Derivation and test of an effective lattice theory for finite density

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Outline of the talk

- Derivation
  - Definition
  - Leading order terms and leading corrections
  - Remarks on the expansion of the fermion determinant
- Tests
  - Correlation functions
  - Polyakov loop susceptibility
  - Free energy density
  - Analytical results
- Improvements
  - Nonperturbative effective couplings

Derivation

- Definition

$$e^{S_{\mathrm{eff}}} \equiv \int [dU_k] e^{S_{\mathrm{QCD}}} \equiv e^{S_0 + S[L]}$$

- $S[L]$ depends on Polyakov loops $L = \mathrm{Tr}\, W$ instead of single temporal link variables

$$Z = e^{S_0} \int [dW] e^{S[L]}$$

- $S[L]$ couples all numbers of Polyakov loops to arbitrary powers (or representations) and over arbitrary distances
- Eff. action inherits center symmetry from pure gauge theory

Advantages/problems

- Computationally cheap
  - Dimensional reduction $(3+1)d \to 3d$
  - Complex numbers instead of group matrices:

$$\int [dW] e^{S[L]} = \int [dL] e^{V[L]+S[L]}$$

  - N_τ as a parameter: One 3d simulation $\to$ results for all N_τ
- Truncations
  - Number of interaction terms must be finite
  - Series expansions of effective couplings also finite
  - Works better the smaller the lattice coupling β and the heavier the quarks are, i.e. small κ_q

$\longrightarrow$ Sign problem ameliorated (solved) in certain (not yet physical) parameter regimes
$\longrightarrow$ Task: Try to push the valid parameter values as close to physical ones as is possible

How to compute effective action

- Expand gauge part into characters

$$e^{S_g} = \prod_p \left[1 + \sum_{r\neq 0} d_r a_r(\beta)\chi_r(U_p)\right] \qquad a_f(\beta) = u(\beta) = \frac{\beta}{18} + \ldots$$

- Separate Quark determinant into static and kinetic part

$$\begin{aligned}\det[Q] &= \det[1-T-S] = \det[1-T]\det[1-(1-T)^{-1}S] \\ &\equiv \det[Q_{\mathrm{stat}}]\det[Q_{\mathrm{kin}}]\ ,\end{aligned}$$

and expand the latter in orders of κ_q (Wilson fermions)

- Integration over spatial link variables couples β and κ_q expansions

$$\begin{aligned}\int [dU_k] e^{S_g} \det[Q] &= e^{S_{\mathrm{eff}}[\beta,\kappa_q,N_\tau,\mu_q,(N_c,d,\Theta),\ldots]} \\ &= \exp\left[\sum_n \left(S_n^s + S_n^a\right)\right]\end{aligned}$$

Graphical expansion

- Three parameters and their graphical expression:
  1. $u = u(\beta)$: fundamental plaquette
  2. κ: single quark hop
  3. $c = (2\kappa e^{a\mu})^{N_\tau}$: Polyakov loop; $\bar{c} = (2\kappa e^{-a\mu})^{N_\tau}$: Anti-Loop
- The factor c is a fugacity factor connected with the "fermionic" Polyakov loop
- Selection rule: All links that are integrated over, have to be occupied by at least two nontrivial link variables, due to

$$\int dU \; U_{ij} = 0$$

Examples: Pure gauge theory

Contributions to nearest-neighbour Polaykoy-Loop interactions

Next-to-nearest-neighbour contributions

Contributions that cancel against each other

Examples: Including quark hops

Contributions projecting onto single Polyakov loops

Graphs leading to nearest-neighbour quark-quark interactions

Graphs leading to nearest-neighbour quark-antiquark interactions

Resummations

It is possible to resum interaction terms and effective coupling contributions:

$$2\lambda_1 \mathrm{Re} L_i L_j^* - \frac{1}{2}\left(2\lambda_1 \mathrm{Re} L_i L_j^*\right)^2 + \ldots = \ln\left[1 + 2\lambda_1 \mathrm{Re} L_i L_j^*\right]$$

$$\begin{aligned}
\lambda_1 &= u^{N_\tau}\left[1 + 4N_\tau u^4 + \frac{1}{2} 4N_\tau (4N_\tau - 3) u^8 + \ldots\right] \\
&= u^{N_\tau} \exp\left[4N_\tau u^4 + \ldots\right] \\
h_1 &= (2\kappa e^{a\mu})^{N_\tau}\left[1 + 6N_\tau \kappa^2 u + \frac{1}{2} 6N_\tau (6N_\tau - 3)\kappa^4 u^2 + \ldots\right] \\
&= (2\kappa e^{a\mu})^{N_\tau} \exp\left[6N_\tau \kappa^2 u + \ldots\right]
\end{aligned}$$

Effective partition function

$$Z = \int [dW] \prod_i \det \left[1 + h_1 W_i\right]^2 \left[1 + \bar{h}_1 W_i^\dagger\right]^2 \prod_{<ij>} \left[1 + 2\lambda_1 \mathrm{Re} L_i L_j^*\right]$$

- Effective couplings

$$\begin{aligned} h_1 &= c \exp\left[N_\tau \left(6\kappa^2 u + \ldots\right)\right] \\ \lambda_1 &= u^{N_\tau} \exp\left[N_\tau \left(4u^4 + \ldots\right)\right] \end{aligned}$$

- Higher order interaction terms

$$\begin{aligned} e^{S_2^a} &= \prod_{<ij>} \left[1 - h_2 \mathrm{Tr} \frac{cW_i}{1 + cW_i} \mathrm{Tr} \frac{cW_j^\dagger}{1 + cW_j^\dagger}\right] \\ e^{S_2^s} &= \prod_{[kl]} \left[1 + 2\lambda_2 \mathrm{Re} L_k L_l^*\right] \end{aligned}$$

Critical couplings

SU(2)

Receipt: Solve eff. theory for critical $\left(\lambda_i^c(\mu), h_i^c(\mu)\right)$ and convert back to critical $\left(\beta_c(\mu, N_\tau), \kappa_c(\mu, N_\tau)\right)$ via known relations

N_τ	3d Eff. Th. β_c	4d YM β_c
2	2.1929(13)	2.1768(30)
4	2.3102(08)	2.2991(02)
6	2.4297(05)	2.4265(30)
8	2.4836(03)	2.5104(02)
12	2.5341(02)	2.6355(10)
16	2.5582(02)	2.7310(20)

4d Monte Carlo results taken from [Fingberg et al. (1992), Bogolubsky et al. (2004) and Velytsky (2007)]

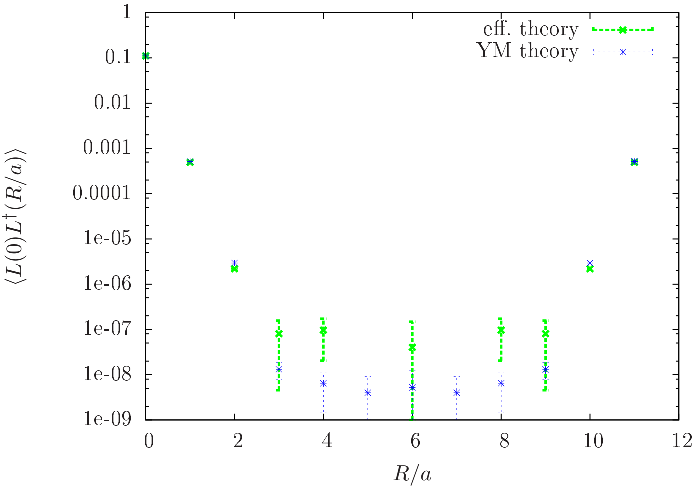
Critical couplings

SU(3)

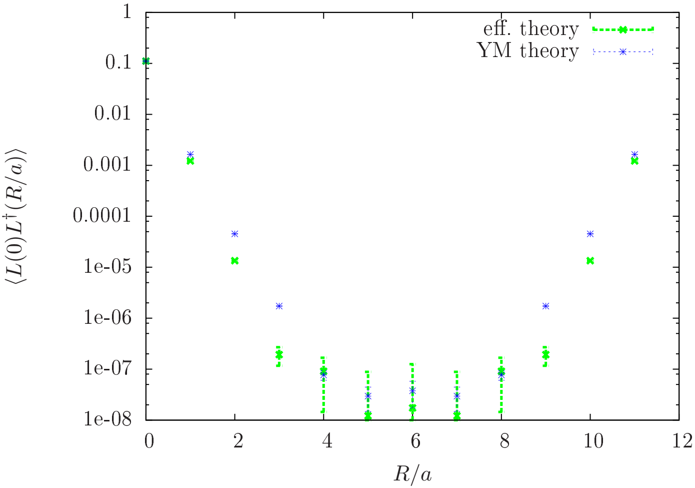
N_τ	3d Eff. Th. β_c	4d YM β_c
2	5.1839(2)	5.10(5)
4	6.09871(7)	5.6925(2)
6	6.32625(4)	5.8941(5)
8	6.43045(3)	6.001(25)
12	6.52875(2)	6.268(12)
16	6.57588(1)	6.45(5)

4d Monte Carlo results taken from [Fingberg et al. (1992)]

Comparison of correlators at $\beta = 5.0$

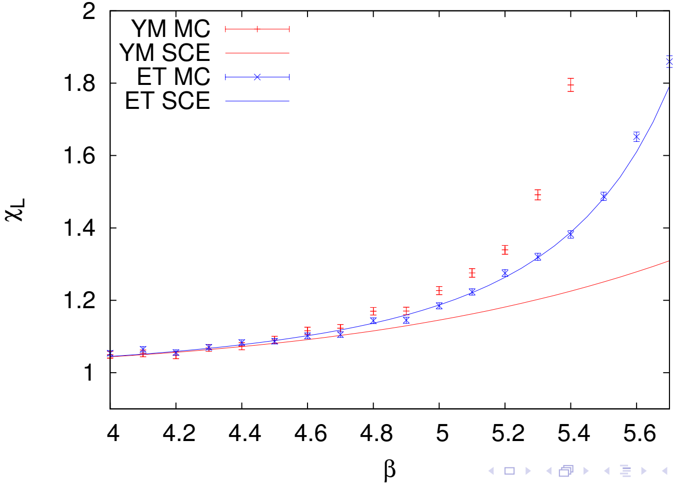


Comparison of correlators at $\beta = 5.4$



Polyakov Loop Susceptibility

$$\chi_L(\beta) = \frac{1}{2} N_s^3 \left[\langle L^2 \rangle - \langle L \rangle^2 \right] \qquad L = \frac{1}{N_s^3} \sum_i \left[L_i + L_i^* \right]$$



Free energy density

How to compute it in a MC Simulation [1312.7823]

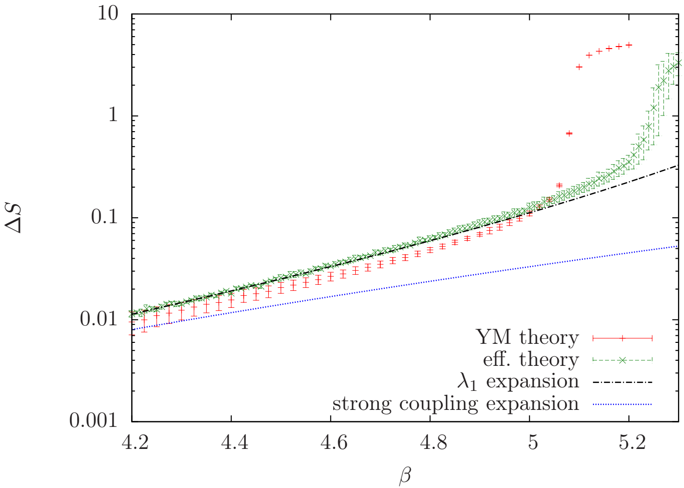
$$a^4 f\Big|_{\beta_0}^{\beta} = -\int_{\beta_0}^{\beta} d\beta' \Delta S(\beta') \, ,$$

with the interaction measure

$$\Delta S(\beta) \;=\; \frac{6}{N_c}\left(\left\langle \mathrm{Re}\mathrm{Tr}\, U_P \right\rangle\Big|_T - \left\langle \mathrm{Re}\mathrm{Tr}\, U_P \right\rangle\Big|_{T=0}\right)$$

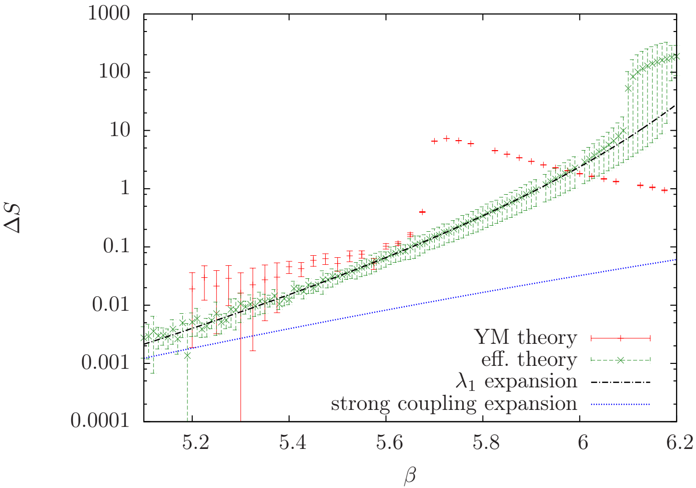
Free energy density

Interaction measure 2×8^3



Free energy density

Interaction measure 4×16^3



Analytic tests: Static strong coupling limit

- Consider only static determinant for large μ

$$Z(N_f) = \int [dW] \prod_{i,f} \det \left[1 + c_f W_i\right]^2$$

- Single-site problem, exactly solvable

$$\begin{aligned} Z(1) &= \left[1 + 4c^3 + c^6\right]^{N_s^3} \\ a^3 n_q(1) &= \frac{1}{N_\tau N_s^3} \frac{\partial}{\partial a\mu} \ln Z = \frac{12c^3 + 6c^6}{1 + 4c^3 + c^6} \end{aligned}$$

- Saturation density is $2N_c$ quarks per site
- $T = 0$ onset transition at c=1, i.e. $\mu = \ln(2\kappa) = \frac{m_b^{\rm stat}}{3}$

Analytic tests: Beyond the static limit

- Consider the following variant of the eff. th.

$$Z = \int [dW] \prod_i \det \left[1 + cW_i\right]^2 \prod_{<ij>} \left[1 + 2\lambda_1 \mathrm{Re} L_i L_j^*\right]$$
$$\times \prod_{<ij>} \left[1 - h_2 \mathrm{Tr} \frac{cW_i}{1 + cW_i} \mathrm{Tr} \frac{cW_j^\dagger}{1 + cW_j^\dagger}\right]$$

  Good approximation for heavy quarks, small temperatures and large chemical potentials.

- In leading orders it is

$$\frac{1}{N_s^3} \ln Z = \ln z_0 + 6\lambda_1 \frac{z_1 z_2}{z_0^2} - 6h_2 \frac{z_3^2}{z_0^2}$$
$$z_0 = 1 + 4c^3 + c^6 \qquad z_1 = 3c^2 + 2c^5$$
$$z_2 = 2c + 3c^4 \qquad z_3 = 6c^3 + 3c^6 \; .$$

Analytic tests: Beyond the static limit

$$E = -\frac{\partial}{\partial \beta} \ln Z \Big|_z$$

- Lattice version of the dimensionless energy density

$$a^4 e \;=\; -\frac{a}{N_\tau N_s^3}\left(\frac{\partial h_1}{\partial a}\right)\Big|_z \frac{\partial}{\partial h_1} \ln Z + \frac{6a}{N_\tau}\left(\frac{\partial h_2}{\partial a}\right)\left(\frac{z_3}{z_0}\right)^2$$

- After a few more steps

$$a^4 e = am_b a^3 n_b - \frac{am_m}{2} e^{-am_m}\left(\frac{z_3}{z_0}\right)^2$$

- Binding energy density

$$a^4 e_B = -\frac{am_m}{2} e^{-am_m}\left(\frac{z_3}{z_0}\right)^2$$

Nonperturbative effective couplings

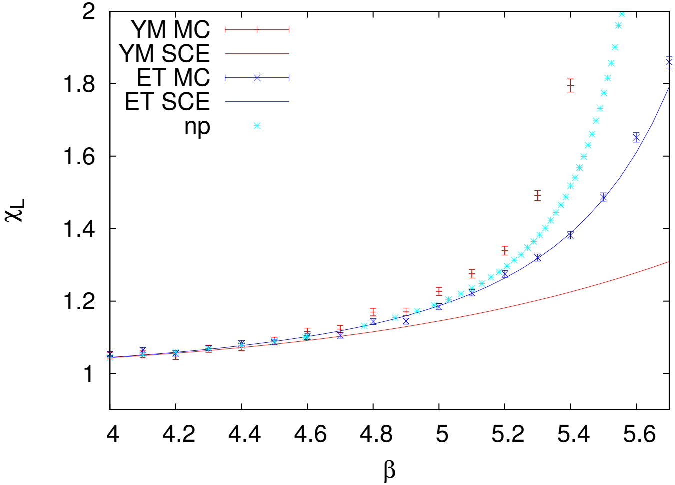
- Completely general Ansatz for SU(N) effective theory

$$e^{S_{\text{eff}}} = \sum_{\{r_i\}} \tilde{\lambda}(\{r_i\}) \prod_i \chi_{r_i}(W_i)$$

- Determine $\tilde{\lambda}_i$ nonperturbatively by computing correlation functions

More information: See Georg Bergner's Poster

Polyakov Loop Susceptibility with nonperturbatively improved effective coupling



Conclusions/Summary

- Presented derivation and several tests of effective PL action
- Effective theory able to describe physics in the confinement phase up to the transition region
- Works best for local observables
- Room for improvements: going from Taylor expansions to nonperturbatively computed effective couplings