Inhomogeneous chiral phases in QCD and implication on compact stars

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In collaboration with K. Nishiyama, S. Karasawa and T. Muto

I. Introduction

Inhomogeneous (spatially dependent) order parameter in the vicinity of the phase transition

LOFF state in superconductor $\Delta \to \Delta(\mathbf{r})\ (\to 0)$

Spin density wave (SDW) from ferromagnetic phase $\mathbf{M} \to \mathbf{M}(\mathbf{r})\ (\to 0)$

Chiral symmetry (SSB) $\langle \bar{q}\hat{O}q \rangle \to \langle \bar{q}\hat{O}q \rangle(\mathbf{r})\ (\to 0)$

Magnetic field $(H \neq 0)$

- High-energy H.I. Collisions $O(m_\pi^2 \sim 10^{17} G)$
- Compact stars $10^{12-15} G$
- Early Universe

Generalized chiral order-parameter:

$$M \equiv \langle \bar{q}q \rangle + i \langle \bar{q} i \gamma_5 \tau_3 q \rangle = \Delta(\mathbf{r}) \exp(i\theta(\mathbf{r})) \in \mathbb{C}$$

NJL studies have shown inhomogeneous chiral phases in the vicinity of the chiral transition :

Critical point (CCP) as Lifshitz point (LP)

Two typical forms of the condensates Have been studied:

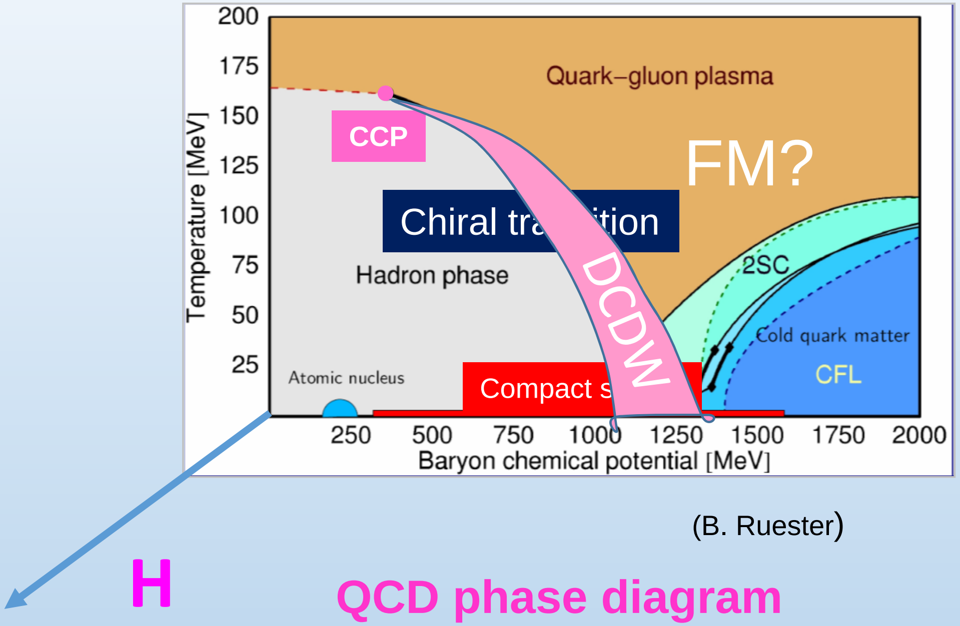
Dual chiral density wave (DCDW):

$$\Delta = \text{const.}, \theta = \mathbf{q} \cdot \mathbf{r}$$

(T. T. and E. Nakano, hep-ph/0408294.
E. Nakano and T. T., PR**D 71** (2005) 114006.)

Real kink crystal (RKC): $\Delta(z), \theta = 0$

(D.Nickel, PRL 103(2009) 072301; PR**D** 80(2009) 074025.)



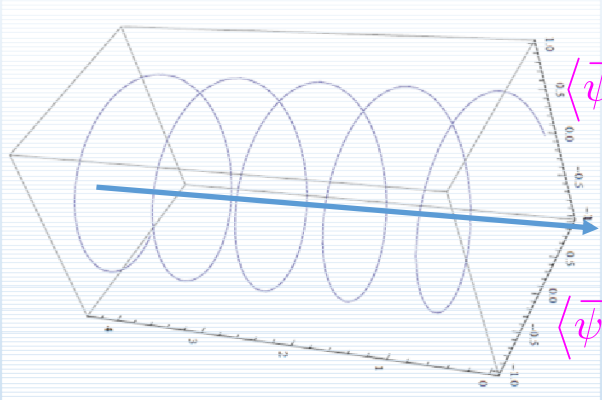
(B. Ruester)

QCD phase diagram

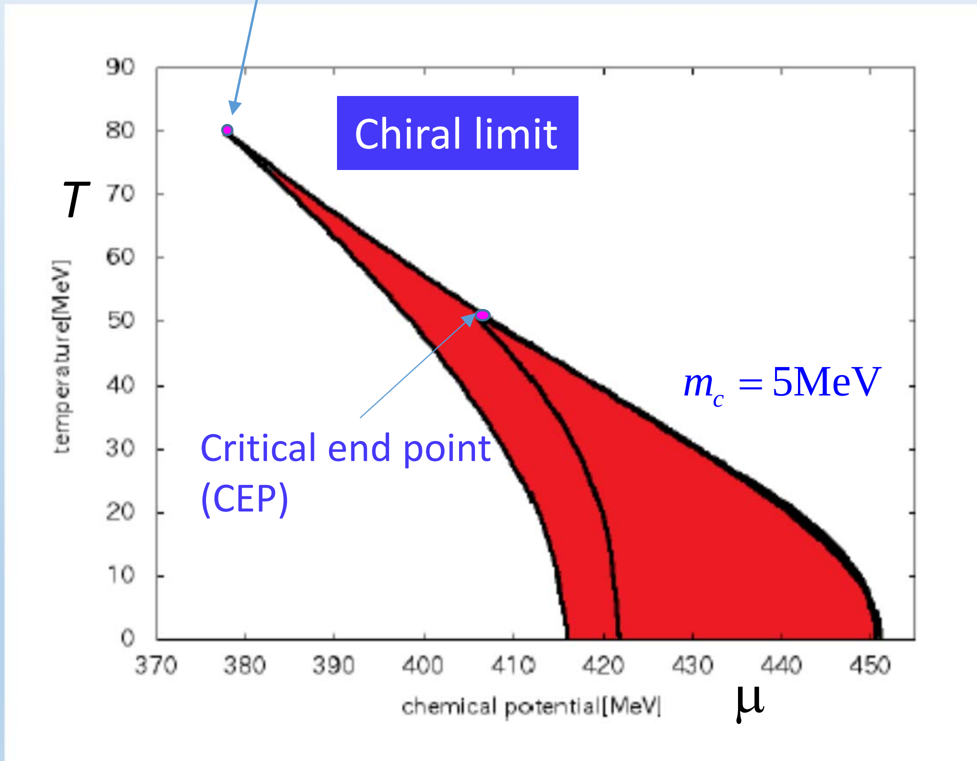
DCDW(dual chiral density wave) (2flavor NJL model)

$$\langle \bar{q}q \rangle = \Delta \cos(qz)$$

$$\langle \bar{q}i\gamma_5\tau_3 q \rangle = \Delta \sin(qz)$$

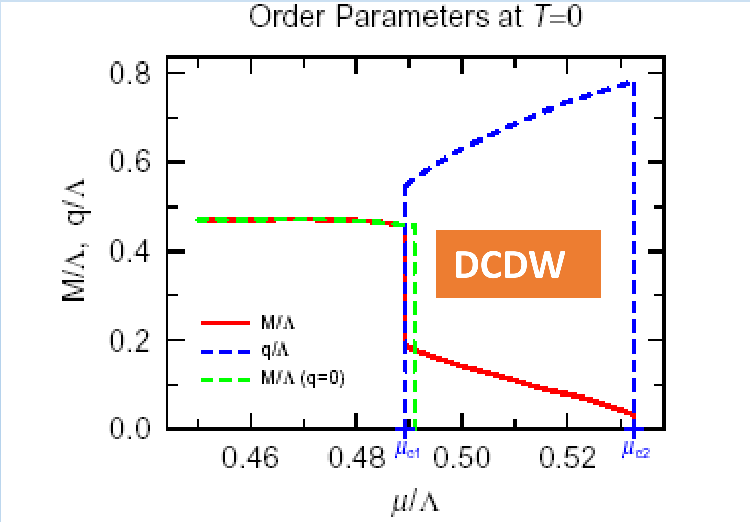


Tricritical point=Lifshitz point



S. Karasawa and T.T., arXiv:1307.6448.

Chiral limit



ref. T.T. and E. Nakano, hep-ph/0408294
PRD71(2005)114006.

Some results in 1+1 dimensional models:

(i) tricritical point is the Lifshitz point for RKC

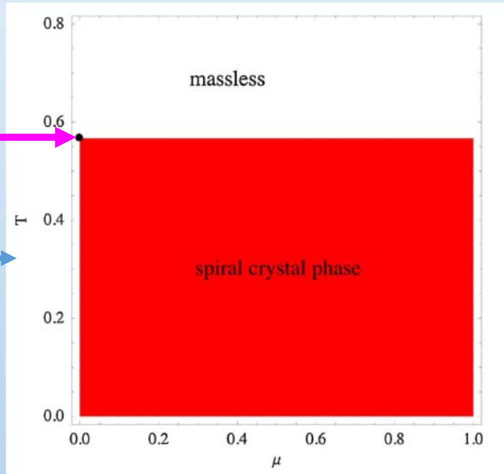
(ii) $(T_c, \mu = 0)$ is the tricritical point, and T_c is independent of μ for chiral spiral.

(iii) q=2μ for chiral spiral (*nesting is complete*)

(iv) Chiral spiral the most favorite configuration

tricritical point

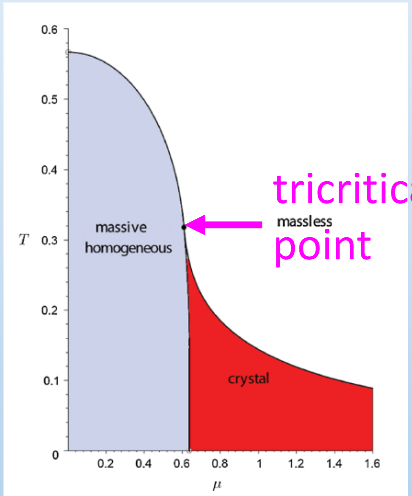
DCDW



Chiral spiral

$$\langle \bar{q}q \rangle = \Delta \cos(\mathbf{q} \cdot \mathbf{r})$$

$$\langle \bar{q}i\gamma^5 q \rangle = \Delta \sin(\mathbf{q} \cdot \mathbf{r})$$



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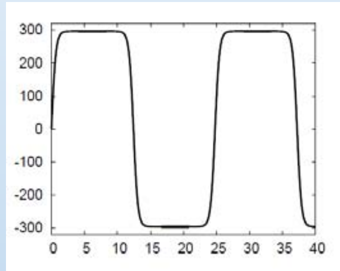
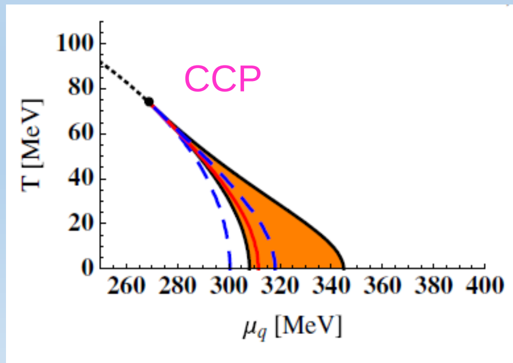


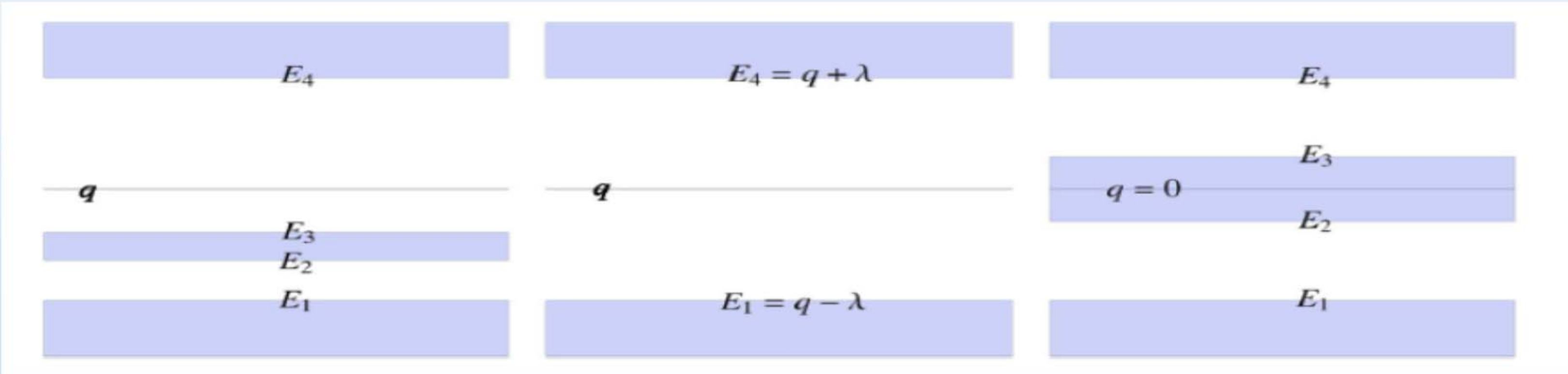
$$\langle \bar{q}q \rangle = \sqrt{\nu} q \,\mathrm{sn}(qz;\nu)$$

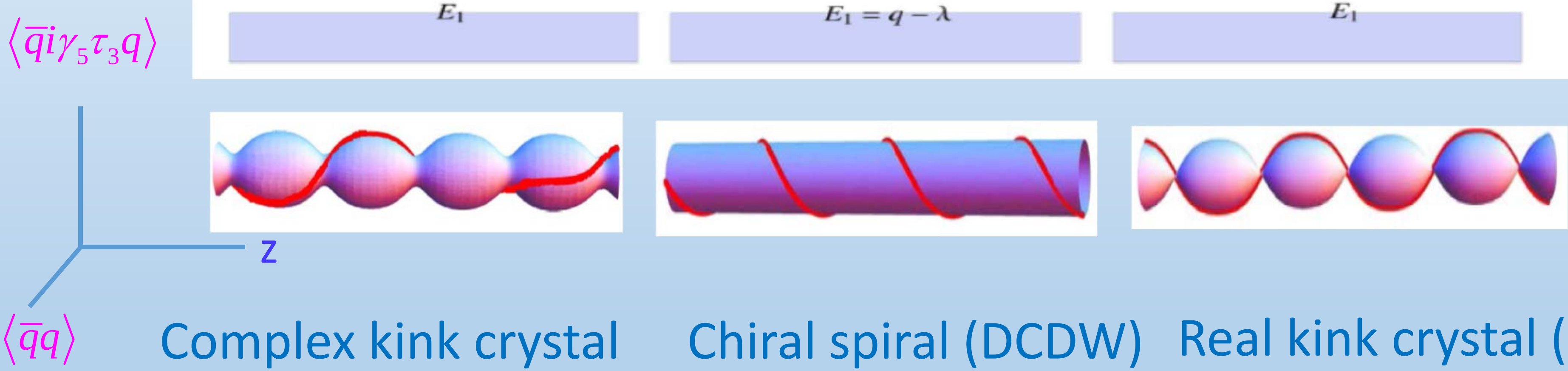
$$\to \sqrt{\nu} q \sin(qz) \quad \text{as } \nu \to 0$$



II Topological aspect of the inhomogeneous chiral phase

Energy spectrum (1+1 chiral Gross-Neveu model)





$:M(z)$

Complex kink crystal (CKC) | Chiral spiral (DCDW) | Real kink crystal (RKC)

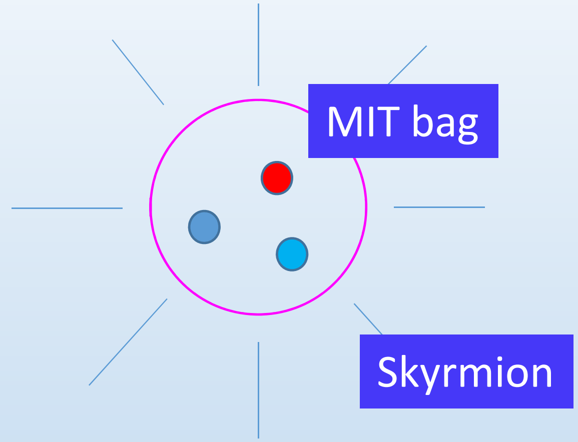
$$E_{p,\varepsilon} = \varepsilon\sqrt{\lambda^2 + p^2} + q/2$$

(G.Basar et al. PRD 79(2009) 105012)

Spectral asymmetry ← phase degree of freedom of *M(z)*

It is known that spectral asymmetry induces the anomalous baryon number.

e.g. Chiral bag (Goldstone & Jaffe, PRL 51(1983) 1518.)



Baryon number operator is defined by

$$\hat{N} = \frac{1}{2}\int d^D x\left[\psi^\dagger(x), \psi(x)\right].$$

Atiyah-Patodi-Singer η invariant (Atiyah et al, Proc.Cambridge Phils. Soc. 77(1975) 42;78(1975) 405;79 (1976) 71):

$$\eta_H = \sum_{E_\lambda > 0} 1 - \sum_{E_\lambda < 0} 1$$

$$= \lim_{s \to 0} \sum_\lambda \left|E_\lambda\right|^{-s} \mathrm{sign}(E_\lambda)$$

("Baryon number in the vacuum")

Baryon number in the thermodynamic limit: $N \to \infty, \quad V \to \infty$ with fixed $\rho = N / V$.

$$N = -\frac{1}{2}\eta_H + \sum_\lambda \mathrm{sign}(E_\lambda)\left[\frac{\theta(E_\lambda)}{e^{\beta(E_\lambda - \mu)} + 1} + \frac{\theta(-E_\lambda)}{e^{-\beta(E_\lambda - \mu)} + 1}\right],$$

(A.J. Niemi and G.W. Semenoff, Phys. Reports 135 (1986) 99.)

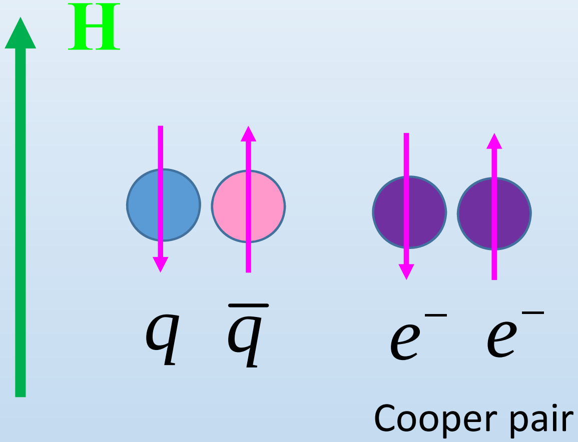
Topological one

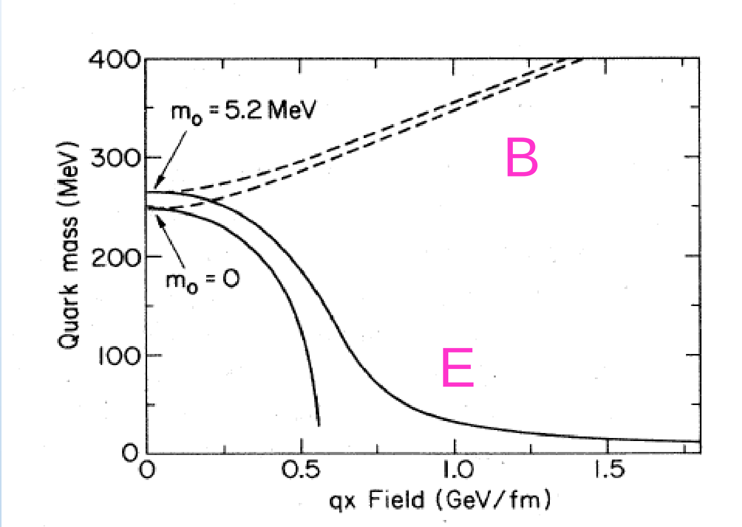
which must be consistent with the thermodynamic relation, $N = -\partial\Omega / \partial\mu.$

III Inhomogeneous chiral phase in the magnetic field

Phase transition

・Enhancement of SSB
or Magnetic catalysis





S.P. Klevansky and R.H. Lemmer,
PRD 39(1989) 3478.

S. P. Klevansky and R.H. Lemmer, PRD 39 (1989) 3478.
H. Suganuma and T.T., Ann.Phys. 208 (1991) 371.
V.P. Gusynin, V.A. Milansky, I.A. Shovkovy, NPB 462 (1996) 249

・Dimensional reduction due to the Landau levels

Note: Chiral spiral (DCDW) is the most favorable phase in 1+1 dimensions.

(G.Baser et al., 2009)

Dirac operator in the magnetic field: $\mathbf{H} = (0, 0, H)$

Using the Landau gauge, $\mathbf{A} = (0, Hx, 0)$

Dirac Hamiltonian with the inhomogeneous condensate:

$$H_D = \boldsymbol{\alpha} \cdot \mathbf{P} + \beta \left[\frac{1+\gamma_5 \tau_3}{2} M(z) + \frac{1-\gamma_5 \tau_3}{2} M^*(z) \right] = H_D{}^{\dagger}, \quad \mathbf{P} = -i\nabla + e\mathbf{A}$$

$$\tau_3 = 1$$

can be partially diagonalized as

$$H_D = \begin{pmatrix} -i\partial_z & M(z) & 0 & \sqrt{2eHn} \\ M^*(z) & i\partial_z & -\sqrt{2eHn} & 0 \\ 0 & -\sqrt{2eHn} & -i\partial_z & M^*(z) \\ \sqrt{2eHn} & 0 & M(z) & i\partial_z \end{pmatrix}$$

(n: Landau levels)

$$\left(\times e^{iky} u_n(\xi) \right)$$

on the basis of the orthonormal Hermite functions $\{u_n(\xi)\}$

with $\xi = \sqrt{eH}\, x + k / \sqrt{eH}$, for $n \neq 0$.

$$\left(\frac{d}{d\xi} - \xi \right) u_n(\xi) = -\sqrt{2(n+1)}\, u_{n+1}(\xi),$$

$$\left(\frac{d}{d\xi} + \xi \right) u_n(\xi) = -\sqrt{2n}\, u_{n-1}(\xi).$$

For the lowest Landau level (LLL), *n=0,*

H_D is reduced to 2×2 matrix,

$$H_{LLL}=\begin{pmatrix} -i\partial_z & M(z) \\ M^*(z) & i\partial_z \end{pmatrix}$$

from the property $u_{-1}(\xi)=0.$

It should be interesting to see that

$$H_D=\begin{pmatrix} H_{LLL}(M(z)) & 0 \\ 0 & H_{LLL}(M^*(z)) \end{pmatrix}$$

in the absence of the magnetic field, before Lorentz boost [Nickel '09].
These features of the Dirac operator give direct consequences.
Also, note that complex structure of the order parameter *M(z)* is important.

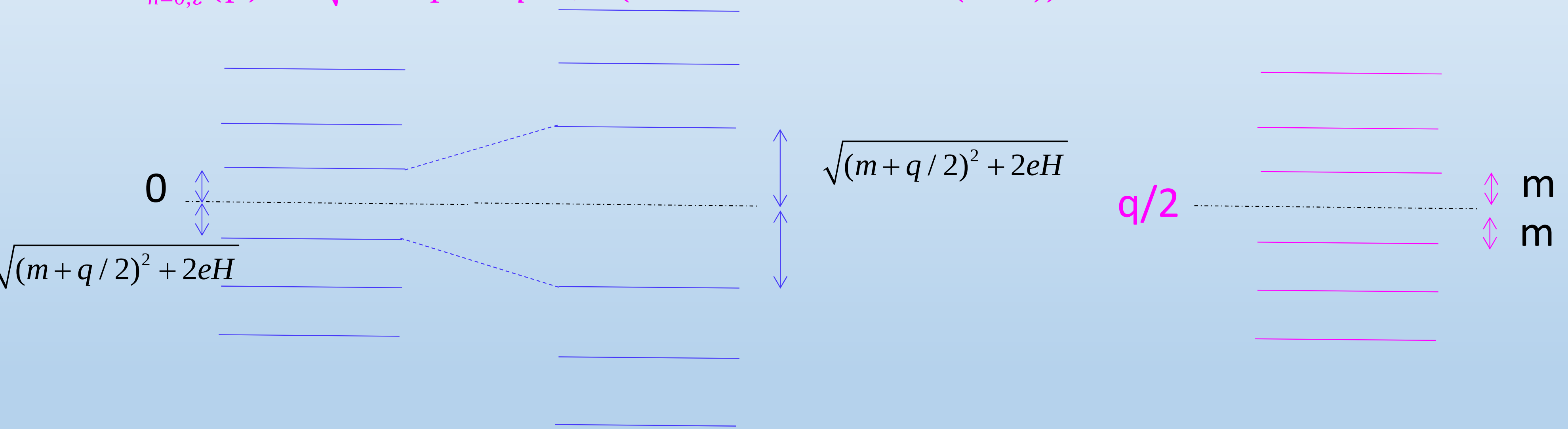
ex) In the case of DCDW $M(z) = -2G\Delta \exp(iqz)$

Energy spectrum in the presence of the magnetic field: [I.E. Frolov et al., Phys. Rev. D82 (2010) 076002]

$$E_{n,\zeta,\varepsilon}(p) = \varepsilon\sqrt{\left(\zeta\sqrt{m^2+p^2}+q/2\right)^2+2eHn}, \quad n=1,2,3,\ldots$$

$$E_{n=0,\varepsilon}(p) = \varepsilon\sqrt{m^2+p^2}+q/2, \quad \text{(Lowest Landau level(LLL))}$$

← Spectral asymmetry



Cf: Landau levels: $E_{p\varepsilon} = \varepsilon\sqrt{m^2+|eH|(2n+1+s)+p^2}, \quad n=0,1,2,\ldots,$

DCDW $E_{p,\zeta,\varepsilon} = \varepsilon\sqrt{p_\perp^2+\left(\zeta\sqrt{m^2+p_z^2}+q/2\right)^2}$

Spectrum is symmetric about 0

Spectral asymmetry or η invariant vanishes for the real order parameter

Proof:

Consider the Dirac operator

$$H_D = \boldsymbol{\alpha}\cdot\mathbf{P} + \beta\left[\frac{1+\gamma_5\tau_3}{2}M(z) + \frac{1-\gamma_5\tau_3}{2}M^*(z)\right], \quad \mathbf{P} = -i\nabla + eA$$

and the eigenvalues,

$$H_D q_\lambda = E_\lambda(M) q_\lambda$$

Under the CT transformation,

$$q_\lambda \to i\gamma^0\gamma^5 q_\lambda,$$

$$E_\lambda(M) \to -E_\lambda(M^*)$$

Therefore

$$\eta_H(M) = -\eta_H(M^*)$$

from which anomalous baryon number vanishes for real configurations

s.t. $M = M^*$.

Direct evaluation of η_H for DCDW (LLL only)

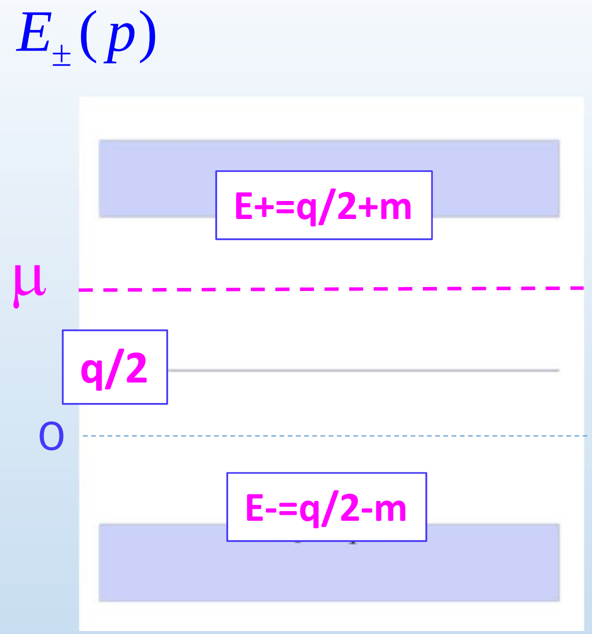
Using the Mellin transform

$$|E|^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty d\omega \omega^{s-1} \exp\left(-|E|\omega\right),$$

$$\eta_H(s) = \frac{L}{\Gamma(s)} \int \frac{dp}{2\pi} \int_0^\infty d\omega \omega^{s-1} \sum_\varepsilon \mathrm{sign}(E_\varepsilon) \exp\left(-|E_\varepsilon|\omega\right)$$

$$= -\frac{q}{\pi} \frac{L}{\Gamma(s)} m^{-s} 2^{s-1} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s}{2}+1\right) {}_2F_1\left(1+\frac{s}{2}, \frac{s}{2}, \frac{3}{2}; \left(\frac{q}{2m}\right)^2\right)$$

Gauss hypergeometric function



removable singularity at $s = 0$

$$\eta_H = \lim_{s\to 0+} \eta_H(s) = -\frac{qL}{\pi}.$$

Thus η_H is independent of m.

Thus topological baryon-number density can be written as

$$\rho_B^{\text{anomaly}} = -\frac{eH}{4\pi L} \eta_H = \frac{eH}{2\pi} \frac{q}{2\pi},$$

for $q/2 < m, \quad \mu < q/2 + m$

Accordingly an anomalous term appears in the thermodynamic potential

$$\Omega = \Omega_{vac} + \Omega_{\mu} + \Omega_{T},$$

$$\Omega_{vac} = -\frac{1}{2}\frac{eH}{2\pi}\int \frac{dp}{2\pi}\sum_{n,\zeta,\varepsilon}|E|,$$

$$\Omega_{\mu} = -\frac{1}{2}\frac{eH}{2\pi}\int \frac{dp}{2\pi}\sum_{n,\zeta,\varepsilon}\left(|E-\mu|-|E|\right), \rightarrow -\frac{1}{2}\frac{eH}{2\pi}\lim_{s\to 0+}\sum_{LLL}\left(|E-\mu|-|E|\right)|E|^{-s} - \frac{1}{2}\frac{eH}{2\pi}\int \frac{dp}{2\pi}\sum_{n\neq 0,\zeta,\varepsilon}\left(|E-\mu|-|E|\right)$$

$$\Omega_{T} = -T\frac{eH}{2\pi}\int \frac{dp}{2\pi}\sum_{n,\zeta,\varepsilon}\ln\left(1+e^{-\beta|E-\mu|}\right).$$

We can easily varify the thermodynamic relation,

$$\rho_B = -\frac{\partial \Omega}{\partial \mu},$$

including anomalous baryon number.

IV Novel tricritical point in the presence of the magnetic field

Generalized Ginzburg-Landau (gGL) theory:

$$
\begin{aligned}
\Delta\Omega &= \Omega_{\mathrm{GL}}(M) - \Omega_{\mathrm{GL}}(M=0) \\
&= \frac{\alpha_2}{2}|M|^2 + \frac{\alpha_3}{3}\mathrm{Im}\,(MM'^*) + \frac{\alpha_4}{4}\left(|M|^4 + |M'|^2\right) + \frac{\alpha_5}{5}\mathrm{Im}\,((M'' - 3|M|^2M)\,M'^*) \\
&+ \frac{\alpha_6}{6}\left(|M|^6 + 3|M|^2|M'|^2 + 2|M|^2|M^2|' + \frac{1}{2}|M''|^2\right) + \ldots
\end{aligned}
$$

The coefficients are evaluated in terms of the Green function in the presence of the magnetic field.

Generally the Green function is given by the well-known formula [Schwinger, (1951)], and that for the uniform magnetic field is given by

$$
S_A(x,y) = \exp\left[\frac{ie}{2}(x-y)^\mu A_\mu(x+y)\right]\tilde{S}_A(x-y)
$$

(removable by the suitable gauge trans.)

Fourier transform reads

$$\tilde{S}_A(k) = \int_0^\infty ds \exp\left[is\left(k_0^2 - k_3^2 - \mathbf{k}_\perp^2 \frac{\tan(eHs)}{eHs} \right)\right]\left[(k^0\gamma^0 - k^3\gamma^3)\left(1+\gamma^1\gamma^2 \tan(eHs)\right) - \mathbf{k}_\perp \cdot \boldsymbol{\gamma}_\perp \left(1+\tan^2(eHs)\right)\right]$$

$$= i \exp\left(-\frac{\mathbf{k}_\perp^2}{|eH|} \right) \sum_{n=0}^{\infty} (-1)^n \frac{D_n(eH,k)}{k_0^2 - k_3^2 - 2|eH|n},$$

(sum over the Landau levels)

With the denominator

$$D_n(eH,k) = \left(k^0\gamma^0 - k^3\gamma^3\right)\left[\left(1 - i\gamma^1\gamma^2 \mathrm{sign}(eH)\right) L_n^0\left(2\frac{\mathbf{k}_\perp^2}{|eH|}\right) - \left(1 + i\gamma^1\gamma^2 \mathrm{sign}(eH)\right) L_{n-1}^0\left(2\frac{\mathbf{k}_\perp^2}{|eH|}\right)\right] + 4\left(k^1\gamma^1 + k^2\gamma^2\right) L_{n-1}^1\left(2\frac{\mathbf{k}_\perp^2}{|eH|}\right)$$

For LLL contribution,

($L_n^\alpha(x)$: Generalized Laguerre function)

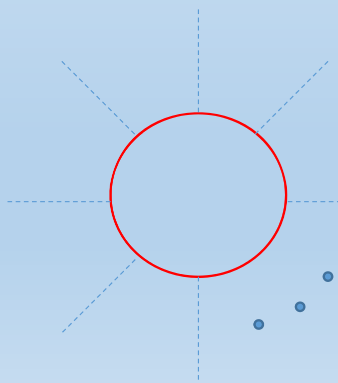
[A.Chodos et al., (1990)]

$$S_A^{(0)}(k) = i \exp\left(-\frac{\mathbf{k}_\perp^2}{|eH|} \right) \frac{k^0\gamma^0 - k^3\gamma^3}{k_0^2 - k_3^2} \left(1 - i\gamma^1\gamma^2 \mathrm{sign}(eH)\right)$$

2-dim. degeneracy

Spin projector

$$\alpha_n = (-1)^{n/2} 8 N_c T \sum_m \int_{\mathrm{reg}} \frac{d^3k}{(2\pi)^3} \mathrm{tr}\left[\tilde{S}_A(k) \right]^n$$



$M(z)$

Derivative expansion

If spectrum is symmetric, the "odd" terms vanish:

$$\alpha_i = 0, \quad \text{for i=odd.}$$

Then the Lifshitz point is given by the conditions:

$$\alpha_2 = \alpha_4 = 0,$$

which meets with the tricritical point of the chiral transition in the chiral limit [Nickel, (2009)].

New tricritical point due to the odd terms

In 1+1 dim. the chiral spiral (DCDW) is the most favorable phase, and the "tricritical point" is defined as the point where

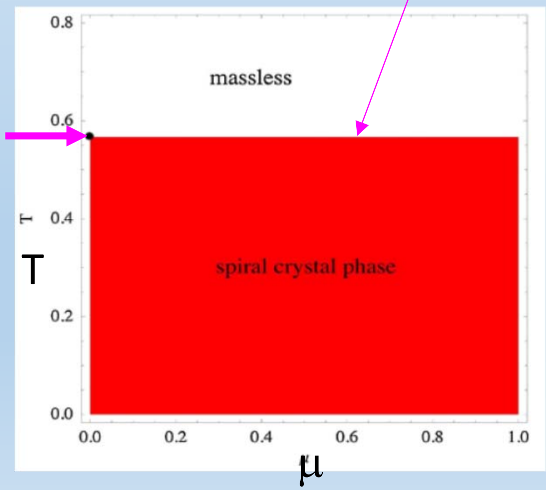
$$\alpha_2(\mu,T) = \alpha_3(\mu,T) = 0$$

which gives

$$T_{\rm tc} = \frac{e^{\gamma}}{\pi} = 0.5669, \mu_{\rm tc} = 0.$$

[Baser et al, (2009)]

nesting $q = 2\mu$



The odd terms survive in the presence of the magnetic field, due to the spectral asymmetry of the LLL levels.

So, we can define the new *tricritical* point (Lifshitz point) by

$$\alpha_2 = \alpha_3 = 0$$

First

(no divergence)

$$\alpha_3(H;T,\mu) = (-1)^{3/2}\, 2N_c \sum_f \frac{|e_f H|}{2\pi} T \sum_m \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{1}{\left[(\omega_m + i\mu)^2 + k^2\right]^{3/2}}$$

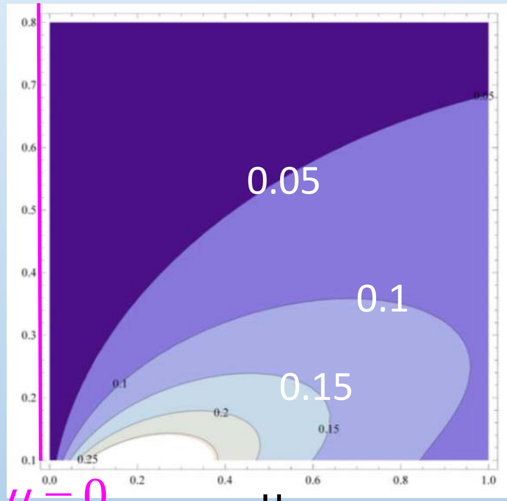
$$= \frac{1}{\pi^3 T} N_c \sum_f \frac{|e_f H|}{2\pi} \mathrm{Im}\,\psi^{(1)}\left(\frac{1}{2} + i\frac{\beta\mu}{2\pi}\right) \to 0, \quad \text{as} \quad \mu \to 0.$$

(tri-gamma function)

（No other levels don't contribute to α_3）

Thus $\alpha_3 = 0 \to \mu = 0$

$\alpha_3(H)$



α_2 includes some divergence, and we regularize it by the proper-time method, with the cut-off parameter Λ.

$$\alpha_2(H) = -2N_c \sum_{f,m} \frac{T|e_f H|}{2\pi} \int \frac{dk_3}{2\pi} \sum_n \frac{2-\delta_{n0}}{(\omega_m + i\mu)^2 + k_3^2 + 2|e_f H|n}$$

$$\alpha_2(H) = -N_c \sum_f \frac{|e_f H|}{2\pi^2} \int_{\Lambda^{-2}}^{\infty} \frac{d\tau}{\tau} \frac{1}{\exp(2|e_f H|\tau)-1} + \underline{\alpha_2^{(0)}(H)}$$

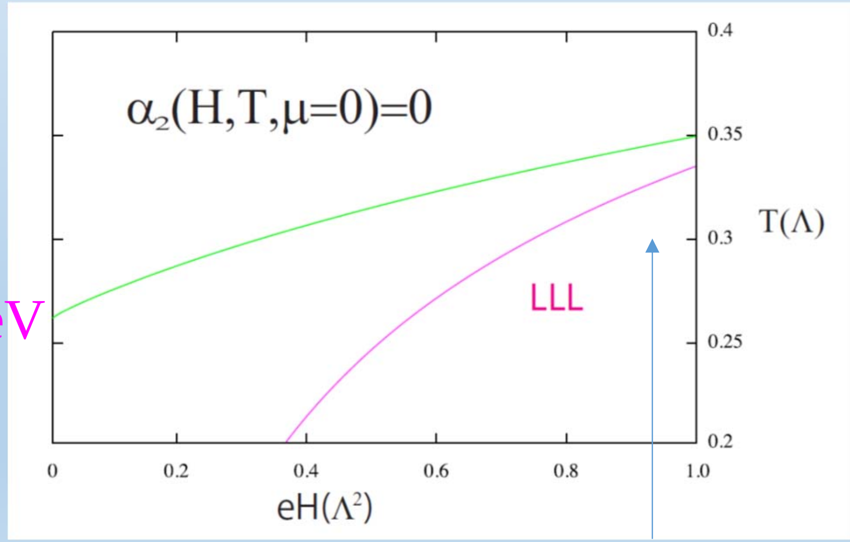
LLL

+ (thermal contibution),

where $\alpha_2^{(0)}(H)$ is the LLL contribution.

Tricritical line

~ 220MeV



Dimensional reduction becomes complete

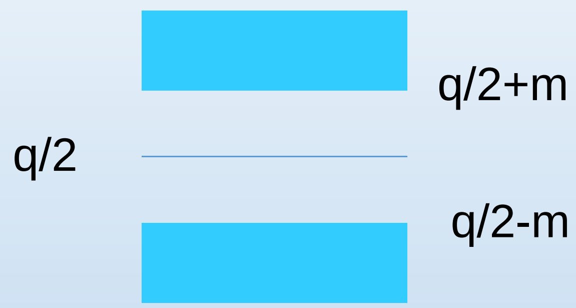
Dimensional reduction

$$7.1\times 10^{17}G\left(\frac{\rho}{\rho_{nucl}}\right)^{2/3}\leq |eH|$$

In the large H limit, only LLL contributes, so that

1+3 $\longrightarrow$ 1+1(complete)

Energy spectrum



$$E_{p\varepsilon}=\varepsilon\sqrt{m^2+p^2}+q/2,\quad \varepsilon=\pm 1$$

This is exactly the same with NJL_2 model in 1+1 dimension and *nesting* is complete.

↓

Chiral spiral is the most favorite solution in 1+1 dim,

(G. Basar, G.V. Dunne, M. Thies, PRD 79(2009)105012)

while RKC is the lowest in 1+3 dim.? (cf. D. Nickel, PRL 103(2009) 072301; PRD 80(2009) 074025.)

We have a model-independent result in the large H limit.

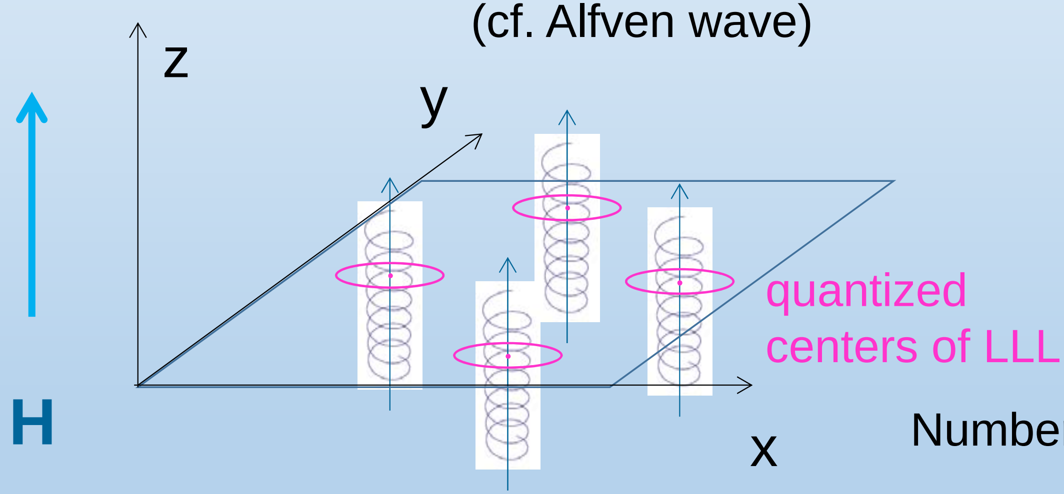
DCDW is the most favorite state

V. Phenomenological implications

Magnetic structure

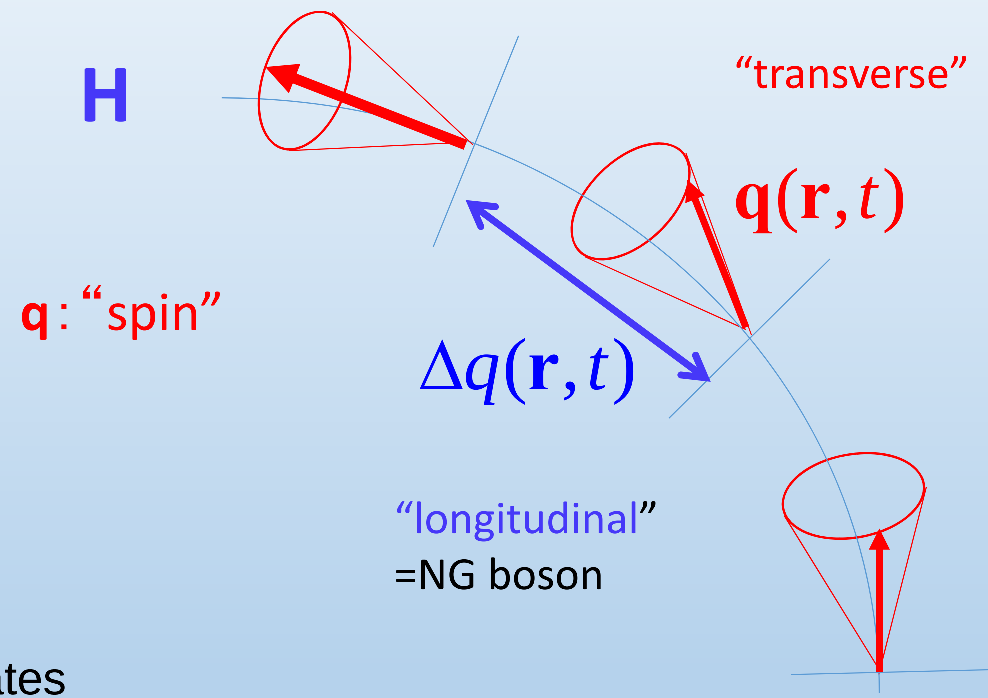
Coupling of H and DCDW

(cf. Alfven wave)



“strands” structure
along the magnetic flux

Local density approximation



Number of states

$$dN_0 = N_C S_{12} \frac{|eH|}{2\pi} L_3 \frac{dk_3}{2\pi}$$

Low-energy excitations

Spontaneous magnetization? (TT, K. Nishiyama, S. Karasawa, in preparation)

For the configuration $M(z) = m(z)e^{i\theta(z)} \in \mathbb{C}$

The η invariant can be easily evaluated:

$$\eta_H(s) = \frac{2}{\pi}\cos\left[s\frac{\pi}{2}\right]\int_0^\infty d\omega\omega^{-s}\int d^D x \mathrm{tr}\left\langle z\left|\frac{H_{LLL}}{H_{LLL}^2+\omega^2}\right|z\right\rangle,$$

Using the derivative expansion, we find

$$\eta_H = \lim_{s\to 0+}\eta_H(s) = -\frac{1}{\pi}\int d^D x\theta'(z).$$

Accordingly we have

$$\Omega_{anom} = -\frac{e\mu}{4\pi^2}\int d^D x \mathbf{H}\nabla\theta$$

c.f. Gauging WZW action by Son & Stephanov,

(D.T. Son and M.A. Stephanov, PRD 77(2008) 014021.)

→ Magnetization? $M_z = \frac{e\mu}{4\pi^2}\theta'$

Cooling of hybrid stars (TT and T. Muto, inpreparation.)

The ground state can be represented as a chirally rotated state

$$|G\rangle = \exp\left(i\mathbf{q}\cdot\int \mathbf{r}A_3^0(\mathbf{r})d^3r\right)|\mathrm{normal}\rangle \left(\equiv U_{\mathrm{DCDW}}(\theta(\mathbf{r}))\right), \quad \theta(\mathbf{r}) \equiv \mathbf{q}\cdot\mathbf{r}$$

Effective Weak Hamiltonian:

$$H_W = \frac{G_F}{\sqrt{2}}\cos\theta_C h_{1+i2}^{\mu} l_{\mu} + h.c.$$

(cf. T. Muto and TT, PTP 80(1988)28.)

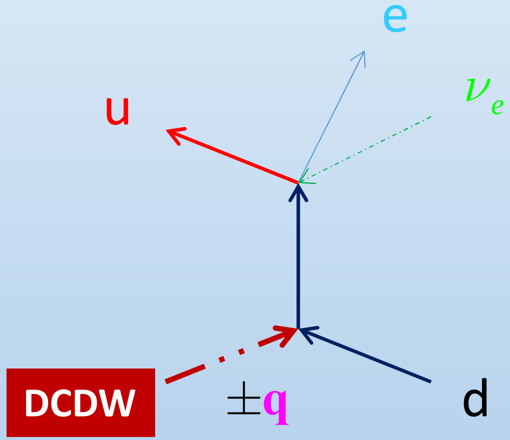
$$\langle u, e, \bar{\nu}|H_W|d\rangle = \langle u_W, e, \bar{\nu}|\tilde{H}_W|d_W\rangle$$

$$H_W \to \tilde{H}_W = U_{\mathrm{DCDW}} H_W U_{\mathrm{DCDW}}^{\dagger}$$

$$= \frac{G_F}{\sqrt{2}}\cos\theta_C U_{\mathrm{DCDW}} h_{1+i2}^{\mu} U_{\mathrm{DCDW}}^{\dagger} l_{\mu} + h.c.$$

$$= \frac{G_F}{\sqrt{2}}\cos\theta_C \tilde{h}_{1+i2}^{\mu} l_{\mu} + h.c.,$$

$$U_{\mathrm{DCDW}} h_{1+i2}^{\mu} U_{\mathrm{DCDW}}^{\dagger} = \exp(i\mathbf{q}\cdot\mathbf{r}) h_{1+i2}^{\mu}$$



"pole" contribution

DCDW catalyzed cooling

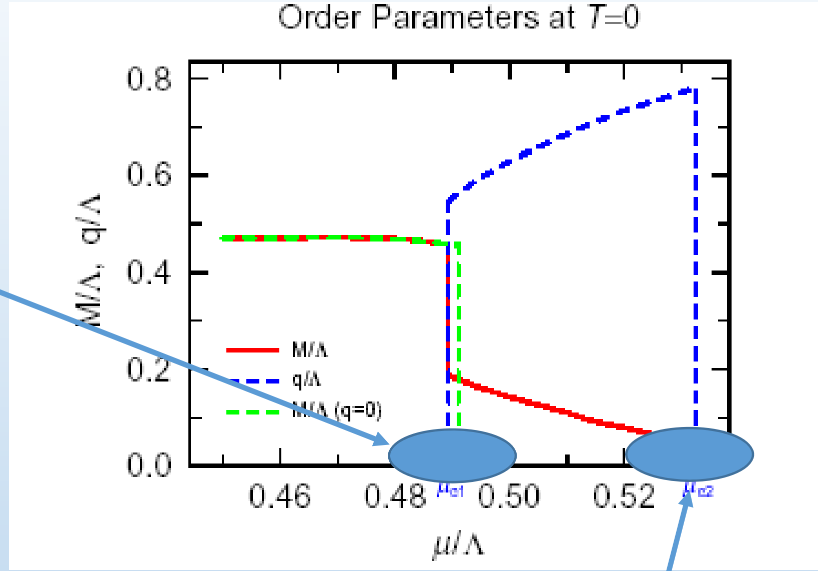
cf. pion cooling

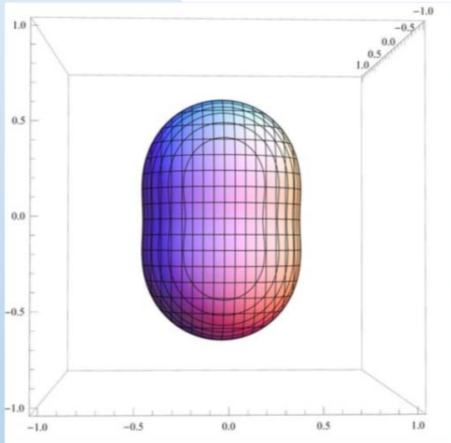
(O.Maxwell et al(MBCDM), Ap.J.216(1977)77)

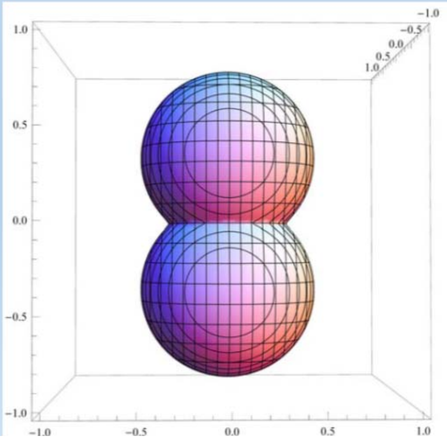
,which means DCDW modifies the momentum conservation at the weak-interaction vertex.

Emissibilities near the phase boundary:

$$\varepsilon_{DCDW} = \frac{457}{1680}\pi G^2 \cos^2\theta_C \mu_d \mu_u \frac{\mu_e^2}{q} T^6$$
$$\simeq 6\times 10^{26}\left(\frac{\rho}{\rho_0}\right) Y_e^{2/3}\ T_9^6\ (\mathrm{erg\ cm^{-3}s^{-1}})$$







$$\varepsilon_{DCDW} = 2\times\frac{3\pi}{8}\frac{457}{5040} G^2 \cos^2\theta_C \frac{\mu_d}{\mu_u} m^2 |\mathbf{q}| \ln\left[\frac{2|\mathbf{q}|}{m}\right] T^6$$
$$\simeq 10^{26}\ \ (\mathrm{erg\ cm^{-3}s^{-1}})$$

VI Summary and concluding remarks:

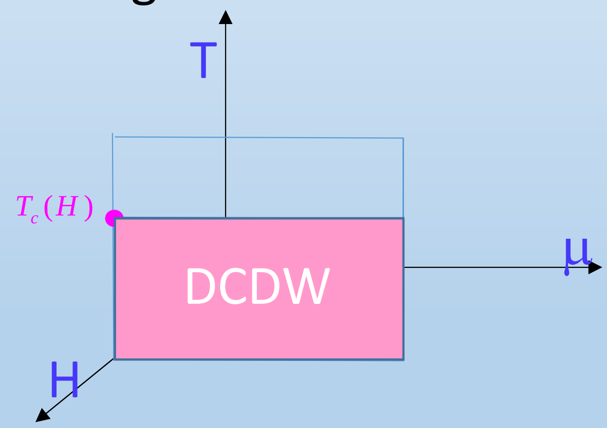
・We have studied dual chiral density wave (DCDW) in the presence of the magnetic field.

・Spectral asymmetry arises in the LLL and anomalous baryon number is induced.

— Consequently, DCDW phase-region is greatly extended .

・Generalized Ginzburg-Landau analysis suggests, that there emerges a novel tricritical point (line) on the H-T plane (μ=0).

— It should be interesting to explore it in the lattice simulations, free from the sign problem.



・Possibility of similar phenomena in condensed-matter physics , e.g., FFLO state.

- Generally dimensional reduction in not complete

  — Phase diagram in the density-temperature-magnetic-field space

- Phenomenological implications

  (i) longitudinal and transverse excitations are possible along the magnetic field.

  — magnetic and dynamic properties of compact stars

  (ii) Spontaneous magnetization is possible?

  — origin of magnetic field in compact stars

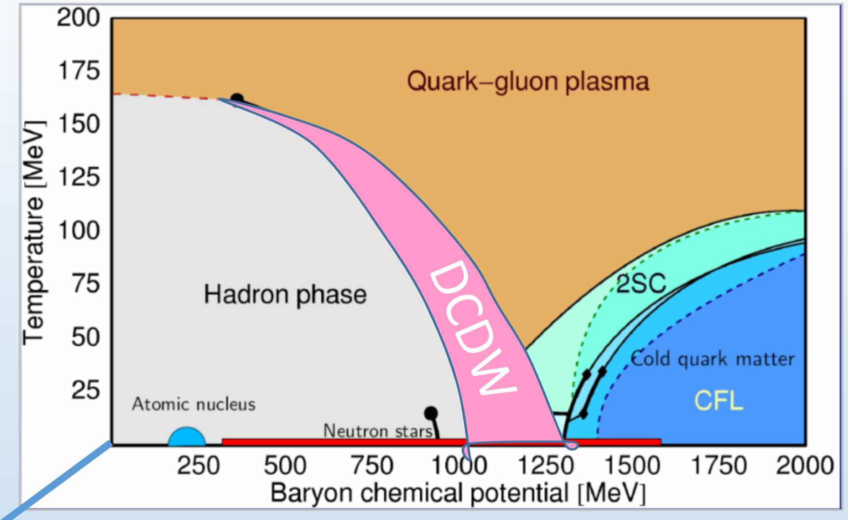
  (iii) Novel cooling mechanism works and emissivity is almost the same as the pion cooling or quark cooling

- High-energy heavy ion collisions

- Ads/CFT correspondence

QCD phase diagram

DCDW ?



NJL as an effective theory of QCD

(Here we only consider one flavor case.)

(B. Ruester)

CCP as the Lifshitz point

H

DCDW ?

D.Nickel, PRL 103(2009) 072301;
PRD 80(2009) 074025.

“Extended” QCD phase diagram??

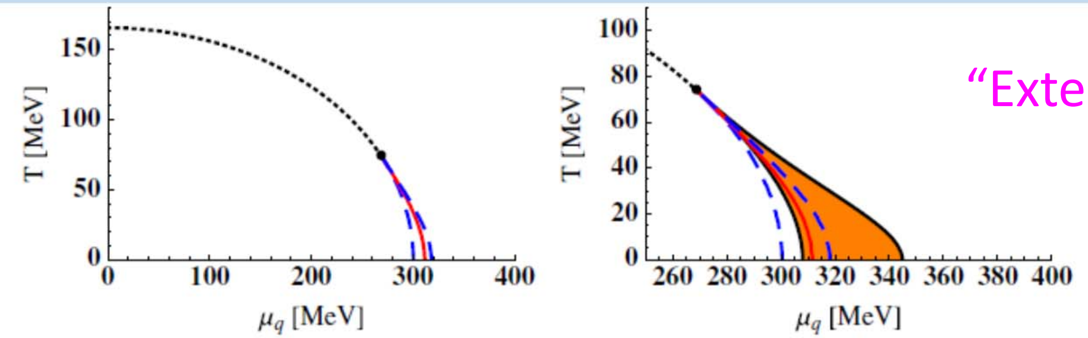


FIG. 1 (color online). Left: Structure of the NJL phase diagram in the chiral limit as a function of temperature T and quark chemical potential μ_q for $M_q = 300$ MeV. The black (short-dashed) line indicates the second order phase transition from the chirally broken to restored phase, the red (solid) line the first order phase transition and the bullet the critical point. The spinodal region is enclosed by the blue (long-dashed) lines. Right: Same plot as on the left including the orange (shaded) domain where the energetically preferred ground state is inhomogeneous.

What is the baryon number or baryon-number density？

$$E = \sqrt{p^2 + m^2} \rightarrow \mu = \sqrt{p_F^2 + m^2}$$

and

$$N/V = n_f \int \frac{d^3 p}{(2\pi)^3} 1. \quad \longrightarrow \quad N \neq 0 \quad \text{for} \quad \mu > m$$

Background soliton can induce *N*. → We shall see manifestation of hidden anomalies in the presence of magnetic field

eg) chiral bag

We shall see that phase degrees of freedom plays an important role

$$M(z) = \Delta e^{i\theta(z)}$$

→ Spectral asymmetry of the Dirac operator

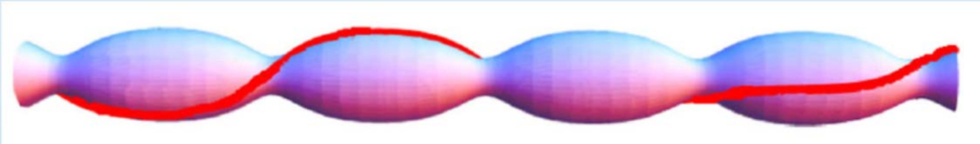
Ex) DCDW (1+3 dim.) or chiral spiral (1+1 dim.)

One dimensional order:

G. Basar et al., PRD 79 (2009) 105012)

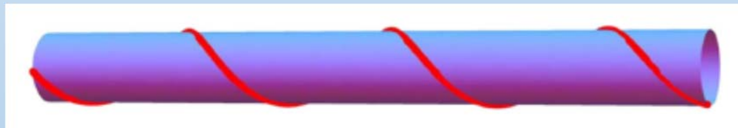
- 1+1 dim. Models:
  Gross-Neveu (Z_2),'t Hooft , NJL2 (O(2)) models
- Large N argument

ex) NJL2 model



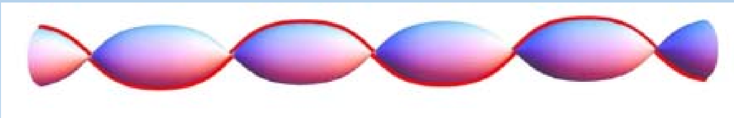
Twisted kink crystal condensate

G.Basar and G.V.Dunne,
PRL 100(2008) 2004004;
PRD 78(2008) 065022.



Spiral condensate

$$\Delta = \lambda e^{iqz}$$



Reak kink crystal

$$\Delta(x) = \sqrt{\nu} q \,\mathrm{sn}(qz;\nu)$$

$$\to \sqrt{\nu} q \sin(qz) \quad \text{as } \nu \to 0$$

Energy spectrum in the presence of the magnetic field: $\mathbf{q} \,||\, \mathbf{H}$

$$E_{n,p,\zeta,\varepsilon} = \varepsilon\sqrt{\left(\zeta\sqrt{m^2+p^2}+q/2\right)^2+2eHn}, \quad n=1,2,3,\dots$$

$$E_{n=0,p,\varepsilon} = \varepsilon\sqrt{m^2+p^2}+q/2, \quad \text{(Lowest Landau level(LLL))}$$

← Spectral asymmetry

Note: for LLL,

$$\mathcal{H}'_D = \begin{pmatrix} H_{1\mathrm{D}}(M(z)) & 0 \\ 0 & H_{1\mathrm{D}}(M^*(z)) \end{pmatrix},$$

with

$$H_{1\mathrm{D}}(M(z)) = \begin{pmatrix} -i\partial_z & M(z) \\ M^*(z) & i\partial_z \end{pmatrix}$$

"1+1 dim." Hamiltonian

Eigenvalues: $E_{n=0,\varepsilon} = \varepsilon\sqrt{p^2+m^2} \pm q/2$

redundant

II. Spectral asymmetry of the Dirac operator in the presence of the magnetic field

Energy spectrum in the presence of the magnetic field: $\mathbf{q} \parallel \mathbf{H}$

$$E_{n,p,\zeta,\varepsilon} = \varepsilon\sqrt{\left(\zeta\sqrt{m^2+p^2}+q\right)^2 + \underline{2eHn}}, \quad n = 1, 2, 3, \ldots$$

$$E_{n=0,p,\varepsilon} = \varepsilon\sqrt{m^2+p^2}+q, \quad \text{(Lowest Landau level(LLL))}$$

← Spectral asymmetry

[I.E. Frolov et al., Phys. Rev. D82 (2010) 076002]

cf

$$E_{p,\zeta,\varepsilon} = \varepsilon\sqrt{\underline{p_\perp^2} + \left(\zeta\sqrt{m^2+p_z^2}+q\right)^2}$$

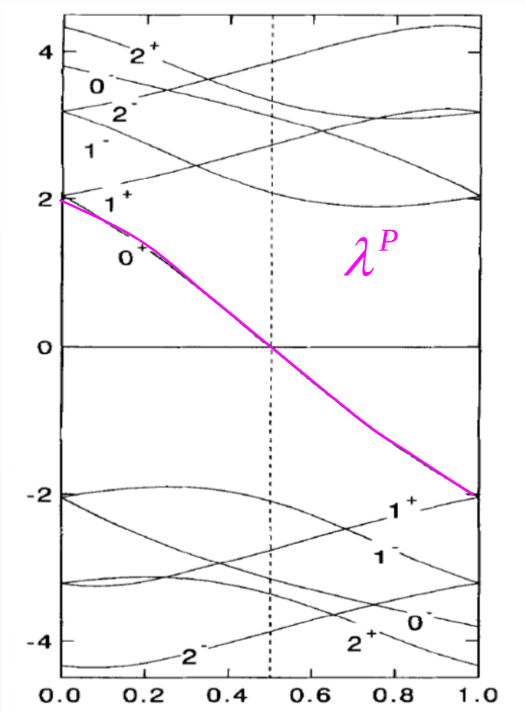
(DCDW)

prolate or oblate deformation of Fermi sea due to spin-orbit coupling

$$E_{p\varepsilon} = \varepsilon\sqrt{m^2 + |eH|(2n+1+s) + p^2}, \quad n = 0, 1, 2, \ldots, \quad s = \pm 1$$

(the Landau levels)

Spectrum is symmetric about 0

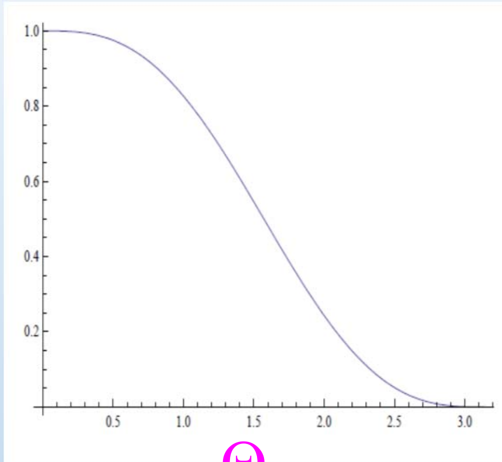


Θ / π

Spectrum of the Dirac operator

[P.J. Mulders (1984)]

N_{in}



Θ

MIT bag $(R \to \infty)$

Skyrmion $(R \to 0)$

Note:

$$-\frac{1}{2}\eta_H(\Theta) = \frac{1}{2}\eta_H(-\Theta)$$

$$\begin{aligned} N_{\mathrm{out}} &= \frac{1}{24\pi^2}\epsilon^{ijk}\int_R^\infty d^3x \mathrm{Tr}\left[U^{-1}\partial_i U U^{-1}\partial_j U U^{-1}\partial_k U\right] \\ &= \frac{1}{\pi}\left[\Theta(R) - \frac{1}{2}\sin 2\Theta(R)\right]. \end{aligned}$$

(mapping degree $\pi_3(SU(2)) = \pi_3(S^3) = Z$)

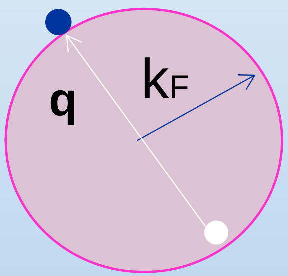
$\rightarrow$ $N_{\mathrm{in}} + N_{\mathrm{out}} = 1$

Origin of inhomogeneous phase is the nesting effect of the Fermi surface

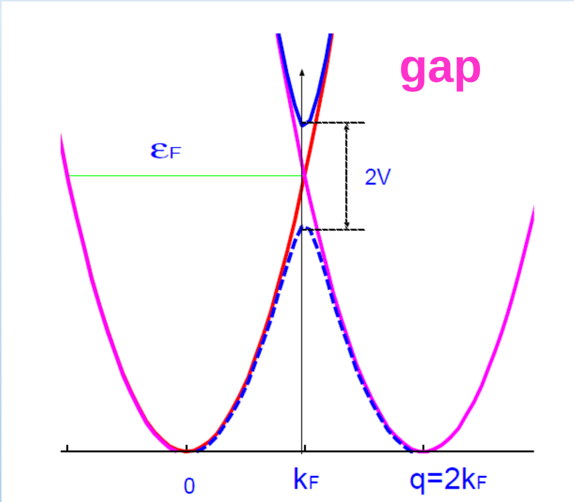
(cf. quantum phase transition)

Nesting (Overhauser, Peiels) is the key mechanism for generating SDW

Level crossing of the energy spectrum near the Fermi surface



Particle-hole instability



Mean-field

$V \cos \mathbf{qx}$

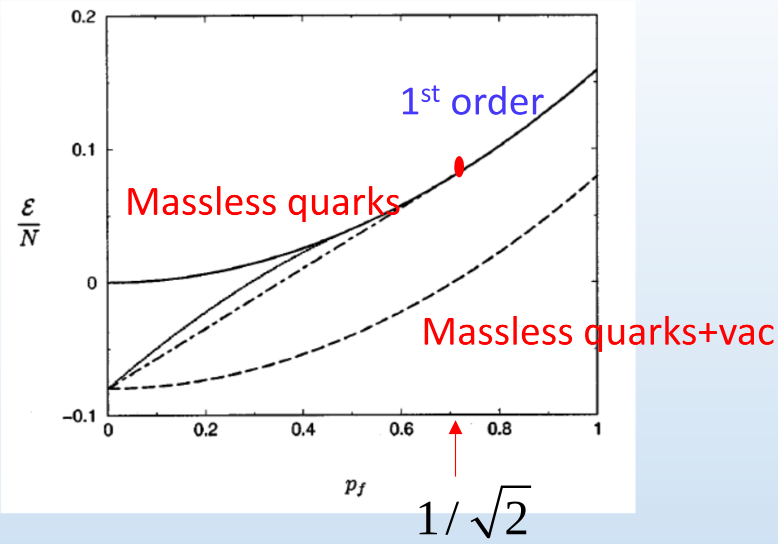
$e^{i\mathbf{kx}} \rightarrow e^{i\mathbf{kx}} + e^{i(\mathbf{k}\pm\mathbf{q})\mathbf{x}}$

$|\mathbf{q}| = 2k_F$

Model indep.

→ SDW,CDW → Floerich superconductivity

A.W. Overhauser, PRL 4(1960) 462.
R.E. Peierls, *Quntum Theory of Solids* (1955)



V. Schoen and M. Thies, PRD 62(2000)096002.

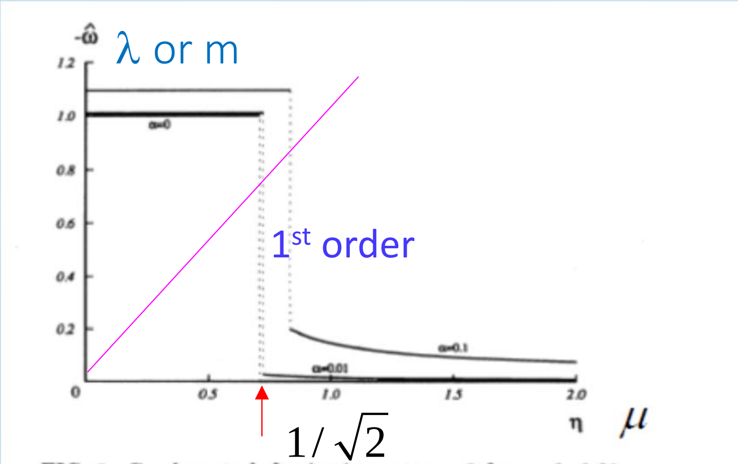
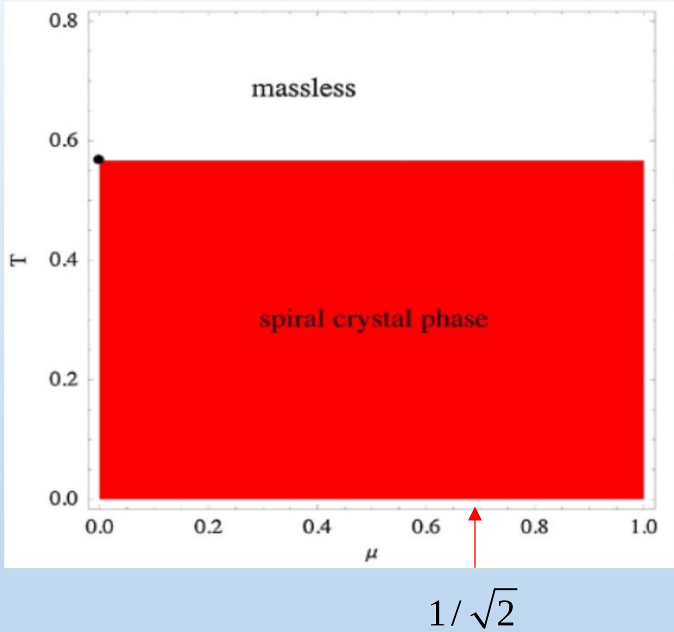


FIG. 5. Condensate behavior in η at $r = 0$ for α=0, 0.01, and 0.1.

A. Barducci et al., PRD 51(1995)3042.



IV Is DCDW the multi-soliton state?

Recall that

$$N_B = \frac{1}{2\pi}\int_{-L}^{L} dx \frac{d\theta}{dx} = \frac{1}{2\pi}\left[\theta(L) - \theta(-L)\right], \quad \theta(x) = qx$$

from WZW anomaly (chiral anomaly),

which reflects the homotopy group

$$\pi_1(U(1)) \simeq \pi_1(S^1)$$

ex)

$$\theta(L) = -\theta(L) = \pi \rightarrow N_B = 1$$

cf

$$\pi_3(SU(2)) \simeq \pi_3(S^3) = \mathbb{Z}$$

for skyrmion.

(J.Goldstone and F. Wilczek (1983),
J.Goldstone and R.L. Jaffe (1984) for chiral bag)

Why $N/V = \dfrac{q}{\pi}$?

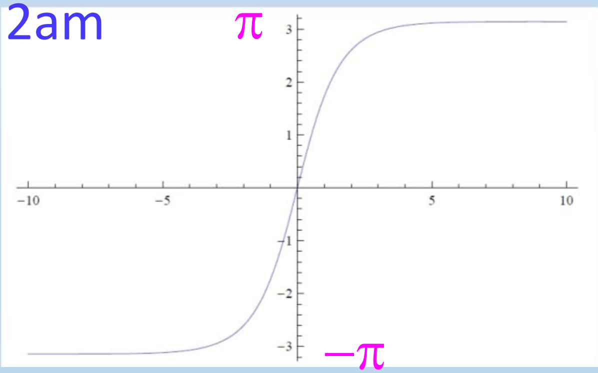
Consider amplitude, ← Sine-Gordon equation

$$2\mathrm{am}(x,k) \rightarrow 2x, \quad k \rightarrow 0$$

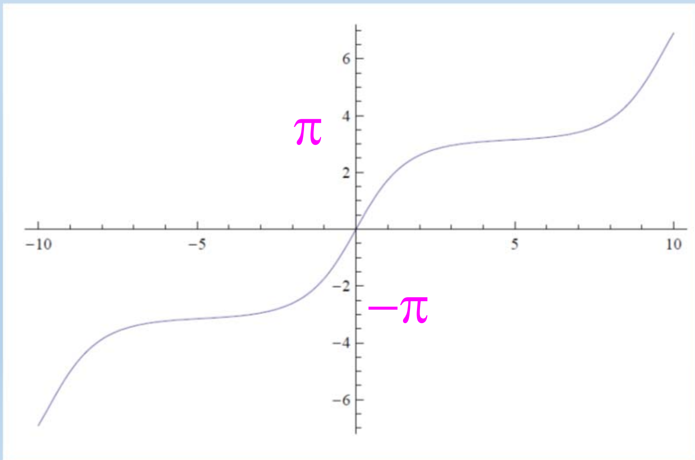
$$\rightarrow 2\arcsin\tanh(x), \quad k \rightarrow 1$$

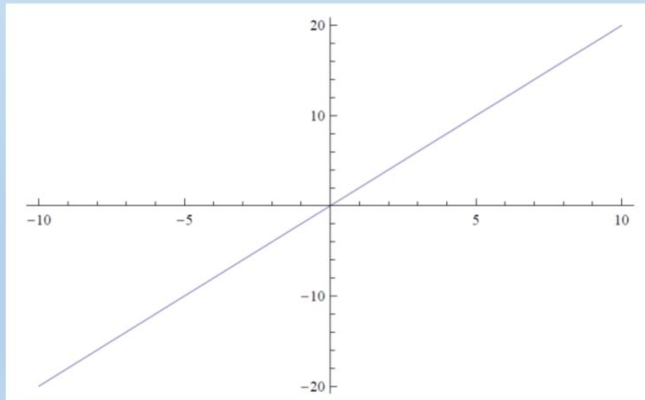
So we can see

$$q \lim_{k\rightarrow 0} \mathrm{am}(x,k) = qx$$



k=1





k=0

The coefficients are evaluated in terms of the Green function in the presence of the magnetic field.
Generally the Green function is given by the well-known formula [Schwinger, (1951)], and that for the uniform magnetic field is given by

$$S_A(x,y) = \exp\left[\frac{ie}{2}(x-y)^\mu A_\mu(x+y)\right]\tilde{S}_A(x-y)$$

(removable by the suitable gauge trans.)

with

$$\begin{aligned}\tilde{S}_A(k) &= \int_0^\infty ds \exp\left[is\left(k_0^2 - k_3^2 - \mathbf{k}_\perp^2\frac{\tan(eBs)}{eBs} - m^2\right)\right] \\ &\times\left[(k^0\gamma^0 - k^3\gamma^3 + m)(1+\gamma^1\gamma^2\tan(eBs)) - \mathbf{k}_\perp\boldsymbol{\gamma}_\perp(1+\tan^2(eBs))\right] \\ &= i\exp\left(-\frac{\mathbf{k}_\perp^2}{|eB|}\right)\sum_{n=0}^\infty(-1)^n\frac{D_n(eB,k)}{k_0^2 - k_3^2 - m^2 - 2|eB|n}\end{aligned}$$

[A.Chodos et al., (1990)]

For LLL contribution

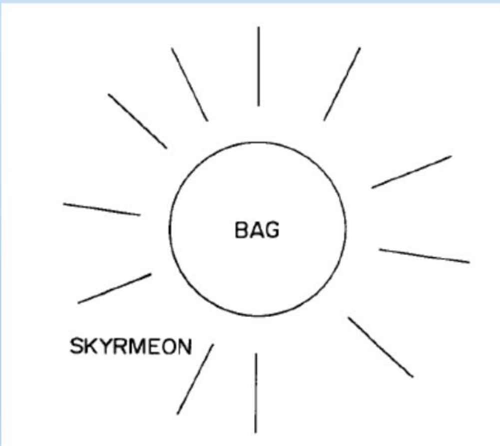
$$\tilde{S}^{(0)}(k) = i\exp\left(-\frac{\mathbf{k}_\perp^2}{|eB|}\right)\frac{k^0\gamma^0 - k^3\gamma^3 + m}{k_0^2 - k_3^2 - m^2}(1 - i\gamma^1\gamma^2\mathrm{sign}(eB))$$

degeneracy Spin projection

Example:

Chiral bag

$$L = \bar{\psi} i \partial\!\!\!/ \psi,$$



$$[i\boldsymbol{\gamma} \cdot \hat{\mathbf{n}} + \exp\left(i\boldsymbol{\tau} \cdot \hat{\mathbf{n}}\gamma_5 \Theta\right)]\,\psi = 0$$ on the bag boundary

The APS η invariant gives [Goldstone & Jaffe, (1983)]

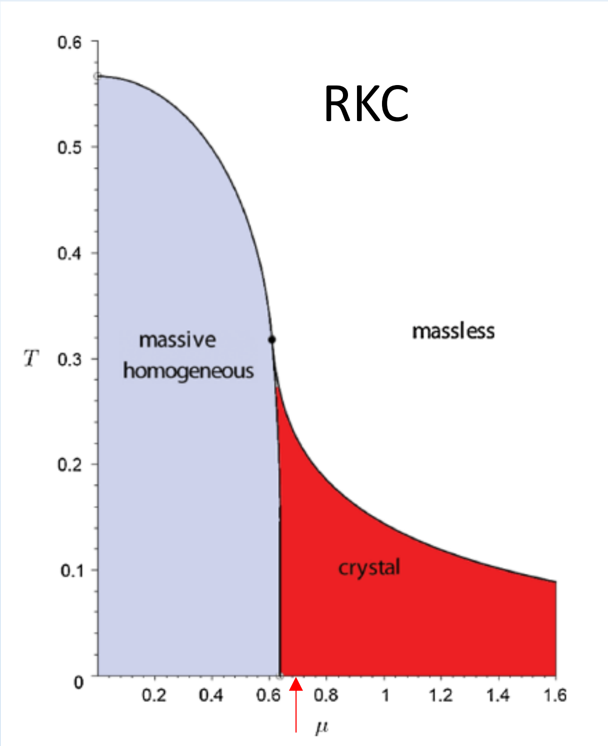
$$N_{\rm in} = 1 - \frac{1}{\pi}\left[\Theta - \frac{1}{2}\sin(2\Theta)\right]$$

valence

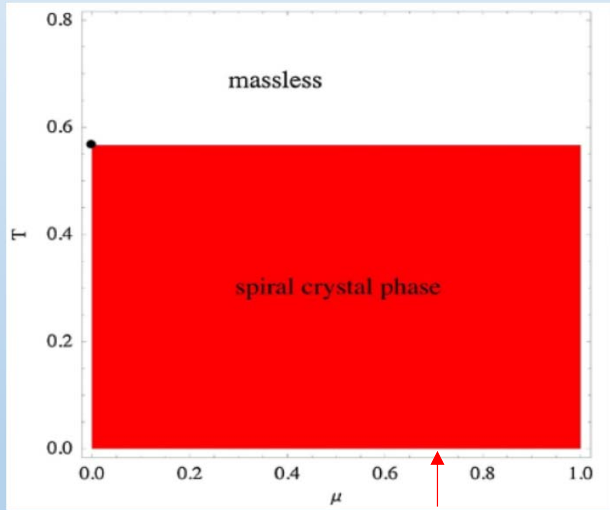
[Niemi & Semenoff, (1986)]

IV Summary and concluding remarks:

$$\rho_B^{\text{anomaly}} = \frac{eH}{2\pi}\frac{q}{2\pi}$$ due to LLL



$1/\sqrt{2}$



$1/\sqrt{2}$

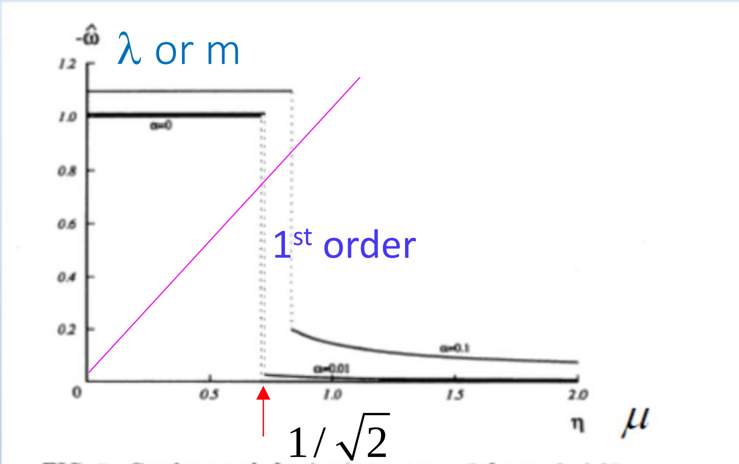


FIG. 5. Condensate behavior in η at $r = 0$ for α=0, 0.01, and 0.1.

A. Barducci et al., PRD 51(1995)3042.

No anomaly due to Z_2 symmetry

Thermodyamic potential

$$\Omega = \Omega_{\rm vac} + \Omega_\mu + \Omega_T,$$

$$\begin{aligned}
\Omega_{\rm vac} &= -\frac{1}{2}\frac{eH}{2\pi}\int\frac{dp}{2\pi}\sum_{n,\zeta,\epsilon}|E|,\\
\Omega_\mu &= -\frac{1}{2}\frac{eH}{2\pi}\int\frac{dp}{2\pi}\sum_{n,\zeta,\epsilon}\left(|E-\mu|-|E|\right),\\
\Omega_T &= -\frac{1}{\beta}\frac{eH}{2\pi}\int\frac{dp}{2\pi}\sum_{n,\zeta,\epsilon}\ln\left(1+e^{-\beta|E-\mu|}\right),
\end{aligned}$$

→

$$-\frac{1}{2}\frac{eH}{2\pi}\lim_{s\to 0+}\int\frac{dp}{2\pi}\sum_{n,\zeta,\epsilon}\left(|E-\mu|-|E|\right)|E|^{-s}$$

$$\begin{aligned}
\rho_B &= -\frac{\partial\Omega}{\partial\mu}\\
&= -\frac{1}{2}\frac{eH}{2\pi L}\sum_\lambda \mathrm{sign}\,(E_\lambda) + \frac{eH}{2\pi L}\sum_\lambda \mathrm{sign}\,(E_\lambda)\left[\frac{\theta(E_\lambda)}{e^{\beta(E_\lambda-\mu)}+1}+\frac{\theta(-E_\lambda)}{e^{-\beta(E_\lambda-\mu)}+1}\right].
\end{aligned}$$

At $T=0$ **or** $\beta\to\infty$**,**

$$\rho_B = -\frac{1}{2}\frac{eH}{2\pi L}\sum_\lambda \mathrm{sign}\,(E_\lambda) + \frac{eH}{2\pi L}\sum_\lambda \theta(E_\lambda)\theta(\mu-E_\lambda).$$

Evaluation of η_H (LLL only)

Using the Mellin transform

$$|E|^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty d\omega\omega^{s-1}\exp\left(-|E|\omega\right),$$

$$\begin{aligned}\eta_{\mathrm{H}}(s) &= \frac{L}{\Gamma(s)}\int\frac{dp}{2\pi}\int_0^\infty d\omega\omega^{s-1}\sum_\epsilon \mathrm{sign}(E_\epsilon)\exp\left(-\left|E_\epsilon\right|\omega\right)\\ &= -\frac{q}{\pi}\frac{L}{\Gamma(s)}m^{-s}2^{s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+1\right){}_2\mathrm{F}_1\left(1+\frac{s}{2},\frac{s}{2},\frac{3}{2};\left(\frac{q}{2m}\right)^2\right)\end{aligned}$$

$$E_\varepsilon = \varepsilon\sqrt{p^2+m^2}+q$$

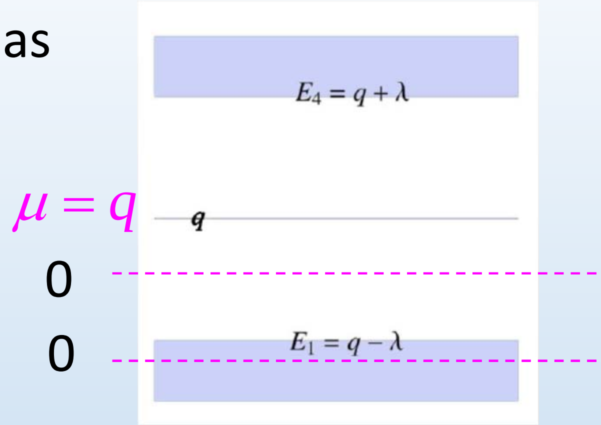
Removable singularity at s=0.

$$\eta_{\mathrm{H}} = \lim_{s\to 0+}\eta_{\mathrm{H}}(s) = -\frac{qL}{\pi}.$$

Thus η_H is independent of m.

Thus topological baryon-number density can be written as

$$\rho_{\mathrm{B}}^{\mathrm{anomaly}} = -\frac{eH}{4\pi L}\eta_H = \frac{eH}{2\pi}\frac{q}{2\pi}, \quad \text{for } q(=\mu) < \lambda.$$



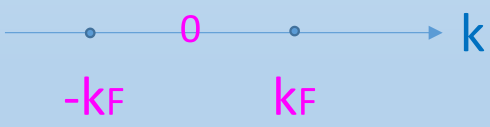
On the other hand, $$\frac{\partial \Omega}{\partial q} = 0 \rightarrow q = 2\mu$$

Note: 1+1 dim.

$\lambda > q$

- Fermi surface is now defined by

$$\rho_B = \frac{2k_F}{2\pi} = \frac{k_F}{\pi}$$



Hence

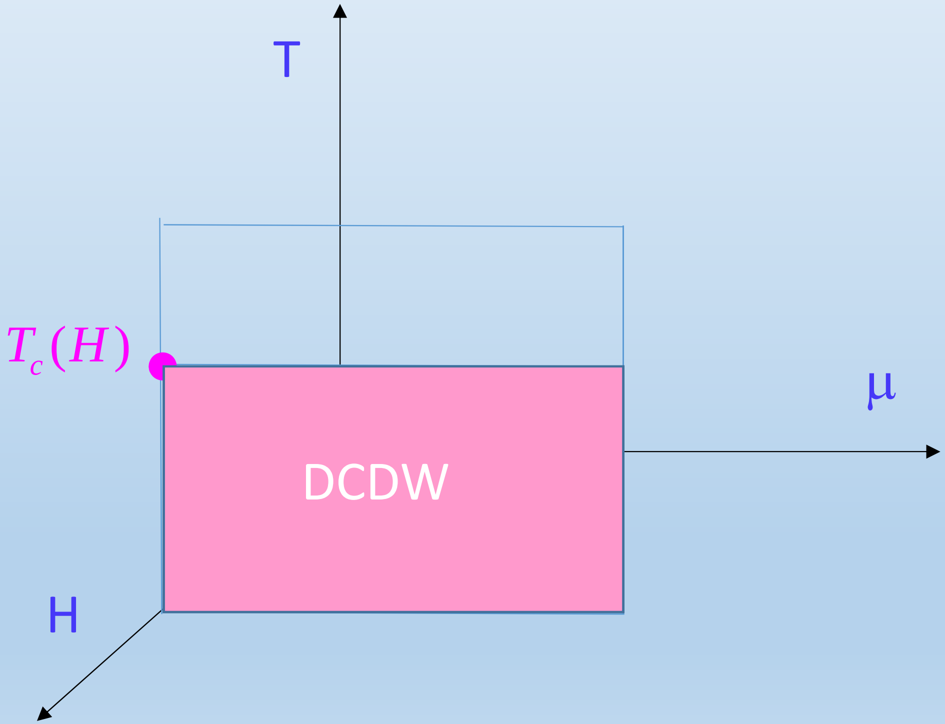
$$2k_F = q$$

Nesting relation!

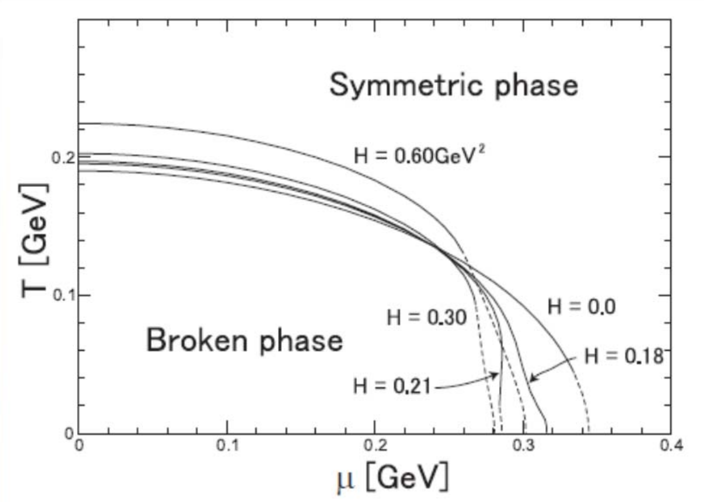
- $\rho_B = \dfrac{\mu}{\pi}$ → Relation for massless quarks!

gGL may gives a novel tricritical point, which should be located near μ=0

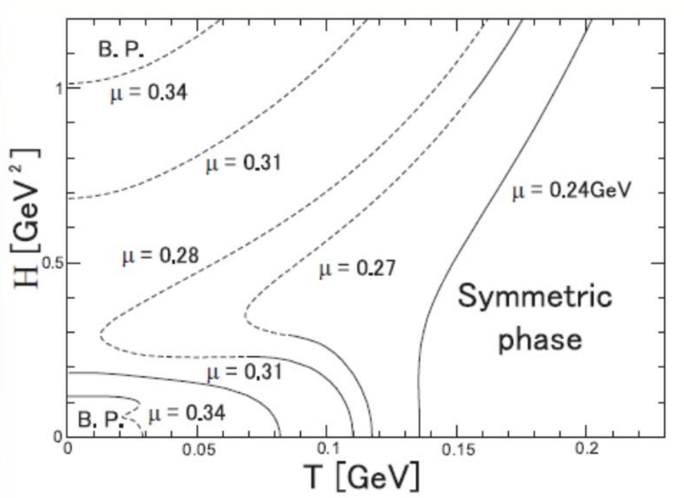
→ Can be explored by the lattice QCD?



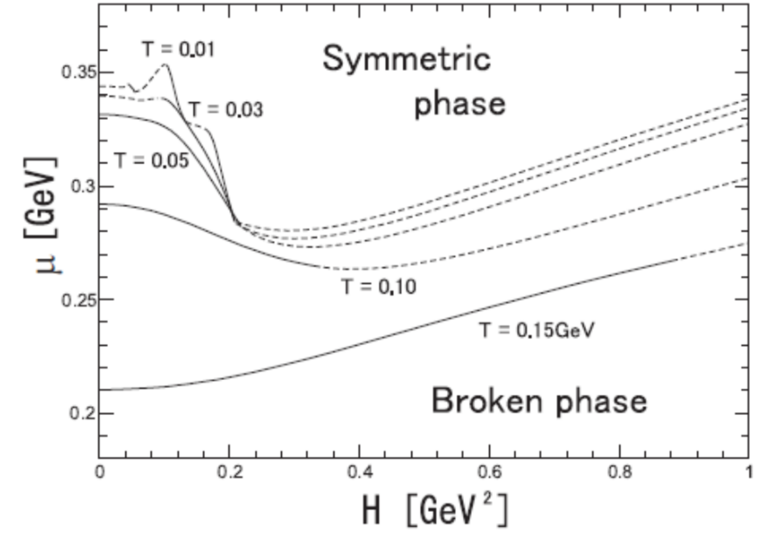
Similar phenomena in condensed matter physics?



(a) The T-μ plot with H fixed.



(b) The H-T plot with μ fixed.



(c) The μ-H plot with T fixed.

Note that spectrum for the Dirac operator is symmetric
even in the presence of the magnetic field.
LLL contribution exhibits 1+1 dim. character.
LLL contribution should be responsible for the odd terms in the gGL expansion

New tricritical point due to the odd terms

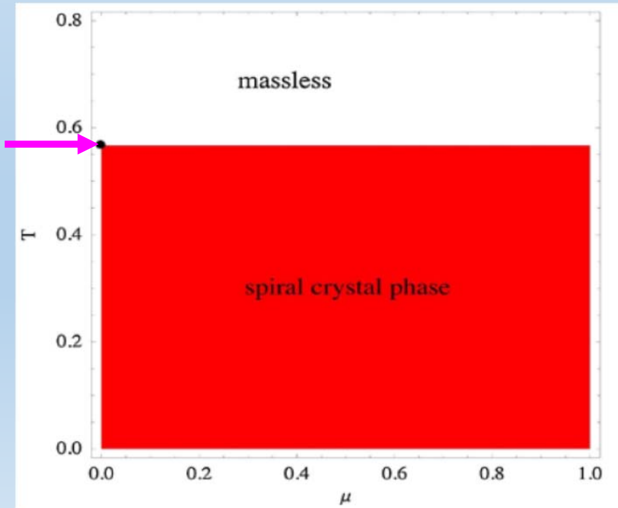
In 1+1 dim. The "tricritical point" is defined as the point where

$$\alpha_2(T,\mu) = \alpha_3(T,\mu) = 0$$

which gives

$$T_{\rm tc} = \frac{e^\gamma}{\pi} = 0.5669, \mu_{\rm tc} = 0.$$

[Dunne et al]



Phase degree of freedom of M(z) is necessary

Consider the Dirac operator

$$H'_D = \boldsymbol{\alpha}\cdot\mathbf{P} + \beta\left[\frac{1+\gamma_5}{2}M(z) + \frac{1-\gamma_5}{2}M^*(z)\right]$$

and the eigenvalues,

$$H'_D q_\lambda = E_\lambda(M) q_\lambda$$

$E_\lambda(M) \to -E_\lambda(M^*)$ under the CT transformation,

Therefore

$$\eta_H(M) = -\eta_H(M^*)$$

from which anomalous baryon number vanishes for real configurations s.t.

$$M = M^*.$$

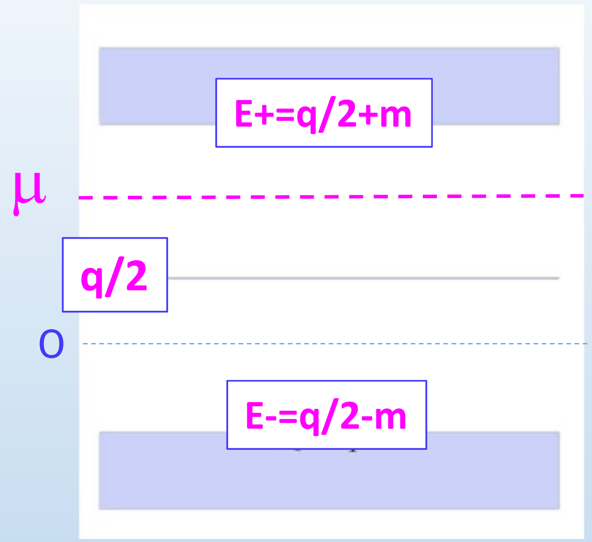
Direct evaluation of η_H for DCDW (LLL only)

$$\eta_H = \lim_{s\to 0+} \eta_H(s)$$

$$= \lim_{s\to 0+} \frac{L}{\Gamma(s)} \int \frac{dp}{2\pi} \int_0^\infty d\omega \omega^{s-1} \sum_{\varepsilon=\pm 1} \mathrm{sign}(E_\varepsilon(p)) \exp(-|E_\varepsilon(p)|\omega)$$

$$= -\frac{qL}{\pi}$$ (Independent of m)

$E_\pm(p)$

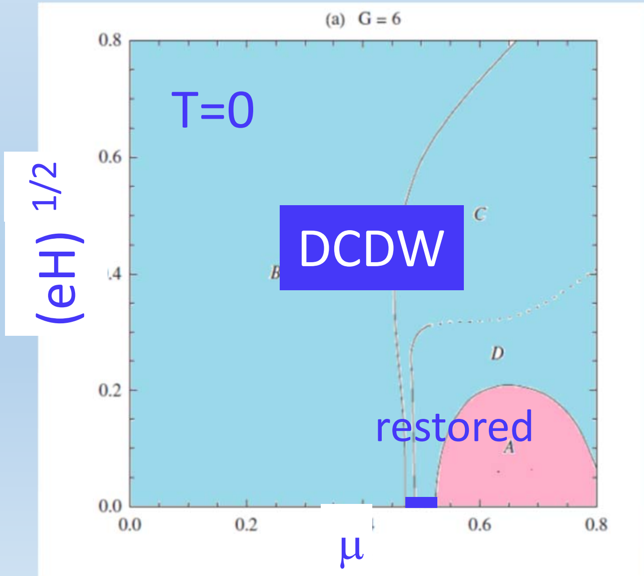


Thus topological baryon-number density can be written as

$$\rho_B^{\text{anomaly}} = -\frac{eH}{4\pi L}\eta_H = \frac{eH}{2\pi}\frac{q}{2\pi},$$

$$\text{for } q/2 < m, \quad \mu < q/2 + m$$

From these considerations, we can say DCDW develops in the very wide region in the presence of the magnetic field



[Florov et al, (2010)]

A “paradox” and anomalous baryon number

Phase degree of freedom of M(z) is necessary

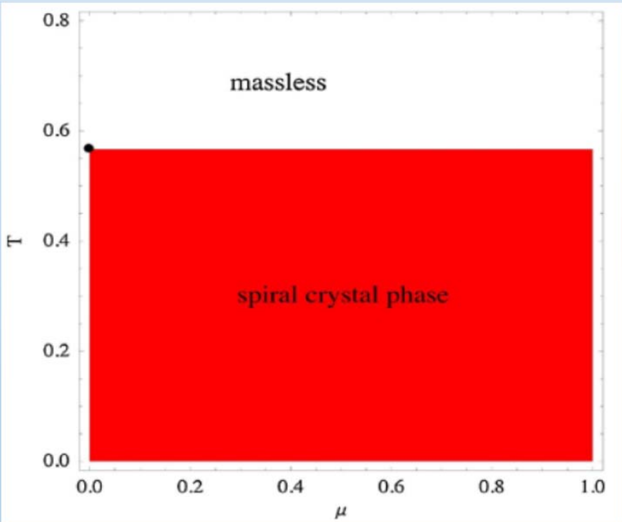
Chiral spiral (DCDW)

$M(z) = \langle \bar{q}q \rangle + i \langle \bar{q}\gamma_5\tau_3 q \rangle = \Delta e^{i\theta(z)}$,

$\theta(z) = qz$

Ex)

Phase diagram of DCDW (chiral spiral) in 1+1 dim.



(Baser et al, (2009))

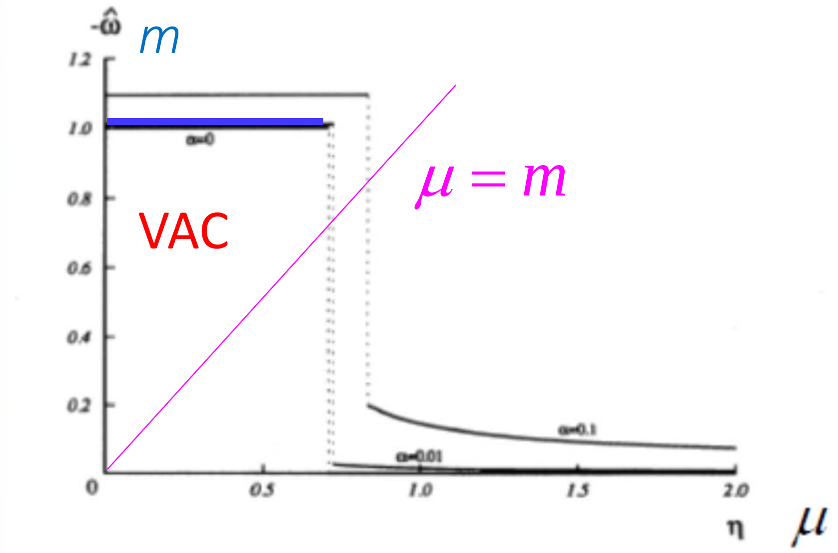


FIG. 5. Condensate behavior in η at $r = 0$ for α=0, 0.01, and 0.1.

A. Barducci et al., PRD 51(1995)3042.

One dimensional order:

1+1 dim. Models:
Gross-Neveu (Z_2),'t Hooft , NJL$_2$ (O(2)) models

Hartree-Fock (self-consistent) solutions

$$\mathcal{L}_{\rm NJL} = \bar\psi i \partial\!\!\!/ \psi + \frac{g^2}{2}\left[(\bar\psi\psi)^2 + (\bar\psi i\gamma^5\psi)^2 \right]$$

$$\text{MFA} \rightarrow \langle\bar\psi\psi\rangle - i\langle\bar\psi i\gamma^5\psi\rangle = -\Delta / g^2$$

Re Im

Bogoliubov-de Gennes (BdG) equation (F. Correa et al.,Ann.Phys.324 (2009) 2522)

Under the Lorentz transformation Λ^μ_ν

$$(\lambda,\mathbf{0}) \rightarrow \left(\lambda\sqrt{1+\mathbf{p}_\perp^2/\lambda^2},\mathbf{p}_\perp\right), \rightarrow \psi_{\left(\lambda\sqrt{1+\mathbf{p}_\perp^2/\lambda^2},\mathbf{p}_\perp\right)} = \left(\sqrt{1+\mathbf{p}_\perp^2/\lambda^2}\right)^{-1/2} S^{-1}(\Lambda)\psi_{(\lambda,\mathbf{0})}$$

with $S(\Lambda)P^\mu S^{-1}(\Lambda) = \Lambda^\mu_\nu P^\nu.$ (D. Nickel 2009)

α_2 includes some divergence, and we regularize it by the proper-time method.

$$\alpha_2(H) = -2N_c \sum_{f,m} \frac{T|e_f H|}{2\pi} \int \frac{dk_3}{2\pi} \sum_n \frac{2-\delta_{n0}}{(\omega_m + i\mu)^2 + k_3^2 + 2|e_f H|n}$$

$$\alpha_2(H) = -N_c \sum_f \frac{|e_f H|}{2\pi^2} \int_{\Lambda^{-2}}^{\infty} \frac{d\tau}{\tau} \frac{1}{\exp(2|e_f H|\tau) - 1} + \underbrace{\alpha_2^{(0)}(H)}_{\text{LLL}}$$

$$+4N_c \sum_f \frac{|e_f H|}{2\pi} \sum_{n\neq 0} \int \frac{dk}{2\pi} \frac{1}{2e_n} \left[n_F^-(e_n) + n_F^+(e_n) \right] + \frac{1}{2G},$$

$$e_n = \sqrt{k^2 + 2|e_f H|n}, n_F^{\pm}(e) \equiv \left[1 + \exp\left(\beta(e \pm \mu)\right)\right]^{-1},$$

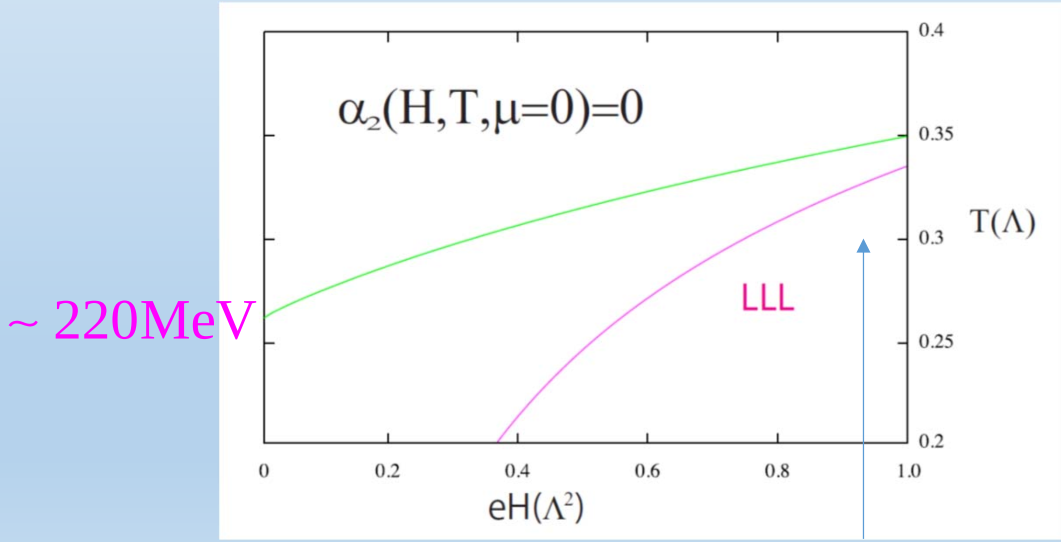
where $\alpha_2^{(0)}(H)$ is the LLL contribution,

$$\alpha_2^{(0)}(H) = -4N_c \sum_f \frac{|e_f H|}{(2\pi)^2} T \,\mathrm{Re} \int_{\Lambda^{-2}}^{\infty} d\tau \sqrt{\frac{\pi}{\tau}} e^{-(\omega_m + i\mu)^2 \tau}$$

$$\rightarrow -4N_c \sum_f \frac{|e_f H|}{(2\pi)^2} T \sum_{m=0}^{\infty} \sqrt{\pi}\, \omega_m^{-1} \Gamma\left(\frac{1}{2}, \frac{\omega_m^2}{\Lambda^2}\right), \mu \rightarrow 0.$$

(incomplete gamma function)

Tricritical line



Dimensional reduction becomes complete