

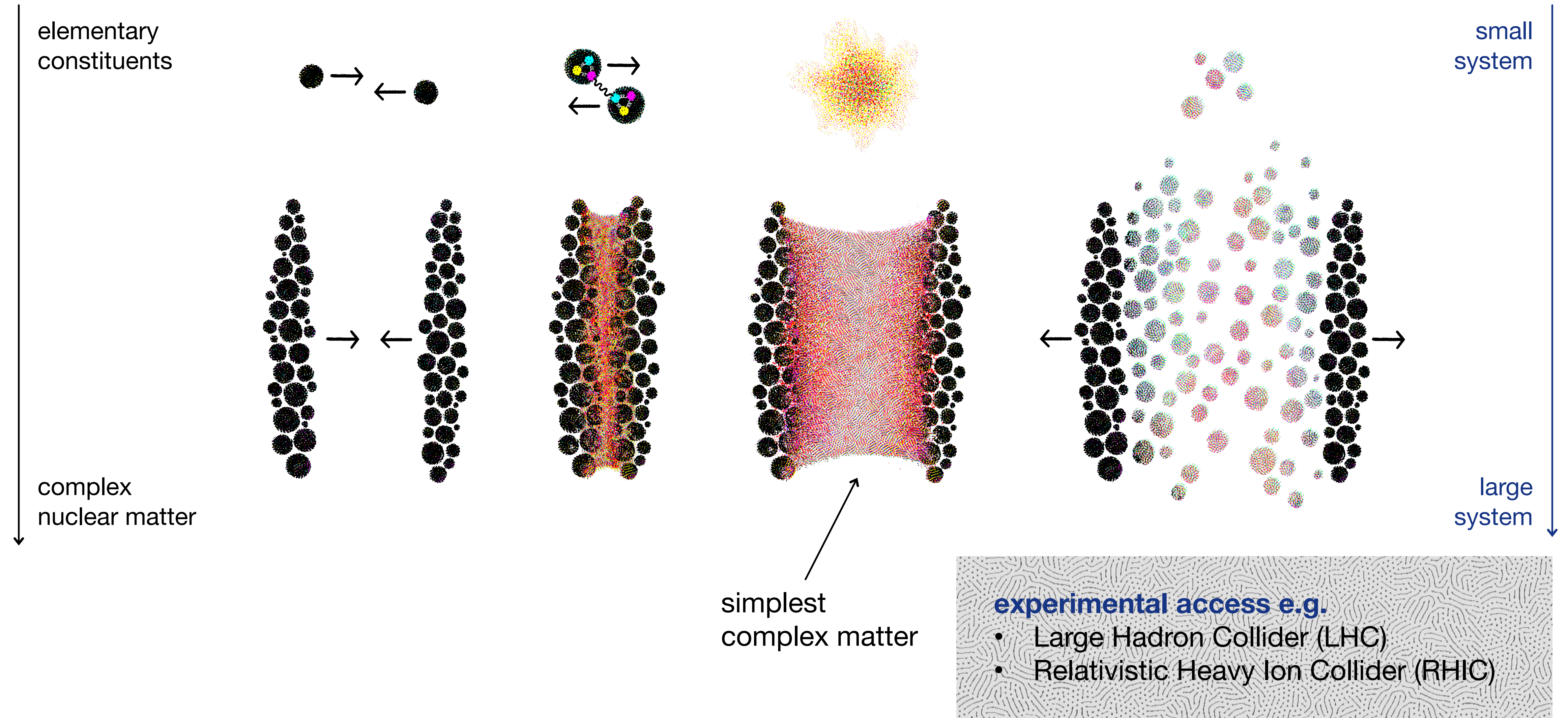


Jet Quenching in Flowing QCD Matter: Masses and Gradient Effects

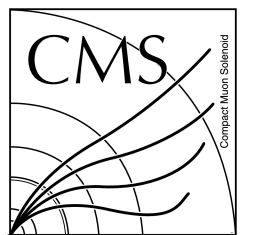
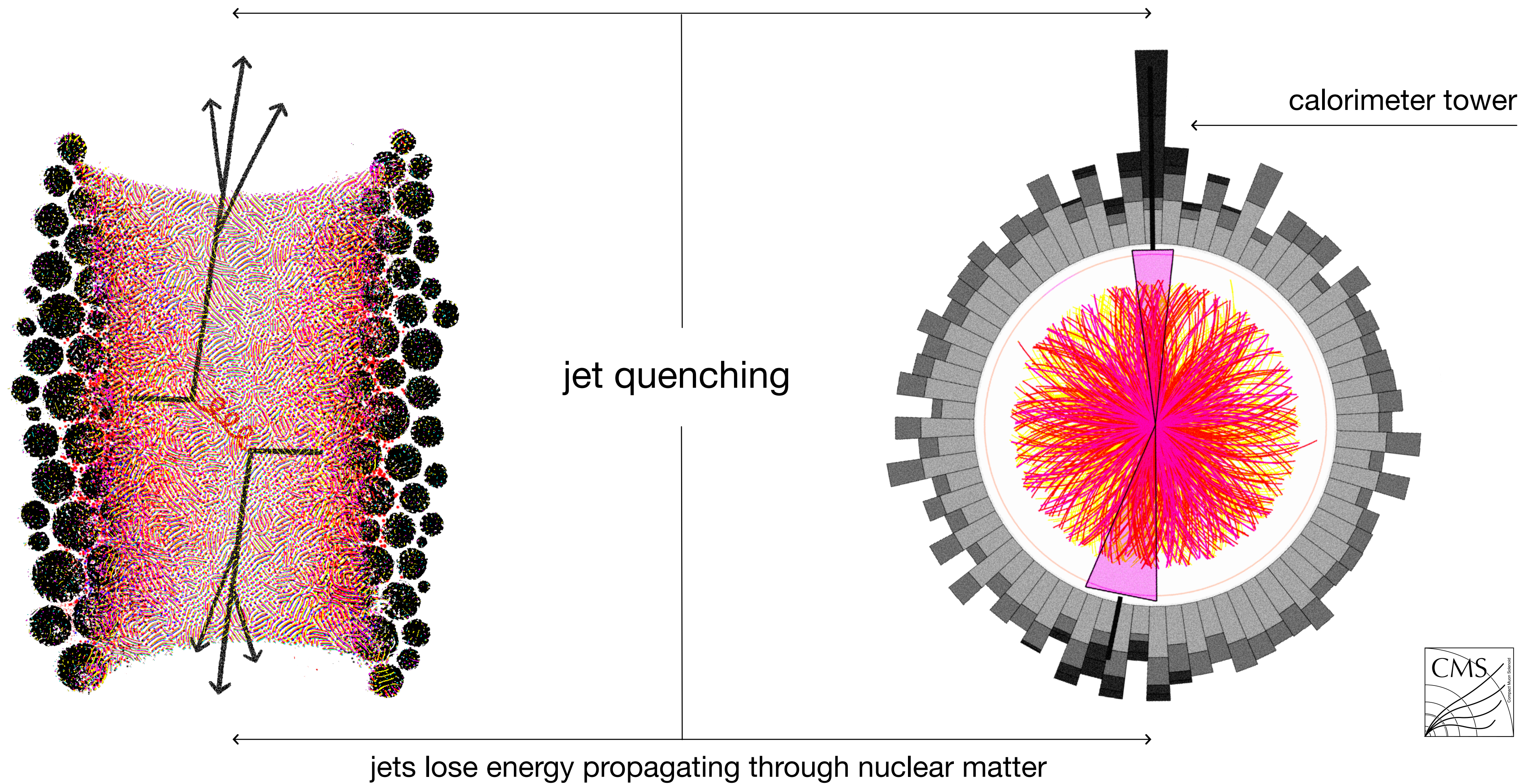
Matvei Kuzmin

University of Oxford

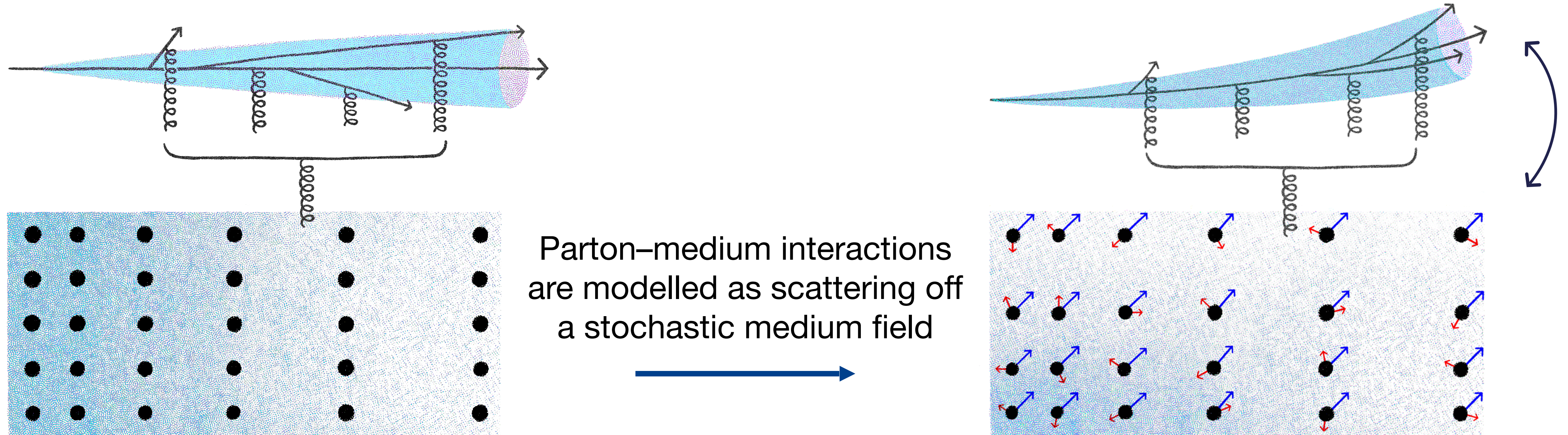
The origin of complex matter



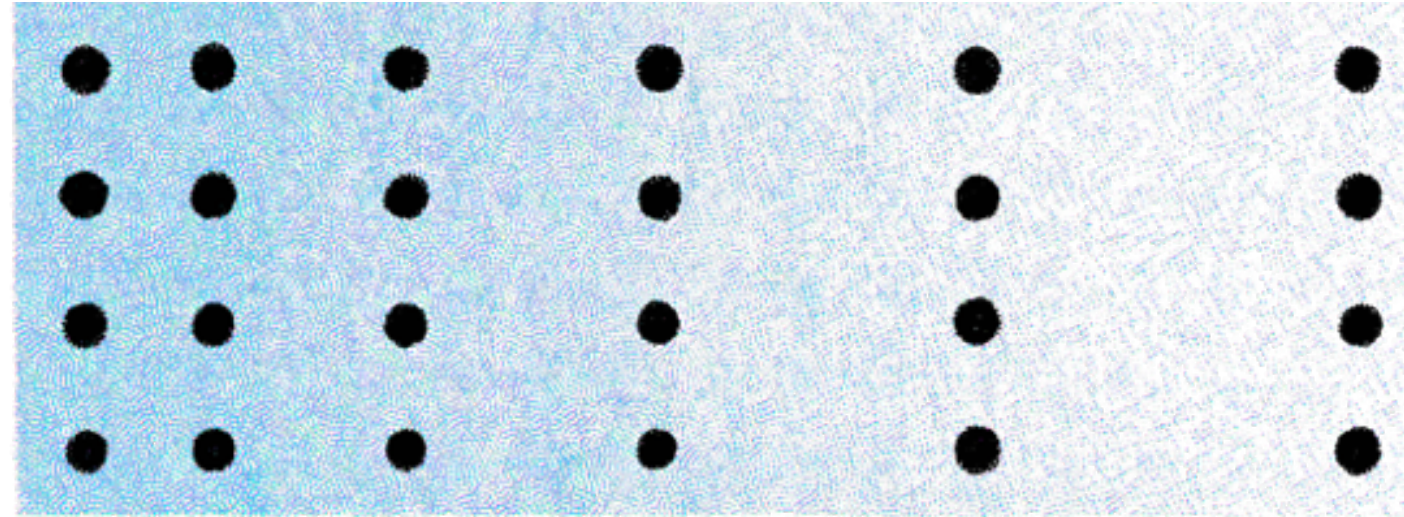
A tool to probe the QGP evolution



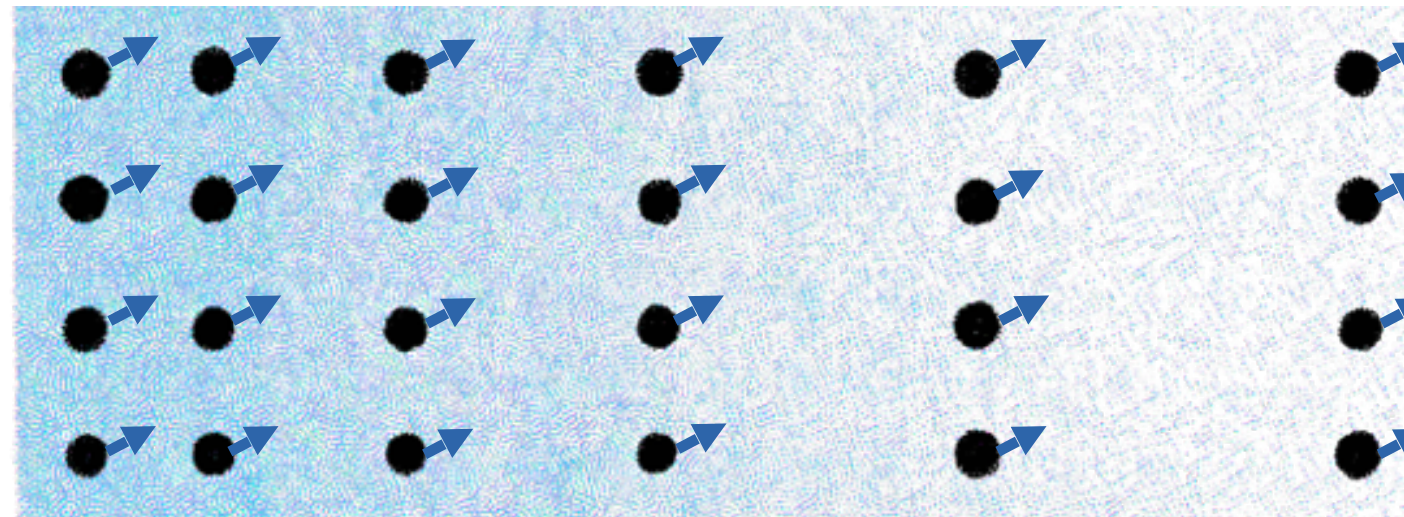
Jet Interaction with a Flowing Medium



Medium Model



QGP brick



Flowing medium

$$u_\mu = (1, \mathbf{u}, u_z)$$

$$t_i^a$$

Colour charge of
an individual source.
Stochastic variable

Model dependent
medium potential

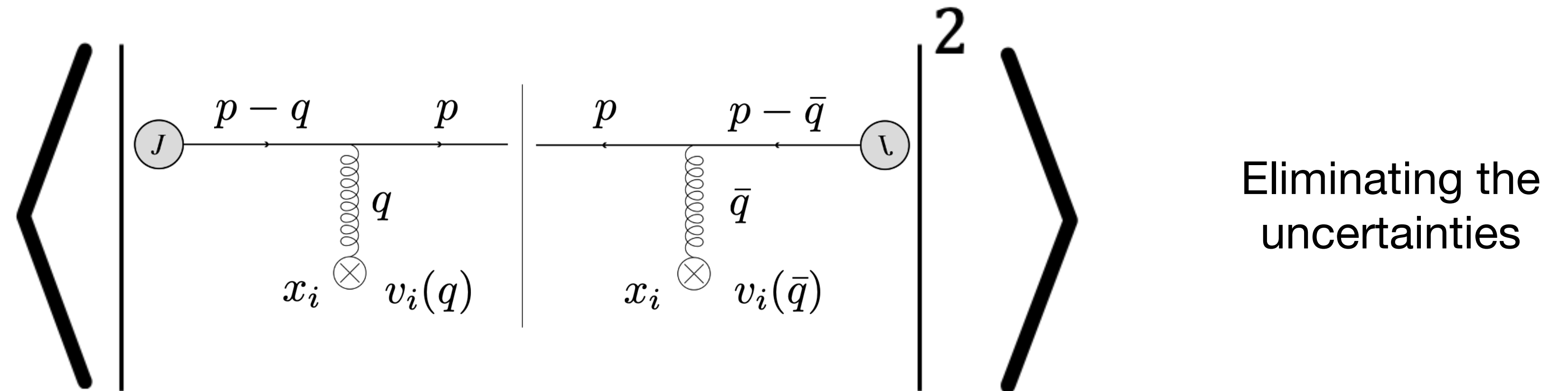
$$gA^{a\mu}(q) = \sum_i g^{\mu 0} e^{-iq \cdot x_i} t_i^a v_i(q) (2\pi) \delta(q_0)$$

$$gA^{a\mu}(q) = \sum_i u_i^\mu e^{-iq \cdot x_i} t_i^a v_i(q) (2\pi) \delta(q_0 - \mathbf{q} \cdot \mathbf{u} - q_z u_z)$$

$$v_i(q) = \frac{g^2}{q^2 - \mu_i^2}$$

GW potential
with μ – Debye
screening

Medium Averaging



Colour averaging

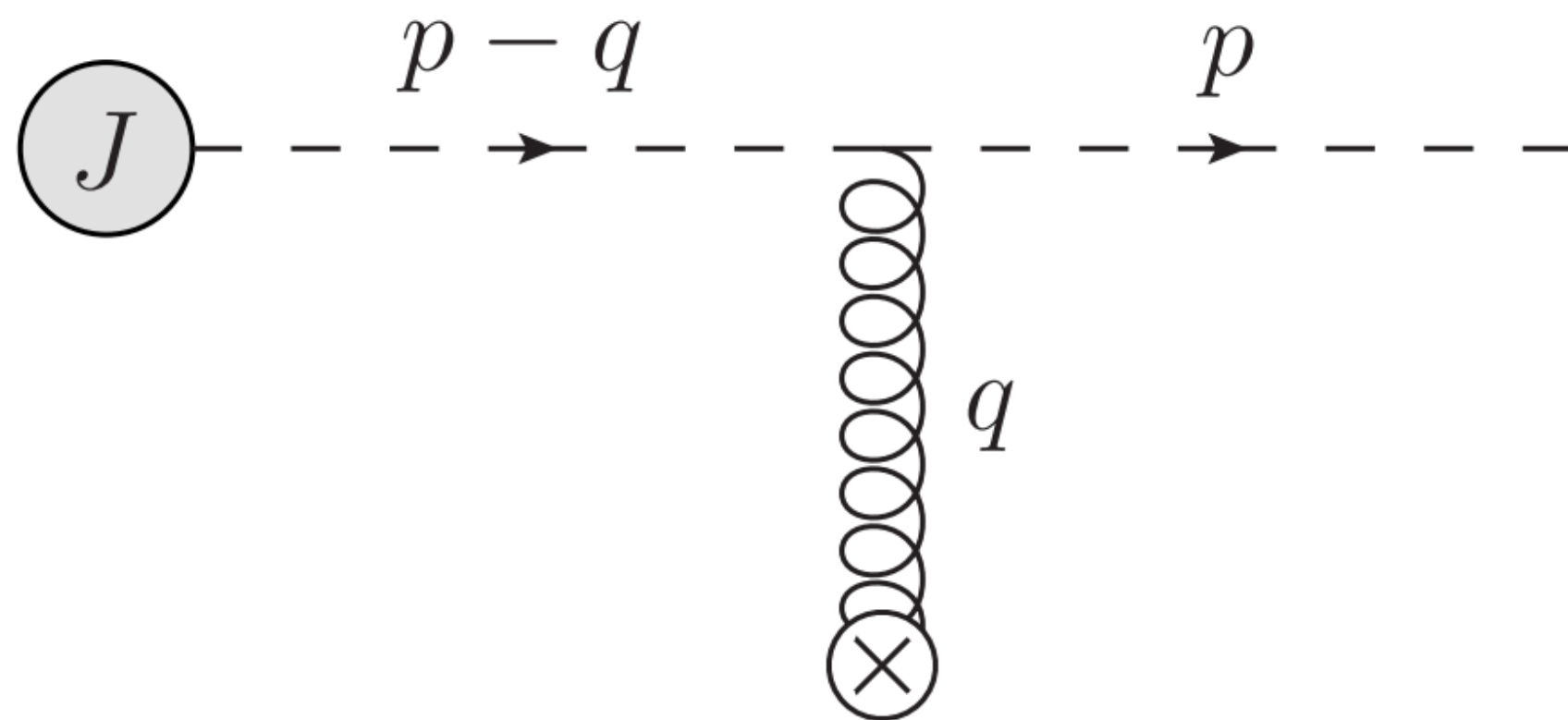
Sum over the
colour sources

$$\langle A^a(q) A^b(\bar{q}) \rangle \sim \langle t_i^a t_j^b \rangle = \mathcal{C} \delta_{ij} \delta^{ab}$$

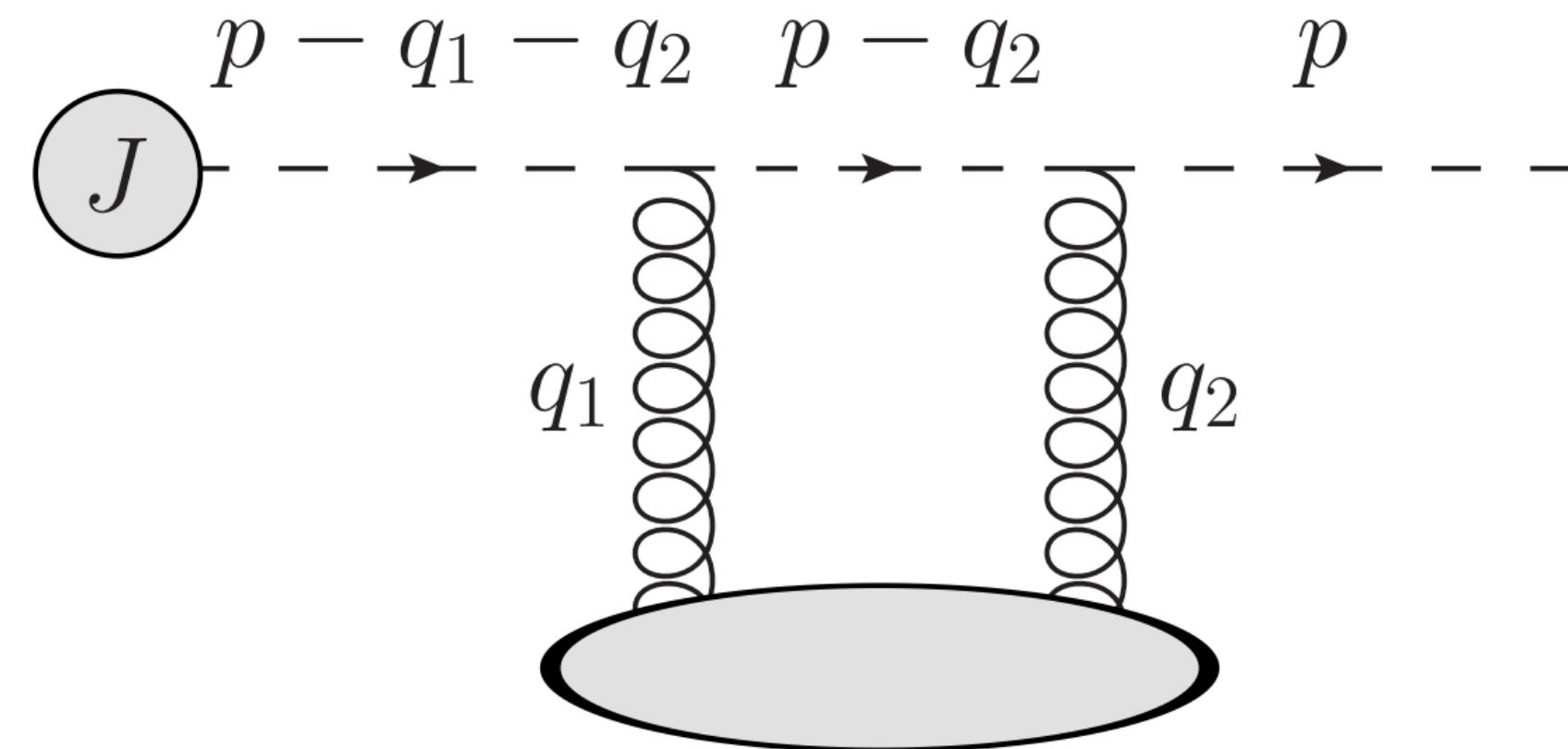
$$\sum_i = \int d^2 \mathbf{x} dz \rho(\mathbf{x}, z)$$

number density
of in-medium sources
x-dependent

GLV Formalism: Opacity Expansion Approach



Single Born Diagram

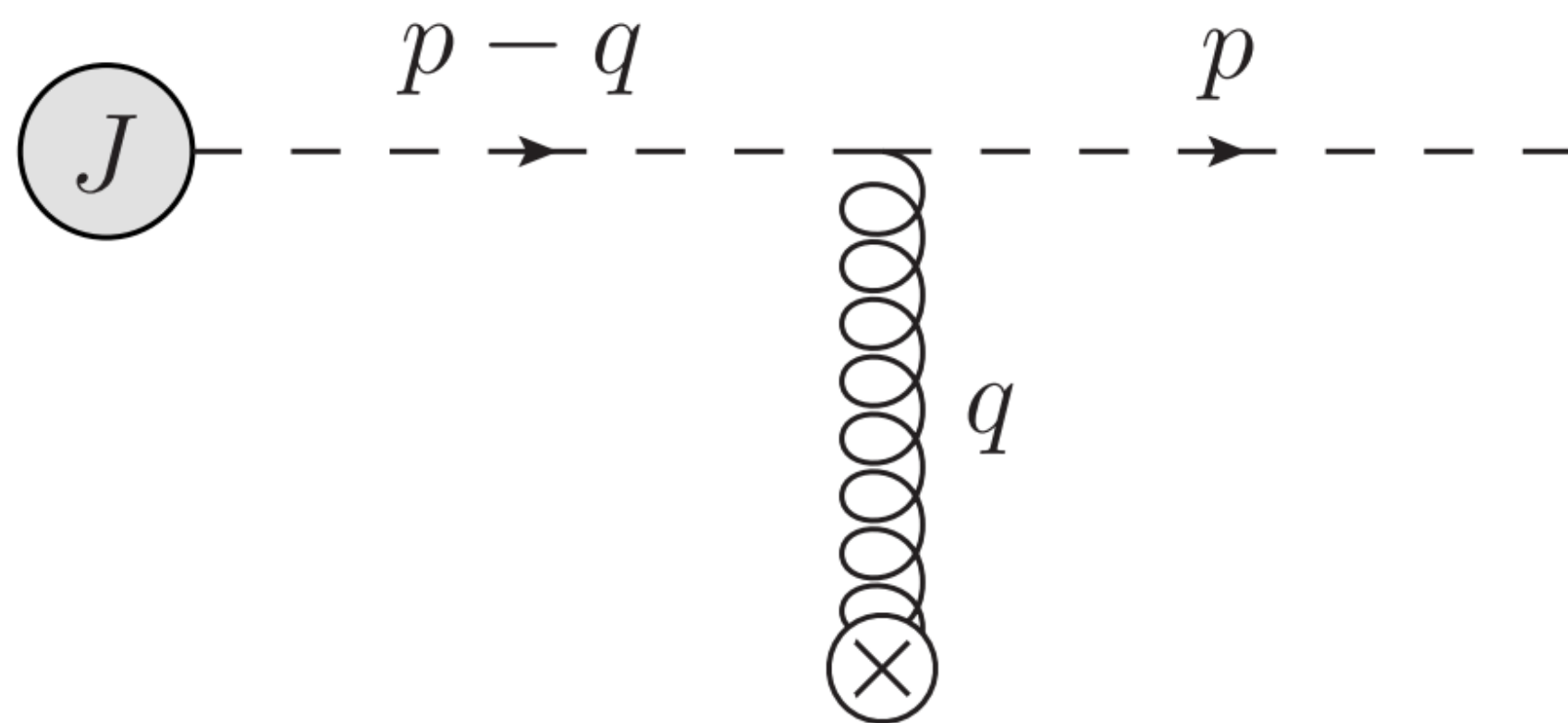


Double Born Diagram

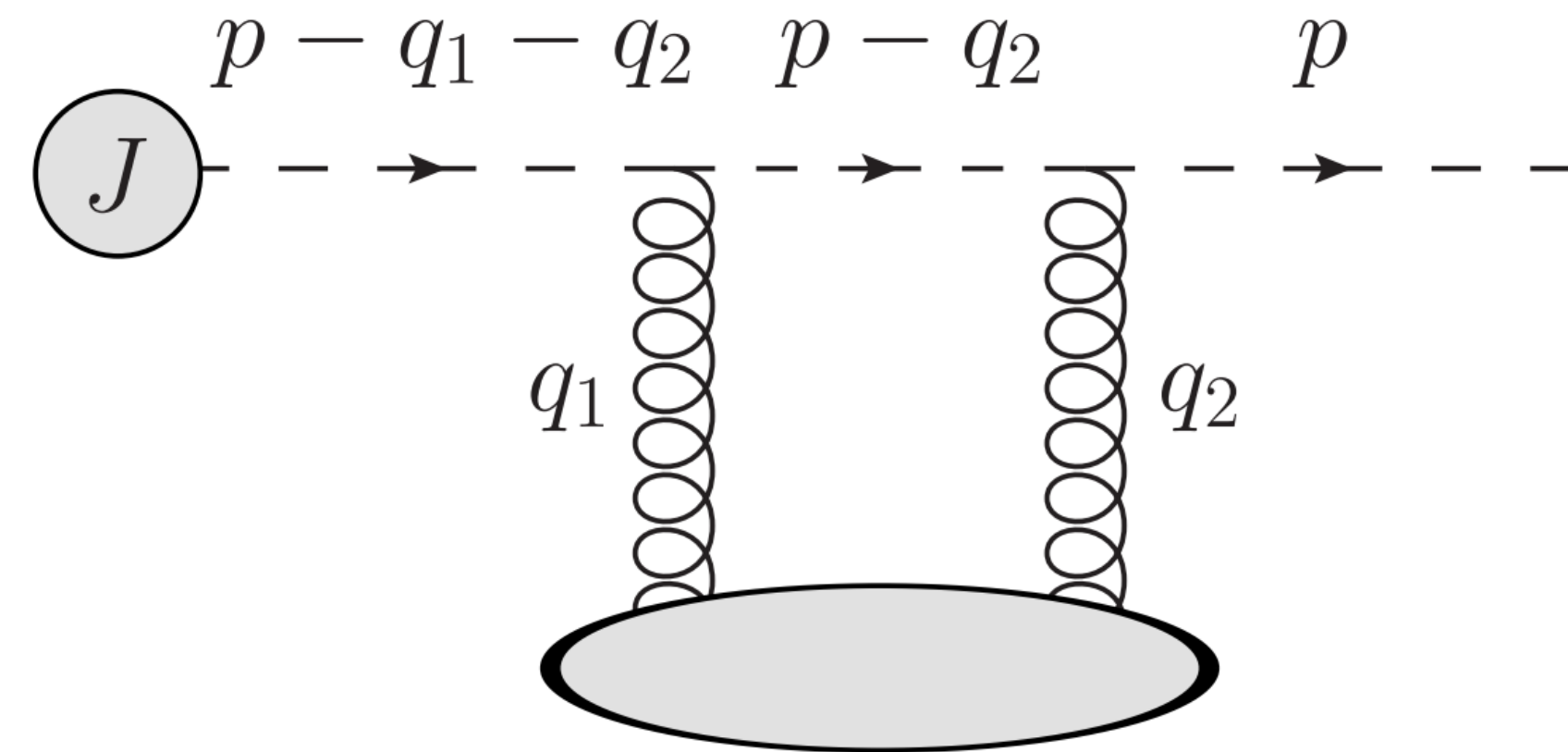
Parameter of the
perturbative expansion:

$$\chi \equiv 4\pi \mathcal{C} \alpha_s^2 \frac{\rho L}{\mu^2} \ll 1$$

Jet Broadening in a Flowing-Anisotropic Medium



Single Born Diagram



Double Born Diagram

Explicit example of
the amplitude

$$iM_1(p) = \int \frac{d^4 q}{(2\pi)^4} \left[ig t_{\text{proj}}^a \underbrace{A_{\text{ext}}^{\mu a}(q)}_{\text{Medium field}} (2p - q)_\mu \right] \left[\frac{i}{(p - q)^2 + i\epsilon} \right] \underbrace{J(p - q)}_{\text{Encodes initial parton distribution}}$$

Jet Broadening in a Flowing-Anisotropic Medium

$$\frac{(2p - q)_\mu A_{ext}^\mu(q)}{(p - q)^2 + i\epsilon} \xrightarrow{\text{Simplifying}} \frac{2u_\mu p^\mu}{(1 - u_z^2)(Q^+ - Q^-)} = 1 - \frac{\mathbf{u} \cdot (\mathbf{p} - \mathbf{q})}{E(1 - u_z)} + \mathcal{O}\left(\frac{p^2}{E^2}\right)$$

$$v(q) = \frac{g^2}{q^2 - \mu^2}$$

$$\mu z \gg 1$$

GW potential poles
can be neglected

$$\text{Integration poles in momentum space} \begin{cases} Q_{p-q}^+ = \frac{2E}{1 + u_z} \left[1 - \frac{\mathbf{u} \cdot \mathbf{q}}{2E} + \mathcal{O}\left(\frac{p^2}{E^2}\right) \right] \\ Q_{p-q}^- = \frac{\mathbf{u} \cdot \mathbf{q}}{1 - u_z} + \frac{(\mathbf{p} - \mathbf{q})^2 - p^2}{2E(1 - u_z)} + \mathcal{O}\left(\frac{p^2}{E^2}\right) \end{cases}$$

Eikonal expansion
facilitates analytic
simplification

$$\frac{\perp}{E} \ll 1$$

Jet Broadening: Final Distribution

$$E \frac{d\mathcal{N}}{d^2\mathbf{p} dE} = \underbrace{E \frac{d\mathcal{N}^{(0)}}{d^2\mathbf{p} dE}}_{\text{Encodes initial parton distribution}} + \mathcal{C} \int_0^L dz \int \frac{d^2\mathbf{q}}{(2\pi)^2} \left(1 - \underbrace{\hat{\mathbf{g}}_\alpha \frac{u_\alpha z}{1 - u_z}}_{\text{Leading flow-gradients corrections}} \right) \left[\underbrace{E \frac{d\mathcal{N}^{(0)}}{d^2(\mathbf{p} - \mathbf{q}) dE}}_{\text{Gradients of hydrodynamic variables}} - \underbrace{E \frac{d\mathcal{N}^{(0)}}{d^2\mathbf{p} dE}}_{\text{Encodes initial parton distribution}} \right] \rho(z) [v(\mathbf{q}^2)]^2.$$

Encodes initial
parton distribution

$$E \frac{d\mathcal{N}^0}{d^2p dE} \equiv \frac{1}{2(2\pi)^3} |J(E, \mathbf{p})|^2$$

Gradients of
hydrodynamic variables

$$\hat{\mathbf{g}}_\alpha \equiv \left(\nabla_\alpha T \frac{\delta}{\delta T} + \nabla_\alpha u_z \frac{\delta}{\delta u_z} + \nabla_\alpha \mathbf{u}_\beta \frac{\delta}{\delta \mathbf{u}_\beta} \right)$$

Jet Broadening: Example Observable

$$\langle \mathbf{p}^2 \rangle = 2w^2 + \mathcal{C} \int_0^L dz \left[1 - z \hat{\mathbf{g}} \cdot \frac{\mathbf{u}}{1 - u_z} \right] \rho(z) \int_{\mathbf{q}} \mathbf{q}^2 [v(\mathbf{q}^2)]^2$$

Jet Broadening: Example Observable

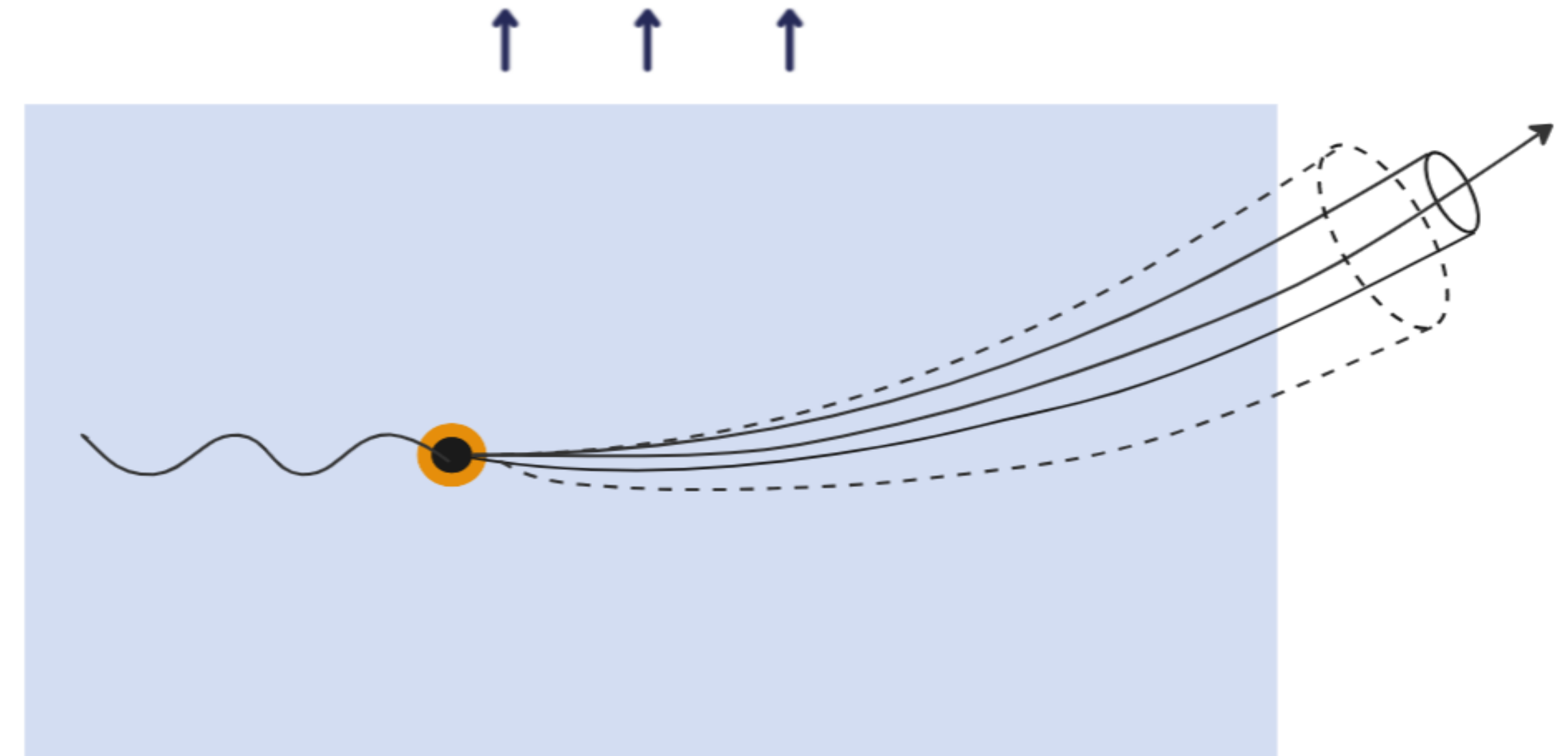
$$\langle \mathbf{p}^2 \rangle = 2w^2 + \mathcal{C} \int_0^L dz \left[1 - z \hat{\mathbf{g}} \cdot \frac{\mathbf{u}}{1 - u_z} \right] \rho(z) \int_{\mathbf{q}} \mathbf{q}^2 [v(\mathbf{q}^2)]^2$$

$$\hat{q} \equiv \frac{d\langle \mathbf{p}^2 \rangle}{dL}$$

Quenching parameter

$$\hat{q}(L) = \left[1 - L \hat{\mathbf{g}} \cdot \frac{\mathbf{u}}{1 - u_z} \right] \hat{q}_0(L),$$

Quenching is
sensitive to
flow and gradients



Jet Broadening: Numerical Estimate

$$\hat{q}(L) = \left[1 - L \hat{\mathbf{g}} \cdot \frac{\mathbf{u}}{1 - u_z} \right] \hat{q}_0(L) ,$$



$$\hat{q}L = \left[1 - \frac{L}{2} \frac{3 \left| \frac{\nabla T}{T^2} \right| T |\mathbf{u}| \cos(\theta)}{1 - u_z} \right] \hat{q}_0 L$$

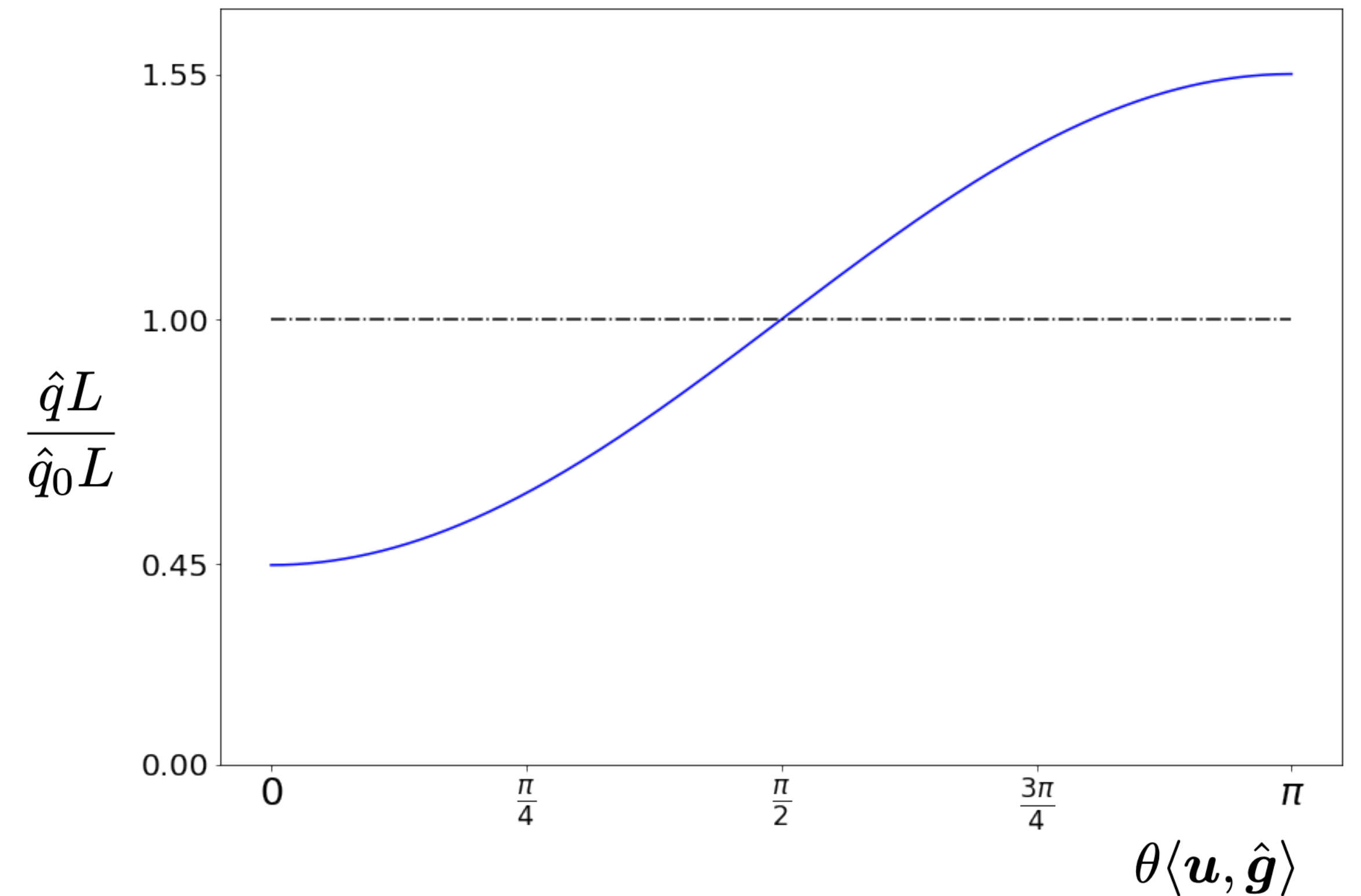
Parameters choice:

$$T \simeq 0.3 \text{ GeV}$$

$$L \simeq 5 \text{ fm}$$

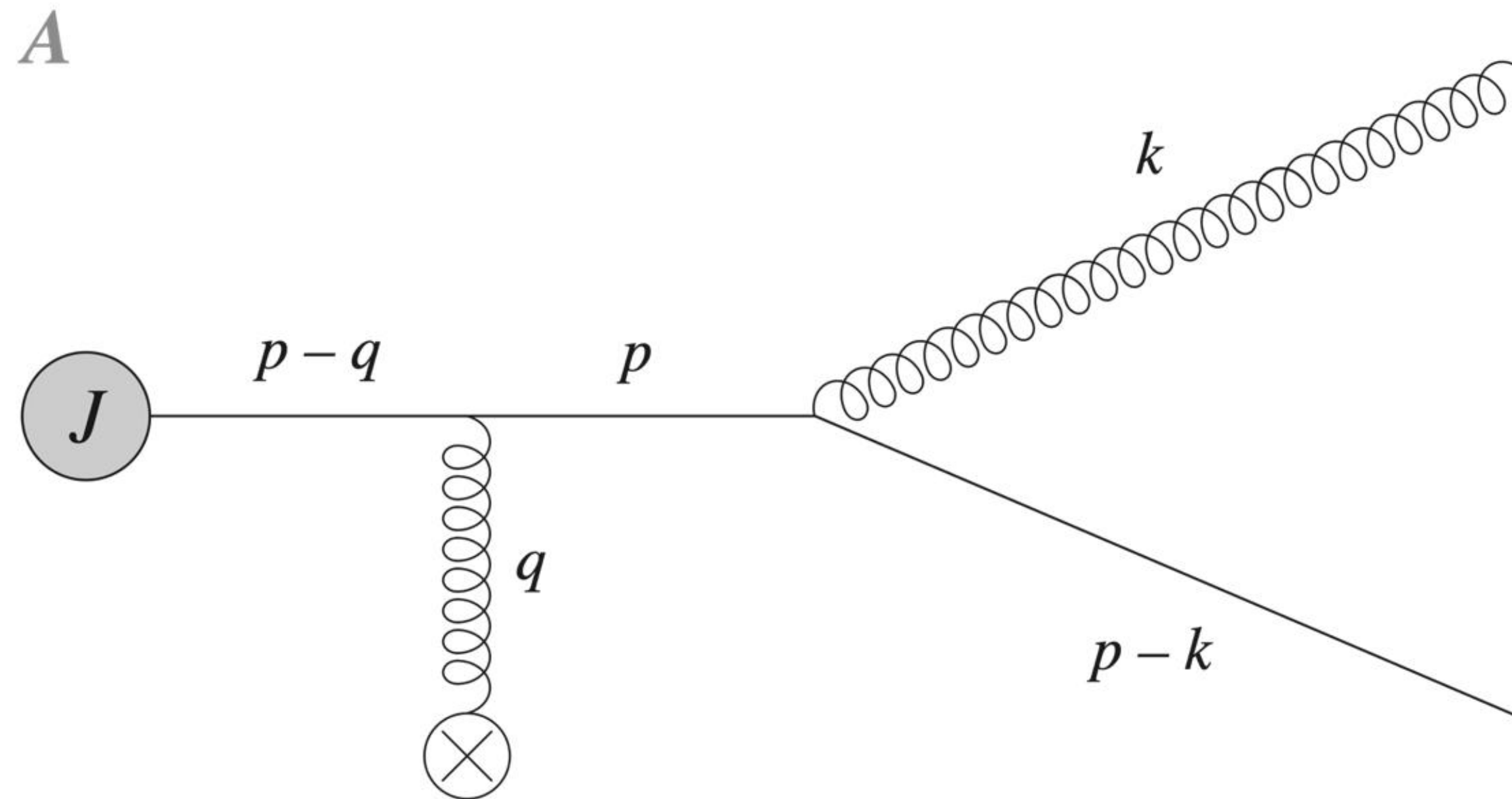
$$u \simeq 0.7 c$$

$$\left| \frac{\nabla T}{T^2} \right| \simeq 0.05$$



May lead to substantial modifications relative to the static limit.

In-Medium Branching



+ 8 more diagrams
To get $\chi^{(1)}$

Example of a Single Born Diagram

In-Medium Branching: Final Distribution

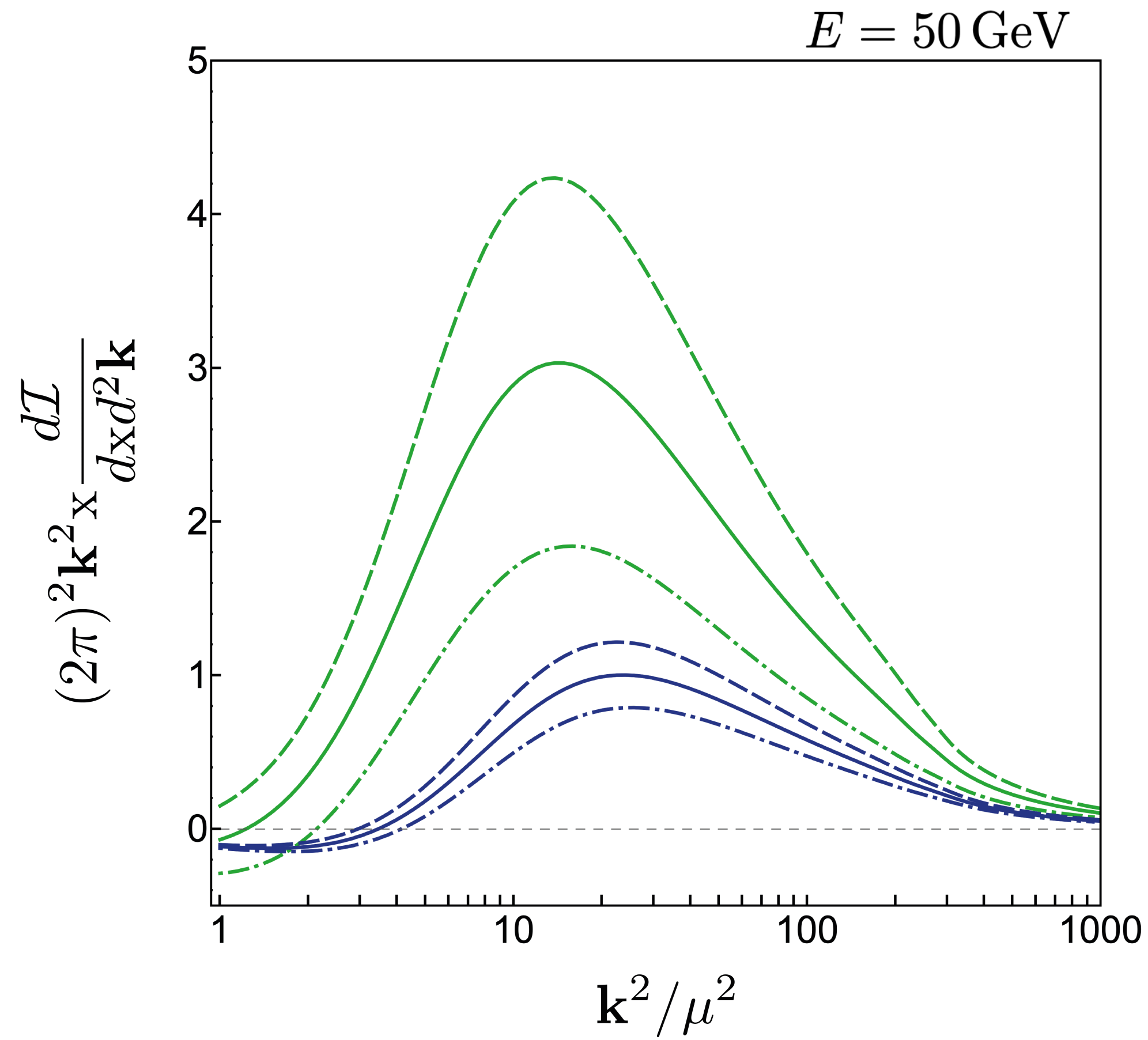
$$E \frac{d\mathcal{N}^{(1)}}{d^2k \, dx \, d^2p \, dE} \equiv \frac{1}{[2(2\pi)^3]^2} \frac{1}{x(1-x)} \langle |\mathcal{R}_{N=1}|^2 \rangle$$

$$\omega \frac{dI}{d^2k \, d\omega} = \frac{g^2 C_F}{(2\pi)^2} \int_0^L dz \int_{\mathbf{q}} [1 - \hat{\mathbf{g}} \cdot \mathbf{u} z] \frac{2 \mathbf{k} \cdot \mathbf{q}}{k^2 (\mathbf{k} - \mathbf{q})^2} \left[1 - \cos \left(\frac{(\mathbf{k} - \mathbf{q})^2}{2x E} z \right) \right] \rho(z) [v(\mathbf{q}^2)]^2$$

Leading
Flow-gradients
Correction

LPM phase factor

In-Medium Branching: Final Distribution



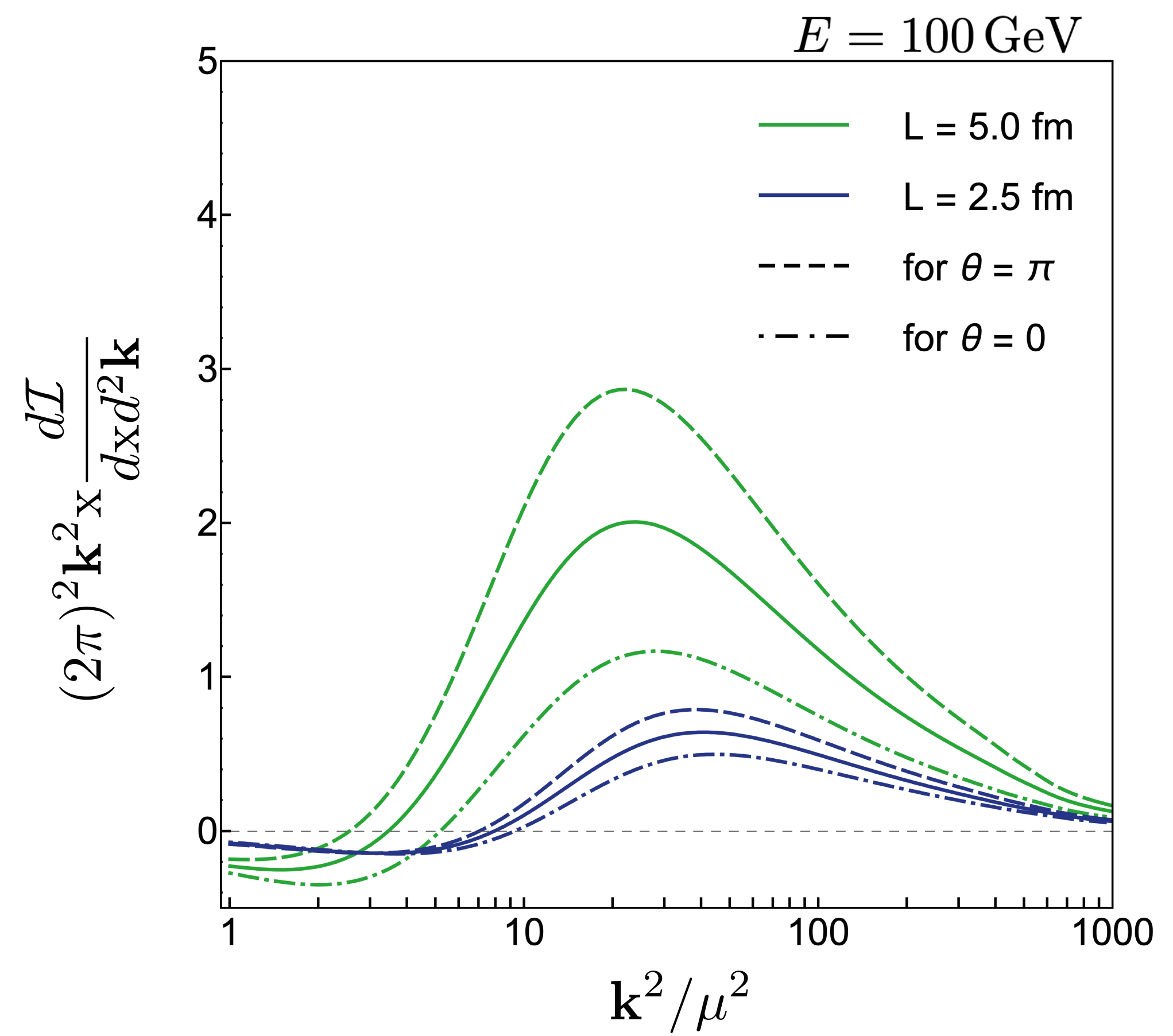
$\alpha_s = 0.3$

$\mu = 0.6 \text{ GeV}$

$x = 0.1$

$u = 0.3 c$

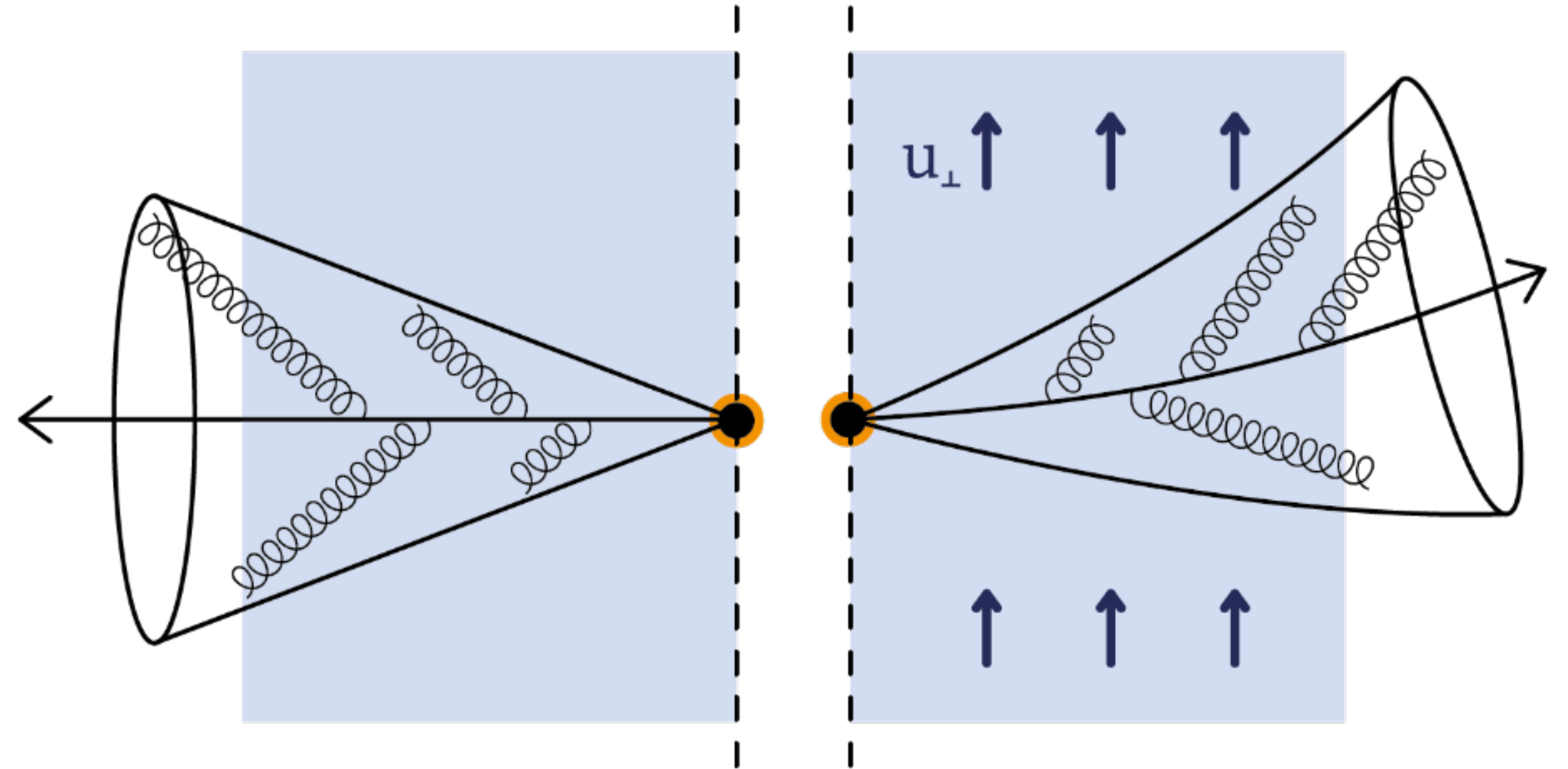
$\left| \frac{\nabla T}{T^2} \right| = 0.1 \quad T = 0.3 \text{ GeV}$



In-Medium Branching: Energy Loss

$$\frac{dE}{dL} = \left[1 - L \hat{\mathbf{g}} \cdot \mathbf{u} \right] \frac{dE_0}{dL}$$

Same type of
flow-gradients
corrections

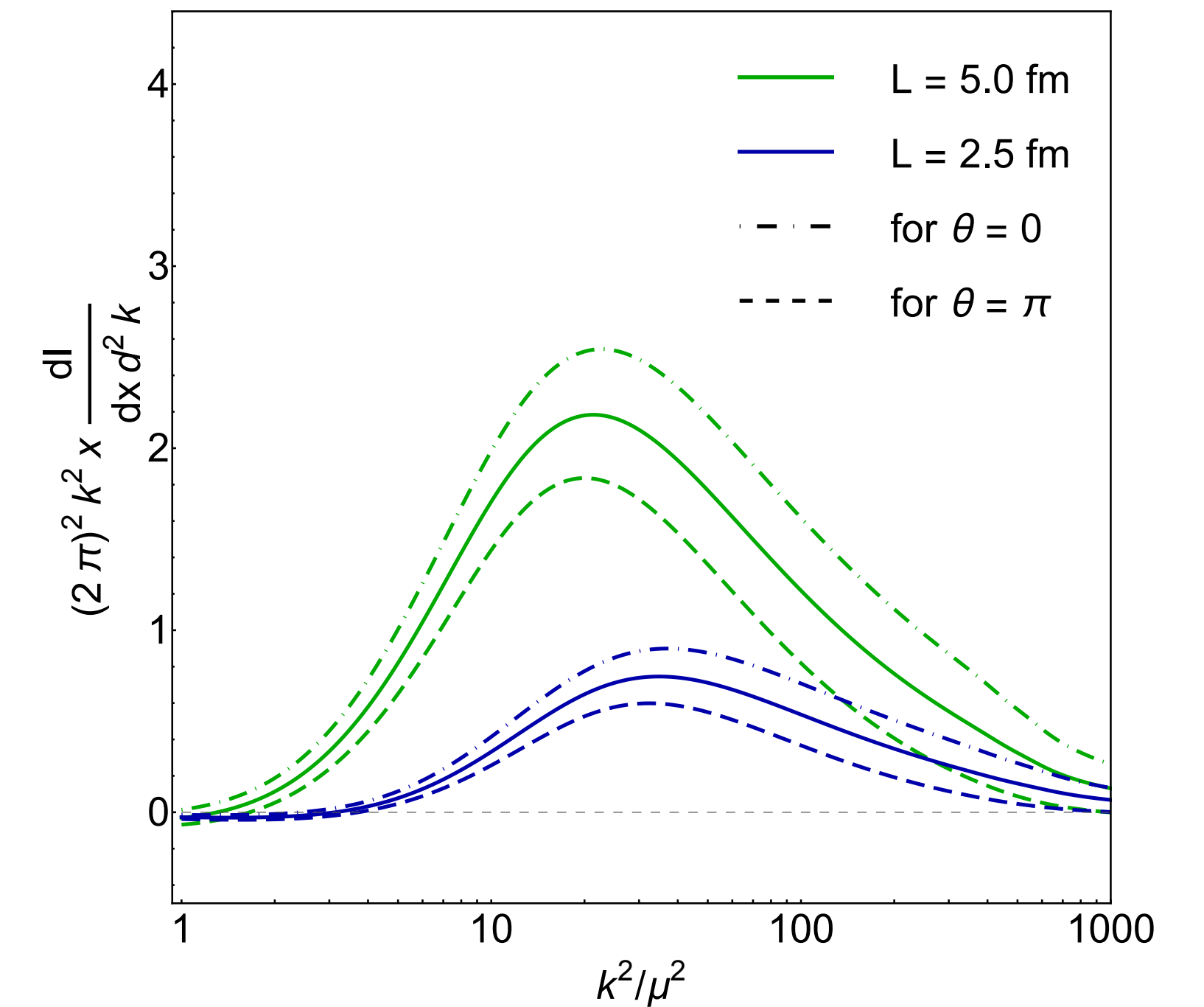
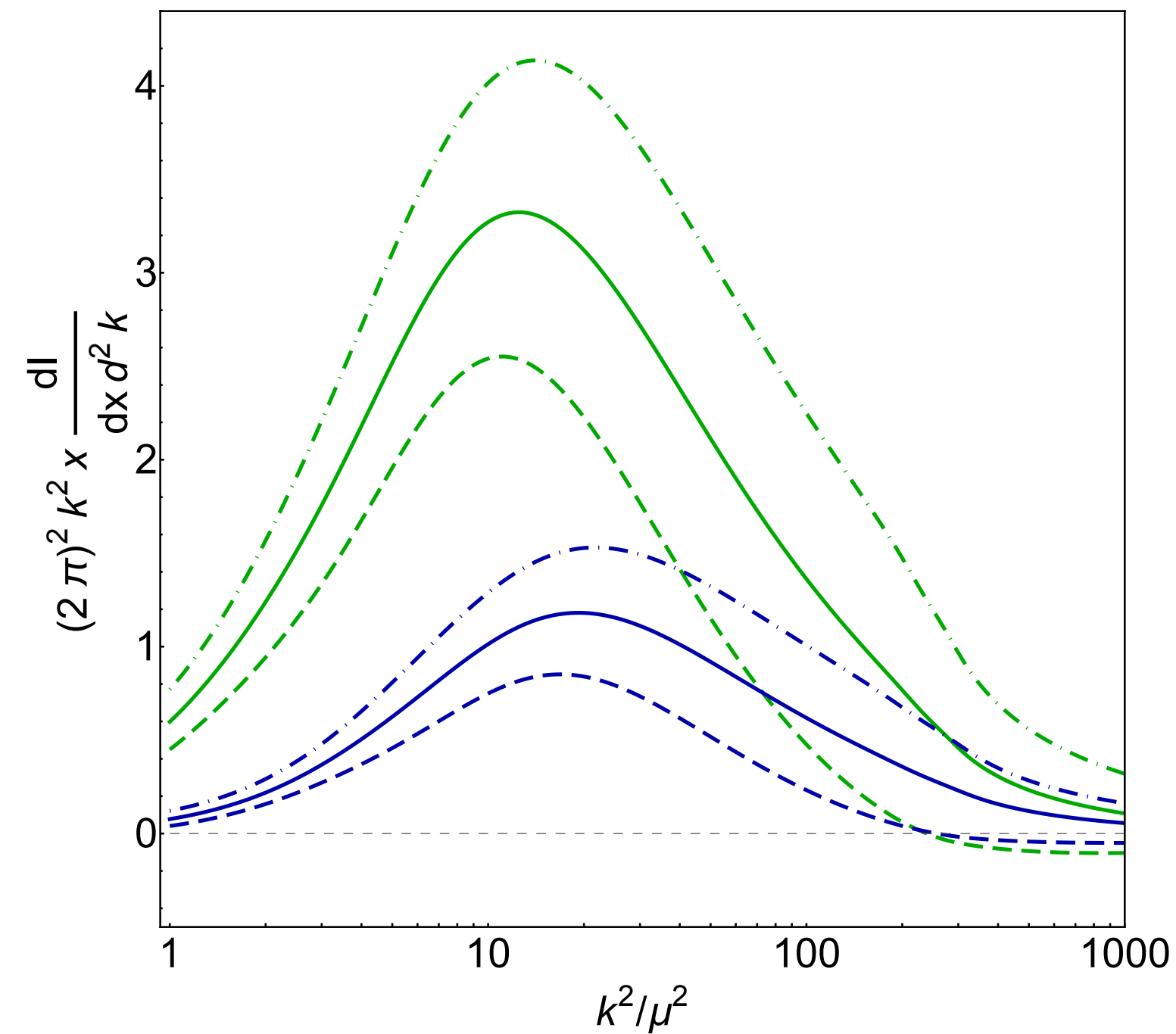


$$\hat{q}(L) = \left[1 - L \hat{\mathbf{g}} \cdot \frac{\mathbf{u}}{1 - u_z} \right] \hat{q}_0(L) ,$$

What About Pure Flow Effects?

Leading flow corrections are subeikonal in the absence of medium gradients $\mathcal{O}\left(\frac{k \cdot u}{\omega}\right)$

But they are significant numerically!

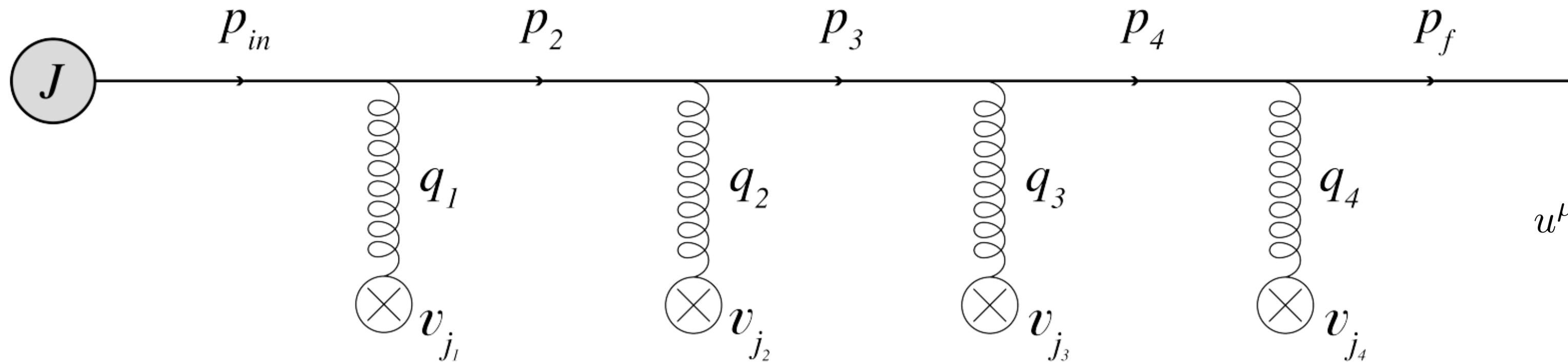


For details see
arXiv:2406.14628

BDMPS-Z Formalism: Resumming all Orders

R. Baier, Y. L. Dokshitzer, A. H. Mueller, S. Peigné, D. Schiff, arXiv:hep-ph/9608322
 R. Baier, Y. L. Dokshitzer, A. H. Mueller, S. Peigné, D. Schiff, arXiv:hep-ph/9607355
 B. G. Zakharov, arXiv:hep-ph/9607440
 X.-N. Wang, X.-f. Guo, arXiv:hep-ph/0102230
 P. B. Arnold, G. D. Moore, L. G. Yaffe, arXiv:hep-ph/0204343

Broadening



$$u^\mu = (u^+, u^-, \mathbf{u})^\mu = \left(\frac{1 - \sqrt{1 - \mathbf{u}^2}}{\sqrt{2}}, \frac{1 + \sqrt{1 - \mathbf{u}^2}}{\sqrt{2}}, \mathbf{u} \right)^\mu.$$



$$gA^{a\mu}(q) = (2\pi) v(-q^2) u^\mu \delta(q \cdot u)$$

$$\times \int dy^+ d^2\mathbf{y} \rho^a(\mathbf{y}, y^+) e^{iq^- y^+} e^{-i\mathbf{q} \cdot \mathbf{y}}$$

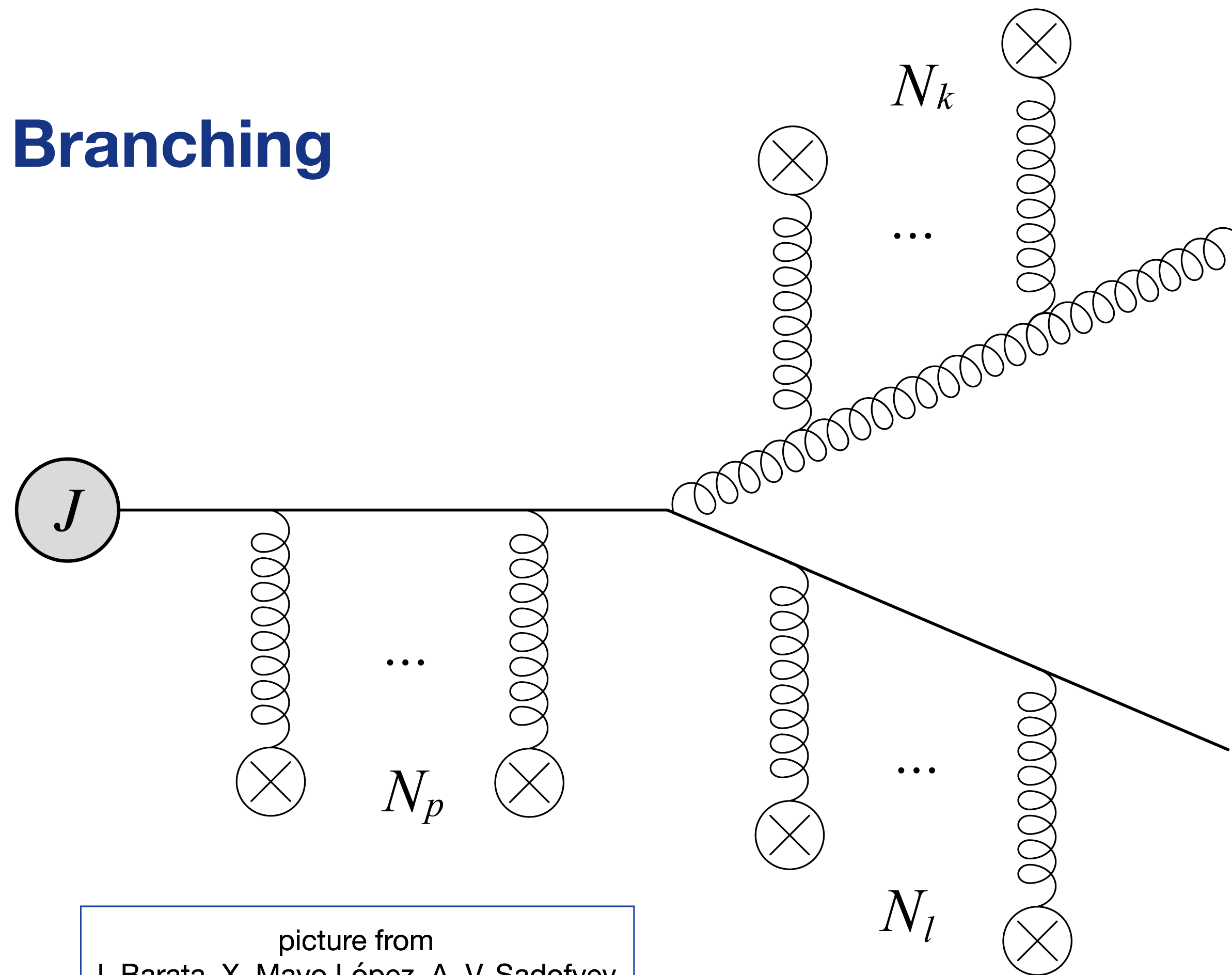
Continuous version of
external medium field

picture from
J. Barata, A. V. Sadofyev, C. A.
Salgado, arXiv:2202.08847

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 P. B. Arnold, G. D. Moore, L. G. Yaffe, arXiv:hep-ph/0204343

Branching



picture from
 J. Barata, X. Mayo López, A. V. Sadofyev,
 C. A. Salgado, arXiv:2304.03712

Continuous version of
 external medium field

$$u^\mu = (u^+, u^-, \mathbf{u})^\mu = \left(\frac{1 - \sqrt{1 - \mathbf{u}^2}}{\sqrt{2}}, \frac{1 + \sqrt{1 - \mathbf{u}^2}}{\sqrt{2}}, \mathbf{u} \right)^\mu.$$



$$gA^{a\mu}(q) = (2\pi) v(-q^2) u^\mu \delta(q \cdot u)$$

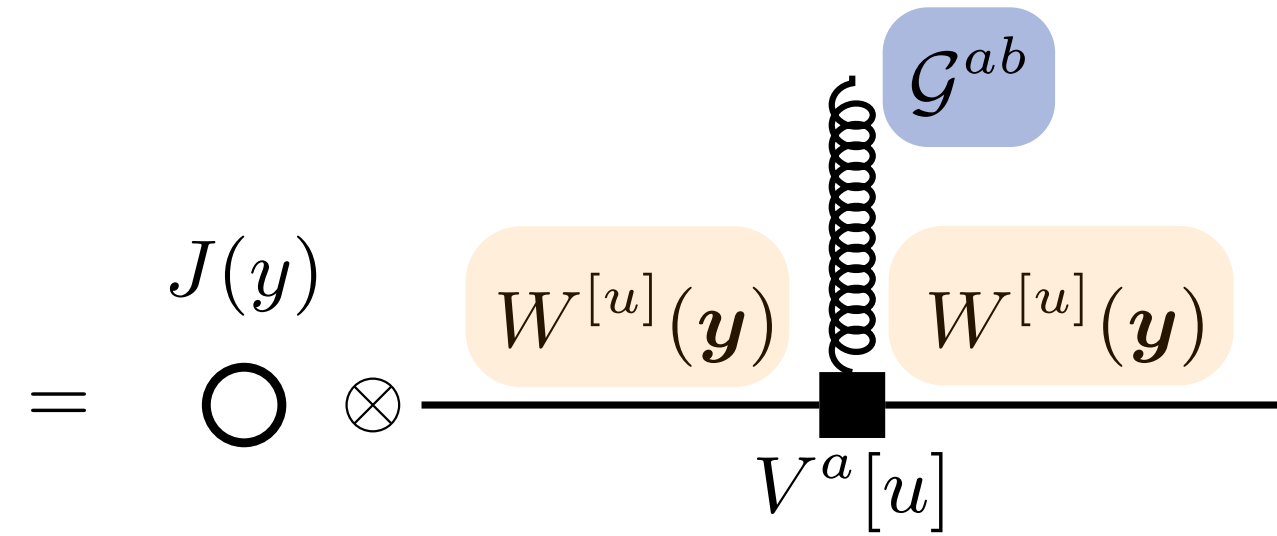
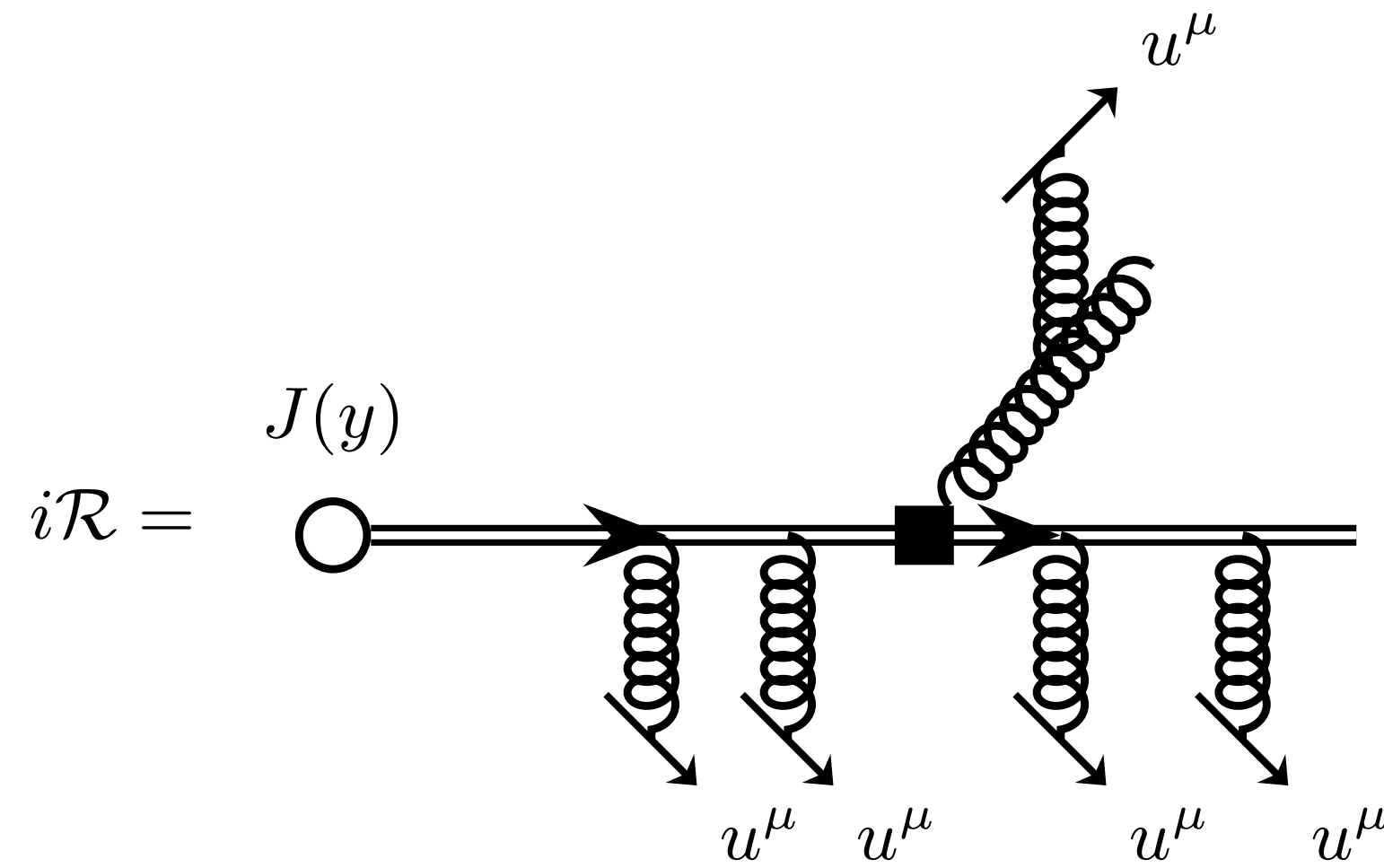
$$\times \int dy^+ d^2\mathbf{y} \rho^a(\mathbf{y}, y^+) e^{iq^- y^+} e^{-i\mathbf{q} \cdot \mathbf{y}}$$

Resummed Branching: Amplitude

$$\begin{aligned}
 i\mathcal{R}_{N_p N_l N_k} = & \prod_{n=1}^{N_p} \left[(-1) \int_{p_n, p_n^+} t^{a_n} g A^{a_n}(p_{n+1} - p_n) \cdot (p_{n+1} + p_n) \frac{1}{p_n^2 - m_Q^2 + i\epsilon} \right] \\
 & \times \int dx_s^+ \frac{d^4 p_s}{(2\pi)^4} (-1) g t^{b_1} \frac{(p_s + l_1)_{\mu_1}}{p_s^2 - m_Q^2 + i\epsilon} (2\pi)^3 \delta^{(3)}(p_s - l_1 - k_1) e^{ix_s^+ (l_1^- + k_1^- - p_s^-)} J(p_1) \\
 & \times \prod_{r=1}^{N_k} \left[\left(-\frac{i}{g} \right) \int_{k_r} \frac{N_{k_r}^{\mu_r \nu_r}}{k_r^2 + i\epsilon} \Gamma_{\nu_r \mu_{r+1} \lambda_r}^{b_r b_{r+1} c_r}(k_r, -k_{r+1}) g A^{c_r \lambda_r}(k_{r+1} - k_r) \right] \epsilon_k^{*\mu_{N_k+1}} \\
 & \times \prod_{m=1}^{N_l} \left[(-1) \int_{l_m} t^{d_m} g A^{d_m}(l_{m+1} - l_m) \cdot (l_{m+1} + l_m) \frac{1}{l_m^2 - m_Q^2 + i\epsilon} \right].
 \end{aligned}$$

m_Q – quark mass

Resummed Branching: Amplitude



Following eikonal approximation

$$E \gg |\mathbf{p}|, \mu$$

Keeping the following correction

$$\sim \left(\frac{m_Q^2}{E^2} \mathbf{u}^\beta \cdot \perp_\beta x^+ \right)^n$$

$$i\mathcal{R} \simeq i \frac{g}{\omega} \lim_{x_f^+ \rightarrow \infty} \int_{x_s^+, \mathbf{k}_1, \mathbf{y}} J(\mathbf{y}) e^{-i\mathbf{y} \cdot (\mathbf{l} + \mathbf{k}_1)} \\ \times e^{ix_s^+ \left(\frac{m_Q^2 x^2}{2\omega} - \frac{m_Q^2}{E^2} \frac{(\mathbf{k} - \mathbf{k}_1) \cdot \mathbf{u}}{2u^-} \right)} \boldsymbol{\epsilon} \cdot \mathbf{k}_1 W^{[u]}(\mathbf{y}; \infty, x_s^+) t^{b_1} W^{[u]}(\mathbf{y}; x_s^+, 0) \\ \times e^{ix_f^+ \frac{\mathbf{k}^2}{2\omega}} \mathcal{G}^{b_f b_1}(\mathbf{k}, x_f^+; \mathbf{k}_1, x_s^+).$$

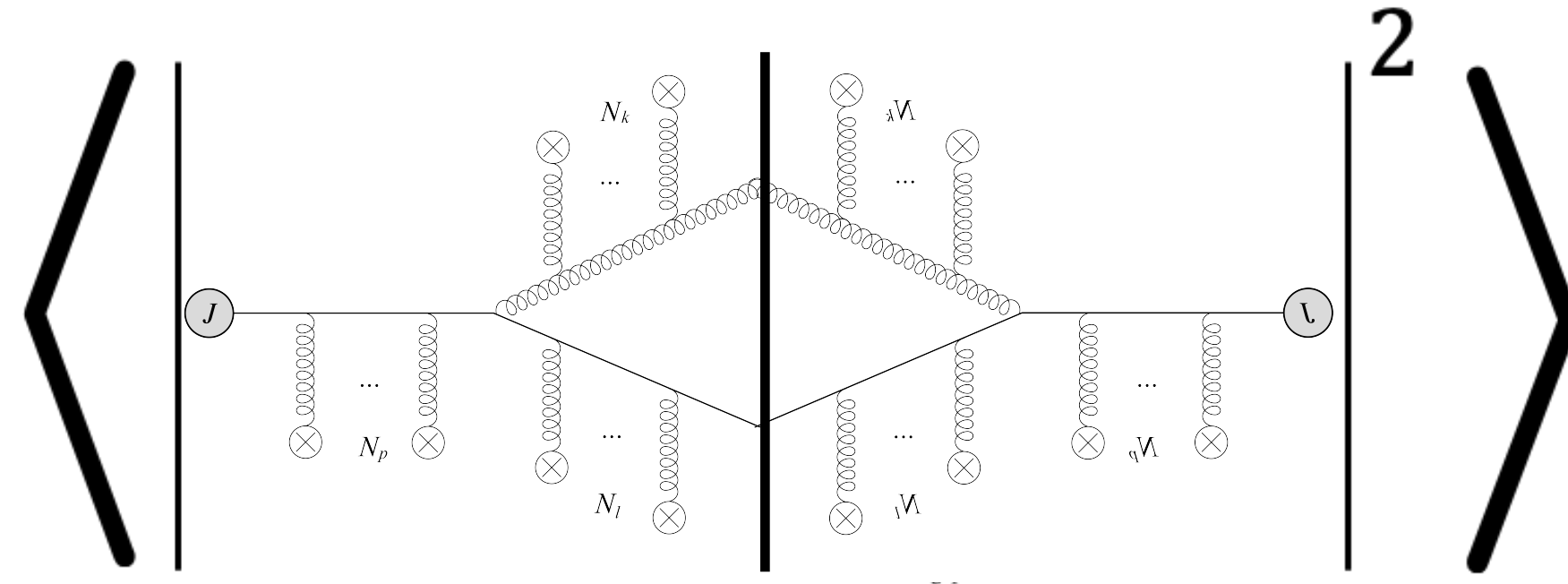
Single-particle propagator

$$\mathcal{G}(\mathbf{x}_f, x_f^+; \mathbf{x}_i, x_i^+) = \int_{\mathbf{x}_i}^{\mathbf{x}_f} \mathcal{D}\mathbf{r} \exp \left\{ i \frac{\omega}{2} \int_{x_i^+}^{x_f^+} d\tau \dot{\mathbf{r}}^2 \right\} \\ \times \mathcal{P} \exp \left\{ i \int_{x_i^+}^{x_f^+} d\tau T^c v^c(\mathbf{r}(\tau), \tau) \right\}$$

Wilson line

$$W^{[u]}(\mathbf{y}; L, 0) = \mathcal{P} \exp \left\{ i \int_0^L d\tau t^a v^a \left(\mathbf{y} - \frac{1}{2} \frac{m_Q^2}{E^2} \frac{\mathbf{u}}{u^-} \tau, \tau \right) \right\}$$

Resummed Branching: Masses and Flow



$$\leftarrow 2C_A \langle \rho^a(\mathbf{x}, x^+) \rho^b(\bar{\mathbf{x}}, \bar{x}^+) \rangle = \rho \delta^{ab} \delta^{(2)}(\mathbf{x} - \bar{\mathbf{x}}) \delta(x^+ - \bar{x}^+)$$

Main mass
contribution

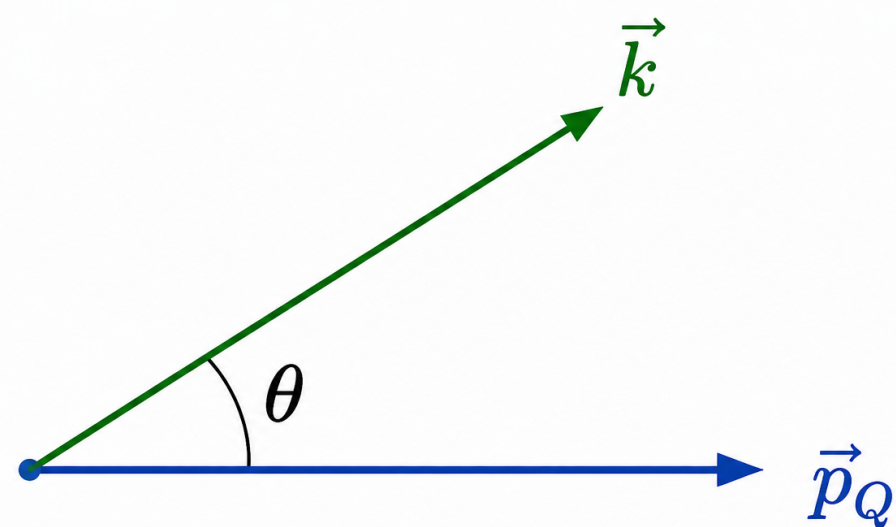
Mass-flow
corrections

$$(2\pi)^2 \omega \frac{dI}{d\omega d^2\mathbf{k}} = \lim_{x_f^+ \rightarrow \infty} \frac{\alpha_s}{N_c \omega^2} \text{Re} \int_0^\infty d\bar{x}_s^+ \int_0^{\bar{x}_s^+} dx_s^+ \int d^2\mathbf{y} e^{-i \frac{m_Q^2 x^2}{2\omega}} \left(1 - \frac{\mathbf{k} \cdot \mathbf{u}}{u - \omega} - \frac{m_Q^2}{E^2} \frac{\mathbf{u}^2}{4u - 2} \right) (\bar{x}_s^+ - x_s^+) \\ \times e^{-i \frac{m_Q^2}{E^2} \omega \frac{\mathbf{y} \cdot \mathbf{u}}{2u -}} S_2(\mathbf{k}, \mathbf{k}, x_f^+; \mathbf{y}, \mathbf{r}, \bar{x}_s^+) \nabla_{\mathbf{y}} \cdot \nabla_{\mathbf{x}} \mathcal{K}(\mathbf{y}, \bar{x}_s^+; \mathbf{x}, x_s^+) \Big|_{\mathbf{r}=\mathbf{x}=\mathbf{0}}.$$

Kernels described by
differential equations

Emission off Massive Quarks: Vacuum

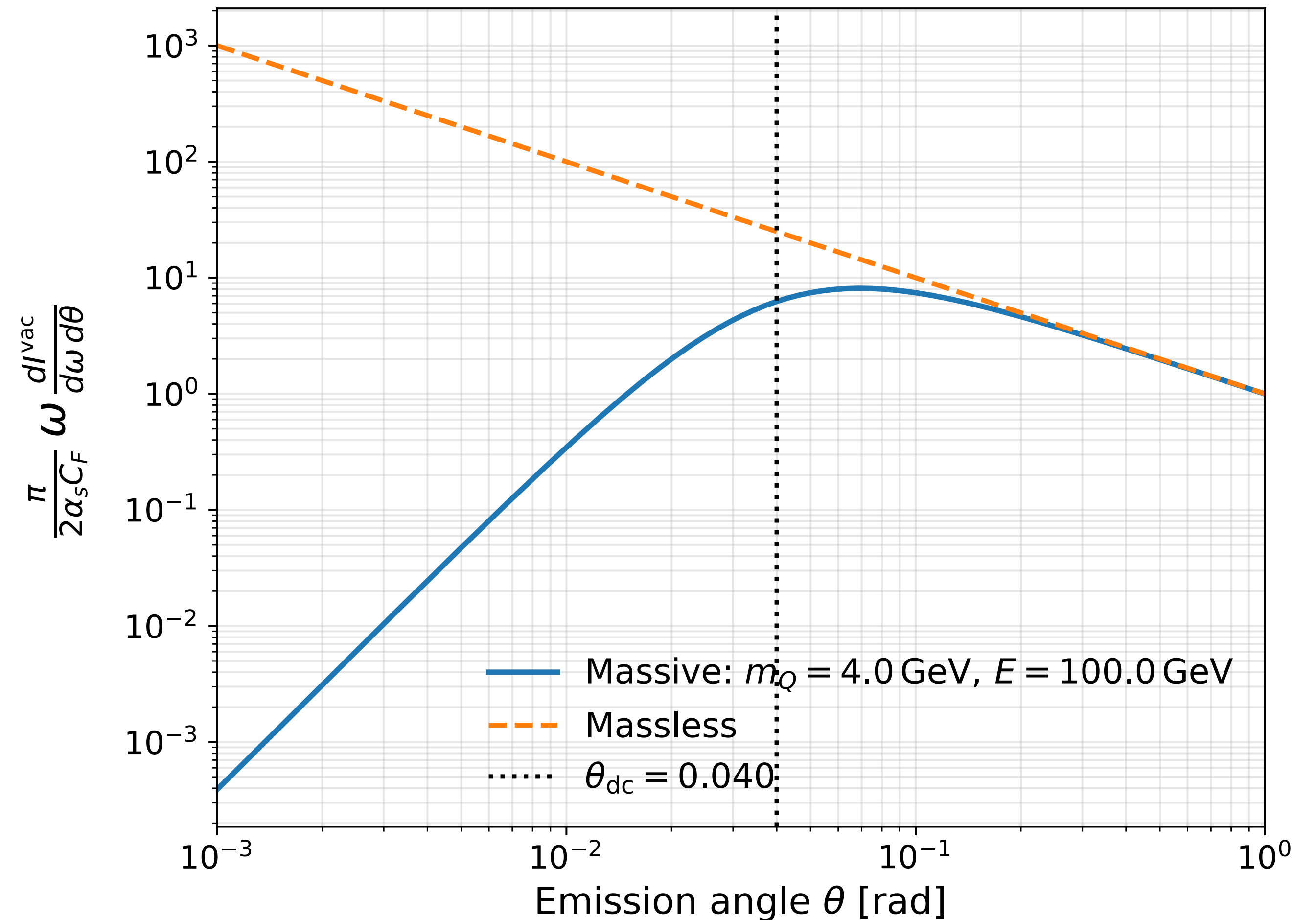
$$\omega \frac{dI^{\text{vac}}}{d\omega d\theta^2} = \frac{\alpha_s C_F}{\pi} \frac{\theta^2}{(\theta^2 + \theta_{\text{dc}}^2)^2}$$



$$\theta_{\text{dc}}^2 = \left(\frac{m_Q}{E} \right)^2$$

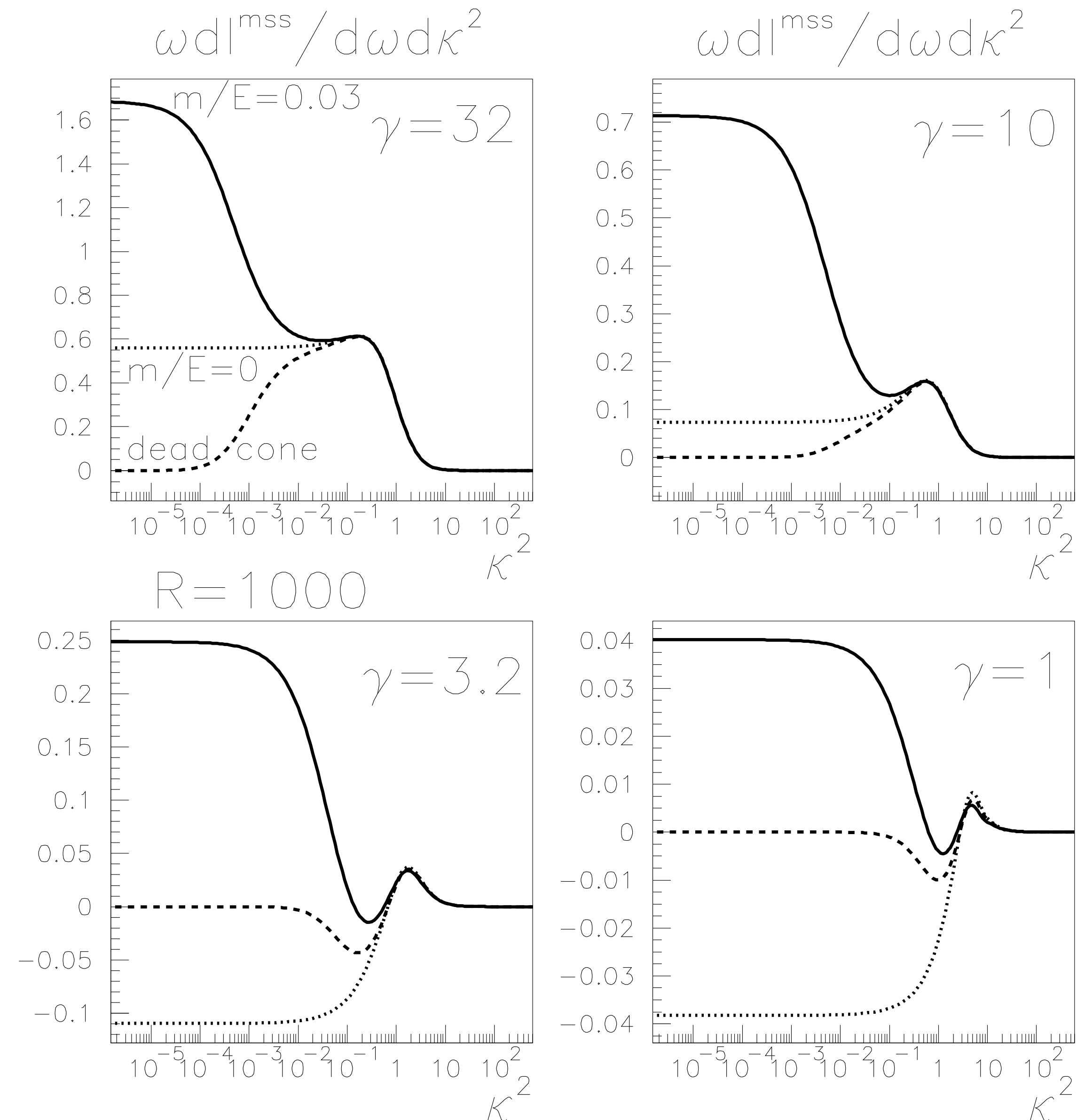
Dead Cone
angle

In vacuum, the dead cone acts as a regulator of the collinear divergence, **suppressing** gluon emission inside the dead cone.



Emission off Massive Quarks: Static Medium

Multiple in-medium scatterings partly **fill the dead-cone** region of the radiation pattern.



Purely medium-induced spectrum

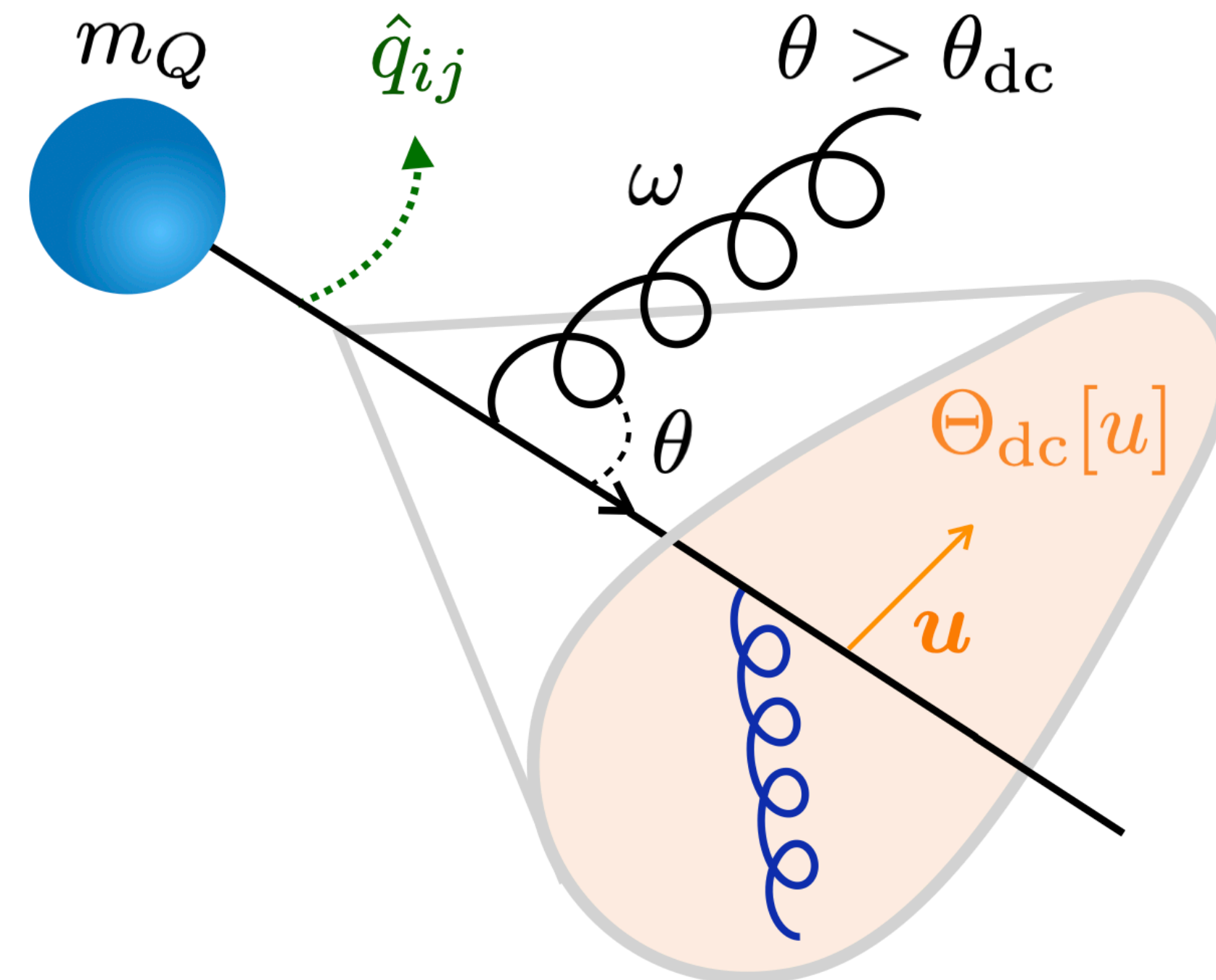
- Solid line – heavy quark
- Dotted line – massless limit
- Dashed line – DC approximation

Emission off Massive Quarks: Flowing Medium

$$(2\pi)^2 \omega \frac{dI}{d\omega d^2\mathbf{k}} = \lim_{x_f^+ \rightarrow \infty} \frac{\alpha_s}{N_c \omega^2} \text{Re} \int_0^\infty d\bar{x}_s^+ \int_0^{\bar{x}_s^+} dx_s^+ \int d^2\mathbf{y} e^{-i \frac{m_Q^2 x^2}{2\omega}} \left(1 - \frac{\mathbf{k} \cdot \mathbf{u}}{u^- \omega} - \frac{m_Q^2}{E^2} \frac{\mathbf{u}^2}{4u^-2} \right) (\bar{x}_s^+ - x_s^+) \\ \times e^{-i \frac{m_Q^2}{E^2} \omega \frac{\mathbf{y} \cdot \mathbf{u}}{2u^-}} S_2(\mathbf{k}, \mathbf{k}, x_f^+; \mathbf{y}, \mathbf{r}, \bar{x}_s^+) \nabla_{\mathbf{y}} \cdot \nabla_{\mathbf{x}} \mathcal{K}(\mathbf{y}, \bar{x}_s^+; \mathbf{x}, x_s^+) \Big|_{\mathbf{r}=\mathbf{x}=0}.$$

$$\Theta_{\text{dc}}^2(\mathbf{k} \cdot \mathbf{u}) = \theta_{\text{dc}}^2 \left(1 - \frac{\mathbf{k} \cdot \mathbf{u}}{u^- \omega} \right).$$

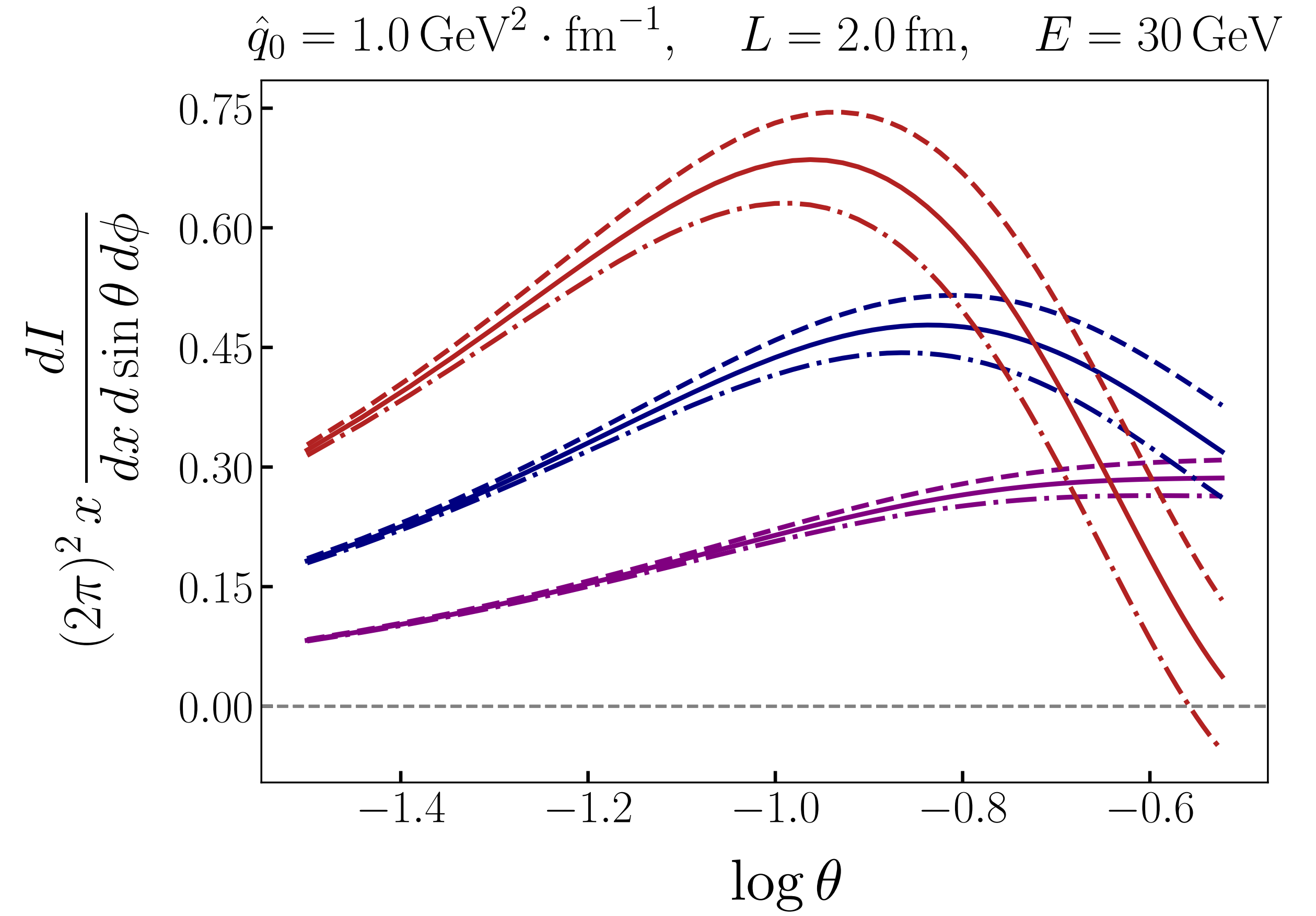
Dead Cone now depends
on the direction of the
emitted gluon with respect
to the flow



Emission off Massive Quarks: Flowing Medium

$$\Theta_{\text{dc}}^2(\mathbf{k} \cdot \mathbf{u}) = \theta_{\text{dc}}^2 \left(1 - \frac{\mathbf{k} \cdot \mathbf{u}}{u^- \omega} \right) .$$

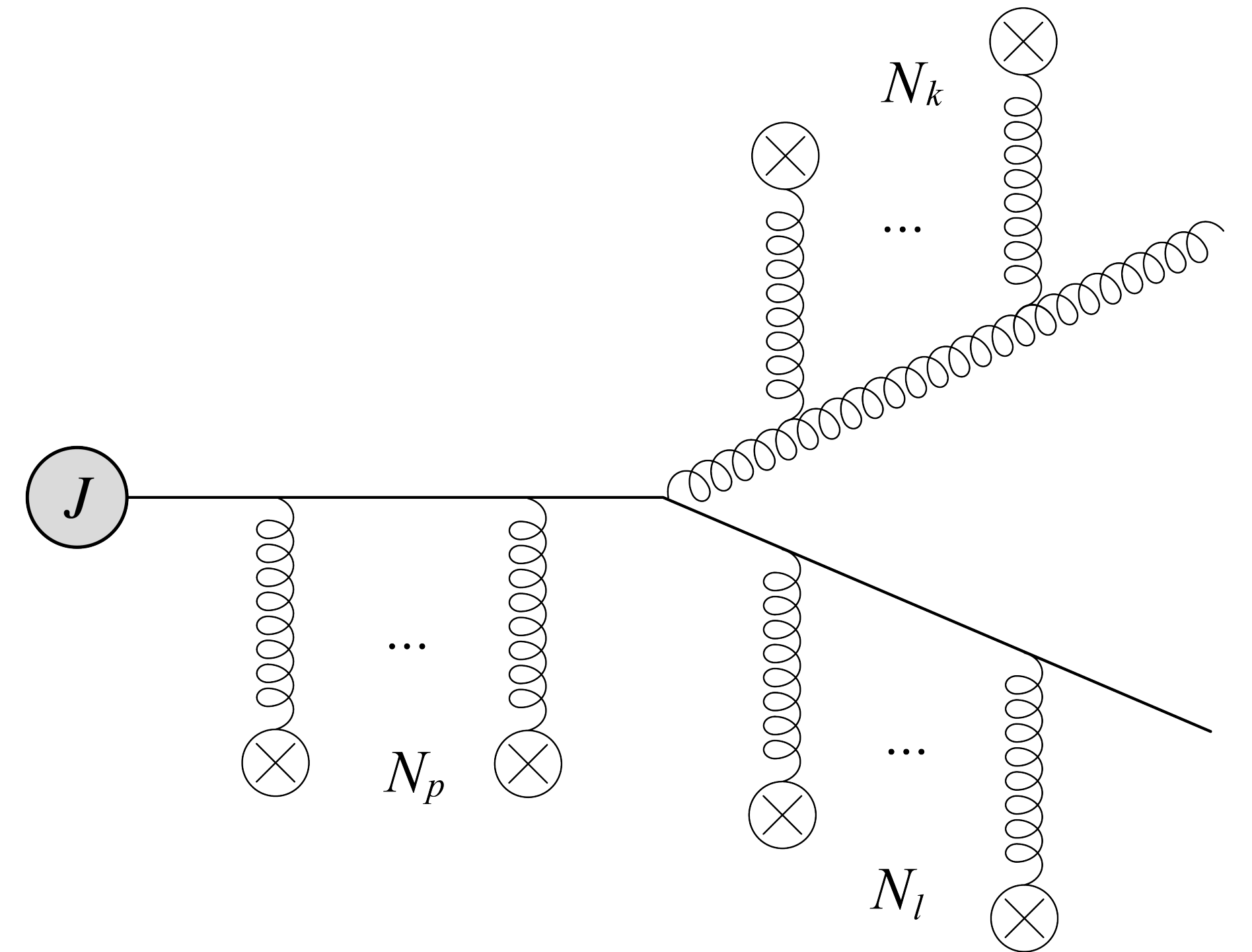
Dead Cone now depends
on the direction of the
emitted gluon with respect
to the flow



BDMPS-Z Formalism and Flow

Conceptual and technical challenges

- No generally robust gauge-invariance check is available for the truncated amplitude-level calculation. Explicit cross-gauge comparison remains the most direct test, although different gauges can lead to substantially different calculational complexity.
- In the BDMPS-Z approach, potential-pole residues are commonly neglected under the assumption $\mu\Delta z \gg 1$. However, double-Born terms probe the $\Delta z = 0$ region, where this suppression argument does not apply. Their treatment beyond simplified local-medium models requires further investigation.
- Mass, subeikonal, flow, and gradient effects can enter at the same parametric order and generate mixed contributions. A tractable calculation requires an explicit hierarchy of scales and a controlled isolation of selected corrections.



Summary and outlook

- Energetic partons are sensitive to the hydrodynamic evolution of the medium, which leaves an imprint on their final-state distributions.
- Medium-induced effects are intrinsically anisotropic and may therefore provide valuable information for jet tomography.
- Further progress in in-medium resummation is needed to incorporate additional hydrodynamic effects into jet-quenching calculations.

Discussion is welcome!



BACKUP



Broadening Effect of Massive Quarks

$$\begin{aligned}
 \hat{q}_{ij} &= -\frac{\partial}{\partial L} \left[\nabla_{(\mathbf{y}-\bar{\mathbf{y}})_i} \nabla_{(\mathbf{y}-\bar{\mathbf{y}})_j} \left\langle W^{[u]}(\mathbf{y}; L, 0) W^{[u]\dagger}(\bar{\mathbf{y}}; L, 0) \right\rangle \right]_{\bar{\mathbf{y}}=\mathbf{y}} \\
 &= \frac{\hat{q}_0}{2} \left(1 + \frac{u^+}{u^-} \frac{m_Q^2}{E^2} \right) \left[\delta_{ij} - \frac{u_i u_j}{(u^-)^2} \frac{m_Q^2}{E^2} \right].
 \end{aligned}$$

Emission off Massive Quarks: Kernels

$$S_2(\mathbf{k}, \mathbf{k}, x_f^+; \mathbf{l}, \bar{\mathbf{k}}_1, \bar{x}_s^+) = \frac{1}{N_c^2 - 1} \left\langle \mathcal{G}^{ba}(\mathbf{k}, x_f^+; \mathbf{l}, \bar{x}_s^+) \mathcal{G}^{\dagger ab}(\mathbf{k}, x_f^+; \bar{\mathbf{k}}_1, \bar{x}_s^+) \right\rangle,$$

$$\mathcal{K}^{[u]}(\mathbf{l}, \mathbf{y}, \bar{x}_s^+; \mathbf{k}_1, \mathbf{y}, x_s^+) = \frac{1}{N_c^2 - 1} \left\langle \mathcal{G}^{cd}(\mathbf{l}, \bar{x}_s^+; \mathbf{k}_1, x_s^+) W_A^{[u]\dagger dc}(\mathbf{y}; \bar{x}_s^+, x_s^+) \right\rangle$$

$$\frac{\partial}{\partial \bar{x}^+} \mathcal{K}^{[u]}(\mathbf{x}_f, \bar{x}^+; \mathbf{x}_i, x_s^+)$$

$$= \left[i \frac{\nabla_{\mathbf{x}_f}^2}{2\omega} - \mathcal{V} \left(\mathbf{x}_f + \frac{m_Q^2}{E^2} \frac{\mathbf{u}}{2u^-} \bar{x}^+, \bar{x}^+ \right) \right] \mathcal{K}^{[u]}(\mathbf{x}_f, \bar{x}^+; \mathbf{x}_i, x_s^+).$$

Emission kernel

$$S_2(\mathbf{k}, \mathbf{k}, x_f^+; \mathbf{x}, \bar{\mathbf{x}}, \bar{x}_s^+) = e^{-i\mathbf{k} \cdot (\mathbf{x} - \bar{\mathbf{x}})} \exp \left[- \int_{\bar{x}_s^+}^{x_s^+} d\tau \mathcal{V}(\mathbf{x} - \bar{\mathbf{x}}, \tau) \right]$$

Broadening kernel

What About Pure Flow Effects?

Leading flow corrections are
subeikonal in the absence of
medium gradients $\mathcal{O}\left(\frac{\mathbf{k} \cdot \mathbf{u}}{\omega}\right)$

Example of a Single Born amplitude
squared and averaged

$$\begin{aligned} \langle |\mathcal{R}_C|^2 \rangle = & C_F g^2 \int_0^L dz \rho(z) \int_{\mathbf{q}} \frac{4}{(\mathbf{k} - \mathbf{q})^2} \left(1 - 2 \frac{(\mathbf{k} - \mathbf{q}) \cdot \mathbf{u}}{xE} \right) \\ & \times \left(1 - \frac{\mathbf{u} \cdot \mathbf{q} (2k^2 - (\mathbf{k} - \mathbf{q})^2)}{2xE} \frac{1}{v^2} \frac{\partial v^2}{\partial \mathbf{q}^2} \right) \left(1 - \cos \left(\frac{(\mathbf{k} - \mathbf{q})^2}{2xE} z \right) \right) v(\mathbf{q})^2 |J(E, \mathbf{p} - \mathbf{q})|^2. \end{aligned}$$

For details see
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