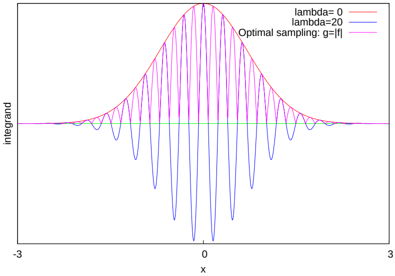
The phase diagram of QCD in the strong-coupling limit, and how to go further

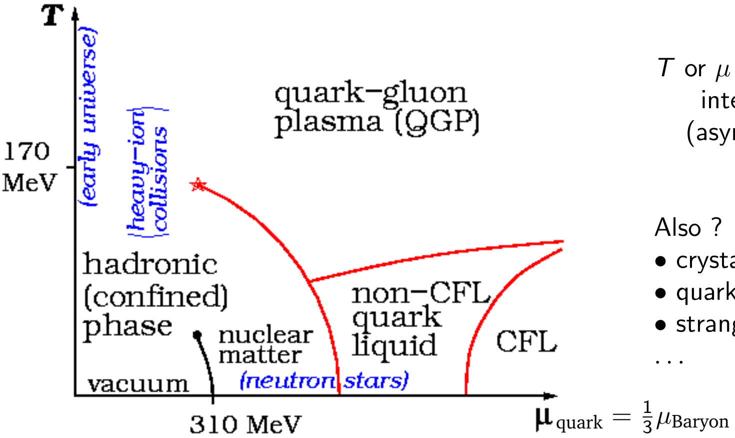
Philippe de Forcrand
ETH Zürich & CERN

Sign 2014, Darmstadt, Feb. 2014



Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

The mandatory slide



T or $\mu \to \infty$:
interaction weak
(asymptotic freedom)

Also ?

- crystal phase(s)
- quarkyonic phase
- strangelets

...

Everything in red is a conjecture

Progress 2012 ⟶ 2014

Sign 2012:

"Solving the sign pb in QCD
is like climbing Mt. Everest"

Progress 2012 ⟶ 2014

Sign 2014:

Large-scale expeditions launched

Progress 2012 $\longrightarrow$ 2014

Sign 2014:

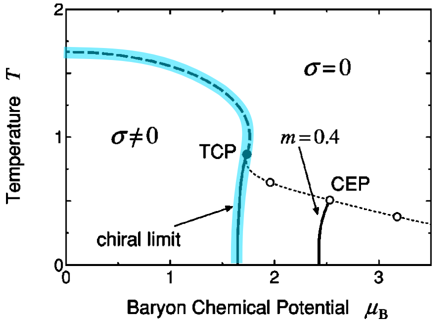
Large-scale expeditions launched including sherpas and tour-operators

Goal here: (μ, T) phase diagram of $SU(3)$ + KS fermions, esp. *in chiral limit*

$$S_F = \sum_x \left[\cancel{am_q \bar{\chi}(x)\chi(x)} + \sum_\nu \frac{\eta_\nu(x)}{2} \left(\bar{\chi}(x) U_\nu(x) \chi(x+\hat{\nu}) - h.c. \right) \right]$$

- Why staggered fermions?
  - simpler than Wilson
  - $U(1)$ remnant of chiral symmetry at finite lattice spacing:

$$\chi(x) \to e^{i\theta_V + i\theta_{55}\epsilon(x)} \chi(x), \; \epsilon(x) = (-1)^{x_1+x_2+x_3+x_4}$$



Mean-field
Nishida, 2004

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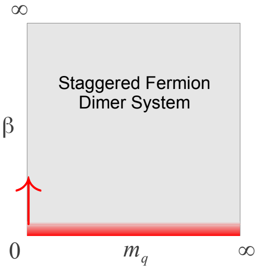
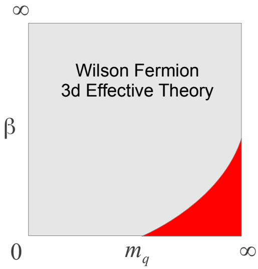
- Contrast with $3d$ Polyakov loop effective theory:
  truncated 1/mass expansion (cf. HQET) $\Longrightarrow$ no chiral physics





Well-defined, classic problem: long history

- Mean-field ($1/d$ expansion):
  1983: development of the technique — Kluberg-Stern et al
  1985: first finite-density analysis — Damgaard et al
  1992: $aT_c(\mu = 0) = 5/3, \; a\mu_c(T = 0) = 0.66$ — Bilic et al
  1995: entropy per baryon — Bilic & Cleymans
  2004: full phase diagram, incl. tricritical point — Nishida
  2009^+: include $\mathcal{O}(\beta)$ corrections — Ohnishi et al

- Monte Carlo:
  1984: formulation as a dimer system — Rossi & Wolff
  1989: first finite-density results ($aT_c(\mu = 0) = 1.4, \; a\mu_c(T = 0) = 0.63$) — Karsch & Mütter
  2003: first worm algorithm for $U(3)$: fast, even in chiral limit — Adams & Chandrasekharan
  2010: full phase diagram and nuclear potential for $SU(3)$ — Fromm & PdF
  2011: continuous Euclidean time — Unger & PdF
  2011: include $\mathcal{O}(\beta)$ corrections for $U(3)$ — Unger, Langelage et al

"Dual" variables in strong-coupling limit (I)

$$Z_{QCD} = \int d\bar{\chi} d\chi dU \; e^{S_G + S_F}; \quad S_G = \frac{\beta}{2N_c} \sum_P \mathrm{Tr}(U_P + U_P^\dagger); \quad \beta = \mathbf{0}$$

$$S_F = \sum_x am_q \bar{\chi}_x \chi_x + \sum_{x,\nu} \frac{\eta_\nu(x)}{2} \gamma^{\delta_{\nu 4}} \left[\bar{\chi}_x e^{\mathbf{a}_\tau \mu \delta_{\nu 4}} U_\nu(x) \chi_{x+\hat{\nu}} - \bar{\chi}_{x+\hat{\nu}} e^{-\mathbf{a}_\tau \mu \delta_{\nu 4}} U_\nu^\dagger(x) \chi_x \right]$$

$\int dU$ *factorizes* into 1-link $SU(N_c)$ integrals $\longrightarrow$ colorless d.o.f.

$$\begin{aligned} Z_{QCD} &= \int \prod_x d\bar{\chi} d\chi e^{2am_q M(x)} \prod_\nu \Big[\sum_{k_\nu(x)=0}^{N_c} \frac{(N_c - k_\nu(x))!}{N_c! k_\nu(x)!} (M(x) M(x+\hat{\nu}))^{k_\nu(x)} \\ &+ \rho_{x,x+\hat{\nu}}^{N_c} \bar{B}(x) B(x+\hat{\nu}) + (-\rho_{x,x-\hat{\nu}})^{N_c} \bar{B}(x+\hat{\nu}) B(x) \Big] \end{aligned}$$

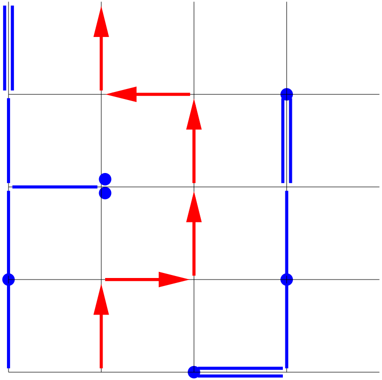
- Mesons $M(x) = \bar{\chi}_x \chi_x$
- Baryons $B(x) = \frac{1}{N_c!} \varepsilon_{i_1 \cdots i_{N_c}} \chi_{i_1}(x) \cdots \chi_{i_{N_c}}(x)$, weight $\rho_{x,x\pm\hat{\nu}} = \eta_\nu(x) e^{\pm a_\tau \mu \delta_{\nu 4}}$

Now, Grassmann integral $\longrightarrow$ worldline constraints

“Dual” variables in strong-coupling limit (II)

$$Z(m_q,\mu,\gamma) = \sum_{n_x,k_b,l_b} \underbrace{\prod_x \frac{N_c!}{n_x!}(2am_q)^{n_x}}_{\text{chiral condensate } M_x} \underbrace{\prod_{b=(x,\nu)} \frac{(N_c-k_b)!}{N_c!k_b!}\gamma^{2k_b\delta_{\nu 4}}}_{\text{meson hopping } M_x M_{x\pm\hat\nu}} \underbrace{\prod_{b=(x,\nu)} w(l_b,\mu)}_{\text{baryon hopping } \bar{B}_x B_{x\pm\hat\nu}}$$

$n_x \in 0,..,N_c,\quad k_b \in 0,..,N_c,\quad l_b \in 0,\pm 1$



Exact rewriting at $\beta = 0$

- Constraint at every site (Grassmann): 3 meson symbols (• $\bar\psi\psi$, meson hop) or a baryon loop

Point-like, hard-core baryons in pion bath

- Sign pb.: Δf reduced by $\mathcal{O}(10^4)$

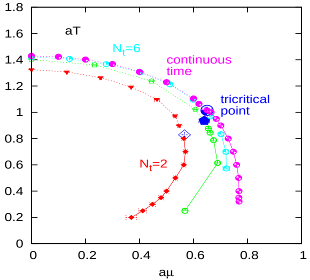
$\rightarrow$ full (μ, T) phase diagram

Phase diagram at $\beta = 0$

- Need to increase temperature above $T = \frac{1}{2a}$ to see chiral symmetry restoration
  $\Longrightarrow$ asymmetry γ in Dirac operator

- But resulting anisotropy $\frac{a}{a_\tau} = f(\gamma)$ unknown
  - Mean-field $\frac{a}{a_\tau} = \gamma^2$
  - Not continuum field theory anyway

Phase diagram depends - mildly - on N_τ and approaches *continuous time* result



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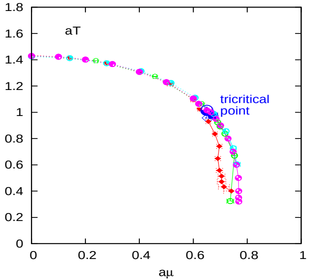
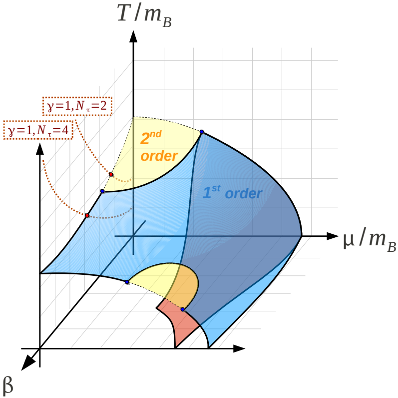
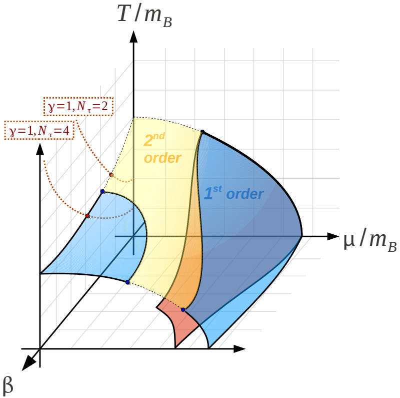


$\frac{a}{a_\tau} = \gamma^2(1 + \mathcal{O}(\frac{1}{\gamma^2}))$

Moving away from strong-coupling limit $\beta = 0$

- In continuum limit $\beta \to +\infty$, recover phase diagram of $N_f = 4$





- "Trajectory" of $\beta = 0$ tricritical point ?
- Appearance of second low-T phase transition ?

Strategy ? Sign problem ??

2012: How to make the sign problem milder?

- Severity of sign pb. is representation dependent:

$$Z = \mathrm{Tr} e^{-\beta H} = \mathrm{Tr}\left[e^{-\frac{\beta}{N}H} \left(\sum |\psi\rangle\langle\psi|\right) e^{-\frac{\beta}{N}H} \left(\sum |\psi\rangle\langle\psi|\right) \cdots \right]$$

Any complete set $\{|\psi\rangle\}$ will do

If $\{|\psi\rangle\}$ form an eigenbasis of H, then $\langle\psi_k|e^{-\frac{\beta}{N}H}|\psi_l\rangle = e^{-\frac{\beta}{N}E_k}\delta_{kl} \geq 0 \rightarrow$ no sign pb

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QCD physical states are color singlets $\rightarrow$ Monte Carlo on colored gluon links is bad idea

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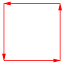
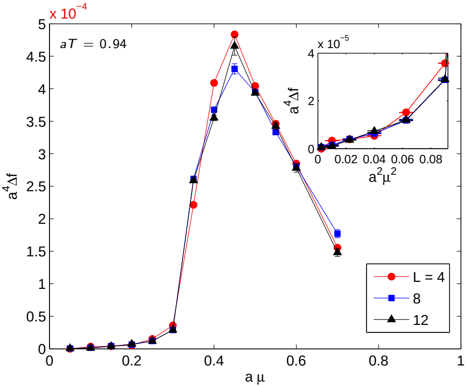
Usual:
- integrate over quarks analytically $\rightarrow \det(\{U\})$
- Monte Carlo over gluon fields $\{U\}$

Reverse order:
- integrate over gluons $\{U\}$ analytically
- Monte Carlo over quark color singlets (hadrons)

- Caveat: so far, turn off 4-link coupling  in $\beta \sum_P \mathrm{ReTr} U_P$ by setting $\beta = 0$

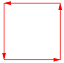
$\beta = 0$: strong-coupling limit $\longleftrightarrow$ continuum limit ($\beta \rightarrow \infty$)

2012: Sign problem? Monitor $-\frac{1}{V}\log\langle\text{sign}\rangle$



- $\langle\text{sign}\rangle = \frac{Z}{Z_{||}} \sim \exp(-\frac{V}{T}\Delta f(\mu^2))$ as expected; $\Delta f \sim \mu^2 + \mathcal{O}(\mu^4)$
- Determinant method $\rightarrow \Delta f \sim \mathcal{O}(1)$. Gain $\mathcal{O}(10^4)$ in the exponent!
  - heuristic argument correct: color singlets closer to eigenbasis
  - negative sign caused by spatial baryon hopping:
    - no baryon $\rightarrow$ no sign pb (no silver blaze pb.)
    - saturated with baryons $\rightarrow$ no sign pb

2012: Going beyond strong coupling limit $\beta = 0$?

Problem: $S_{YM} = \beta \sum_P \mathrm{Re}\mathrm{Tr} U_P$ prevents factorization of link integral 

1 $\Rightarrow$ • Truncate $e^{S_{YM}}$ to $\mathcal{O}(\beta)$: in progress w/O. Philipsen, W. Unger et al.

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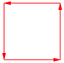
Problem: $S_{YM} = \beta \sum_P \mathrm{ReTr} U_P$ prevents factorization of link integral 

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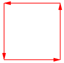
Problem: $S_{YM} = \beta \sum_P \mathrm{ReTr}\, U_P$ prevents factorization of link integral 

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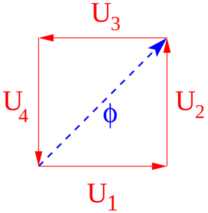
3 $\Rightarrow$ • More original: decouple the 4 links in each plaquette by *auxiliary fields*?
Hubbard-Stratonovich variant:

$$\exp(\beta\ \mathrm{ReTr}\, U_P)$$

$$\Updownarrow$$

$$\int d\phi^* d\phi \exp\left[-\beta\ \mathrm{ReTr}\left(|\phi|^2 - \phi^\dagger U_1 U_2 - U_3 U_4 \phi\right)\right]$$

ie. "2-link + random field" action



Further decoupling to "1-link" action $\rightarrow$ link integration possible $\forall \beta$

2014: see talk by Helvio Vairinhos

How to truncate $e^{S_{YM}}$ to $\mathcal{O}(\beta)$ (I)

$$Z_F(\bar{\chi},\chi) \equiv \int dU\ e^{-S_F} = \prod_{l=(x,\nu)} z_l\ , \quad \langle W \rangle_{Z_F} \equiv \frac{1}{Z_F} \int dU\ W\ e^{-S_F} \quad (1)$$

$$Z_{QCD}(\beta) = \int d\bar{\chi} d\chi\ dU\ e^{-S_F - S_G} = \int d\bar{\chi} d\chi\ Z_F\ \langle e^{-S_G} \rangle_{Z_F} \quad (2)$$

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$$J_{ij} \equiv \int dU\ U_{ij}\exp(\bar\psi U\phi - \bar\phi U^\dagger\psi)$$
$$= -\sum_{k=1}^{3}\frac{(3-k)!}{3!(k-1)!}\left[M_\psi M_\phi\right]^{k-1}\bar\phi_j\psi_i + \frac{1}{12}\varepsilon_{ii_2i_3}\varepsilon_{jj_2j_3}\bar\psi_{i_2}\phi_{j_2}\bar\psi_{i_3}\phi_{j_3} - \frac{1}{3}\bar B_\psi B_\phi\bar\phi_j\psi_i$$

i.e. hopping of $(\bar q g)$ or (qqg) + colorless

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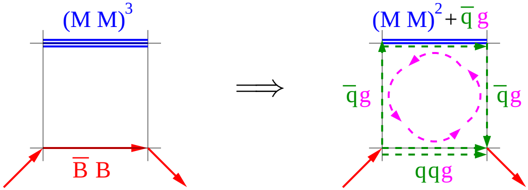
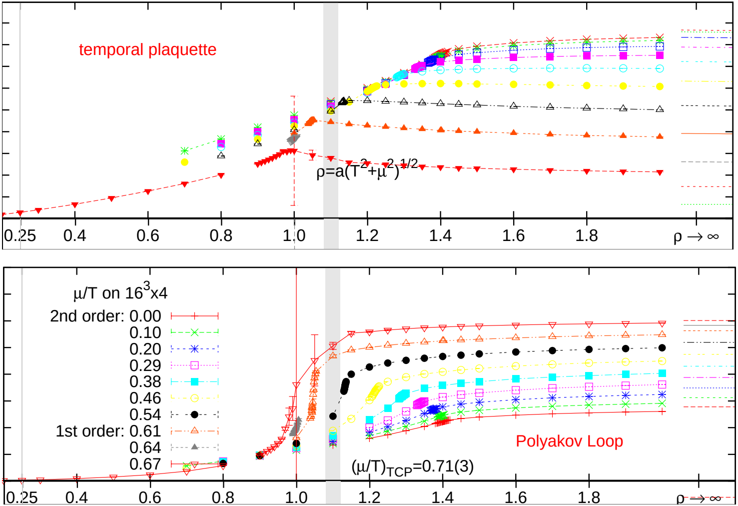
$\mathrm{Tr} J_{ij}J_{jk}J_{kl}J_{li}$ + Grassmann integration $\rightarrow \langle\mathrm{Tr}[U_P + U_P^\dagger]\rangle_{\beta=0} = \langle\sum_{s=1}^{19} F_P^s(M,B,\bar B)\rangle_{\beta=0}$
Each F_P^s modifies $\beta=0$ worldline configuration to insert a plaquette of gluons:



ie. *quark exchange*

Examples: Wilson loop vev's at $\beta = 0$

Plaquette and Polyakov loop sensitive to *chiral* transition



$$\sqrt{(a\mu)^2 + (aT)^2}$$

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$$Z_{QCD}(\beta) = \int d\bar\chi d\chi\ dU\ e^{-S_F-S_G} = \int d\bar\chi d\chi\ Z_F\ \langle e^{-S_G}\rangle_{Z_F}$$

$$e^{-S_G} = \prod_P e^{\frac{\beta}{2N_c}\mathrm{Tr}[U_P+U_P^\dagger]} \approx \prod_P \left(1+\frac{\beta}{2N_c}\mathrm{Tr}[U_P+U_P^\dagger]\right)$$

- Exact at $\mathcal{O}(\beta)$
- Resummation scheme, "almost correct" at $\mathcal{O}(\beta^2)$:
  - Correct weight for $\mathrm{Tr}U_{P_i}\mathrm{Tr}U_{P_j}$
  - Except when P_i and P_j share a link
  - Missing $(\mathrm{Tr}U_P)^2$
- Sample stochastically via $q_P = \{0,1\}$ and

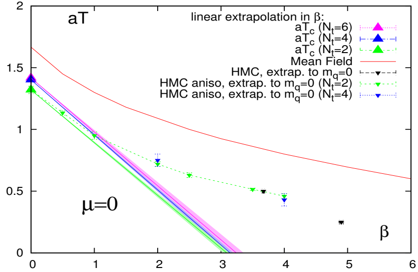
$$1+\frac{\beta}{2N_c}\mathrm{Tr}[U_P+U_P^\dagger] = \sum_{q_P=0,1}\left(\delta_{q_P,0}+\delta_{q_P,1}\frac{\beta}{2N_c}\mathrm{Tr}[U_P+U_P^\dagger]\right)$$

Example: (μ, T) phase diagram, at $\mathcal{O}(\beta)$ and better

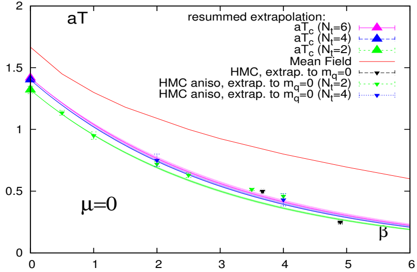
- Generate Monte Carlo ensemble at $\beta = 0$
- Reweight stochastically: choose $q_P = \{0, 1\}, s = \{1, .., 19\}$ for each plaquette

  weight $w \propto \prod_P (1 + q_P F_P^s)$
- Measure observable at $\beta = 0$, eg. chiral susceptibility $(\bar{\chi}\chi)^2$ (for $m_q = 0$)
- Reweight observable: $\langle (\bar{\chi}\chi)^2 \rangle_{\beta > \mathbf{0}} = \frac{\langle w (\bar{\chi}\chi)^2 \rangle_{\beta=0}}{\langle w \rangle_{\beta=0}}$
- Phase boundary from finite-size scaling of chiral susceptibility

Benchmark: $aT_c(\mu = 0)$ versus β $\quad (am_q = 0)$



Linear reweighting



Exponential extrapolation

Conclusions

- No universal solution to sign pb
- Sufficient goal:
  make sign pb mild *enough* to study large *enough* $\{\mu, V, \frac{1}{T}\}$
- Guiding principle: use basis of near-eigenstates
- QCD: integrate out gluons first
- So far: • $\langle$ Wilson loops $\rangle$ at $\beta = 0$
  • (μ, T) phase diagram at $\mathcal{O}(\beta)$ + resummation