Effective actions for SU(3) gauge theories

and mean-field solutions at finite density

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Start with lattice gauge theory and integrate out all d.o.f. subject to the constraint that the Polyakov line holonomies are held fixed. In temporal gauge

$$e^{S_P[U_{\mathbf{x}}]} = \int DU_0(\mathbf{x},0) DU_k D\phi \left\{ \prod_{\mathbf{x}} \delta[U_{\mathbf{x}} - U_0(\mathbf{x},0)] \right\} e^{S_L}$$

At leading order in the strong coupling/hopping parameter expansion S_P has the form of an SU(3) spin model

$$\begin{aligned} S_{spin} &= J \sum_x \sum_{k=1}^{3} \left(\text{Tr}[U_x] \text{Tr}[U^\dagger_{x+\hat{k}}] + \text{c.c.} \right) \\ &\quad + h \sum_x \left(e^{\mu/T} \text{Tr}[U_x] + e^{-\mu/T} \text{Tr}[U_x] \right) \end{aligned}$$

The SU(3) spin model has been solved successfully, for a wide range of parameters J, h, μ, in several different ways:

Methods

1. flux representation *(Gattringer and Mercado)*
2. stochastic quantization *(Aarts and James)*
3. reweighting *(Fromm, Langelage, Lottini and Philipsen)*
4. mean field *(Splittorff and JG)*

Since these methods work for the simple SU(3) spin model S_{spin}, perhaps they also work for the more complicated effective action S_P.

The problem is to find the effective action S_P, corresponding to lattice gauge theory at weaker couplings, finite μ, and light quark masses.

Avoid dynamical fermion simulations for now, work instead with an SU(3) gauge-Higgs model

$$S_L = \frac{\beta}{3} \sum_p \mathrm{ReTr}[U(p)] + \frac{\kappa}{3} \sum_x \sum_{\mu=1}^{4} \mathrm{ReTr}\left[\Omega^\dagger(x) U_\mu(x) \Omega(x+\hat{\mu})\right]$$

If we can derive S_P at $\mu = 0$, then we also have S_P at $\mu > 0$ by the following identity:

$$S_P^{\mu}[U_{\mathbf{x}}, U_{\mathbf{x}}^\dagger] = S_P^{\mu=0}\left[e^{N_t \mu} U_{\mathbf{x}}, e^{-N_t \mu} U_{\mathbf{x}}^\dagger\right]$$

which is true to all orders in the strong coupling/hopping parameter expansion.

How to compute S_P at $\mu = 0$?

- strong-coupling expansions *(Philipsen et al.)*
- inverse Monte Carlo *(Heinzl et al.)*
- relative weights *(this talk)*

And how do we know that we have derived S_P correctly?

One test: compare Polyakov line correlators

$$G(R) = \frac{1}{N_c^2}\left\langle \mathrm{Tr}[U_{\mathbf{x}}]\mathrm{Tr}[U_{\mathbf{y}}^\dagger] \right\rangle \quad , \quad R = |\mathbf{x} - \mathbf{y}|$$

computed for the effective action, and in the underlying lattice gauge theory.

Agreement has not been demonstrated in other approaches to deriving S_P beyond $R = 2$ or 3 lattice spacings (see, e.g., *Bergner et al., arXiv:1311.6745*)

In previous papers we worked out S_P for pure SU(2) gauge theory:

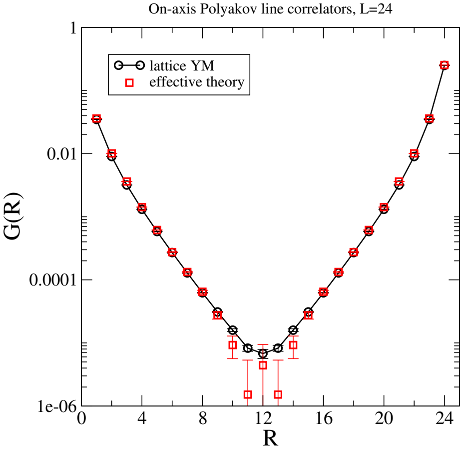
$$S_P = \sum_{\mathbf{x},\mathbf{y}} P_\mathbf{x} K(\mathbf{x}-\mathbf{y}) P_\mathbf{y}$$

where

$$P_\mathbf{x} = \frac{1}{2}\mathrm{Tr}\, U_\mathbf{x}$$

Here is the correlator comparison for

$$G(R) = \langle P_\mathbf{x} P_\mathbf{y} \rangle$$



The underlying lattice gauge theory is at $\beta = 2.2$ on a $24^3 \times 4$ lattice.

The Relative Weights Method

Let S'_L be the lattice action in temporal gauge with $U_0(\mathbf{x}, 0)$ fixed to $U'_{\mathbf{x}}$. It is not so easy to compute

$$\exp\left[S_P[U'_{\mathbf{x}}]\right] = \int DU_k D\phi \; e^{S'_L}$$

directly. But the ratio ("relative weights")

$$e^{\Delta S_P} = \frac{\exp[S_P[U'_{\mathbf{x}}]]}{\exp[S_P[U''_{\mathbf{x}}]]}$$

is easily computed as an expectation value

$$\begin{aligned} \exp[\Delta S_P] &= \frac{\int DU_k D\phi \; e^{S'_L}}{\int DU_k D\phi \; e^{S''_L}} \\ &= \frac{\int DU_k D\phi \; \exp[S'_L - S''_L] e^{S''_L}}{\int DU_k D\phi \; e^{S''_L}} \\ &= \left\langle \exp[S'_L - S''_L] \right\rangle'' \end{aligned}$$

where $\langle ... \rangle''$ means the VEV in the Boltzman weight $\propto e^{S''_L}$.

Suppose $U_{\mathbf{x}}(\lambda)$ is some path through configuration space parametrized by λ, and suppose $U'_{\mathbf{x}}$ and $U''_{\mathbf{x}}$ differ by a small change in that parameter, i.e.

$$U'_{\mathbf{x}} = U_{\mathbf{x}}(\lambda_0 - \tfrac{1}{2}\Delta\lambda) \quad , \quad U''_{\mathbf{x}} = U_{\mathbf{x}}(\lambda_0 + \tfrac{1}{2}\Delta\lambda)$$

Then the relative weights method gives us the derivative of the true effective action S_P along the path:

$$\left(\frac{dS_P}{d\lambda}\right)_{\lambda=\lambda_0} \approx \frac{\Delta S}{\Delta\lambda}$$

The question is: which derivatives will help us to determine S_P itself?

$$P_{\mathbf{x}} \equiv \frac{1}{N_c} \text{Tr}\, U_{\mathbf{x}} = \sum_{\mathbf{k}} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$$

We first set a particular momentum mode $a_{\mathbf{k}}$ to zero. Call the resulting configuration $\widetilde{P}_x$. Then define ($f \approx 1$)

$$\begin{aligned} P''_{\mathbf{x}} &= \left(\alpha - \frac{1}{2}\Delta\alpha\right) e^{i\mathbf{k}\cdot\mathbf{x}} + f\widetilde{P}_x \\ P'_{\mathbf{x}} &= \left(\alpha + \frac{1}{2}\Delta\alpha\right) e^{i\mathbf{k}\cdot\mathbf{x}} + f\widetilde{P}_x \end{aligned}$$

which uniquely determine (in SU(2) and SU(3)) the eigenvalues of the corresponding holonomies $U'_{\mathbf{x}}, U''_{\mathbf{x}}$.

S_P has a remnant local symmetry $U_{\mathbf{x}} \to g_{\mathbf{x}} U_{\mathbf{x}} g^\dagger_{\mathbf{x}}$, so the holonomies $U'_{\mathbf{x}}, U''_{\mathbf{x}}$ can be taken to be diagonal. We then compute

$$\frac{1}{L^3}\left(\frac{\partial S_P}{\partial a^R_{\mathbf{k}}}\right)_{a_{\mathbf{k}}=\alpha}$$

by the relative weights simulation ($a^R_{\mathbf{k}}$ is the real part of $a_{\mathbf{k}}$).

For a pure gauge theory, the part of S_P bilinear in P_x is constrained to have the form

$$S_P = \sum_{\mathbf{xy}} P_{\mathbf{x}} P^\dagger_{\mathbf{y}} K(\mathbf{x}-\mathbf{y})$$

Then, going over to Fourier modes

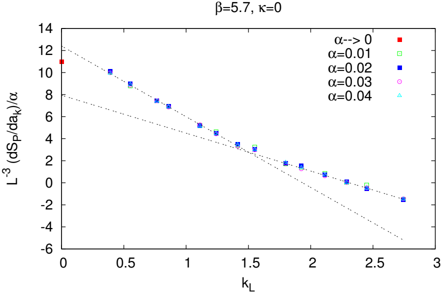
$$\frac{1}{\alpha}\frac{1}{L^3}\left(\frac{\partial S_P}{\partial a^R_{\mathbf{k}}}\right)_{a_{\mathbf{k}}=\alpha} = 2\widetilde{K}(\mathbf{k})$$

We work on a $16^3 \times 6$ lattice volume; there is a deconfinement transition at $\beta = 5.89$, but we are interested in the confinement (or, with matter, the "confinement-like") regime. Here are the relative weights results at $\beta = 5.7$:

1. Rotation invariance: data points only depend on **k** through k_L

$$k_L = 2\sqrt{\sum_{i=1}^{3} \sin^2(k_i/2)}$$

2. Except at $k_L = 0$, the data points fall on one of two straight lines, with different slopes.
3. Deviation at $k_L = 0$ is handled by a long range cutoff in the kernel $K(\mathbf{x}-\mathbf{y})$, which would otherwise be proportional to $\sqrt{-\nabla^2}$.



To compute $K(\mathbf{x}-\mathbf{y})$

1. Fit the $\widetilde{K}(k_L)$ data to

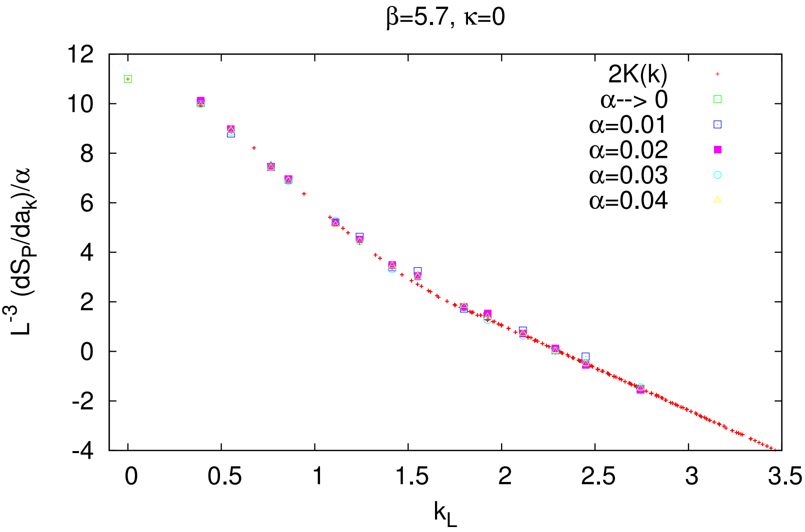
$$\widetilde{K}^{fit}(k_L) = \begin{cases} c_1 - c_2 k_L & k_L \le k_0 \\ b_1 - b_2 k_L & k_L > k_0 \end{cases}$$

2. Introduce a long-range cutoff r_{max}

$$K(\mathbf{x}-\mathbf{y}) = \begin{cases} \frac{1}{L^3}\sum_{\mathbf{k}} \widetilde{K}^{fit}(k_L) e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} & |\mathbf{x}-\mathbf{y}| \le r_{max} \\ 0 & |\mathbf{x}-\mathbf{y}| > r_{max} \end{cases}$$

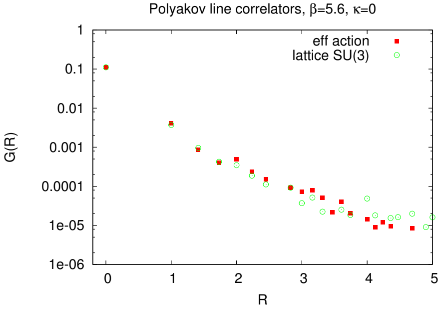
3. Transform back to momentum space. Choose cutoff r_{max} so that $\widetilde{K}(0)$ matches the data point at $k_L = 0$.

The red points are the Fourier transform of $K(\mathbf{x} - \mathbf{y})$, which gives us the effective action S_P

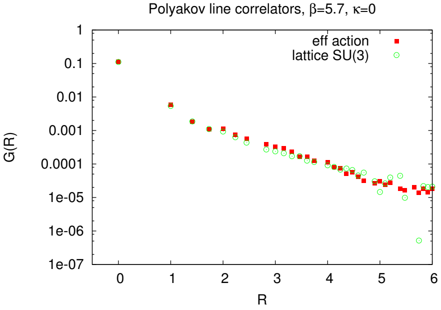


$$S_P = \sum_{\mathbf{xy}} P_{\mathbf{x}} P_{\mathbf{y}}^{\dagger} K(\mathbf{x} - \mathbf{y})$$

Simulate the effective theory in the usual way, and compare the Polyakov line correlators in the effective theory with the correlators in the underlying pure gauge theory



$\beta = 5.6$



$\beta = 5.7$

Gauge-Higgs model: General remarks

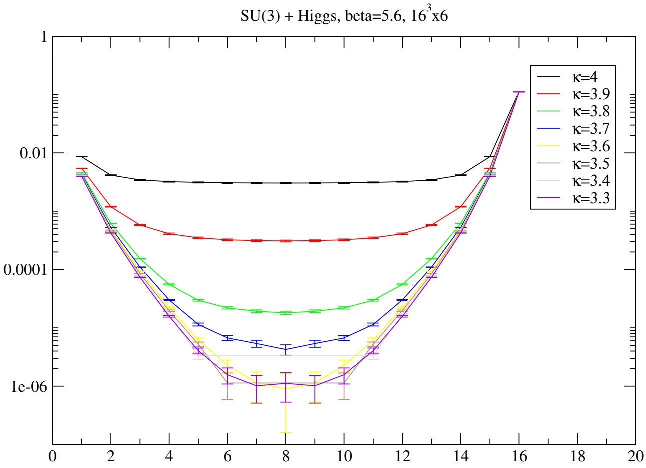
Fradkin-Shenker-Osterwalder-Seiler Theorem

In an SU(N) lattice gauge theory with matter in the fundamental representation, there is no absolute separation in coupling-constant space between a confining and a Higgs phase.

We are considering the SU(3) gauge-Higgs action

$$S_L = \frac{\beta}{3} \sum_p \mathrm{ReTr}[U(p)] + \frac{\kappa}{3} \sum_x \sum_{\mu=1}^4 \mathrm{ReTr}\left[\Omega^\dagger(x) U_\mu(x) \Omega(x+\hat{\mu})\right]$$

In our case, keeping $\beta = 5.6$ fixed and varying κ, there is a rapid crossover from a “confinement-like” to a “Higgs-like” region at $\kappa \approx 4.0$.



This plot shows the Polyakov line correlator $G(R) = \langle P_{\mathbf{x}} P_{\mathbf{y}} \rangle$ vs. R for the SU(3) gauge-Higgs model, computed by standard lattice Monte Carlo (+ Lüscher-Weisz noise reduction), at $\beta = 5.6$ and various κ.

Finite density: General remarks

Introducing matter fields introduces a dependence on chemical potential in S_P:

$$S_P = \sum_s e^{s\mu/T} S_P^{(s)}[U_{\mathbf{x}}, U_{\mathbf{x}}^\dagger]$$

- Truncation is inevitable.
- But terms which are negligible at $\mu = 0$ can become significant at large enough μ.
- The hope is to calculate enough of S_P so that the approximation works in the region of interest in the $\mu - T$ plane.
- For now we will determine S_P up to 2nd order in fugacity, and 2nd order in products of Polyakov lines.

The starting point is to include, in the center symmetry-breaking terms, $\text{Tr}U_{\mathbf{x}}, \text{Tr}U_{\mathbf{x}}^2$ (+ complex conjugates), and products of no more than two of these terms.

1. Write down all possible terms in S_P involving $\text{Tr}U_{\mathbf{x}}, \text{Tr}U_{\mathbf{x}}^2, \text{Tr}U_{\mathbf{x}}^\dagger, \text{Tr}U_{\mathbf{x}}^{\dagger 2}$ and nonlocal products of any two of these terms.

2. Introduce a finite chemical potential via the transformation

$$U_{\mathbf{x}} \to e^{N_t \mu} U_{\mathbf{x}}, \;\; U_{\mathbf{x}}^\dagger \to e^{-N_t \mu} U_v^\dagger x$$

3. Make use of the SU(3) identities

$$\text{Tr}[U_{\mathbf{x}}^2] = 9P_{\mathbf{x}}^2 - 6P_{\mathbf{x}}^\dagger \quad , \quad \text{Tr}[U_{\mathbf{x}}^{\dagger 2}] = 9P_{\mathbf{x}}^{\dagger 2} - 6P_{\mathbf{x}}$$

to express everything in terms of the $P_{\mathbf{x}}$ variables.

4. Discard terms involving a product of three or more $P_{\mathbf{x}}$'s.

We end up with the bilinear action

$$
\begin{aligned}
S_P &= \sum_{\mathbf{xy}} P_{\mathbf{x}} P^{\dagger}_{\mathbf{y}} K(\mathbf{x}-\mathbf{y}) + \sum_{\mathbf{xy}} (P_{\mathbf{x}} P_{\mathbf{y}} Q(\mathbf{x}-\mathbf{y},\mu) + P^{\dagger}_{\mathbf{x}} P^{\dagger}_{\mathbf{y}} Q(\mathbf{x}-\mathbf{y};-\mu)) \\
&\quad + \sum_{\mathbf{x}} \Big\{ (d_1 e^{\mu/T} - d_2 e^{-2\mu/T}) P_{\mathbf{x}} + (d_1 e^{-\mu/T} - d_2 e^{2\mu/T}) P^{\dagger}_{\mathbf{x}} \Big\}
\end{aligned}
$$

where

$$
Q(\mathbf{x}-\mathbf{y};\mu) = Q^{(1)}(\mathbf{x}-\mathbf{y}) e^{-\mu/T} + Q^{(2)}(\mathbf{x}-\mathbf{y}) e^{2\mu/T} + Q^{(4)}(\mathbf{x}-\mathbf{y}) e^{-4\mu/T}
$$

The problem is to determine $K(\mathbf{x}-\mathbf{y}), d_1, d_2, Q(\mathbf{x}-\mathbf{y};\mu)$.

Use of the imaginary chemical potential $\mu/T = i\theta$

In terms of Fourier amplitudes

$$\frac{1}{L^3} S_P = \sum_{\mathbf{k}} a_{\mathbf{k}} a_{\mathbf{k}}^* \widetilde{K}(k_L) + a_0 \left(d_1 e^{i\theta} - d_2 e^{-2i\theta} \right) + a_0^* \left(d_1 e^{-i\theta} - d_2 e^{2i\theta} \right) + \sum_{\mathbf{k}} \left(a_{\mathbf{k}} a_{-\mathbf{k}} \widetilde{Q}(k_L, \theta) + a_{\mathbf{k}}^* a_{-\mathbf{k}}^* \widetilde{Q}(k_L, \theta) \right)$$

Then

$$\frac{1}{L^3} \left(\frac{\partial S_P}{\partial a_0^R} \right)_{a_0 = \alpha} = 2\widetilde{K}(0)\alpha + 2d_1 \cos(\theta) - (2d_2 - 4\widetilde{Q}(0)\alpha) \cos(2\theta)$$

Fit to

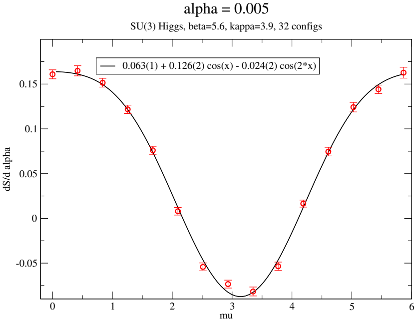
$$\frac{1}{L^3} \left(\frac{\partial S_P}{\partial a_0^R} \right)_{a_0^R = \alpha} = A(\alpha) + B(\alpha) \cos(\theta) - C(\alpha) \cos(2\theta)$$

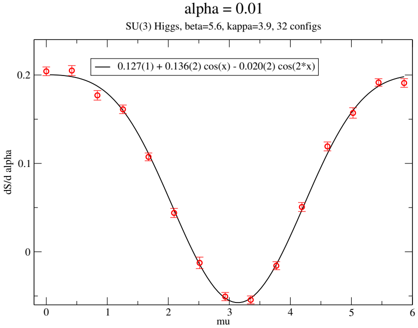
Compare the data to the fit, and we find $d_1, d_2, \widetilde{K}(0), \widetilde{Q}(0)$.

Gauge-Higgs theory at $\beta = 5.6$, $\kappa = 3.9$ on a $16^3 \times 6$ lattice. Calculate (lhs) and fit (rhs)

$$\frac{1}{L^3}\left(\frac{\partial S_P}{\partial a_0^R}\right)_{a_0^R=\alpha} = A(\alpha) + B(\alpha)\cos(\theta) - C(\alpha)\cos(2\theta)$$

at 15 values of θ and several α values:





We can then extract coefficients of center symmetry-breaking terms (in this case $d_1 = 0.0585$, $d_2 = 0.0115$), as well as $\widetilde{K}(0)$ and $\widetilde{Q}(0)$.

For $\mathbf{k} \neq 0$, the derivative wrt $a_\mathbf{k}$ has terms proportional to $a_{-\mathbf{k}}$. We set $a_{-\mathbf{k}}$ to some constant real value $a_{-\mathbf{k}} = \sigma$. Then

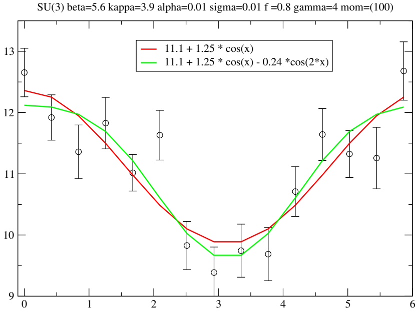
$$\frac{1}{L^3}\left(\frac{\partial S_P}{\partial a^R_\mathbf{k}}\right)^{a_{-\mathbf{k}}=\sigma}_{a_\mathbf{k}=\alpha} = 2\widetilde{K}(k_L)\alpha + 4\Big(\widetilde{Q}^{(1)}(k_L)\cos(\theta) + \widetilde{Q}^{(2)}(k_L)\cos(2\theta) + \widetilde{Q}^{(4)}(k_L)\cos(4\theta)\Big)\sigma$$

First, setting $\sigma = 0$, we have

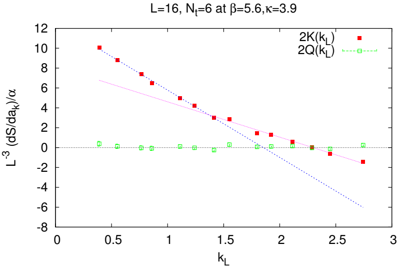
$$\widetilde{K}(k_L) = \frac{1}{2L^3}\frac{1}{\alpha}\left(\frac{\partial S_P}{\partial a^R_\mathbf{k}}\right)^{\alpha_{-\mathbf{k}}=0}_{a_\mathbf{k}=\alpha}$$

Then, at small but finite σ, we can determine the $\widetilde{Q}^{(n)}(k_L)$ from the θ-dependence of the data.

$\widetilde{Q}(k_L, \mu)$ seems calculable, but the magnitude is small and the errorbars are large:



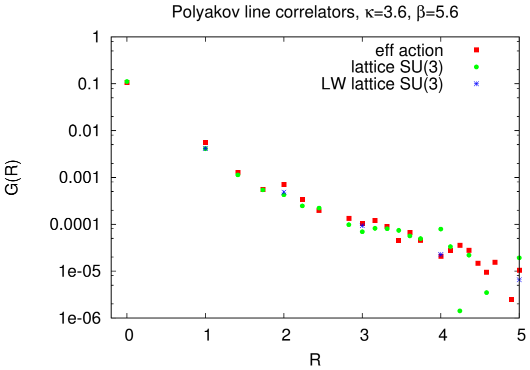
(a) $a_{\mathbf{k}}$ derivative at smallest $k_L \neq 0$, vs. θ



(b) $\widetilde{K}(k_L)$ and (estimate of) $\widetilde{Q}_1(k_L)$ vs. k_L

For now we will ignore the $Q(\mathbf{x} - \mathbf{y}; \mu)$ term in the action.

The underlying lattice gauge-Higgs theory is at $\beta = 5.6$, $\mu = 0$ and $\kappa = 3.6, 3.8, 3.9$ on a $16^3 \times 6$ lattice volume.



(a) $\kappa = 3.6$

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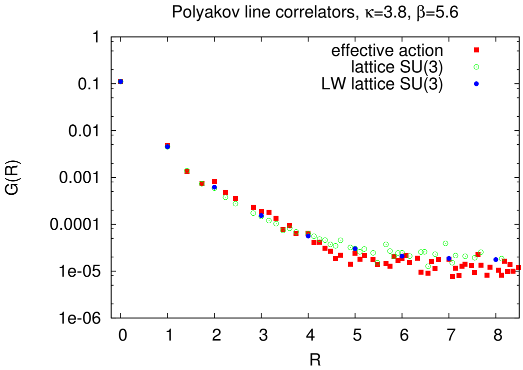
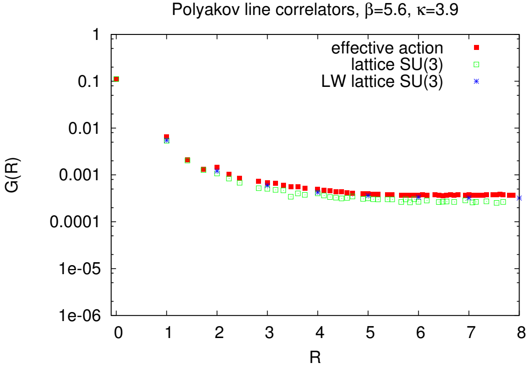


(b) $\kappa = 3.8$

The underlying lattice gauge-Higgs theory is at $\beta = 5.6$, $\mu = 0$ and $\kappa = 3.6, 3.8, 3.9$ on a $16^3 \times 6$ lattice volume.



(c) $\kappa = 3.9$

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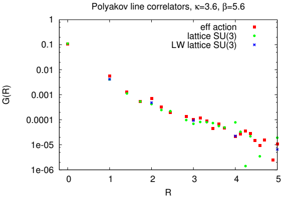
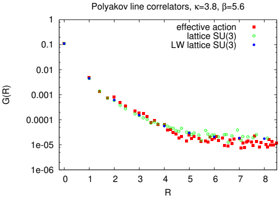
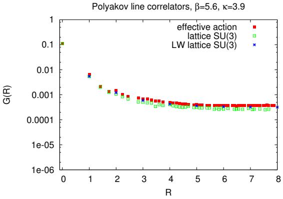


(a) $\kappa = 3.6$



(b) $\kappa = 3.8$

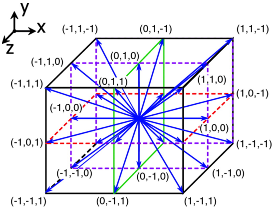


(c) $\kappa = 3.9$

Mean Field solution at $\mu > 0$

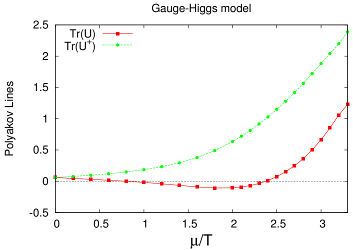
- S_P still has a sign problem.
- It can be addressed in various ways: flux representation, stochastic quantization, reweighting, and mean field.
- In general $-$ mean field becomes more reliable the more spins are coupled to a given spin. But for S_P, *many* spins are coupled to any given spin, especially for light scalar masses, through the non-local kernel $K(\mathbf{x} - \mathbf{y})$.
- *Perhaps the mean field method is more reliable, when applied to S_P at finite μ, than one might expect.*

Whether or not that is true, we have applied mean field to S_P, following the treatment in *Splittorff and JG, arXiv:1206.1159* for the SU(3) spin model.

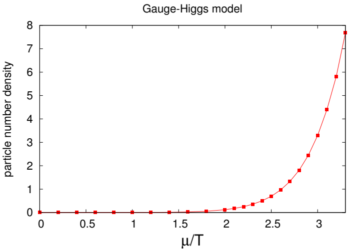


Gauge-Higgs model

Solution of S_P for $\langle \text{Tr} U_{\mathbf{x}} \rangle$, $\langle \text{Tr} U^{\dagger}_{\mathbf{x}} \rangle$ and particle number/site n, for an underlying lattice gauge-Higgs theory at $\beta = 5.6$ and $\kappa = 3.9$, $16^3 \times 6$ lattice volume, varying μ.



(a) $\langle \text{Tr} U_{\mathbf{x}} \rangle$, $\langle \text{Tr} U^{\dagger}_{\mathbf{x}} \rangle$



(b) particle number density

Heavy Quark model

In a certain limit where the inverse mass (staggered) or hopping parameter (Wilson) is very small, and the chemical potential μ is large, the fermion determinant simplifies. In temporal gauge, the lattice action is

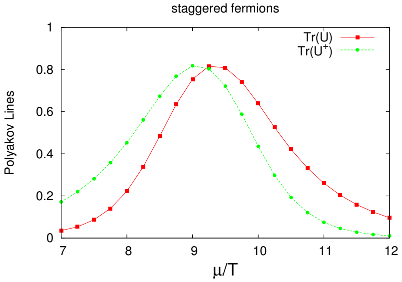
$$e^{S_L} = \prod_{\mathbf{x}} \det\left[1 + he^{\mu/T}U_0(\mathbf{x},0)\right]^p \det\left[1 + he^{-\mu/T}U^\dagger(\mathbf{x},0)\right]^p e^{S_{plaq}}$$

where S_{plaq} is the plaquette action, $p = 1$ for staggered fermions, $p = 2N_f$ for Wilson fermions. If we know the Polyakov line action for the pure gauge theory S_P^{pg}, then the Polyakov line action in this heavy quark limit is obtained immediately:

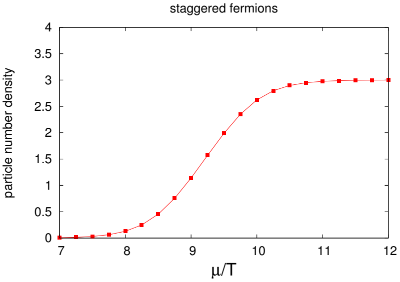
$$e^{S_P} = \prod_{\mathbf{x}} \det\left[1 + he^{\mu/T}U_{\mathbf{x}}\right]^p \det\left[1 + he^{-\mu/T}U_{\mathbf{x}}^\dagger\right]^p e^{S_P^{pg}}$$

This action is also amenable to a mean field solution.

Here are some mean field results for staggered fermions at $\beta = 5.6$ and $h = 0.0001 \rightarrow m = 2.32/a$. Note the saturation in number density at large μ.



Polyakov lines



number density

We have determined the effective Polyakov line actions S_P, up to terms bilinear in $P_{\mathbf{x}}$, corresponding to SU(3) pure gauge theory, to SU(3) gauge-Higgs theory, and to SU(3) with heavy quarks, at finite chemical potential.

Next Steps:

1. Beyond bilinear: determine contributions to S_P involving products of three or four Polyakov line variables $P_{\mathbf{x}}$.
2. Beyond mean field: reweighting, stochastic quantization, flux representation...
3. Beyond scalars: relative weights for lighter dynamical fermions.