The Dual Representation as a Tool to Check Expansion Methods

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Sign Problem in QCD and Beyond
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Motivation: Sign problem of QCD

- Method: **Dual representation.**
  Write the partition sum in terms of dual variables.

- We start using simpler models:
  - $\mathbb{Z}_3$ spin model (this talk)
    [Y. D., H. G. Evertz, C. Gattringer (PRL 2011, CPC 2012)]
  - SU(3) spin model [Y. D., C. Gattringer (NPB 2011, NPB 2012)]
  - Relativistic Bose gas [C. Gattringer, T. Kloiber (NPB 2012, PLB 2012)]
  - $\mathbb{Z}_3$ gauge-Higgs model [C. Gattringer, A. Schmidt (PRD 2012)]
  - U(1) gauge-Higgs model [Y .D., C. Gattringer, A. Schmidt (CPC 2012, PRL 2013)]

An effective theory for QCD thermodynamics

$$S_{eff} = -\sum_x \left(\tau \sum_{\nu=1}^{3} \Big[P(x) P(x+\hat{\nu})^* + c.c. \Big] \; + \; \kappa \Big[e^{\mu} P(x) \; + \; e^{-\mu} P(x)^* \Big] \right)$$

- The degrees of freedom are:

$$P(x) \in \mathbb{Z}(3) = \{1, e^{+i2\pi/3}, e^{-i2\pi/3}\}$$

- P is related to the Polyakov loop, which is a static source quark.

  $T < T_c$: $\langle P \rangle = 0 \;\rightarrow$ quarks confined

  $T > T_c$: $\langle P \rangle \neq 0 \;\rightarrow$ quarks deconfined

- τ increases with the temperature, and κ decreases with the quark mass.

Remarks

- The deconfined transition of pure gluodynamics can be understood through the spontaneous breaking of center symmetry.

$$\sum_x \tau \sum_{\nu=1}^{3} \Big[P(x) P(x+\hat{\nu})^* + c.c. \Big]$$

- Description by an effective 3d center symmetric spin model. Yaffe and Svetitsky (1981).

- Our theory also contains center symmetry breaking terms and chemical potential. These terms come from the fermion determinant.

$$\sum_x \kappa \Big[e^{\mu} P(x) \; + \; e^{-\mu} P(x)^* \Big]$$

- The structure of the new terms can be obtained from hopping expansion.

Dual representation - 1

- Effective center model still has complex action ⇒ new variables!
- For the neighbor interaction and magnetic term, we use the Ansatz:

$$e^{\,\tau[P(x)P(x+\hat{\nu})^*+c.c.]} \;=\; C \sum_{b_{x,\nu}=-1}^{+1} B^{|b_{x,\nu}|}\,\left[\,P(x)P(x+\hat{\nu})^*\,\right]^{b_{x,\nu}}$$

$$e^{\,\kappa e^{\mu}P(x)+\kappa e^{-\mu}P(x)^*} \;=\; \sum_{s_x=-1}^{+1} M_{s_x}\,P(x)^{s_x}$$

- C, B and M_s are real and positive numbers.
- New variables:
  - **dimers** : $b_{x,\nu} \in \{-1, 0, +1\}$ on the link (x, ν).
  - **monomers**: $s_x \in \{-1, 0, +1\}$ on the site x.

Dual representation - 2

- The partition function in the flux representation:

$$Z = \sum_{\{b,s\}} \left(\prod_{x,\nu} B^{|b_{x,\nu}|} \right) \left(\prod_{x} M_{s_x} \right) \prod_{x} T \left(\sum_{\nu} [b_{x,\nu} - b_{x-\hat{\nu},\nu}] + s_x \right)$$

- Only real and positive contributions.

- Constraint $T(n)$: flux conservation modulo 3 at every site.

F. Karsch et al.(1984)

A. Patel,T. DeGrand,C. DeTar(1983)

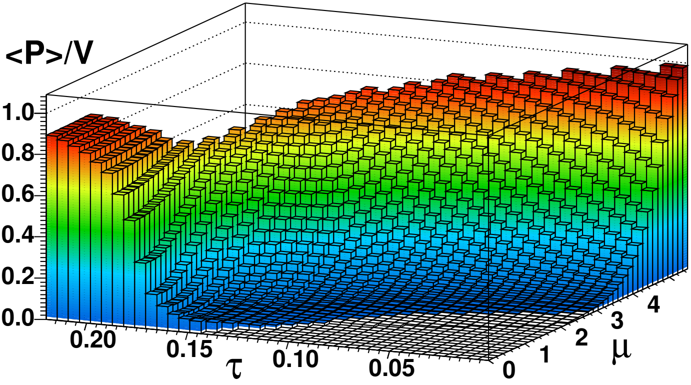
Y. Delgado, H.G. Evertz, C. Gattringer (2011)

Numerical analysis

- Study the $\tau - \mu$ phase diagram.
- Identify the phase boundaries between confinement and deconfinement.
- Analyze the nature of the transitions.
- Location of the transition lines determined from the maxima of the Polyakov loop susceptibility χ_P and the heat capacity C.
- Cross check expansion methods.
- **MC simulation:** Worm Algorithm

  [N. Prokof'ev and B. Svistunov (2001); Y. Delgado, H.G. Evertz, C. Gattringer (2012)]

Order parameter $\langle P \rangle / V$

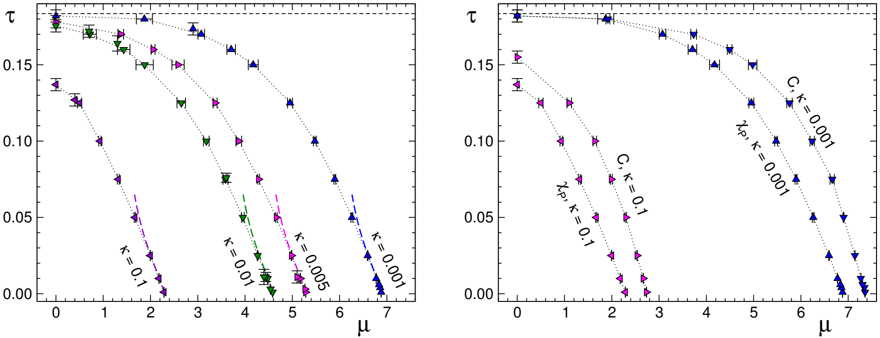


- $\langle P \rangle / V \;\; \to 0$, center symmetry is broken very mildly $\Rightarrow$ confined phase
- $\langle P \rangle / V \;\; \to 1$, center symmetry broken $\Rightarrow$ deconfined phase

Y. Delgado, H.G. Evertz, C. Gattringer (2011)

Determination of phase boundaries

- Location of phase boundaries determined by the maxima of χ_P and C.



Y. Delgado, H.G. Evertz, C. Gattringer (2011)

Expansion Methods

- We compare exact results obtained from the dual representation with different expansion techniques:
  - Fugacity expansion
  - Regular Taylor expansion
  - Improved Taylor expansion

- Goal: Find a better method that could be applied to full QCD.

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Fugacity Expansion - 1

- Rewriting the partition sum:

$$Z = \sum_{q \in \mathbb{Z}} e^{\mu q} Z_q$$

The sum runs over all particle number q.

- The canonical partition sum Z_q

$$Z_q = \sum_P e^{\tau \sum_{x,\nu}[P(x)P(x+\hat{\nu})*+c.c.]} D_q$$

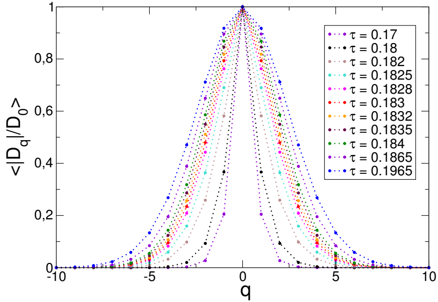
- D_q is the “fermion part” of the action projected to a fixed quark number.

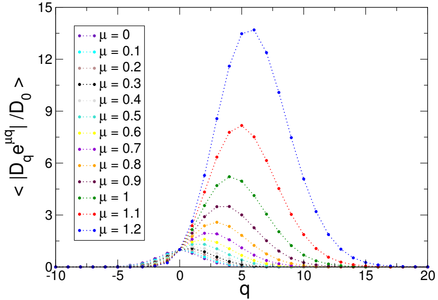
$$D_q = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi e^{i\phi q} \exp(\kappa[e^{i\phi} H + e^{-i\phi} H^*]) = e^{i\theta q} I_q(2\kappa R)$$

where $H = \sum_x P(x) = Re^{i\theta}$

Fugacity Expansion - 2

- Distribution of the coefficients in the fugacity series.





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Taylor expansions

- **Regular Taylor Expansion (RTE)**:
  - Expansion parameter: μ

$$\ln Z = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \left(\frac{\partial}{\partial \mu} \right)^n \ln Z$$

- **Improved Taylor Expansion (ITE)**:
  - Expansion parameter: $\rho = \kappa(e^{\mu} - 1)$ and $\overline{\rho} = \kappa(e^{-\mu} - 1)$
  - The action is given by:

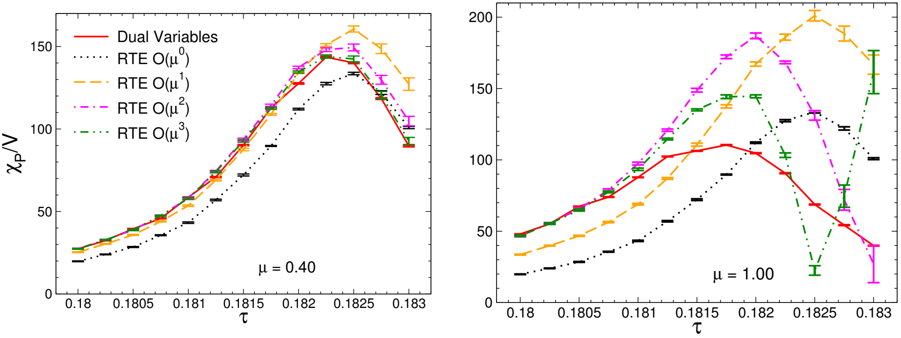
$$e^{-S_\mu} = e^{-S_0} e^{\rho H + \overline{\rho} H^*}$$

  - Expanding the second factor:

$$Z(\mu) = Z(0) \left[1 + \langle \rho H + \overline{\rho} H^* \rangle_0 + \frac{1}{2} \langle (\rho H + \overline{\rho} H^*)^2 \rangle_0 + \dots \right]$$

Regular Taylor expansion

- $\kappa = 0.001$



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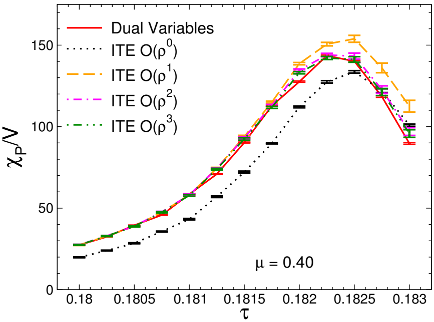
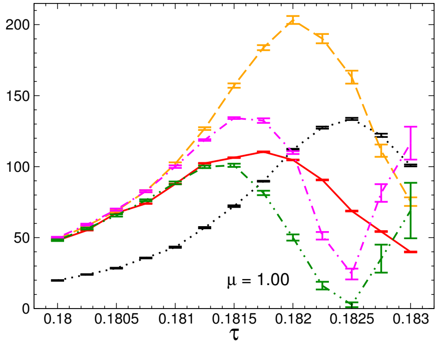
Improved Taylor expansion

- $\kappa = 0.001$





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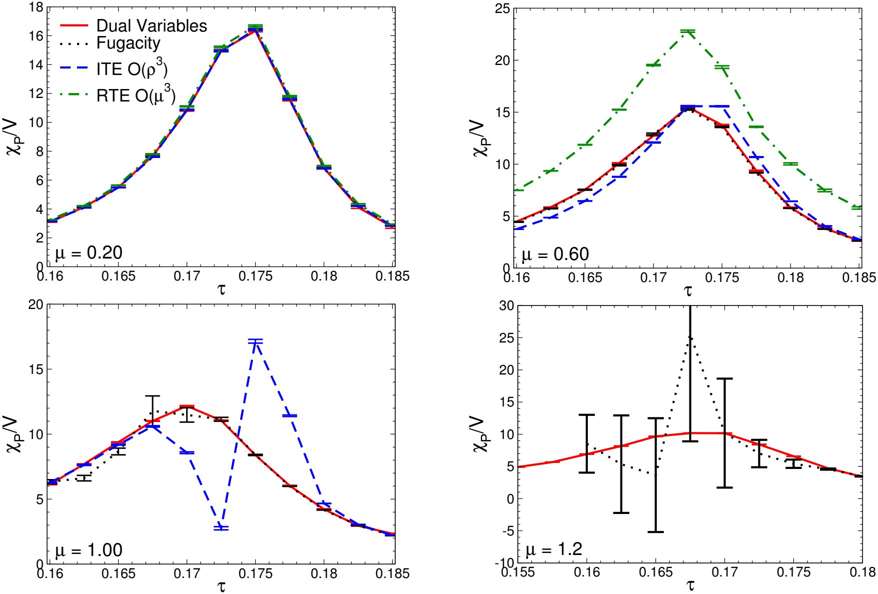
Fugacity Exp. vs. ITE vs. RTE



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Summary

- Considerable progress was made towards rewriting several systems in the dual representation, where the sign problem is solved exactly.
- Our results can be used as reference to test other approaches:
  - Complex Langevin [G. Aarts, F. A. James (2012)]
  - Fugacity expansion and improved Taylor expansion [Y. D., E. Grünwald, C. Gattringer (2013)]

Outlook

- Dual representation of non-Abelian theories ??
- Comparison of the fugacity expansion and the improved Taylor expansion for full QCD. [Y. D., M. Wilfling, C. Gattringer (2013)]

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Thank you for your attention!