# Solution to Sign Problems in p-h symmetric spin-less fermion systems

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Work supported by US Department of Energy

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# Outline

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  - Single species of Hamiltonian staggered fermions are harder.
- Can we solve sign problems in systems containing odd numbers of NR fermions OR minimally doubled R fermions?

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Here we will show that
p-h symmetry can help solve some sign problems.

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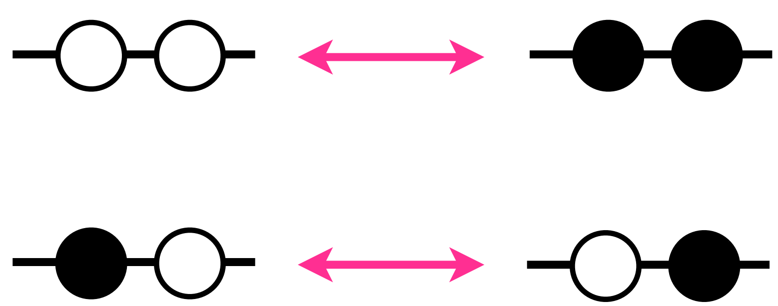

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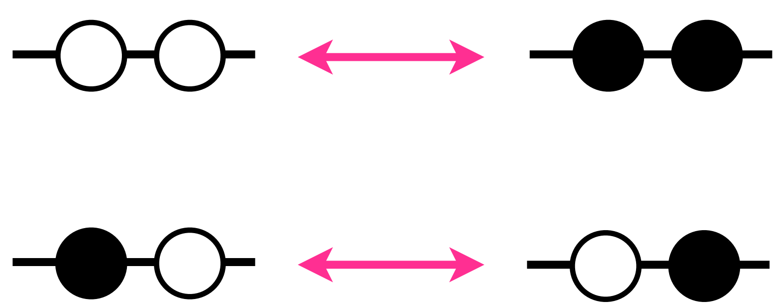

Thermal Average

$$\langle C_i^\dagger C_i \rangle_T = \frac{1}{2}$$

partition function

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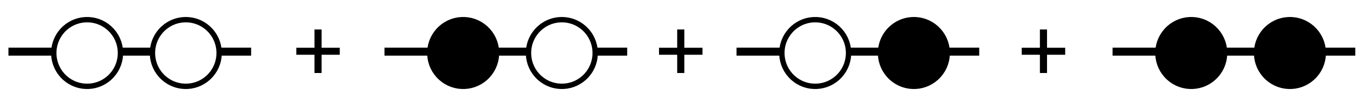

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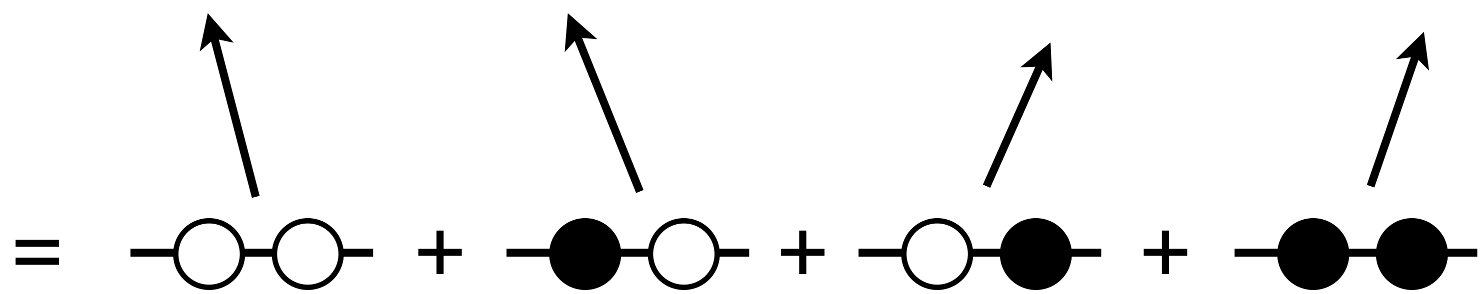

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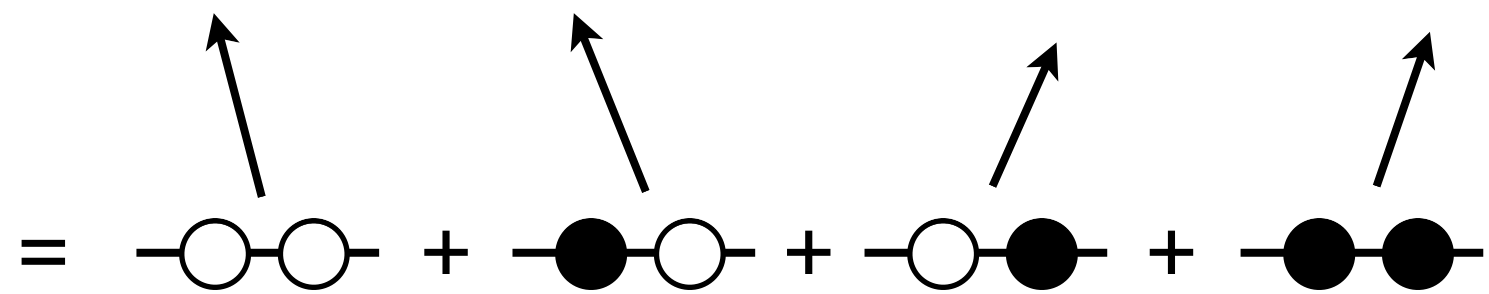

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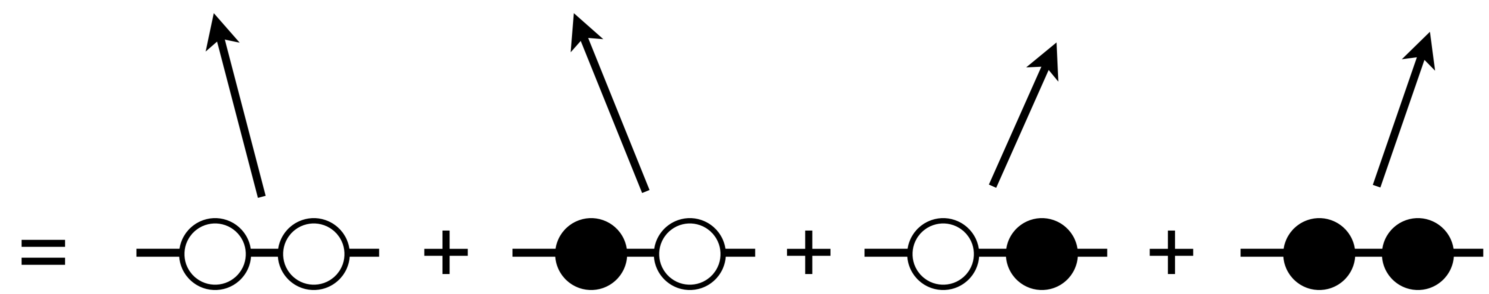

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Easy to see why $\langle C_i^\dagger C_i \rangle_T = \dfrac{1}{2}$

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$$S = -\sum_t \;\{\sum_i(\overline{\psi}_{i,t+1} - \overline{\psi}_{i,t})\psi_{i,t} + \Delta \sum_{i,j} \overline{\psi}_{i,t} M_{ij}\; \psi_{j,t}\}$$

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discrete time formulations preserving
p-h symmetry are indeed possible.

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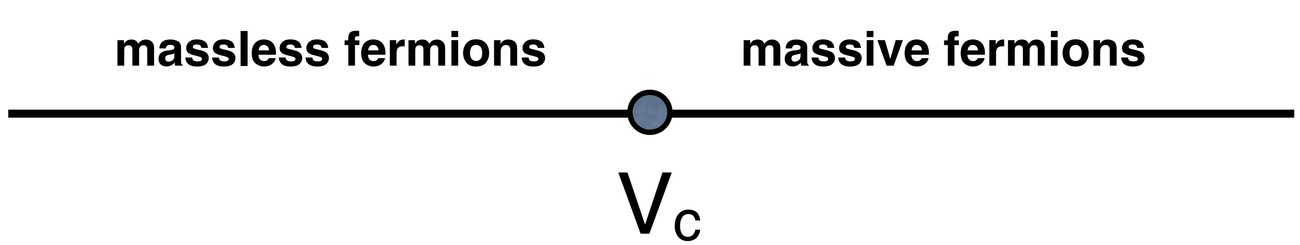

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What is the p-h symmetry here?

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Potential term invariant due to the fact that

$$\left(n_i - \frac{1}{2}\right) \to -\left(n_i - \frac{1}{2}\right)$$

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Instead note that

$$\left(n_i - \frac{1}{2}\right) \; = \; \frac{1}{2}\left(c_i^\dagger c_i - c_i c_i^\dagger\right)$$

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under p-h symmetry $s \rightarrow -s$

We write

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$$H_0 \;=\; \sum_{ij} C_i^\dagger \; M_{ij} \; C_j \qquad\qquad H_{\text{int}} \;=\; \frac{V}{4} \sum_{b=\langle ij \rangle, s_i, s_j} s_i \; n_i^{s_i} \; s_j \; n_j^{s_j}$$

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Using standard techniques can then write

$$Z = \sum_k \int [dt_1 ... dt_k] \; (-1)^k \; \mathrm{Tr}\Big( \mathrm{e}^{-(\beta - t_1) H_0} \; H_{int} \mathrm{e}^{-(t_1 - t_2) H_0} \; H_{int} ... \mathrm{e}^{-t_k \; H_0} \Big)$$

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Continuous time Monte Carlo: **Beard, Wiese(1996), Sandvik (1998), Prokof'ev, Svistunov (1998), Rubtsov, Savkin Lichtenstein (2005), many others in CM community**

We insert $H_{\text{int}} = \dfrac{V}{4} \sum_{b=\langle ij \rangle, s_i, s_j} s_i \; n_i^{s_i} \; s_j \; n_j^{s_j}$ into

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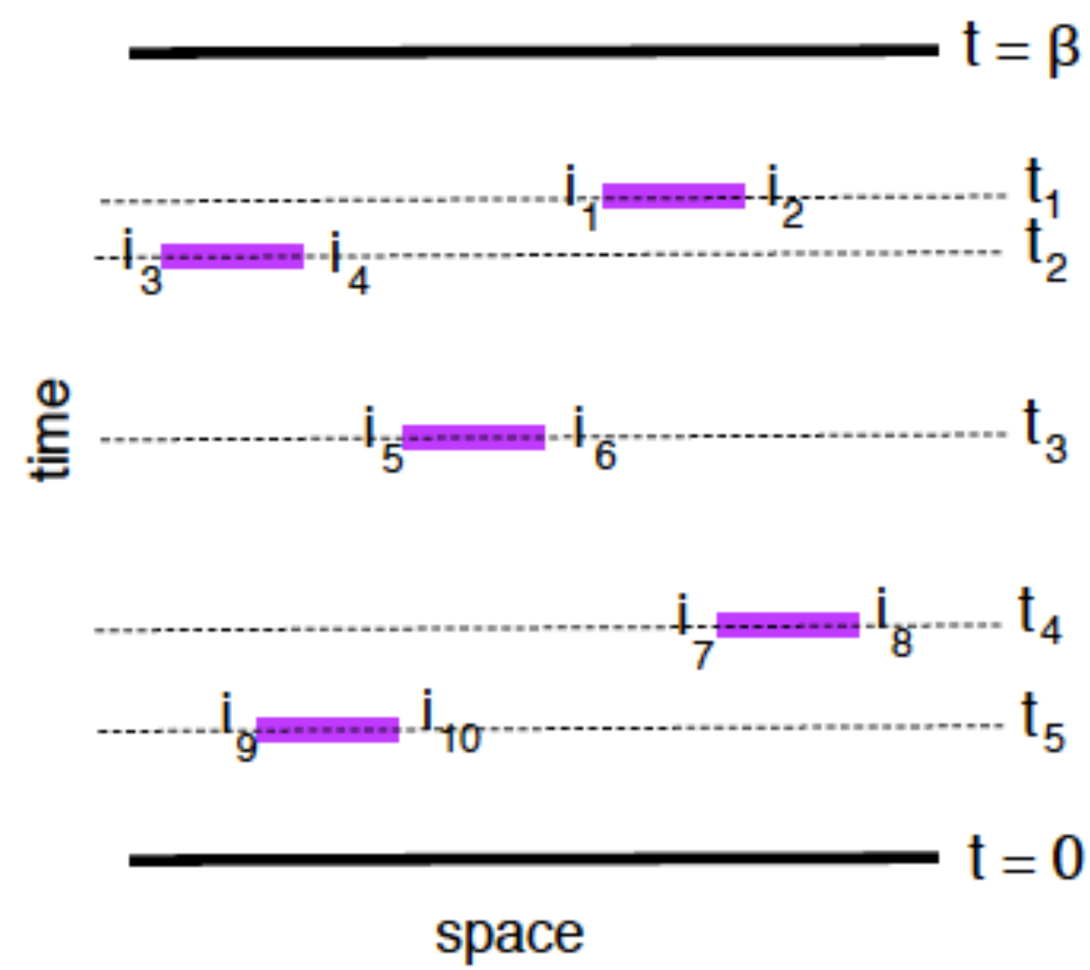

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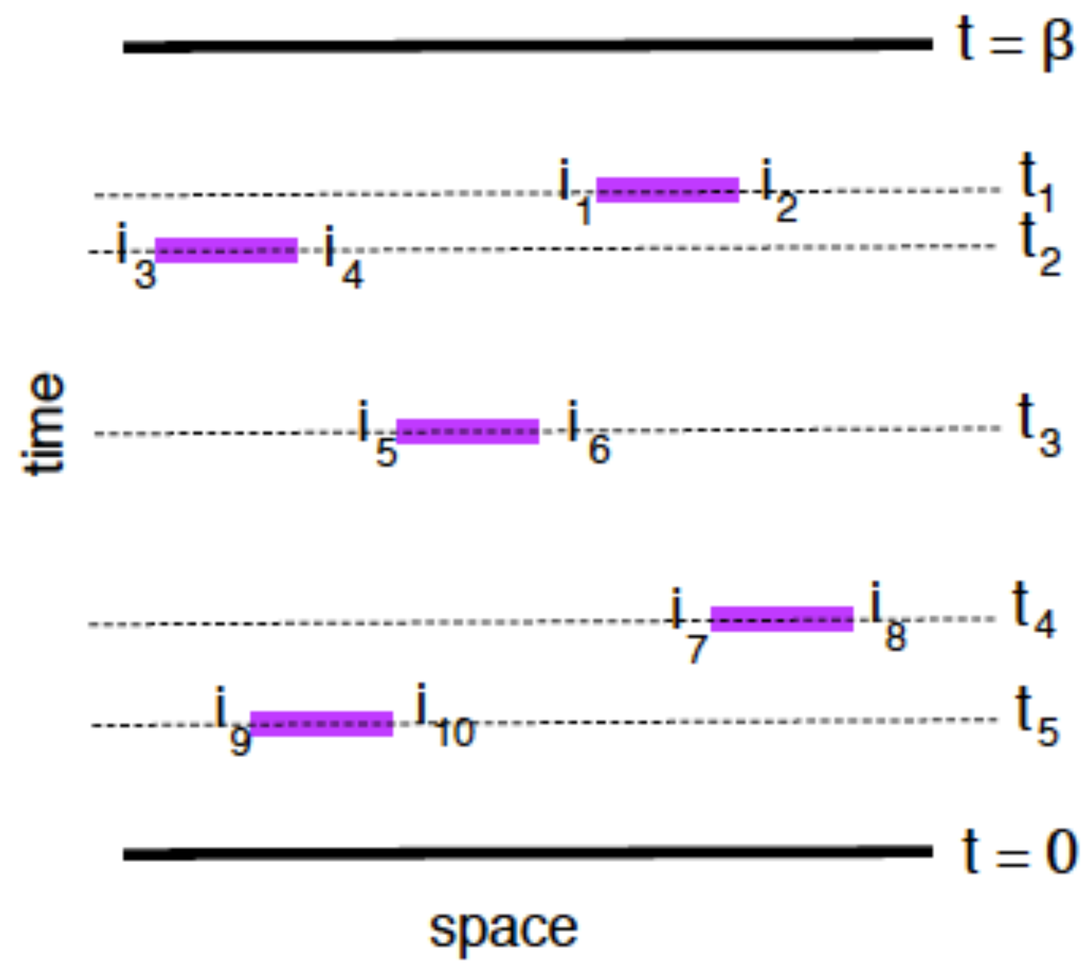

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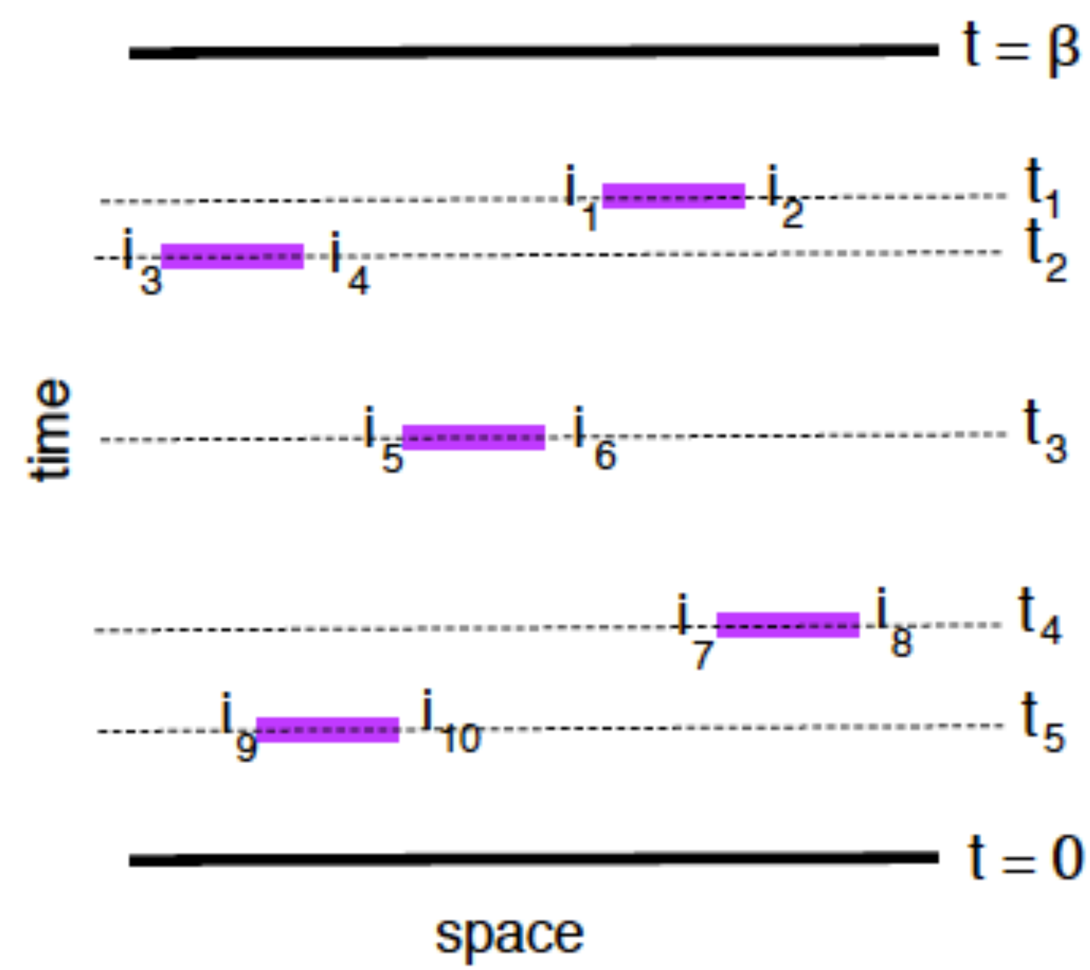


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Surprise: The [s] dependence is only through diagonal terms!

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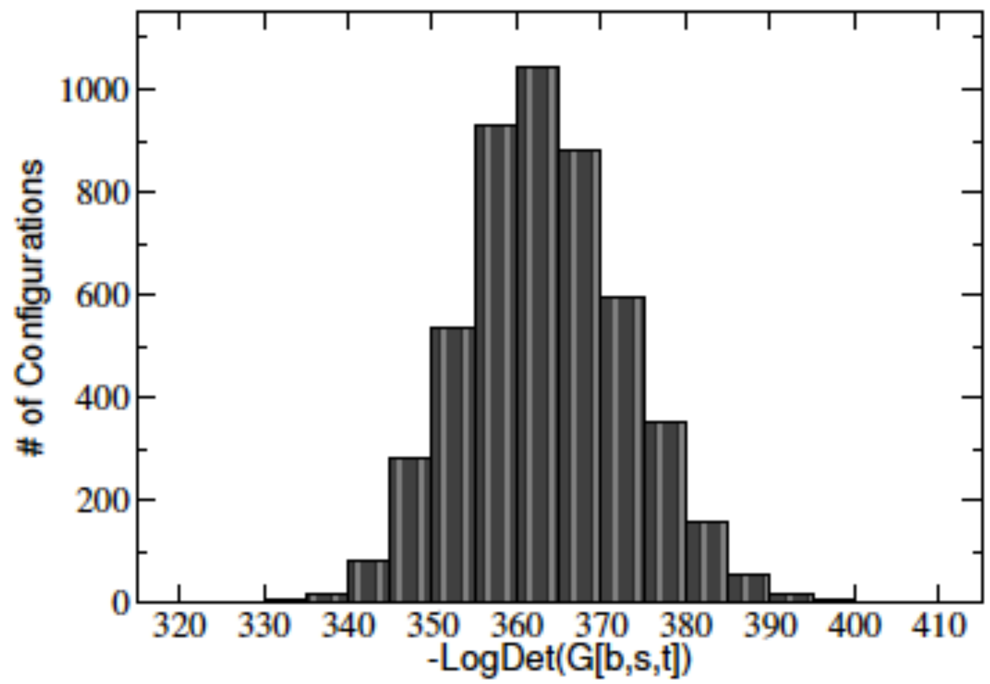


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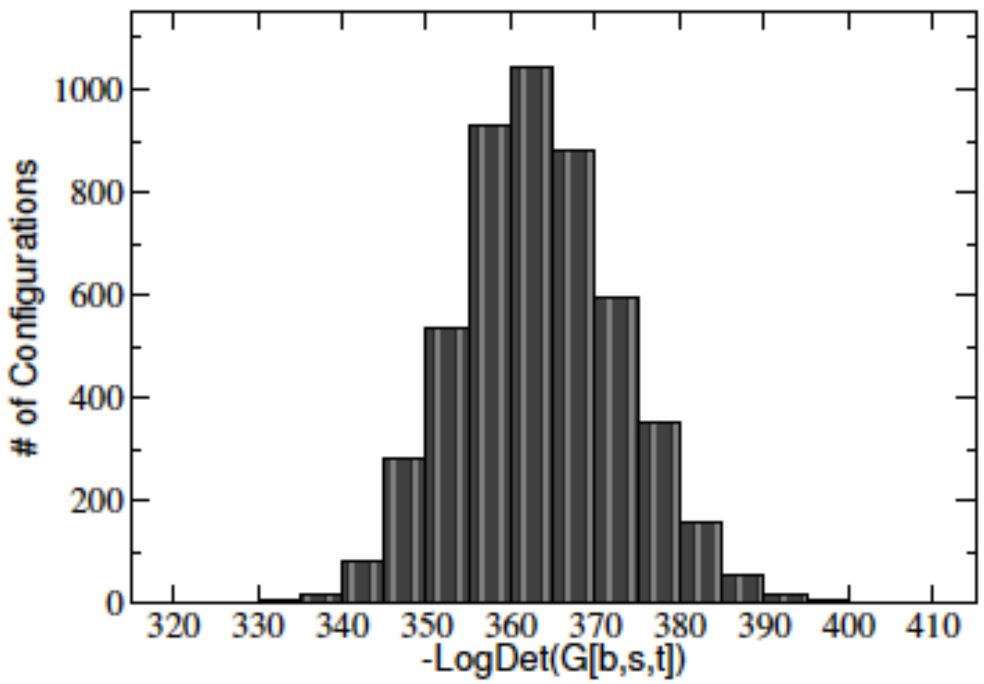


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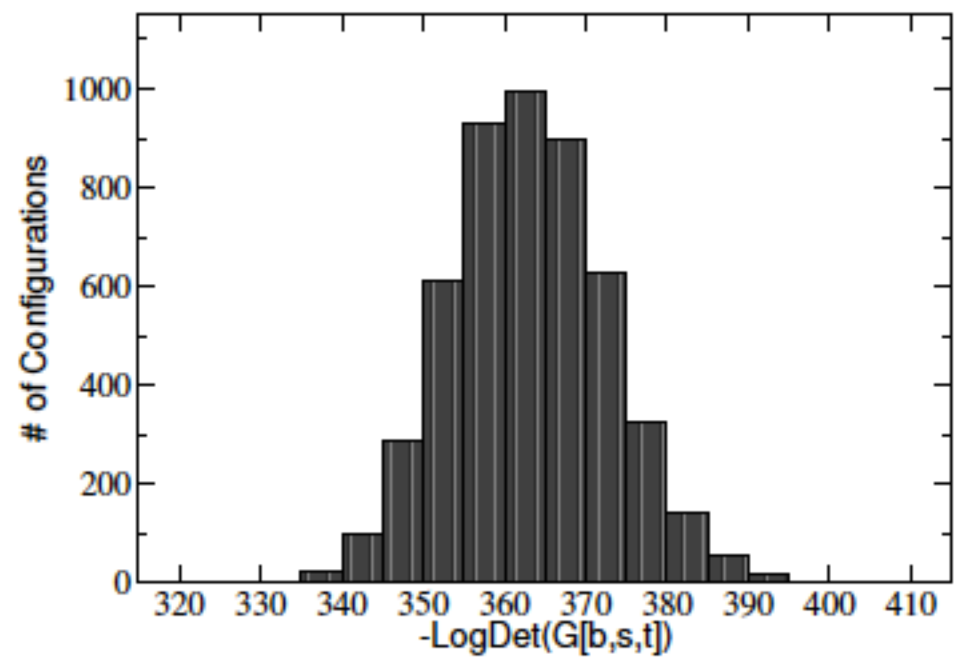


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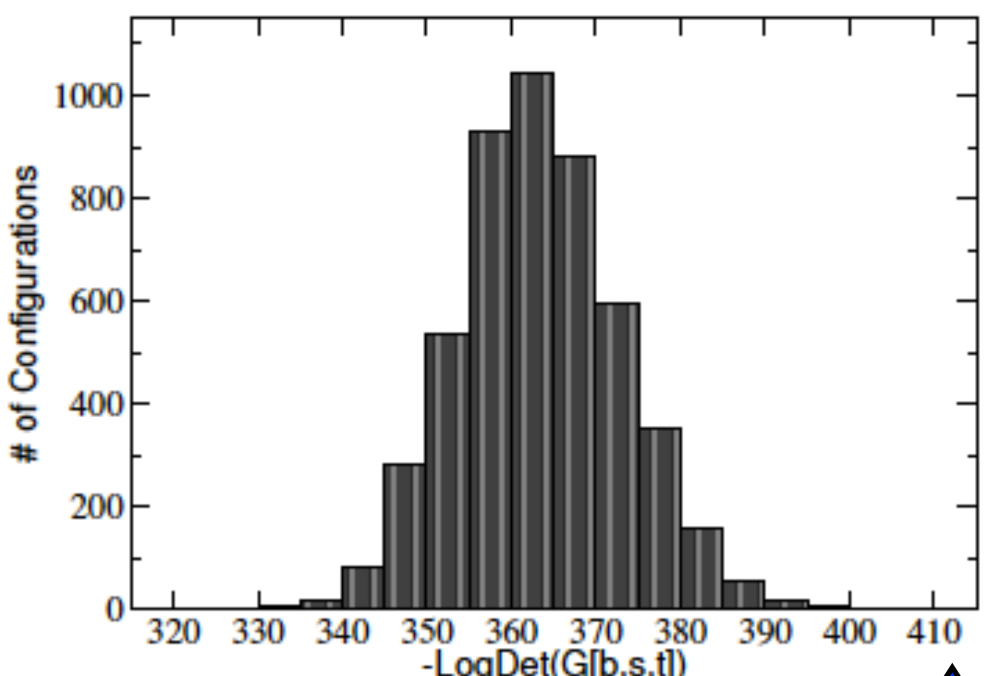


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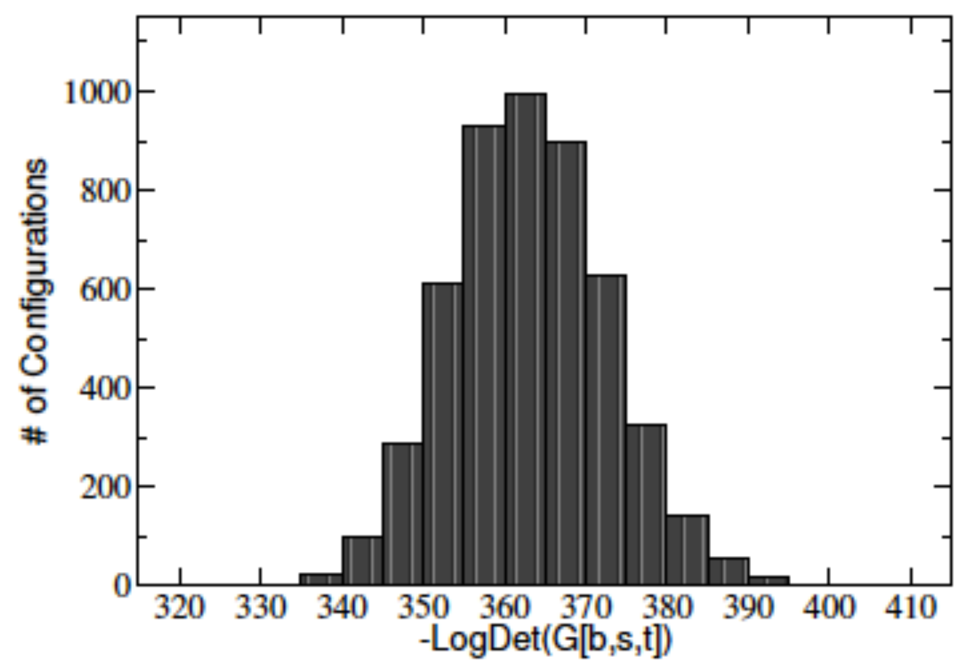


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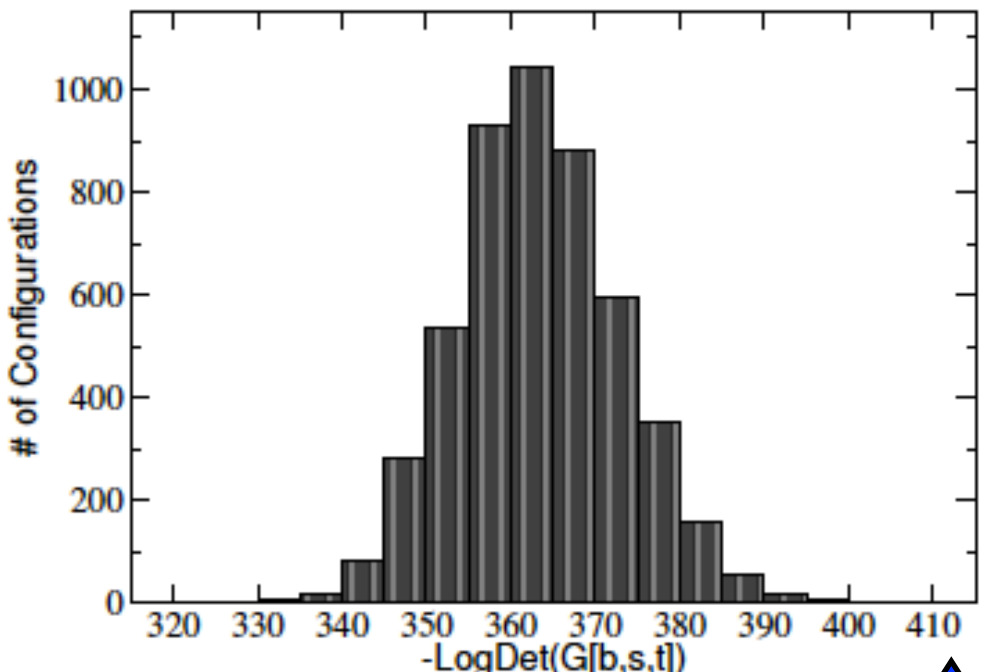


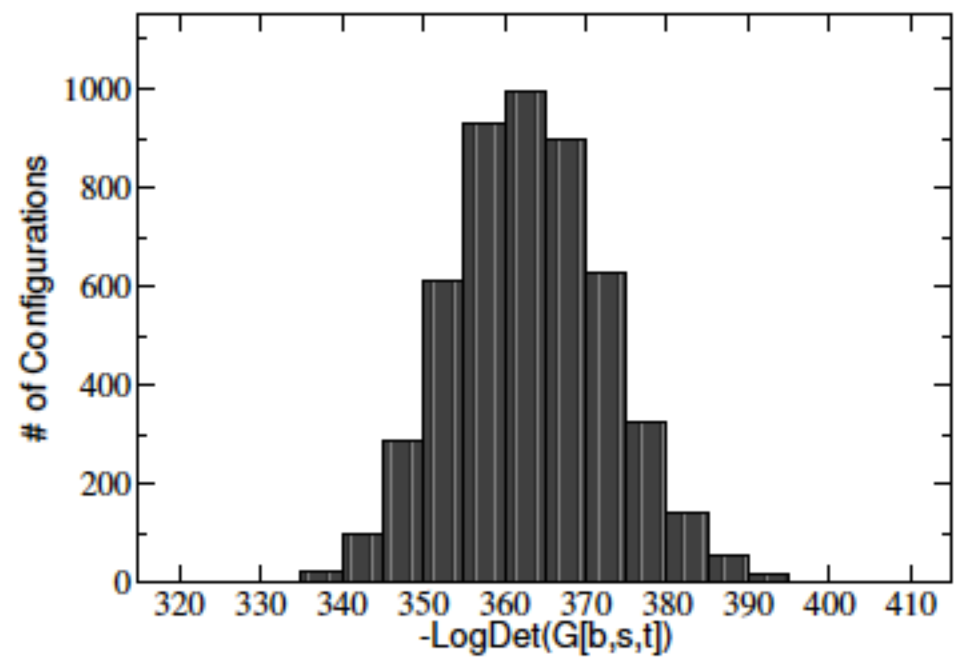


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Severe Sign Problem!

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**insight from fermion bag approach**

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**Sign problem solved??**

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$\tilde{D}A$ is real $\qquad (\tilde{D}A)^T \ = \ -\tilde{D}A$

$$\mathrm{Det}\left(\tilde{D}A\right) \ = \ (-1)^k \mathrm{Det}\left(A([b,t]\right) \geq 0$$

Thus, finally

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The sign problem remains unsolved
for V < 0 (attractive interations)!

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