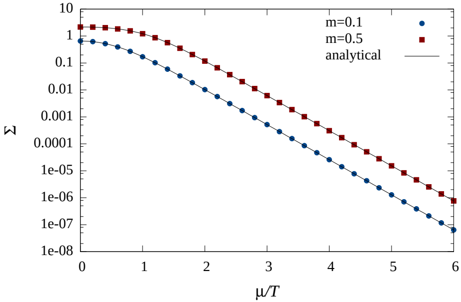


- computation:

$$\langle O\rangle = \frac{1}{Z(\mu)}\int \mathrm{d}P\, \det(P;\mu)\cdot O(P)$$

$$= \frac{1}{Z(\mu)}\int \mathrm{d}P\, \underbrace{\frac{1}{3}\sum_{k=0}^{2}\det(e^{2\pi i k/3}P;\mu)}_{\sigma(\{P\};\mu)} \cdot \underbrace{\sum_{l=0}^{2}\underbrace{\frac{\det(e^{2\pi i l/3}P;\mu)}{\sigma(\{P\};\mu)}}_{\text{relative weight of config. in subset}} O(e^{2\pi i l/3}P)}_{\text{subset observable}}$$

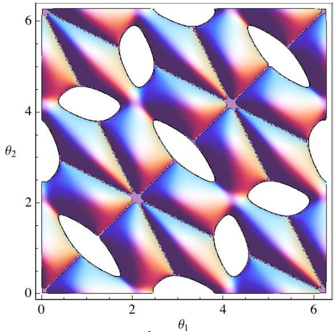
subset weight: sampling

1d QCD with many flavors

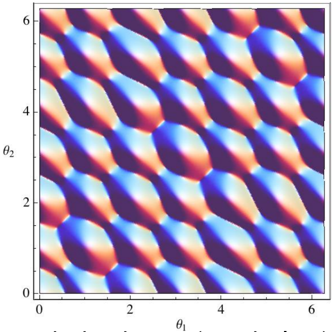
- sign problem reappears with more than 5 flavors even with subsets:

$$Z(\mu) = \int \mathrm{d}_{\mathrm{Haar}} P(\underbrace{\ldots e^{0\mu/T} + \ldots e^{3\mu/T} + \ldots e^{6\mu/T} + \ldots + 1 \cdot e^{3N_f\mu/T}}_{\text{not positive definite for some } (P,\mu)})$$

integrand over config. space (φ_1, φ_2) in $P \sim \mathrm{diag}(e^{i\varphi_1}, e^{i\varphi_2}, e^{-i(\varphi_1+\varphi_2)})$ for $N_f = 24, \mu = 2.6$ ($m = 0$), logarithmically:



center subsets



center + extended subsets (see below)

Extended subsets

- beyond the center:

group multiplication: $U_0 \to G_k U_0 \quad G_k \in SU(3) \quad k = 0, \ldots, N_s - 1$

(0) links remain in $SU(3)$ ✓
translations in the (φ_1, φ_2)-diagram

(1) invertible, N_s-fold covering ✓

(2) plaquettes: not present in 1d

(3) effort? which G's??

our finding:

7 G's solve/significantly attenuate
the sign problem with $N_f = 12, 24, 48$
(see above)

systematic? “the not-so-obvious”

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Lemma

assume:

- path integral over groups with *real* weights $w(U)$
- group multiplication subsets: $\bar{w}(U) = \frac{1}{N_s} \sum_{k=0}^{N_s-1} w(G_k U)$

subset reweighting factor larger and its relative error smaller than for conventional sign quenching

proof:

$$r_{\text{subsets}} = \frac{\int dU \sum w}{\int dU \,|\sum w|} \overset{\text{Schwarz}}{\geq} \frac{\int dU \sum w}{\int dU \sum |w|} \equiv \frac{\sum_k \int dU\, w(G_k U)}{\sum_k \int dU\, |w(G_k U)|}$$

$$\overset{\text{Haar}}{=} \frac{\sum_k \int dU\, w(U)}{\sum_k \int dU\, |w(U)|} = \frac{\int dU\, w(U)}{\int dU\, |w(U)|} = r_{\text{conv}}$$

and $\left(\frac{\sigma_r}{r}\right)^2 = \frac{1}{r^2} - 1$

also for nested subsets, but does not specify the numerical effort

Summary and outlook

explicit summation over a subset of configurations $\overset{!?}{\Leftarrow}$ positive weight

then remaining integration left for numerics using importance sampling

- subsets with phases in $U(1)$ or center Z_3 removes sign problem

  in random matrix model and QCD in 1d with $N_f \leq 5$

  (and in $U(N_c)$ gauge theories)

- ∘ interpretation as imag. $\mu \Rightarrow$ projection onto particular q-sectors

  only $q = 0$ $q \mod 3 = 0$

- extended subsets remove/attenuate sign problem for QCD in 1d with $N_f \geq 6$
- ∘ no interpretation as shift in μ, still improved weights, cf. Lemma

outlook:

- subset method for other sign problems, some thoughts below
- QCD in $\geq$ 2d: see the talk by J. Bloch

More subset methods

- (ungauged) lattice Higgs model with chem. potential:

$$S(\phi) = \sum_x \Big(V(|\phi|) + \text{ spat. hoppings } + e^{a\mu}\phi^*_x\phi_{x+\hat{0}} + e^{-a\mu}\phi_x\phi^*_{x+\hat{0}} \Big)$$

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subsets: since $\phi_x\in\mathbb{C}$ one stays in the config. space with $\phi_x\to e^{i\theta x_0}\phi_x$

interpretation as $a\mu\to a\mu+i\theta$

periodic bc.s constraint: $\theta_k=2\pi k/N_0$

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periodic bc.s constraint: $\theta_k = 2\pi k / N_0$

- 2d Yang-Mills theory with theta term:

$$S(U) = \beta \sum_{\text{plaq.s } \square} \Big(e^{\eta} \,\mathrm{tr} U_\square + e^{-\eta} \,\mathrm{tr} U_\square^\dagger \Big) \qquad \eta \sim \Theta g^2 a^2$$

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subsets: from magn. field setup one can modify: $\mathrm{tr} U_\Box \to e^{i\theta} \mathrm{tr} U_\Box$

interpretation as $\eta \to \eta + i\theta$

flux quantization constraint: $\theta_k = 2\pi k / N_x N_y$

$SU(N_c)$: center, too