In-medium QCD splittings

Beyond the soft, large N_c and harmonic-oscillator approximations all at once

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EMMI Workshop: Probing early stages with jets

GSI, Darmstadt
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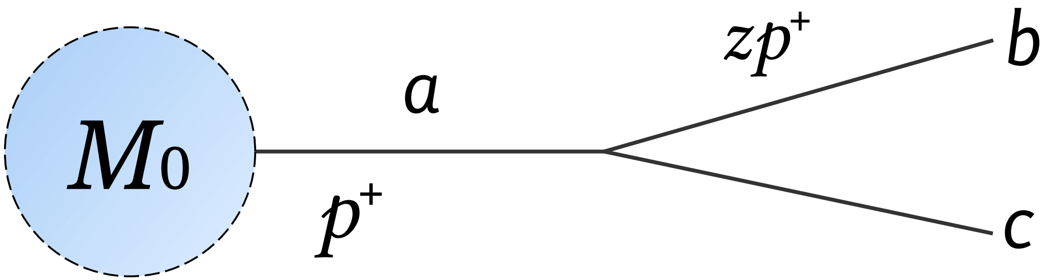
Based on: `arXiv:2606.26039`

In-medium splittings

The **differential probability** of a parton ***a*** to **branch** into partons ***b*** and **c** is the fundamental **building block** for **predictions of observables*** and **Monte Carlo simulations** in **QCD**



$$\sim \mathrm{d}\sigma_0 \times \boxed{\mathrm{d}\mathcal{P}}$$

$$\boxed{\mathrm{d}\mathcal{P} = \frac{\alpha_S}{2\pi}\frac{\mathrm{d}k_T^2}{k_T^2}\frac{\mathrm{d}z}{z}\left[zP_{a\to bc}(z)\right]}$$

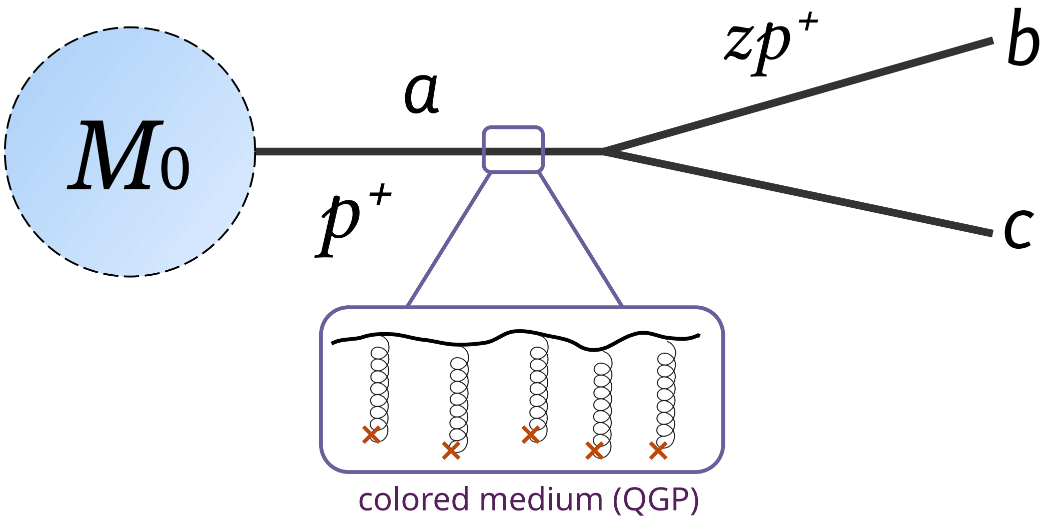
(vacuum)

***e.g.** thrust, multiplicity, jet broadening, energy correlations… very high precision in vacuum.

In-medium splittings

Baier et.al. Nucl.Phys.B483:291-320,1997
Zakharov, JETP Lett.63:952-957,1996
Apolinário et. al. JHEP 02 (2015) 119
Isaksen et. al. JHEP 21 (2020) 125

The **differential probability** of a parton ***a*** to **branch** into partons ***b*** and **c** is the fundamental **building block** for **predictions of observables** and **Monte Carlo simulations** in **QCD**



$$\sim \mathrm{d}\sigma_0 \times \boxed{\mathrm{d}\mathcal{P}}$$

$$\boxed{\mathrm{d}\mathcal{P}_{\mathrm{med}} = ?}$$

Questions: How to compute in-medium splitting probabilities? Example of observables?

(Extra question: how to interface these probabilities in Monte-Carlo simulations?)

Applications (examples)

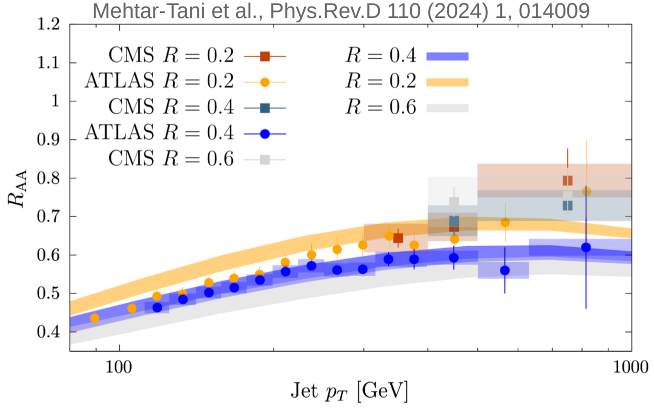
Energy loss (via R_{AA})

$$\frac{d\sigma_{\rm med}}{dE} = \int d\varepsilon\, \boxed{P(\varepsilon)} \frac{d\sigma_{\rm vac}(E+\varepsilon)}{dE}$$

$$\boxed{P_1(\varepsilon) = \int_0^1 dz \int_{\omega R}^{k_t^{\rm max}} dk_t \boxed{\frac{dI_{\rm med}}{dz\, dk_t}} [\delta(\omega(k_t,z)-\varepsilon) - \delta(\varepsilon)]}$$

(one emission, $O(\alpha_s)$)

Mehtar-Tani et al., Phys.Rev.D 110 (2024) 1, 014009



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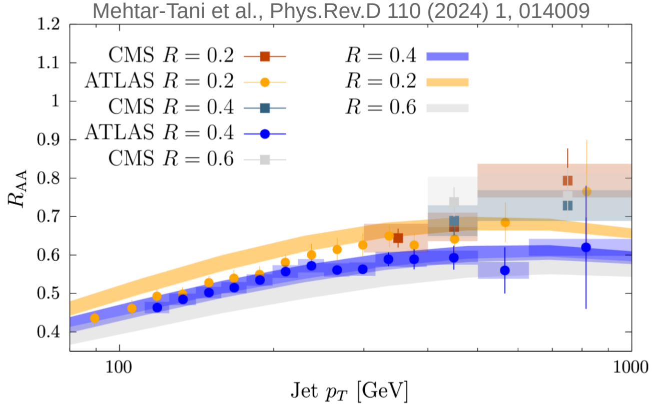
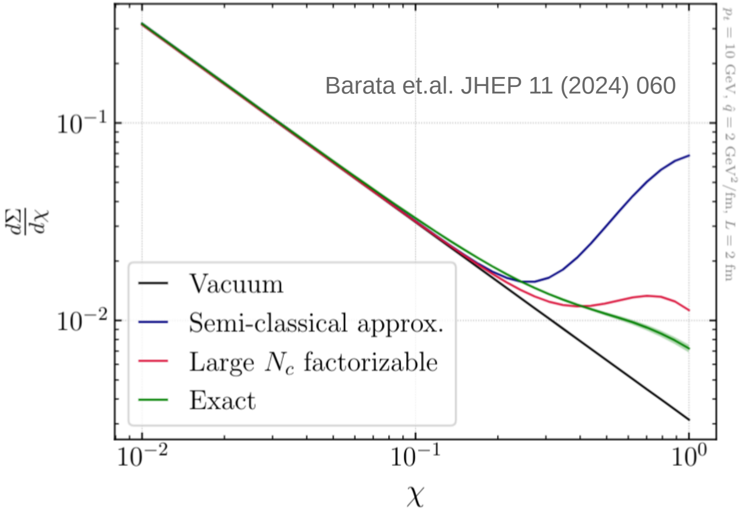


Energy-energy correlators

$$\frac{\mathrm{d}\Sigma_{\mathrm{EEC}}}{\mathrm{d}\theta\,\mathrm{d}p_T} = \int_0^1 \mathrm{d}z \left[z(1-z)\boxed{\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}z\mathrm{d}\theta}}\right] \frac{\mathrm{d}\sigma}{\mathrm{d}p_t}$$



Recent developments

Calculations have been restricted to lifting one (or at most two) of these approximations, often with very limited precision and phase-space coverage.

- **Beyond harmonic-oscillator**, soft limit, large N_c: *Andres et. al.*, JHEP07(2020)114, *Barata et. al.*, JHEP09(2021)153(IOE) *Feal et al.*, Phys.Rev.D 98 (2018) 7, 074029
- Harmonic-oscillator, **finite-z**, **finite N_c**: *Isaksen and Tywoniuk,* JHEP 09 (2023) 049 **(numerics restricted to low energies, small media and $\gamma \to qq$)**
- Harmonic-oscillator, **finite-z**, large N_c: *Andres and Domínguez,* arXiv:2603.27846 **(full solution, semi-analytical)**

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- **Beyond harmonic-oscillator, finite-z, finite** $\boldsymbol{N_c}$: *Barata et. al.*, arXiv:2604.11616 **(foundational quantum circuit approach; very limited numerics, high uncertainties and γ → qq)**

Ingredients of BDMPS-Z

Wang and Gyulassy, PRL 68, 1480(1992)
Aurenche et.al. JHEP 05, 043

i) Medium: modelled as a classical **stochastic background gluon field** with gaussian statistics "spanning" a light-cone time interval $[0, L]$ with linear density $n(t)$.

$$\langle A^{-,a}(\boldsymbol{r},t)A^{-,b}(\boldsymbol{r}',t')\rangle = n(t)\delta^{ab}\delta(t-t')\boxed{\gamma(\boldsymbol{r}-\boldsymbol{r}')}$$

The function

$$\boxed{\gamma(\boldsymbol{x}) = \int_{\boldsymbol{q}} e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\frac{d\sigma}{d^2\boldsymbol{q}}.}$$

encodes the **2 → 2 elastic cross-sections** between **probe** and **medium**. Most widely adopted models: **Gyulassy-Wang (GW)** and the **Hard Thermal Loop (HTL)**:

$$\frac{\mathrm{d}\sigma_{\mathrm{GW}}}{\mathrm{d}^2\boldsymbol{q}} = \frac{g^4}{|\boldsymbol{q}^2+\mu^2|^2}$$

$$n\frac{\mathrm{d}\sigma_{\mathrm{HTL}}}{\mathrm{d}^2\boldsymbol{q}} = \frac{g^2\mu_D^2 T}{\boldsymbol{q}^2(\boldsymbol{q}^2+\mu_D^2)},$$

Both **GW** and **HTL** models can be cast into a common form at **Leading Power (LP)**:

$$n(t)\sigma(\boldsymbol{r}) = g^2 \int_{\boldsymbol{q}} (1 - e^{i\boldsymbol{q}\cdot\boldsymbol{r}}) \frac{\mathrm{d}\sigma}{\mathrm{d}^2\boldsymbol{q}}$$

$$= \frac{1}{4}\tilde{q}\,\boldsymbol{x}^2 \ln\frac{1}{\mu_\star \boldsymbol{x}^2} + \mathcal{O}(\mu_\star^2 \boldsymbol{x}^4),$$

$$\tilde{q} = ng^2 \int^Q \frac{d^2q}{(2\pi)^2} q^2 \frac{d\sigma}{d^2\boldsymbol{q}},$$

Jet quenching parameter

Harmonic-oscillator (HO) amounts to **dropping the logarithm**, yielding

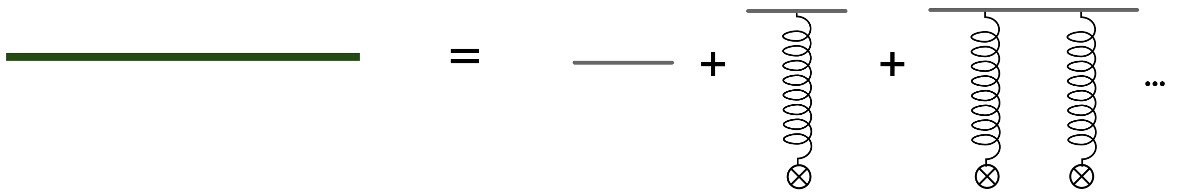
$$n(t)\sigma(\boldsymbol{r}) \simeq \frac{1}{4}\tilde{q}\,\boldsymbol{r}^2,$$

GW and **HTL** **do not admit** a Taylor expansion as $\boldsymbol{r} \to \mathbf{0}$, hence HO will miss qualitative behaviours in the perturbative tail.

ii) Probes (partons): massless, close-to-eikonal propagation, obeying the scale hierarchies

$$p^- \ll |\boldsymbol{p}_\perp| \ll p^+ \qquad\qquad x^+ \simeq t \simeq z$$

In-medium propagator:



$$= \mathcal{G}_{ab}(t, t_0|\boldsymbol{x}, \boldsymbol{x}_0)$$

$$\mathcal{G}_{ab}(t, t_0|\boldsymbol{x}, \boldsymbol{x}_0) = \int_{\boldsymbol{x}_0}^{\boldsymbol{x}} \mathcal{D}\bar{\boldsymbol{x}} \exp\left(i\int \mathrm{d}s\, \frac{p^+}{2}\dot{\bar{\boldsymbol{x}}}(s)^2\right) \boxed{U_{R,ab}(L, t_0|\bar{\boldsymbol{x}})}$$

Transverse space broadening

Wilson line

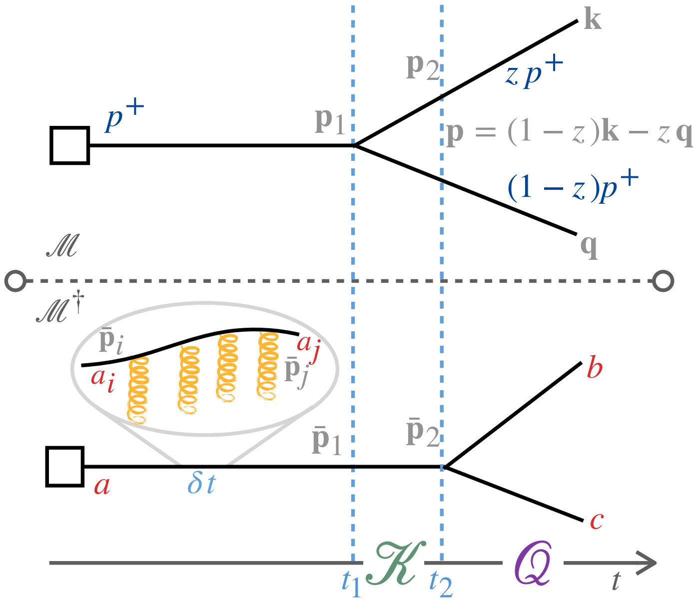
color rotations

$$\boxed{U_R = \exp\left(ig\int_{t_0}^{t} \mathrm{d}s A^c(s) t_R^c\right)}$$

medium interactions are encoded here!

In-medium spectrum

Integration of matrix element squared in the light-cone time, with **distinct time variables** for the **amplitude (t_1)** and the **conjugate amplitude (t_2).**



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$$\frac{\mathrm{d}I}{\mathrm{d}z\mathrm{d}^2\boldsymbol{p}} = \frac{\alpha_s P_{ba}(z)}{4\pi^2\omega^2}\Re e\Big[\int_{\boldsymbol{p}_1,\boldsymbol{p}_2,\bar{\boldsymbol{p}}_2,\bar{\boldsymbol{p}}}\int_0^\infty \mathrm{d}t_1 \int_{t_1}^\infty \mathrm{d}t_2\, \delta^{(2)}_{\bar{\boldsymbol{p}}+\boldsymbol{p}}$$
$$\times\,(\boldsymbol{p}_1\cdot\bar{\boldsymbol{p}}_2)\,\mathcal{K}(\boldsymbol{p_2},\boldsymbol{p_1}|t_2,t_1)\,\mathcal{Q}(\boldsymbol{p},\bar{\boldsymbol{p}},\boldsymbol{p_2},-\bar{\boldsymbol{p}}_2|t_\infty,t_2)\Big]\,,$$

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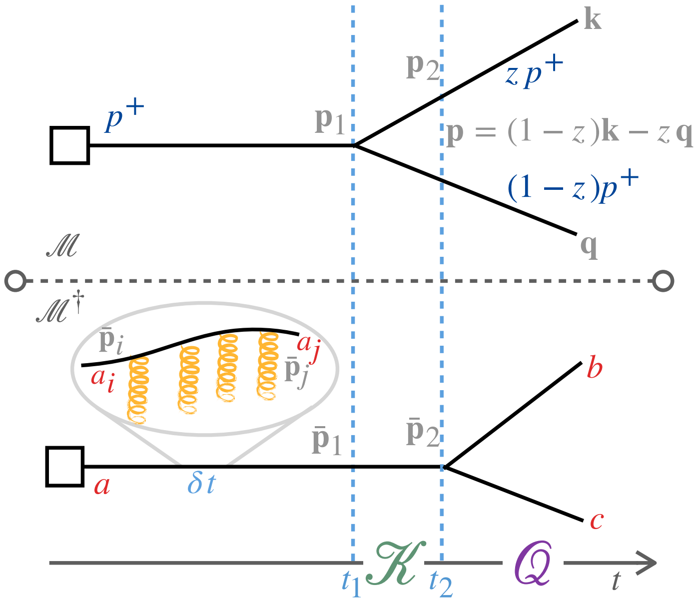

© Alba Soto-Ontoso

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Kernel (3-pt)

$$\boxed{\mathcal{K}(t_2,t_1|\boldsymbol{u}_2,\boldsymbol{u}_1) = \int\mathcal{D}\boldsymbol{u}\,e^{i\int_{t_1}^{t_2}\mathrm{d}s\frac{\omega}{2}\dot{\boldsymbol{u}}^2}\,\mathbb{C}^{(3)}(t_2,t_1|\boldsymbol{u})}$$

$$\mathbb{C}^{(3)} \sim \langle U_{R,b}U_{R,c}U^\dagger_{R,\bar{a}}\rangle$$

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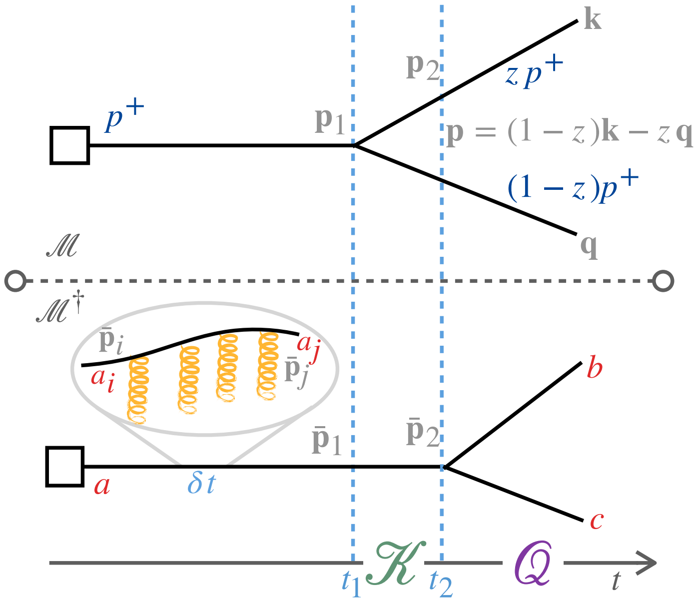
Integration of matrix element squared in the light-cone time, with **distinct time variables** for the **amplitude (t_1)** and the **conjugate amplitude (t_2).**



$$\frac{\mathrm{d}I}{\mathrm{d}z\mathrm{d}^2\boldsymbol{p}} = \frac{\alpha_s P_{ba}(z)}{4\pi^2\omega^2}\Re e\Big[\int_{\boldsymbol{p}_1,\boldsymbol{p}_2,\bar{\boldsymbol{p}}_2,\bar{\boldsymbol{p}}}\int_0^\infty \mathrm{d}t_1\int_{t_1}^\infty \mathrm{d}t_2\,\delta^{(2)}_{\bar{\boldsymbol{p}}+\boldsymbol{p}}$$

$$\times\,(\boldsymbol{p}_1\cdot\bar{\boldsymbol{p}}_2)\underbrace{\mathcal{K}(\boldsymbol{p_2},\boldsymbol{p_1}|t_2,t_1)}_{\text{Kernel (3-pt)}}\underbrace{\mathcal{Q}(\boldsymbol{p},\bar{\boldsymbol{p}},\boldsymbol{p_2},-\bar{\boldsymbol{p}}_2|t_\infty,t_2)}_{\text{Quadrupole (4-pt)}}\Big]\,,$$

$$\boxed{\mathcal{K}(t_2,t_1|\boldsymbol{u}_2,\boldsymbol{u}_1) = \int\mathcal{D}\boldsymbol{u}\,e^{i\int_{t_1}^{t_2}\mathrm{d}s\frac{\omega}{2}\dot{\boldsymbol{u}}^2}\,\mathbb{C}^{(3)}(t_2,t_1|\boldsymbol{u})}$$

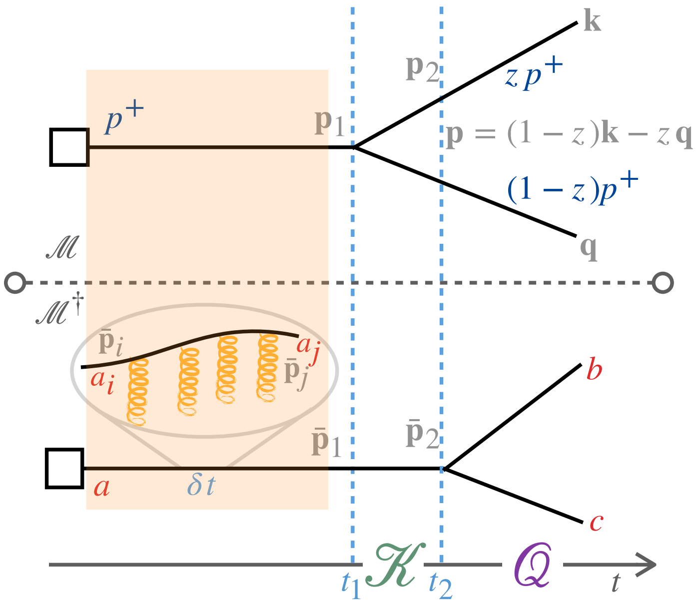
$$\boxed{\mathcal{Q}(L,t_2|\boldsymbol{u},\bar{\boldsymbol{u}},\boldsymbol{u}_2,\bar{\boldsymbol{u}}_2) = \int_{\boldsymbol{u}_2}^{\boldsymbol{u}}\mathcal{D}\boldsymbol{u}'\int_{\bar{\boldsymbol{u}}_2}^{\bar{\boldsymbol{u}}}\mathcal{D}\bar{\boldsymbol{u}}'\,e^{i\int_{t_1}^{t_2}\mathrm{d}s\frac{\omega}{2}(\dot{\boldsymbol{u}}'^2-\dot{\bar{\boldsymbol{u}}}'^2)}\,\mathbb{C}^{(4)}(L,t_2|\boldsymbol{u}',\bar{\boldsymbol{u}}')}$$

$$\mathbb{C}^{(3)} \sim \langle U_{R,b}U_{R,c}U^\dagger_{R,\bar{a}}\rangle \text{ and } \mathbb{C}^{(4)} \sim \langle U_{R,b}U_{R,c}U^\dagger_{R,\bar{c}}U^\dagger_{R,\bar{b}}\rangle$$

$$\frac{\mathrm{d}I}{\mathrm{d}z\mathrm{d}^2\boldsymbol{p}} = \frac{\alpha_s P_{ba}(z)}{4\pi^2\omega^2}\Re e\Big[\int_{\boldsymbol{p}_1,\boldsymbol{p}_2,\bar{\boldsymbol{p}}_2,\bar{\boldsymbol{p}}}\int_0^\infty \mathrm{d}t_1\int_{t_1}^\infty \mathrm{d}t_2\,\delta^{(2)}_{\bar{\boldsymbol{p}}+\boldsymbol{p}}(\boldsymbol{p}_1\cdot\bar{\boldsymbol{p}}_2)\,\mathcal{K}(\boldsymbol{p_2},\boldsymbol{p_1}|t_2,t_1)\mathcal{Q}(\boldsymbol{p},\bar{\boldsymbol{p}},\boldsymbol{p_2},-\bar{\boldsymbol{p}}_2|t_\infty,t_2)\Big]$$

out-out

Spectrum = $\boxed{L < t_1 < t_2}$



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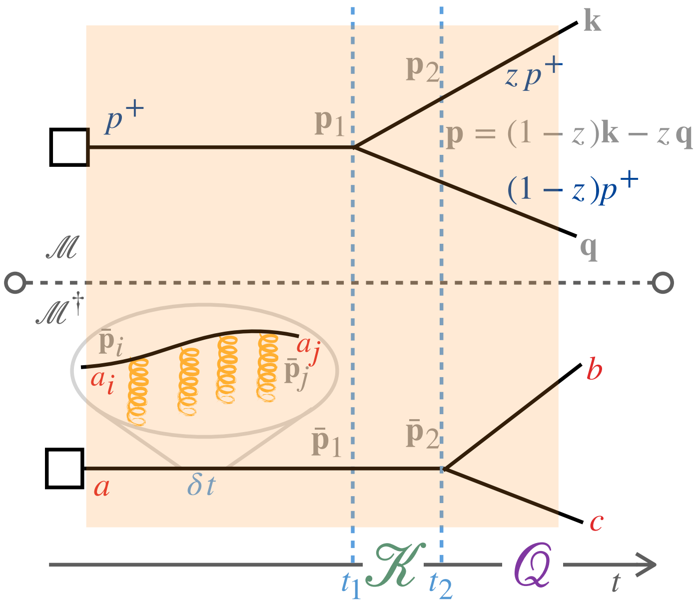
Spectrum = **out-out** $\boxed{L < t_1 < t_2}$ + **in-in** $\boxed{t_1 < t_2 < L}$



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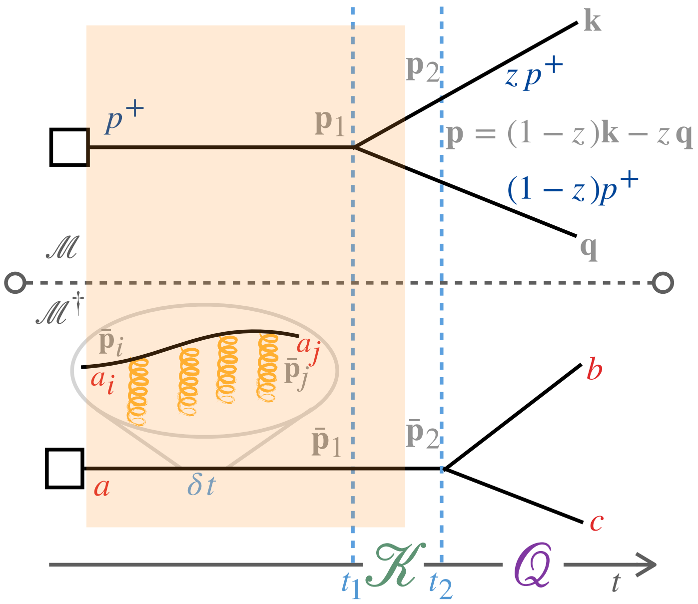

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Spectrum = **out-out** $\boxed{L < t_1 < t_2}$ + **in-in** $\boxed{t_1 < t_2 < L}$ + **in-out** $\boxed{t_1 < L < t_2}$

$$\frac{dI}{dzd^2\boldsymbol{p}} = \frac{dI^{\text{vac}}}{dzd^2\boldsymbol{p}}\underbrace{\left[\boxed{1}+\boxed{\mathcal{R}^{\text{in-in}}(z,\boldsymbol{p})}+\boxed{\mathcal{R}^{\text{in-out}}(z,\boldsymbol{p})}\right]}_{\mathcal{R}_{\text{med}}(z,\boldsymbol{p})}$$

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$$\mathcal{R}^{\text{in-in}}(z,\boldsymbol{p}) = \frac{\boldsymbol{p}^2}{\omega^2}\Re e\left[\mathcal{B}'(t=L,\boldsymbol{p},\bar{\boldsymbol{p}})\big|_{\bar{\boldsymbol{p}}=-\boldsymbol{p}}\right]$$
$$\equiv \frac{\boldsymbol{p}^2}{\omega^2}\Re e\left[\mathcal{B}(t=L,\boldsymbol{k}=\boldsymbol{p},\boldsymbol{l}=\boldsymbol{0})\right]$$

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$$\mathcal{R}^{\text{in-out}}(z,\boldsymbol{p}) = -\frac{1}{\omega}\mathrm{Re}\left[\int_0^L dt_1\int_{\boldsymbol{p}_1} i\,(\boldsymbol{p}\cdot\boldsymbol{p}_1)\,\mathcal{K}(\boldsymbol{p},\boldsymbol{p}_1|L,t_1)\right]$$
$$\equiv \frac{1}{\omega}\mathrm{Re}[\boldsymbol{p}\cdot\mathcal{A}(t=L,\boldsymbol{p})],$$

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$$\equiv \frac{1}{\omega}\mathrm{Re}[\boldsymbol{p}\cdot\mathcal{A}(t=L,\boldsymbol{p})]$$

How to compute $\mathcal{A}$ and $\mathcal{B}$?

Evolution equations

Using the previously known equations for the Kernel and the Quadrupole, one can make some manipulations to find the evolution equations for $\mathcal{A}$ and $\mathcal{B}$, which are **2+1D** and **4+1D**, respectively.

$$i\partial_t \mathcal{A}(t, \boldsymbol{p}') = \left[\frac{\boldsymbol{p}'^2}{2\omega} + i(\boxed{\tilde{v}_{ba}} * \cdot)\right]\mathcal{A}(t, \boldsymbol{p}') - \boldsymbol{p}'$$

Dipole scattering potential

$$\mathcal{R}^{\text{in-out}}(z, \boldsymbol{p}) = \frac{1}{\omega}\text{Re}[\boldsymbol{p}\cdot\mathcal{A}(t = L, \boldsymbol{p})]\,,$$

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Using the previously known equations for the Kernel and the Quadrupole, one can make some manipulations to find the evolution equations for $\mathcal{A}$ and $\mathcal{B}$, which are **2+1D** and **4+1D**, respectively.

$$i\partial_t \mathcal{A}(t,\boldsymbol{p}') = \left[\frac{\boldsymbol{p}'^2}{2\omega} + i\left(\underbrace{\tilde{v}_{ba}}_{\text{Dipole scattering potential}} * \cdot\right)\right]\mathcal{A}(t,\boldsymbol{p}') - \boldsymbol{p}'$$

$$\mathcal{R}^{\text{in-out}}(z,\boldsymbol{p}) = \frac{1}{\omega}\text{Re}[\boldsymbol{p}\cdot\mathcal{A}(t=L,\boldsymbol{p})]\,.$$

Dipole scattering potential

$$i\partial_t \mathcal{B}_\sigma(t,\boldsymbol{k}',\boldsymbol{l}') = \left[\frac{2\boldsymbol{k}'\cdot\boldsymbol{l}'}{\omega}\delta_\sigma^{\sigma'} + i\left(\tilde{\mathbb{M}}_\sigma^{\sigma'} * \cdot\right)\right]\mathcal{B}_{\sigma'}(t,\boldsymbol{k}',\boldsymbol{l}') + (\boldsymbol{k}'-\boldsymbol{l}')\cdot\mathcal{A}(\boldsymbol{k}'+\boldsymbol{l}')\,\xi_\sigma$$

Color-mixing matrix
2x2 for ***g → qq***,
3x3 for ***q → gq***,
6x6 for ***g → gg***

$$\mathcal{R}^{\text{in-in}}(z,\boldsymbol{p}) \equiv \frac{\boldsymbol{p}^2}{\omega^2}\,\text{Re}\,[\mathcal{B}(t=L,\boldsymbol{p},\boldsymbol{l}=0)]\,.$$

(σ = 1 corresponds to the configuration entering in-in)

Isotropic media

Isotropy of medium allows for dimensionality reduction: (2+1)D to **(1+1)D** in equation for $\mathcal{A}$ and (4+1)D to **(3+1)D** in equation for $\mathcal{B}$.

Adopting polar coordinates: $\boldsymbol{p} = (p, \theta)$, $\boldsymbol{k} = (k, \phi)$, $\boldsymbol{l} = (l, \eta)$, $\psi = \phi - \eta$

The reduced equations are:

$$i\partial_t a(t,\, p) = \frac{p^2}{2\omega} a(t,\, p) - i[\tilde{v}_{ba} *_{\rm pol} a](p) - p,$$

$$\mathcal{A}(t, \boldsymbol{p}) = a(p)\hat{\boldsymbol{p}}$$

$$i\partial_t \mathcal{B}_\sigma(t,\, k, l, \psi) = \left[\frac{2kl\cos\psi}{\omega}\, \delta_\sigma^{\sigma'} + i\left(\tilde{\mathbb{M}}_\sigma^{\sigma'} *_{\rm pol} \cdot\right)\right] \mathcal{B}_{\sigma'}(t, k, l, \psi) + S(t, k, l, \psi)\, \xi_\sigma$$

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$$\boxed{S(k, l, \psi \equiv \phi - \eta) = \frac{k^2 - l^2}{\sqrt{k^2 + l^2 + 2kl\cos\psi}} a(\sqrt{k^2 + l^2 + 2kl\cos\psi}),}$$

Numerical time evolution

Borisov et.al. JCP 216.1 (2006), pp. 391–402

Numerical validations in slides 20-22

$$i\partial_t \boldsymbol{f} = \hat{\mathcal{H}}\boldsymbol{f} + \boldsymbol{s},$$

Recursive solution:

$$\boldsymbol{f}(t+\delta t) = e^{-i\hat{\mathcal{H}}\delta t}\boldsymbol{f}(t) - i\int_t^{t+\delta t} dt'\, e^{-i\hat{\mathcal{H}}(t'-t)}\,\boldsymbol{s}(t'),$$

Need to compute terms of the form $e^{-i\mathcal{H}\delta\tilde{t}}\boldsymbol{f}$

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Faber polynomial expansion

$$e^{-i\mathcal{H}\delta\tilde{t}}\boldsymbol{f} = \sum_{n=0}^{\infty} \boxed{c_n(\delta\tau)}\,\boxed{\Phi_n(\mathcal{H}_{\text{sc}})\boldsymbol{f}},$$

Recursion relations of Faber Polynomials

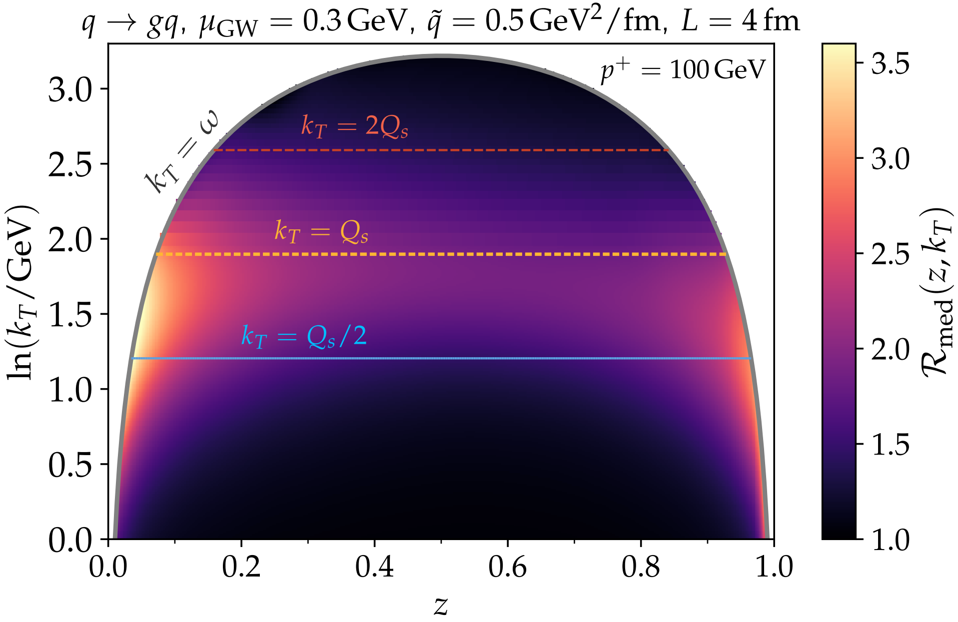
$$\mathbf{f}_1 = (\mathbb{H}_{\text{sc}} - \gamma_0)\,\mathbf{f}_0$$
$$\mathbf{f}_2 = (\mathbb{H}_{\text{sc}} - \gamma_0)\,\mathbf{f}_1 - 2\gamma_1\mathbf{f}_0$$
$$\mathbf{f}_n = (\mathbb{H}_{\text{sc}} - \gamma_0)\,\mathbf{f}_{n-1} - \gamma_1\mathbf{f}_{n-2}, \quad n \geq 3.$$

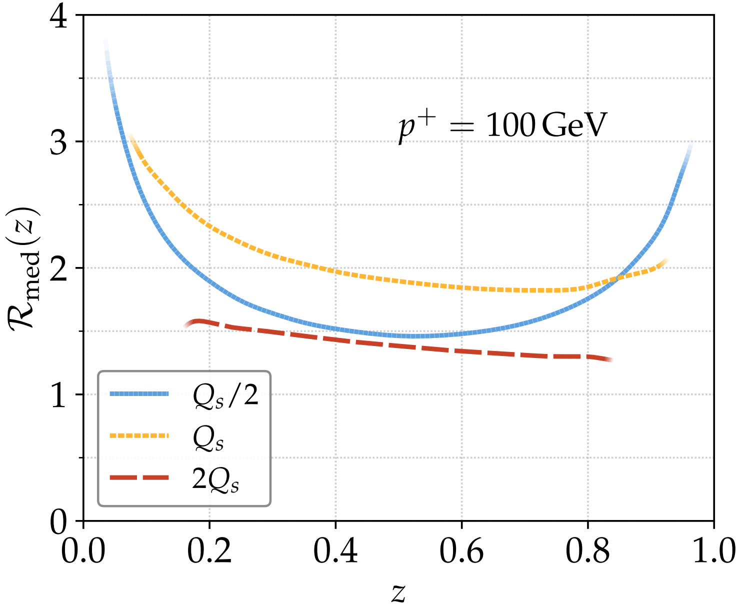
Spectral properties of *H*

★ Provides a very stable time evolution!

Lund-style plane of R_{med} in $q \to gq$

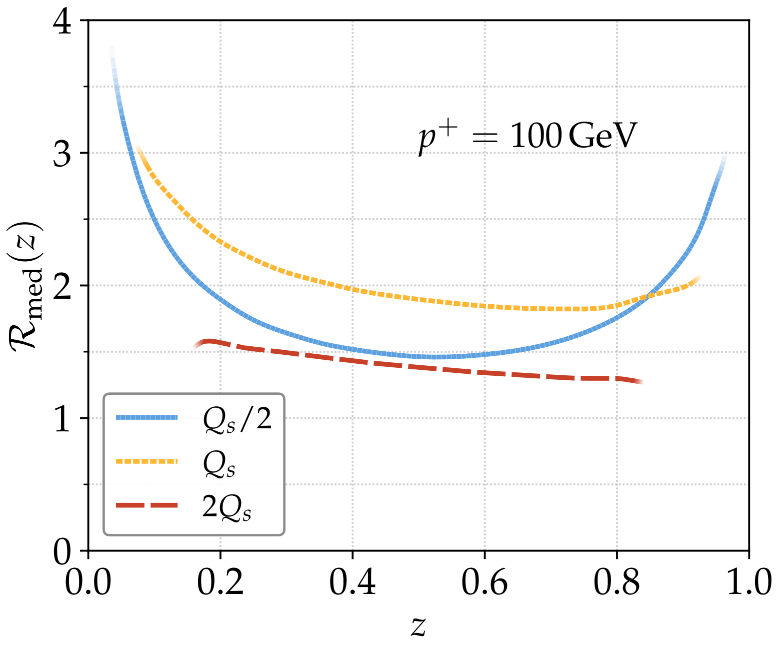
$$\tilde{q} = 4\pi\,\alpha_s^2\,n$$

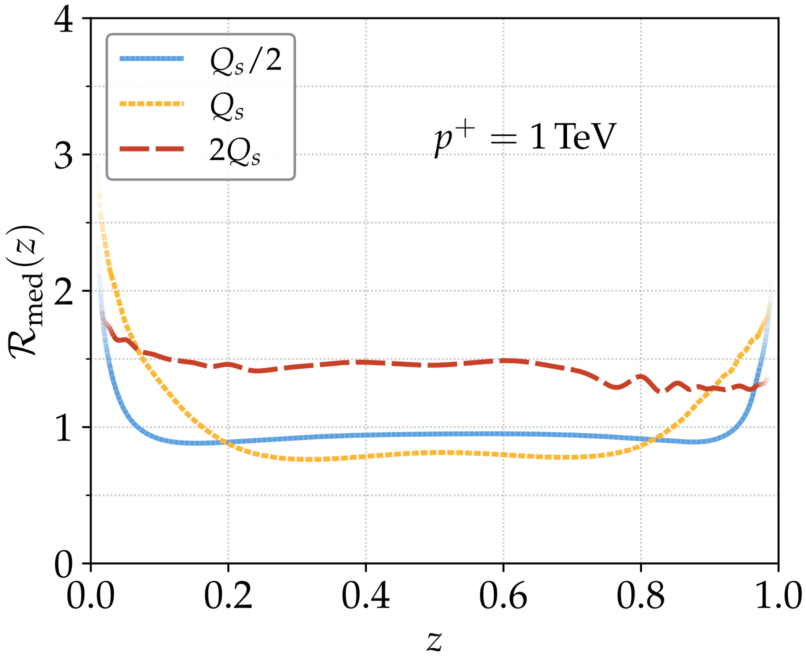




Higher modifications in **soft (hard) limits**, with **R ~3.5 (3)**. **Significant** modifications in the **bulk (R ~ 2)**. **Vacuum-like** emissions for ***k*T << *Q*s** and ***k*T >> *Q*s.**

100 GeV vs 1 TeV



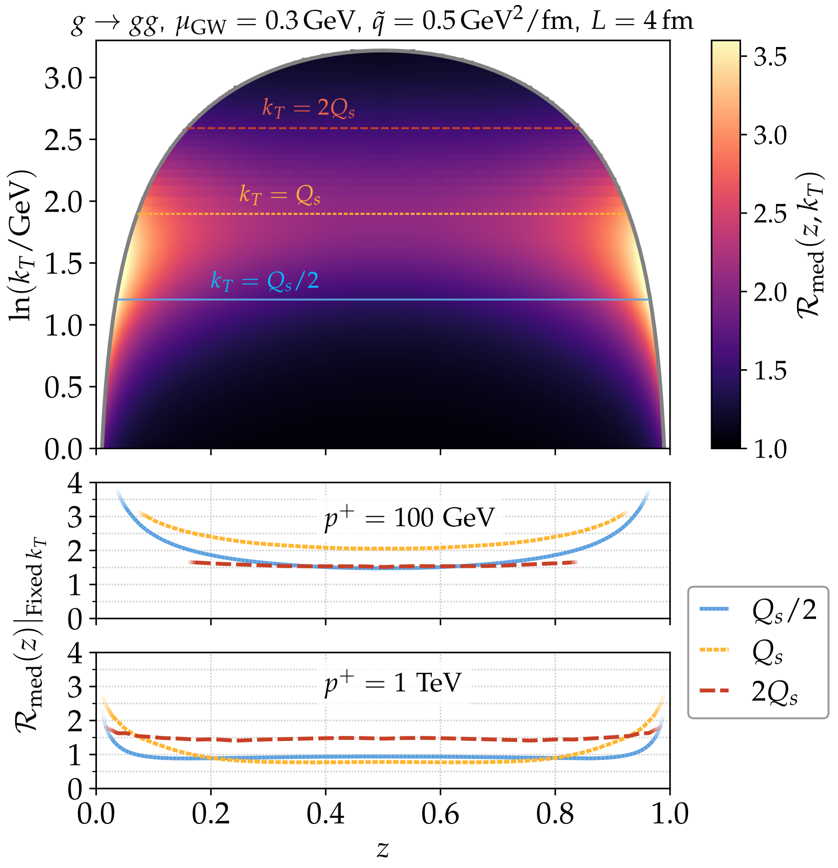


Modifications tend to **decrease** with higher energies, and so emissions are more **vacuum-like**

Other vertices

$g \to gg$

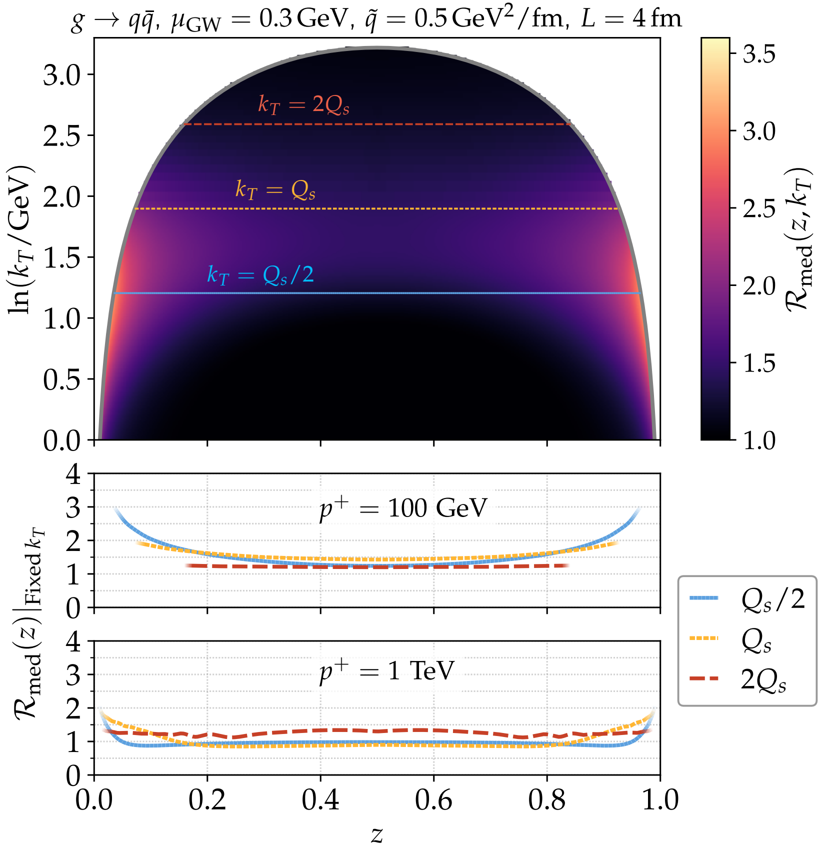
- Results are qualitatively similar to the $q \to gq$ vertex
- Both agree on the **soft limit** (as analytically expected)
- Differ in the $z \leftrightarrow 1\text{-}z$ (a)symmetry



Other vertices

$g \to q\bar{q}$

- Qualitatively similar to the other vertices
- Smaller modifications overall
- Admits the $z \leftrightarrow 1\text{-}z$ symmetry



Finite N_c corrections

Large N_c corresponds to taking $C_F \longrightarrow C_A/2$ and neglecting $O\left(1/N_c\right)$ terms in color-mixing matrix, reducing its complexity: 3x3 for $q \rightarrow gq$ and 6x6 for $g \rightarrow gg$ reduce both to **2x2**! **How relevant are finite N_c corrections?**

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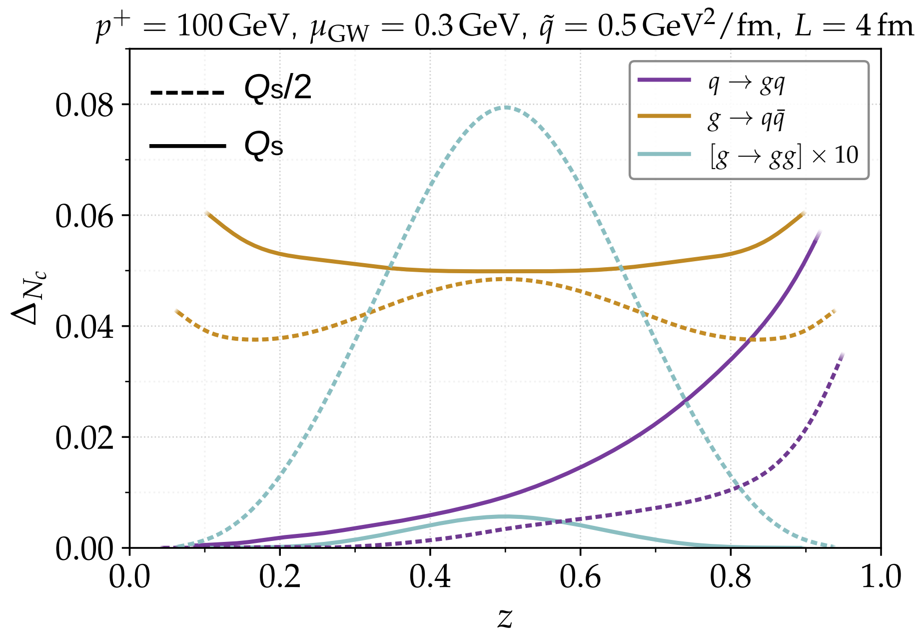


From $\mathbf{0\%}$ to $\mathbf{6\%}$ in $\boldsymbol{q \longrightarrow gq}$

$z \longrightarrow 0$ is dominated by C_A and $z \longrightarrow 1$ by C_F

~$\mathbf{5\%}$ in $\boldsymbol{g \longrightarrow q\bar{q}}$

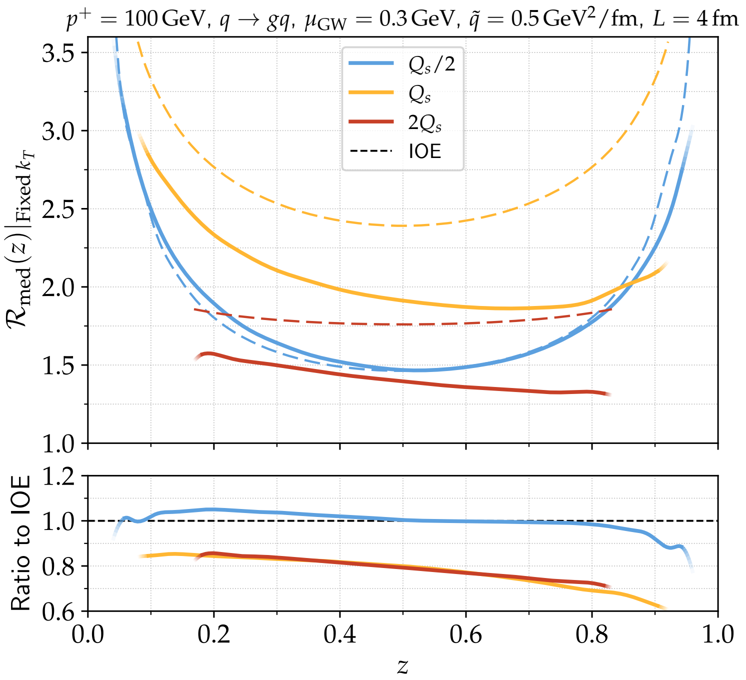
Vertex always dominated by C_F

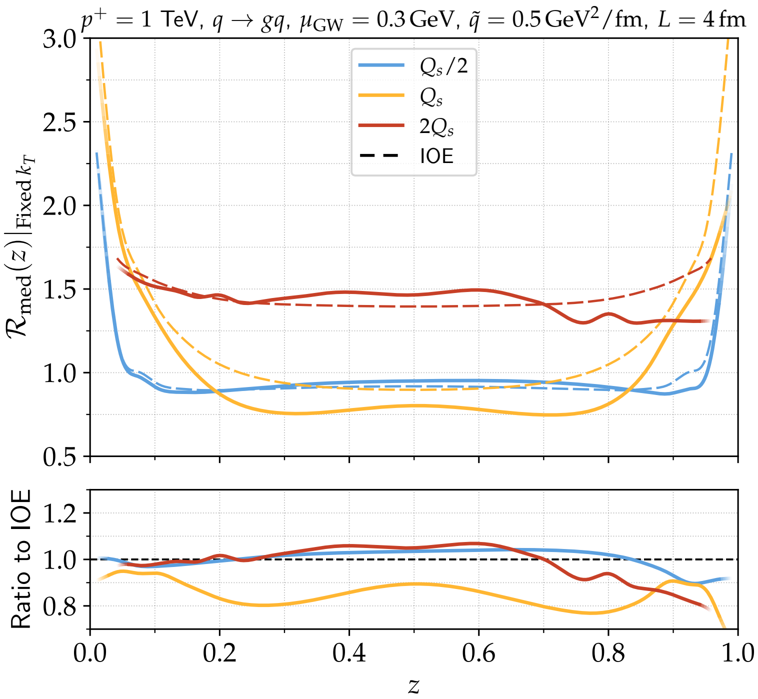
<$\mathbf{1\ \%}$ in $\boldsymbol{g \longrightarrow gg}$**!**

Vertex always dominated by C_A

Improved Opacity Expansion (IOE)

Barata et. al., JHEP09(2021)153 (IOE)





Significant **deviations** in the **bulk** of the phase space, good **agreement** in the **soft limit**

Numerical validation

We test the numerical procedure in the **Harmonic-Oscillator (HO)** potential, which admits semi-analytical solutions for *large-N_c*.

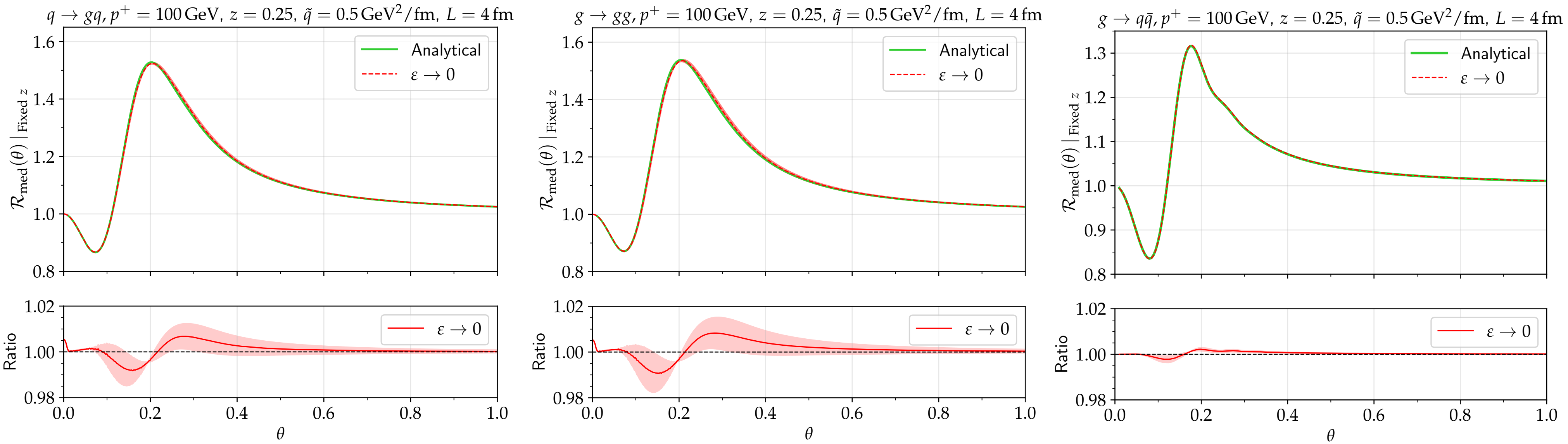
Caveat: algorithm designed for continuous potential; HO in momentum space corresponds to taking Laplacians.

Workaround: Construct continuous surrogate using the identity

$$\nabla_{\boldsymbol{p}}^2 f(\boldsymbol{p}) = \lim_{\varepsilon \to 0^+} \int \frac{d^2\boldsymbol{p}'}{2\pi\varepsilon^4} \left(\frac{\boldsymbol{p}'^2}{2\varepsilon^2} - 1 \right) e^{-\frac{\boldsymbol{p}'^2}{2\varepsilon^2}} f(\boldsymbol{p} - \boldsymbol{p}')$$

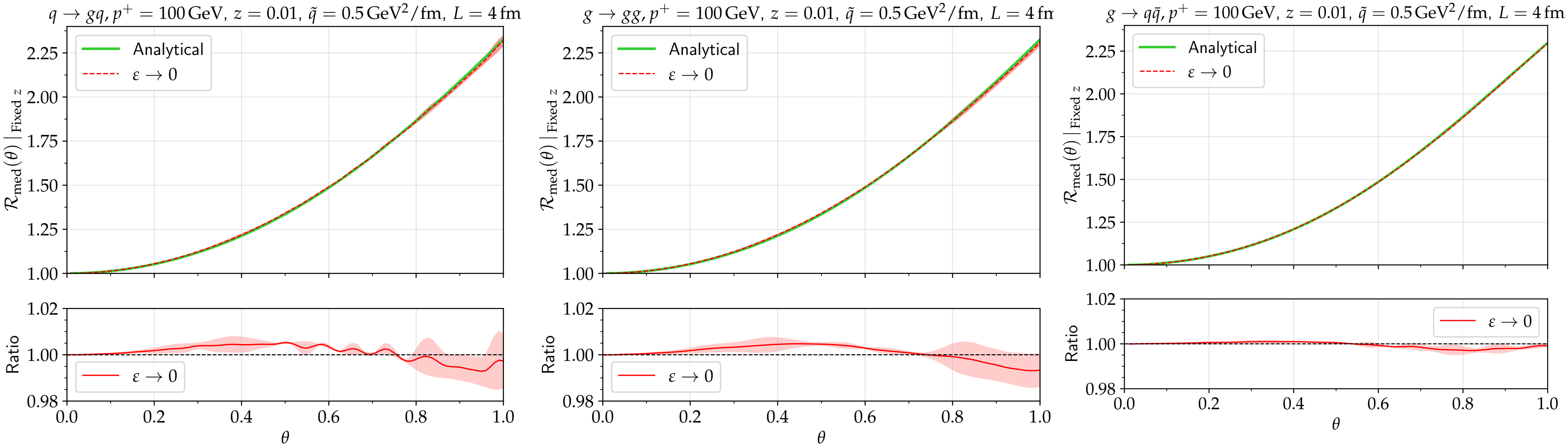
and extrapolate $\varepsilon \longrightarrow 0$.

z = 0.25



Agreement with semi-analytical result within 1-2% accuracy.

BDMPS-Z limit ($z \to 0$)



Agreement with BDMPS-Z within 1-2% accuracy.

Conclusions

- We have **computed for the first time** the fully-differential emission probability in the BDMPS-Z formalism in its **full generality**, for all **QCD** $1 \rightarrow 2$ processes
- We developed a **stable numerical procedure** that is capable of computing the splittings for **arbitrary potentials** and the entirety of the relevant phase space.
- The N_c corrections have been quantified, finding **small deviations overall**
- These results provide **powerful new constraints** on medium parameters, and serve as a solid **baseline** for **analytical calculations** of observables and **Monte Carlos**.
- **Future work** may include: calculation of observables, interface to Monte Carlo parton showers, extension to evolving media, massive quarks.

Thank you!

Acknowledgements

Equations in large-Nc limit

- $g \to q\bar{q}$:

$$\frac{d\mathcal{C}_1^{gq\bar{q}}}{dt} \overset{N_c \gg 1}{\simeq} \frac{N_c}{2} n(t)\, \Sigma_\pm \mathcal{C}_1^{gq\bar{q}} \,.$$

- $q \to gq$:

$$\frac{d\mathcal{C}_1^{qgq}}{dt} \overset{N_c \gg 1}{\simeq} \frac{N_c\, n(t)}{2} \left[(\Sigma_+ + 2\,\Sigma_-)\, \mathcal{C}_1^{qgq} + (\Sigma_0 - \Sigma_{zs})\, \mathcal{C}_2^{qgq} \right] ,$$

$$\frac{d\mathcal{C}_2^{qgq}}{dt} \overset{N_c \gg 1}{\simeq} \frac{N_c\, n(t)}{2} (\Sigma_0 + \Sigma_-) \mathcal{C}_2^{qgq} \,.$$

- $g \to gg$:

$$\frac{d\mathcal{C}_1^{ggg}}{dt} \overset{N_c \gg 1}{\simeq} \frac{N_c\, n(t)}{2} \left[2\Sigma_\pm \mathcal{C}_1^{ggg} + (\Sigma_0 - \Sigma_{zs})\, \mathcal{C}_2^{ggg} \right] ,$$

$$\frac{d\mathcal{C}_2^{ggg}}{dt} \overset{N_c \gg 1}{\simeq} \frac{N_c\, n(t)}{2} (\Sigma_\pm + \Sigma_0)\, \mathcal{C}_2^{ggg} .$$

Equations in soft-limit

$$\frac{d\mathcal{C}_1^{\gamma q\bar{q}}}{dt} \overset{z\to 0}{\simeq} C_F\, n(t)\, \Sigma_d\, \mathcal{C}_1^{\gamma q\bar{q}}\,,$$

$$\frac{d\mathcal{C}_1^{gq\bar{q}}}{dt} \overset{z\to 0}{\simeq} C_F\, n(t)\, \Sigma_d\, \mathcal{C}_1^{gq\bar{q}}\,,$$

$$\frac{d\mathcal{C}_1^{qgq}}{dt} \overset{z\to 0}{\simeq} N_c\, n(t)\, \Sigma_d\, \mathcal{C}_1^{qgq}\,,$$

$$\frac{d\mathcal{C}_1^{ggg}}{dt} \overset{z\to 0}{\simeq} N_c\, n(t)\, \Sigma_d\, \mathcal{C}_1^{ggg}\,.$$

Faber expansion test