# Improved perturbative calculations for forward physics and saturation

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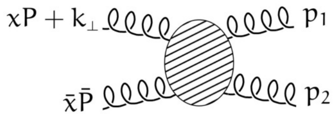
# Contents

- Improved Transverse Momentum Dependent factorization
- $k_T$-type factorization at NLO

# Improved TMD factorization

Kotko, Kutak, Marquet, Petreska, Sapeta, AvH 2015

$$d\sigma_{HH\to X} = \int dk_\perp^2 \int dx \sum_i \int d\bar{x} \sum_b \overbrace{\boxed{\phi_{gb}^{(i)}(x, k_\perp, \mu)}}^{\text{target-side TMD}} \overbrace{\boxed{f_b(\bar{x}, \mu)}}^{\text{projectile-side PDF}} \overbrace{\boxed{d\hat{\sigma}_{gb\to X}^{(i)}(x, {\color{red}k_\perp}, \bar{x}, \mu)}}^{\text{parton-level matrix element}}$$




- approach to saturation in QCD in terms of TMDs and matrix elements
- originally formulated for dijets in hadron scattering,
- $x$ is assumed to be small, so the jets are forward
- parton-level matrix elements explicitly depend on ${\color{red}k_\perp}$, and involve an "off-shell" (reggeized) initial-state gluon
- allows for the final-state jets to be away from the back-to-back limit, so allows for $P_\perp \gtrsim k_\perp \gtrsim Q_s$
- generalization of Generalized TMD factorization (Dominguez, Marquet, Xiao, Yuan 2011)
- obtained from the CGC formalism by including so-called kinematic twist corrections $(k_\perp^2/P_\perp^2)$ (as opposed to genuine twist corrections $Q_s^2/P_\perp^2$) (Antinoluk, Boussarie, Kotko 2019)
- the saturation enters calculations via the TMDs
- the matrix element is "color-decomposed' into gauge-invariant pieces, and each gets its own TMD

# KaTie parton-level Monte Carlo program

The ITMD factorization formula fits in the general shape for hadron-scattering process $Y$ with partonic processes $y = (y_1\ y_1 \to y_3\ y_4\ \ldots)$ contributing to multi-jet final state

$$d\sigma_Y(p_1, p_2; k_3, \ldots, k_{2+n}) = \sum_{y \in Y} \int d^4k_1\, \mathcal{P}_{y_1}(k_1) \int d^4k_2\, \mathcal{P}_{y_2}(k_2)\, d\hat{\sigma}_y(k_1, k_2; k_3, \ldots, k_{2+n})$$

collinear factorization: $\mathcal{P}_{y_i}(k_i) = \int \frac{dx_i}{x_i} f_{y_i}(x_i, \mu)\, \delta^4(k_i - x_i p_i)$

$k_\perp$-dependent factorization: $\mathcal{P}_{y_i}(k_i) = \int \frac{d^2\mathbf{k}_{i\perp}}{\pi} \int \frac{dx_i}{x_i} \mathcal{F}_{y_i}(x_i, |\mathbf{k}_{i\perp}|, \mu)\, \delta^4(k_i - x_i p_i - k_{i\perp})$

partonic cross section: $d\hat{\sigma}_y(k_1, k_2; k_3, \ldots, k_{2+n}) = d\Phi_Y(k_1, k_2; k_3, \ldots, k_{2+n}) \times \mathrm{Cuts}(k_3, \ldots, k_{2+n})$
$\times\, \mathrm{fluxfactor}(k_1, k_2) \times |\mathrm{MatrixElement}_y(k_1, k_2; k_3, \ldots, k_{2+n})|^2$

parton-level phase space: $d\Phi_Y(k_1, k_2; k_3, \ldots, k_{2+n}) = \left( \prod_{i=3}^{n+2} d^4k_i \delta_+(k_i^2 - m_i^2) \right) \delta^4(k_1 + k_2 - k_3 - \cdots - k_{n+2})$

Given $f_{y_i}(x_i, \mu)$ and/or $\mathcal{F}_{y_i}(x_i, |\mathbf{k}_{i\perp}|, \mu)$, KaTie can deal with everything else, including matrix elements with "off-shell" aka "reggeized" initial-state partons, at leading order for now.

# Collinear factorization at NLO

general Sudakov decomposition: $K^\mu = x_K P^\mu + \bar{x}_K \bar{P}^\mu + K_\perp^\mu$

positive-rapidity initial state: $k_i^\mu = x P^\mu$

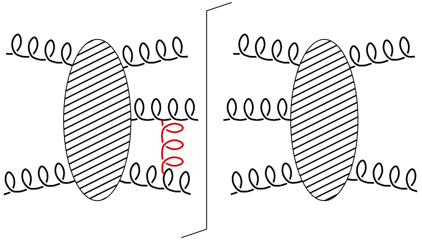
negative-rapidity initial-state: $k_{\bar{\imath}}^\mu = \qquad \bar{x}\bar{P}^\mu$

$$a_\epsilon = \frac{\alpha_s}{2\pi}\,\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}$$

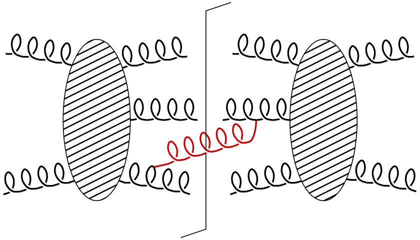
$$d\sigma^{\mathrm{LO}} = \sum_{i,\bar{\imath}} \int dx\,d\bar{x}\, f_i(x)\, f_{\bar{\imath}}(\bar{x})\, dB_{i\bar{\imath}}(x,\bar{x})$$

$$d\sigma^{\mathrm{NLO}} = \sum_{i,\bar{\imath}} \int dx\,d\bar{x}\, f_i(x)\, f_{\bar{\imath}}(\bar{x}) \Big\{ \Big[ a_\epsilon\, dV_{i\bar{\imath}}(x,\bar{x}) \;+\; a_\epsilon\, dR_{i\bar{\imath}}(x,\bar{x}) \Big]$$

Virtual contribution,
containing the interference of
1-loop graphs with tree-level graphs



Real contribution,
containing the square of
real-radiation graphs,
integrated over the radiation



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$$- a_\epsilon \Bigg[ \frac{1}{\epsilon} \sum_{i'} \int_x^1 \frac{dz}{z}\, \mathcal{P}_{ii'}(z)\, \frac{f_{i'}(x/z)}{f_i(x)} + \frac{1}{\epsilon} \sum_{\bar{\imath}'} \int_{\bar{x}}^1 \frac{d\bar{z}}{\bar{z}}\, \mathcal{P}_{\bar{\imath}\bar{\imath}'}(\bar{z})\, \frac{f_{\bar{\imath}'}(\bar{x}/\bar{z})}{f_{\bar{\imath}}(\bar{x})} \Bigg] dB_{i\bar{\imath}}(x,\bar{x})$$

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& - a_\epsilon \left[ \frac{1}{\epsilon} \sum_{i'} \int_x^1 \frac{dz}{z}\, \mathcal{P}_{ii'}(z)\, \frac{f_{i'}(x/z)}{f_i(x)} + \frac{1}{\epsilon} \sum_{\bar{\imath}'} \int_{\bar{x}}^1 \frac{d\bar{z}}{\bar{z}}\, \mathcal{P}_{\bar{\imath}\bar{\imath}'}(\bar{z})\, \frac{f_{\bar{\imath}'}(\bar{x}/\bar{z})}{f_{\bar{\imath}}(\bar{x})} \right] dB_{i\bar{\imath}}(x,\bar{x}) \\
& + a_\epsilon \left[ \frac{\delta f_i\ (x,\mu_F)}{f_i(x)} + \frac{\delta f_{\bar{\imath}}\ (\bar{x},\mu_{\bar{F}})}{f_{\bar{\imath}}(\bar{x})} \right] dB_{i\bar{\imath}}(x,\bar{x}) \Bigg\}
\end{aligned}$$

$$f_i(x) \to f_i(x,\mu_F,\epsilon) = f_i(x) + a_\epsilon\, \delta f_i(x,\mu_F,\epsilon) + \mathcal{O}(\alpha_s^2)$$

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$$f_i(x) \to f_i(x,\mu_F,\epsilon) = f_i(x) + a_\epsilon\, \delta f_i(x,\mu_F,\epsilon) + \mathcal{O}(\alpha_s^2) \qquad \delta f_i(x,\mu_F,\epsilon) = \delta f_i^{\mathrm{fin}}(x,\mu_F) + \frac{1}{\epsilon}\left(\frac{\mu^2}{\mu_F^2}\right)^{\epsilon} \sum_{i'} \int_x^1 \frac{dz}{z}\, \mathcal{P}_{ii'}(z)\, f_{i'}(x/z)$$

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$$\frac{d\sigma}{d\ln\mu_F^2} = 0 \quad\Rightarrow\quad \frac{d\delta f_i^{\mathrm{fin}}(x,\mu_F)}{d\ln\mu_F^2} = \sum_{i'} \int_x^1 \frac{dz}{z}\, \mathcal{P}_{ii'}(z)\, f_{i'}(x/z) \quad\Rightarrow\quad \frac{d f_i^{\mathrm{fin}}(x,\mu_F)}{d\ln\mu_F^2} = a_\epsilon \sum_{i'} \int_x^1 \frac{dz}{z}\, \mathcal{P}_{ii'}(z)\, f_{i'}^{\mathrm{fin}}(x/z,\mu_F)$$

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$$d\sigma^{\mathrm{NLO}} = \sum_{i,\bar{\imath}} \int dx\,d\bar{x}\, f_i \;\cdots\; (\bar{z})\,\frac{f_{\bar{\imath}'}(\bar{x}/\bar{z})}{f_{\bar{\imath}}(\bar{x})}\Bigg]\, dB_{i\bar{\imath}}(x,\bar{x})$$

We want to establish the same for hybrid $k_T$-factorization, which has leading-order formula

$$d\sigma^{\mathrm{LO}} = \sum_{\bar{\imath}} \int dx \int \frac{d^2k_\perp}{\pi} \int d\bar{x}\, F(x,k_\perp)\, f_{\bar{\imath}}(\bar{x})\, dB_{\star\bar{\imath}}(x,k_\perp,\bar{x})$$

and involves both a UPDF and matrix elements explicitly depending on $k_\perp$.

$$f_i(x) \to f_i(x,\mu_F,\epsilon) = f_i(x) + a_\epsilon\, \delta f_i(x,\mu_F,\epsilon) + \mathcal{O}(\alpha_s^2) \qquad \delta f_i(x,\mu_F,\epsilon) = \delta f_i^{\mathrm{fin}}(x,\mu_F) + \frac{-}{\epsilon}\left(\frac{}{\mu_F^2}\right) \sum_{i'}\int_x^1 \frac{dz}{z}\,\mathcal{P}_{ii'}(z)\, f_{i'}(x/z)$$

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# Hybrid $k_T$-factorization at NLO


A. van Hameren, L. Motyka and G. Ziarko,
"Hybrid $k_T$ -factorization and impact factors at NLO,"
JHEP **11** (2022), 103, arXiv:2205.09585,
doi:10.1007/JHEP11(2022)103

E. Blanco, A. Giachino, A. van Hameren and P. Kotko,
"One-loop gauge invariant amplitudes with a space-like gluon,"
Nucl. Phys. B **995** (2023), 116322, arXiv:2212.03572,
doi:10.1016/j.nuclphysb.2023.116322

A. Giachino, A. van Hameren and G. Ziarko,
"A new subtraction scheme at NLO exploiting the privilege of $k_T$-factorization,"
JHEP **06** (2024), 167, arXiv:2312.02808,
doi:10.1007/JHEP06(2024)167

A. van Hameren and M. Nefedov,
"Hybrid high-energy factorization and evolution at NLO from the high-energy limit of collinear factorization,"
JHEP **02** (2025), 160, arXiv:2501.02619,
doi:10.1007/JHEP02(2025)160

A space-like gluon is indicated with "$\star$" and has momentum

$$k_\star^\mu = xP^\mu + k_\perp^\mu \ .$$

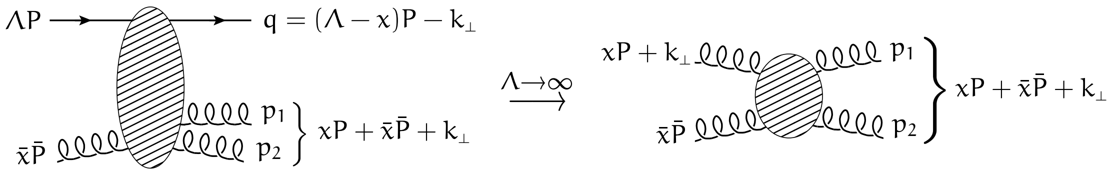
A tree-level matrix element with a space-like initial-state gluon is understood to be defined with the help of auxiliary partons as (AvH, Kotko, Kutak 2012)

$$\frac{x^2|k_\perp|^2}{g_s^2 C_i \Lambda^2}\,\big|\overline{M}_{i\bar{\imath}}\big|^2\big(k_\Lambda, k_{\bar{\imath}}; q_\Lambda, \{p\}_n\big) \ \overset{\Lambda\to\infty}{\longrightarrow}\ \big|\overline{M}_{\star\bar{\imath}}\big|^2\big(k_\star, k_{\bar{\imath}}; \{p\}_n\big)$$

with

$$k_\Lambda^2 = q_\Lambda^2 = 0 \quad , \quad k_\Lambda^\mu - q_\Lambda^\mu \overset{\Lambda\to\infty}{=} xP^\mu + k_\perp^\mu \ .$$

The example of auxiliary quarks, for the gluonic contribution to dijet production, and neglecting momentum components of $\mathcal{O}(\Lambda^{-1})$.



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The matrix element $\left|\overline{M}_{\star\bar{\imath}}\right|^2$ is independent of the type of auxiliary parton $i$ used, partly thanks to the color correction factor $C_i (= 2C_A, 2C_F)$.

The factor $1/g_s^2$ corrects the power of the coupling constant.

The factor $|k_\perp|^2$ assures a smooth on-shell limit $|k_\perp| \to 0$.

The factor $x$ assures that $M_{\star\bar{\imath}}$ only depends on $xP$, not $x$ separately.

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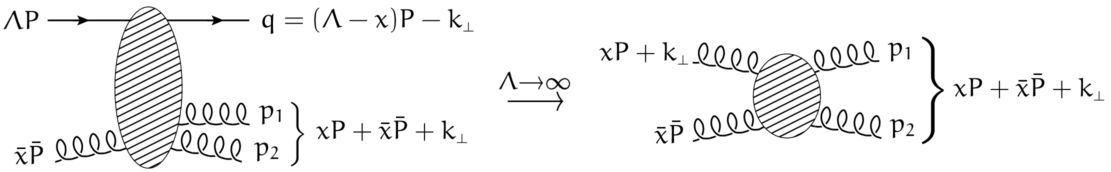


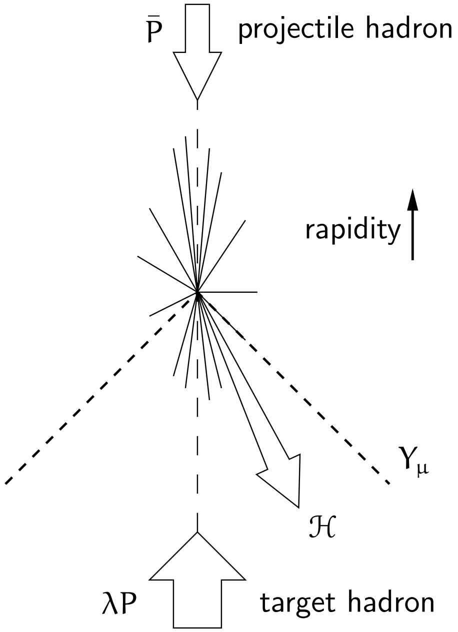
# Embedding in collinear factorization (CF)

Consider hadron collisions, with production of the final state of interest $\mathcal{H}$:

$$h(\lambda P) + h(\bar{P}) \to \mathcal{H} + \mathcal{X}$$

We assume that there is a natural rapidity $Y_\mu$ associated with $\mathcal{H}$, which separates the event into "target" and "projectile" parts. Then we can define

$$x = \sum_j \theta\big(y_j < Y_\mu\big)\,\frac{p_j\cdot\bar{P}}{P\cdot\bar{P}} \quad , \quad k_\perp = -\sum_j \theta\big(y_j < Y_\mu\big)\,p_{j\perp}$$



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We can associate rapidity scale $\mu_Y$ (Collins-Soper scale) to the rapidity $Y_\mu$ as

$$\mu_Y = \nu x\, e^{-Y_\mu} \quad \Leftrightarrow \quad Y_\mu = \ln\frac{\nu x}{\mu_Y} \quad , \quad \nu^2 = (P+\bar{P})^2$$

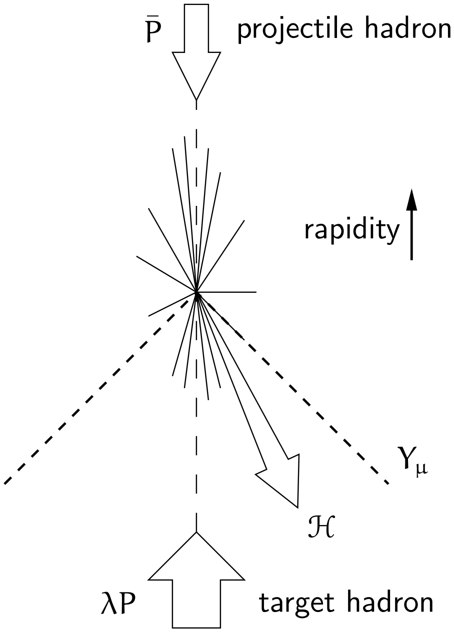
Due to IRC-safety of variables $x$ and $k_\perp$, the hadronic differential cross section

$$\frac{d\sigma_\lambda^{CF}}{dx\,d^2k_\perp}(x,k_\perp,\dots) = \sum_{i,\bar{\imath}}\int_0^1 dX\, f_i(X)\int_0^1 d\bar{x}\, f_{\bar{\imath}}(\bar{x})\,\frac{d\hat{\sigma}_{i\bar{\imath}}^{CF}}{dx\,d^2k_\perp}(\lambda X,\bar{x};x,k_\perp,\dots)$$

$\lambda X = \Lambda$

should be computable in CF, at least up to NLO, and in the limit:

$$\lambda \to \infty \quad , \quad x, k_\perp - \text{fixed}\ .$$



# kT-factorizable cross section

## $\Lambda = \lambda X$ must go to $\infty$

absolute kinematic minimum: $X > x/\lambda$

$$d\sigma^{\mathrm{LO}} = \int_{x/\lambda}^{\delta_0} dX\, f(X)\, d\hat{\sigma}^{\mathrm{LO}}(\lambda X) + \int_{\delta_0}^{1} dX\, f(X)\, d\hat{\sigma}^{\mathrm{LO}}(\lambda X)$$

$\boxed{\text{non } k_T\text{-factorizable}}$

$$\delta_0 = \frac{\lambda_0 x |k_\perp|}{\lambda \mu_Y} \quad , \quad \lambda \succ \lambda_0 \to \infty$$

# kT-factorizable cross section

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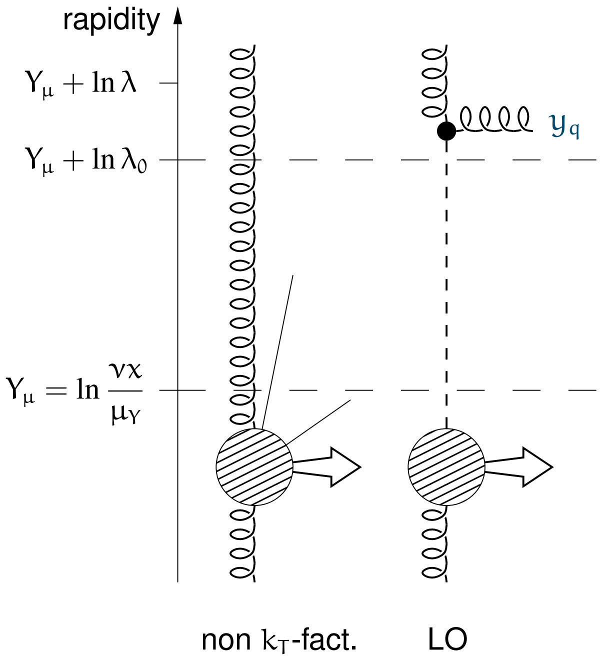


absolute kinematic minimum: $X > x/\lambda$

$$y_q = \ln\frac{\nu(\lambda X - x)}{|k_\perp|} > Y_\mu + \ln\lambda_0 \quad \Rightarrow \quad X > \delta_0$$

The rapidity $y_q$ of the auxiliary parton must be large enough in order to guarantee that $X$ is large enough.

$$d\sigma^{LO} = \underbrace{\int_{x/\lambda}^{\delta_0} dX\, f(X)\, d\hat{\sigma}^{LO}(\lambda X)}_{\text{non } k_T\text{-factorizable}} + \int_{\delta_0}^{1} dX\, f(X)\, d\hat{\sigma}^{LO}(\lambda X)$$

$$\delta_0 = \frac{\lambda_0 x |k_\perp|}{\lambda \mu_Y} \quad , \quad \lambda \succ \lambda_0 \to \infty$$

# kT-factorizable cross section

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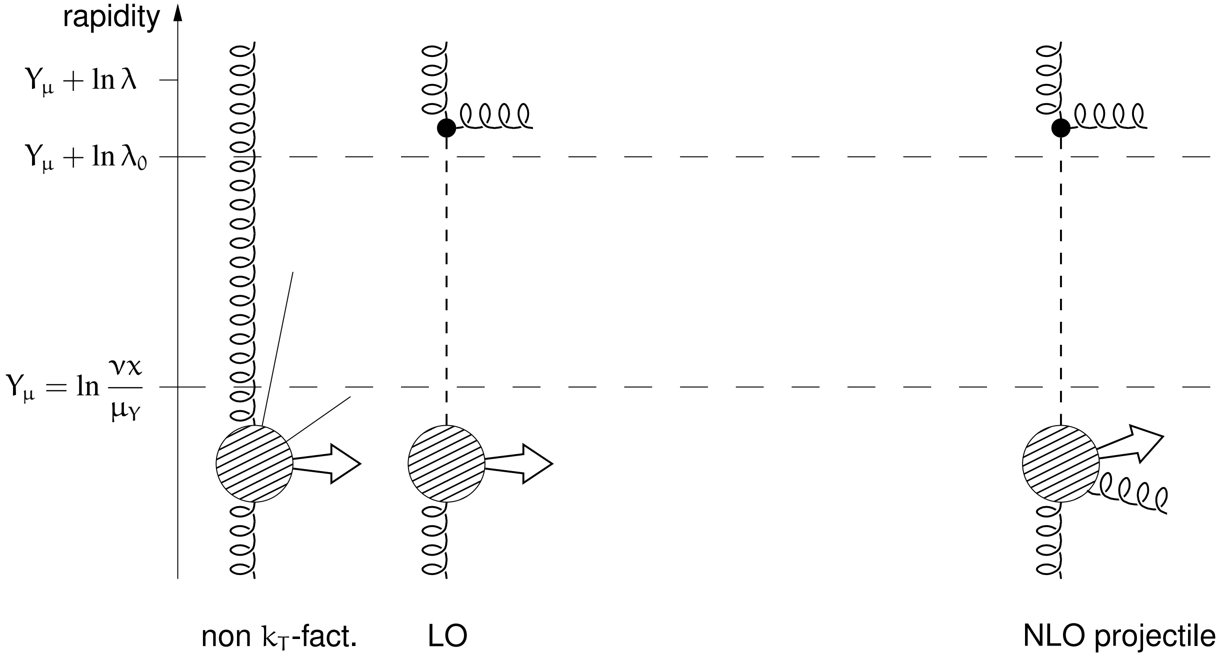
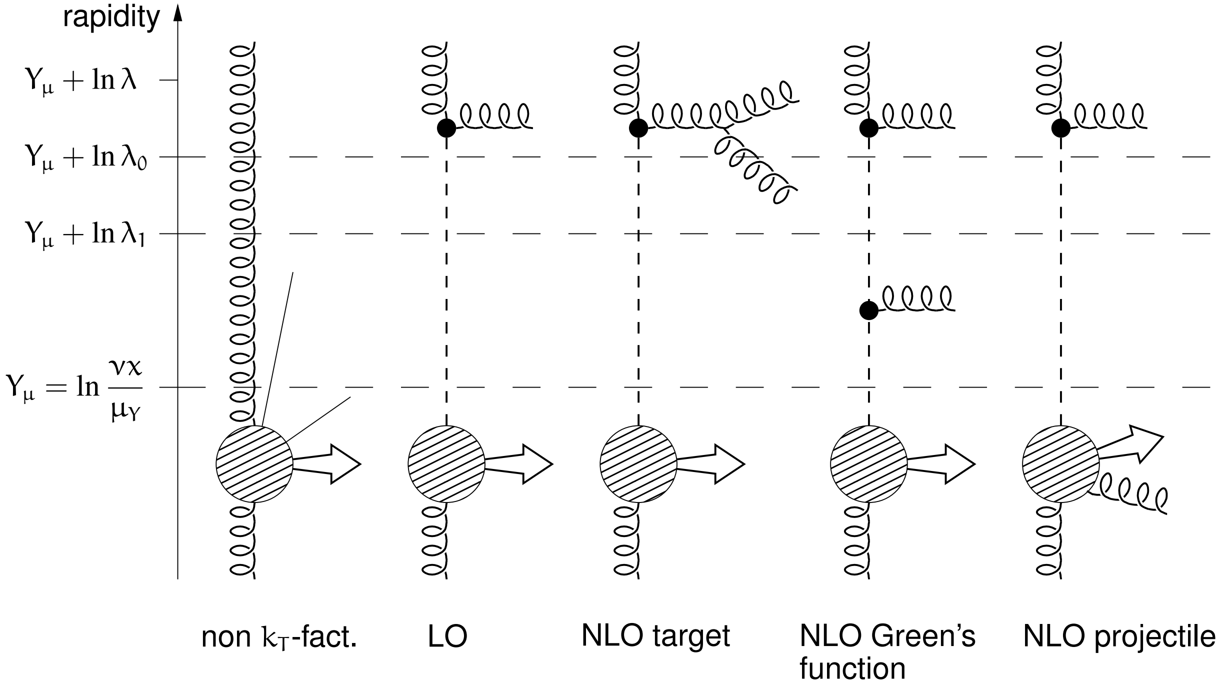


# kT-factorizable cross section

$\Lambda = \lambda X$ must go to $\infty$




$\lambda \succ \lambda_0 \succ \lambda_1 \to \infty$

It turns out that another rapidity separator $\lambda_1$ is needed to consistently define the target and Green's function contributions.

# Born contribution

Given the rapidity separator $Y_\mu$, which separates the event into "target" and "projectile" parts, we define

$$x = \sum_j \theta(y_j < Y_\mu)\,\frac{p_j\cdot\bar{P}}{P\cdot\bar{P}} \quad , \quad k_\perp = -\sum_j \theta(y_j < Y_\mu)\,p_{j\perp}$$

The $k_T$-factorizable Born level differential cross section is defined as

$$\frac{d\sigma^{\mathrm{CF,B}}_{\lambda}(\{p\}_n)}{dx\,d^2k_\perp} = \sum_{i,\bar{\imath}} \int_{\delta_0}^{1} dX\, f_i(X) \int_0^1 d\bar{x}\, f_{\bar{\imath}}(\bar{x})\, \frac{d\hat{\sigma}^{B}_{i\bar{\imath}}(\lambda X, \bar{x}; \{p\}_n)}{dx\,d^2k_\perp}$$

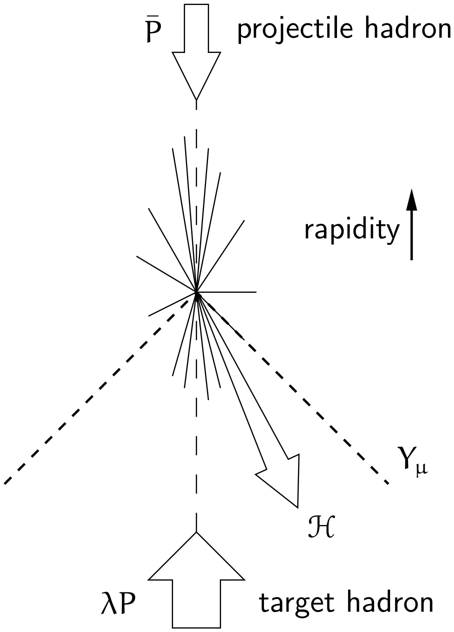
$\lambda \to \infty$ is guaranteed to be equivalent to $\Lambda = \lambda X \to \infty$, and we get

$$\frac{d\sigma^{\mathrm{CF,B}}_{\lambda\to\infty}(\{p\}_n)}{dx\,d^2k_\perp} = F^{\mathrm{LO}}(\delta_0, k_\perp) \sum_{\bar{\imath}} \int_0^1 d\bar{x}\, f_{\bar{\imath}}(\bar{x})\, dB_{\star\bar{\imath}}(x, k_\perp, \bar{x}; \{p\}_n) \ ,$$

where $dB_{\star\bar{\imath}}$ is the usual Born-level "partonic off-shell" cross section, and with

$$F^{\mathrm{LO}}(\delta_0, k_\perp) = \sum_i \int_{\delta_0}^{1} dX\, f_i(X)\, \frac{\alpha_s C_i}{2\pi^2 |k_\perp|^2} \ ,$$

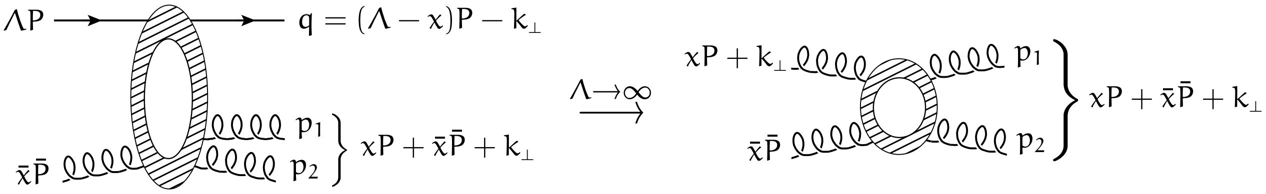
a "proto UPDF". The Born contribution is independent of $Y_\mu$.



# Virtual contribution

Blanco, Giachino, AvH, Kotko 2023, AvH, Motyka, Ziarko 2022

$$\frac{d\sigma_\lambda^{\mathrm{CF,V}}(\{p\}_n)}{dx\,d^2k_\perp} \xrightarrow{\lambda\to\infty} \sum_{i,\bar\imath}\int_{\delta_0}^1 dX\, f_i(X)\,\frac{\alpha_s C_i}{2\pi^2|k_\perp|^2}\int_0^1 d\bar x\, f_{\bar\imath}(\bar x)\Big[\; dV^{\mathrm{unf}}_{i\bar\imath}(\epsilon,\lambda X;x,k_\perp,\bar x;\{p\}_n) \;+\; dV^{\mathrm{fam}}_{\star\bar\imath}(\epsilon;x,k_\perp,\bar x;\{p\}_n)\;\Big]$$



# Virtual contribution

$$\frac{d\sigma_\lambda^{CF,V}(\{p\}_n)}{dx\,d^2k_\perp} \xrightarrow{\lambda\to\infty} \sum_{i,\bar\imath}\int_{\delta_0}^1 dX\,f_i(X)\,\frac{\alpha_s C_i}{2\pi^2|k_\perp|^2}\int_0^1 d\bar x\,f_{\bar\imath}(\bar x)\Big[\boxed{dV_{i\bar\imath}^{unf}(\epsilon,\lambda X;x,k_\perp,\bar x;\{p\}_n)}+\boxed{dV_{\star\bar\imath}^{fam}(\epsilon;x,k_\perp,\bar x;\{p\}_n)}\Big]$$

Contains $\ln\lambda$, $\ln k_\perp$, and depends on auxiliary parton type.

Looks looks similar to if $k_\perp$ were zero.

$$\begin{aligned}
dV_{i\bar\imath}^{targ}(\epsilon,\Lambda,\lambda_1,\mu_Y) &= dV_{i\bar\imath}^{unf}(\epsilon,\Lambda) - dB_{\star\bar\imath}\times a_\epsilon\left[\left(\frac{\mu^2}{\mu_Y^2}\right)^\epsilon\frac{N_c}{\epsilon^2}+\frac{\gamma_g}{\epsilon}+\left(\frac{\mu^2}{|k_\perp|^2}\right)^\epsilon\frac{2N_c\ln\lambda_1}{\epsilon}\right]\\
dV_{\star\bar\imath}^{Green}(\epsilon,\lambda_1) &= dB_{\star\bar\imath}\times a_\epsilon\left[\left(\frac{\mu^2}{|k_\perp|^2}\right)^\epsilon\frac{2N_c\ln\lambda_1}{\epsilon}\right]\\
dV_{\star\bar\imath}^{proj}(\epsilon,\mu_Y) &= dV_{\star\bar\imath}^{fam}(\epsilon) + dB_{\star\bar\imath}\times a_\epsilon\left[\left(\frac{\mu^2}{\mu_Y^2}\right)^\epsilon\frac{N_c}{\epsilon^2}+\frac{\gamma_g}{\epsilon}\right].
\end{aligned}$$

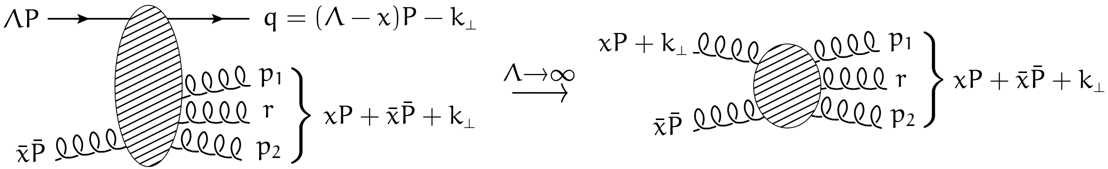
$$\gamma_g = \frac{\beta_0}{2} = \frac{11N_c}{6} - \frac{2T_R n_f}{3}$$

# Real projectile contribution

Familiar real contribution

Includes collinear divergence associated with the positive-rapidity initial state.

No restriction in rapidity on the radiation.




The radiative matrix element in the collinear region $r||P$ is given by

$$\left|\overline{M}_{\star\bar{\imath}}\right|^2(k_\star, k_{\bar{\imath}}; r, \{p\}_n) \xrightarrow{\bar{x}_r\to 0} g_s^2\,\frac{4N_c}{|r_\perp|^2(1-x_r/x)^2}\,\left|\overline{M}_{\star\bar{\imath}}\right|^2(k_\star - x_r P - r_\perp, k_{\bar{\imath}}; \{p\}_n)$$

$$\frac{2N_c}{z(1-z)}$$

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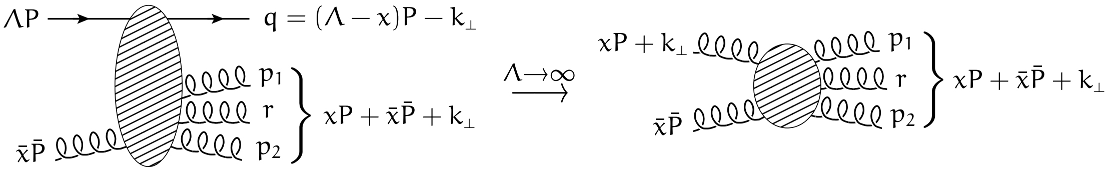
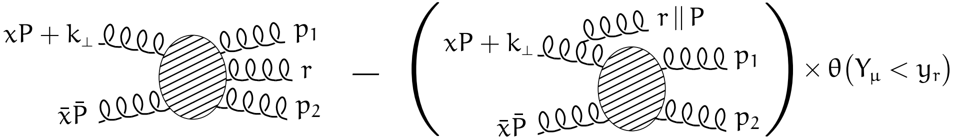
Includes collinear divergence associated with the positive-rapidity initial state.

No restriction in rapidity on the radiation.




## Real projectile contribution

Remove this collinear region, but only for the radiation rapidity above $Y_\mu$.




The radiative matrix element in the collinear region $r\|P$ is given by

$$\left|\overline{M}_{\star\bar{\imath}}\right|^2(k_\star, k_{\bar{\imath}}; r, \{p\}_n) \xrightarrow{\bar{x}_r\to 0} g_s^2 \frac{4N_c}{|r_\perp|^2(1 - x_r/x)^2} \left|\overline{M}_{\star\bar{\imath}}\right|^2(k_\star - x_r P - r_\perp, k_{\bar{\imath}}; \{p\}_n)$$

$$\boxed{\frac{2N_c}{z(1-z)}}$$

# Complete finite projectile contribution

The negative-rapidity-side PDF needs to be renormalized like usual to arrive at a finite result

$$f_{\bar{\imath}}(\bar{x}) \longrightarrow f_{\bar{\imath}}(\bar{x}, \mu_{\bar{F}}) + \frac{a_\epsilon}{\epsilon}\left(\frac{\mu^2}{\mu_{\bar{F}}^2}\right)^\epsilon \sum_{\bar{\imath}'} \left[\mathcal{P}_{\bar{\imath}\bar{\imath}'} \otimes f_{\bar{\imath}'}\right](\bar{x})$$

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$$\begin{aligned}\text{complete finite projectile contribution} = &\Big[\text{resolved contribution}\Big] \\ &+ \Big[\text{unresolved independent of } \mu_{\bar{F}}, \mu_Y\Big] \\ &+ \Big[\text{remnants of collinear cancellations/subtractions}\Big](\mu_{\bar{F}}, \mu_Y)\end{aligned}$$

# Complete finite projectile contribution and evolution equation

The negative-rapidity-side PDF needs to be renormalized like usual to arrive at a finite result

$$f_{\bar{\imath}}(\bar{x}) \to f_{\bar{\imath}}(\bar{x},\mu_{\bar{F}}) + \frac{a_\epsilon}{\epsilon}\left(\frac{\mu^2}{\mu_{\bar{F}}^2}\right)^{\epsilon}\sum_{\bar{\imath}'}\left[\mathcal{P}_{\bar{\imath}\bar{\imath}'}\otimes f_{\bar{\imath}'}\right](\bar{x})$$

$$\text{complete finite projectile contribution} = \Big[\text{resolved contribution}\Big]$$

$$+\Big[\text{unresolved independent of } \mu_{\bar{F}},\mu_Y\Big]$$

$$\boxed{+\Big[\text{remnants of collinear cancellations/subtractions}\Big](\mu_{\bar{F}},\mu_Y)}$$

contract $(x,k_\perp)$-dependent differential cross section with "test UPDF" $\hat{F}(x,k_\perp;\mu_Y)/x$, and demand cross section is independent of $\mu_Y$

$$\frac{d\hat{F}(x,k_\perp;\mu_Y)}{d\ln\mu_Y^2} = \frac{\alpha_s N_c}{2\pi^2}\int\frac{d^2r_\perp}{|r_\perp|^2}\left\{\hat{F}\left(x\left[1+\frac{|r_\perp|}{\mu_Y}\right],k_\perp+r_\perp;\mu_Y\right)\theta\left(|r_\perp|<\mu_Y\frac{1-x}{x}\right)-\theta\big(\mu_Y-|r_\perp|\big)\,\hat{F}(x,k_\perp;\mu_Y)\right\}$$

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contract $(x,k_\perp)$-dependent differential cross section with "test UPDF" $\hat{F}(x,k_\perp;\mu_Y)/x$, and demand cross section is independent of $\mu_Y$

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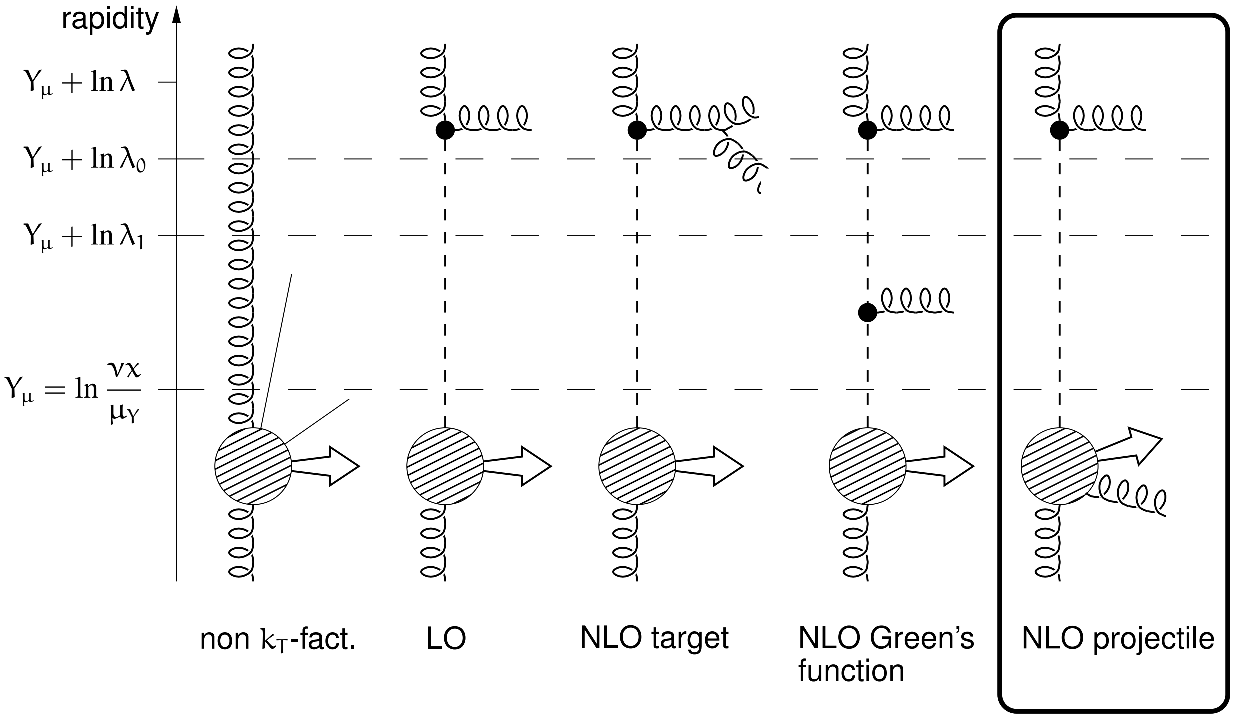
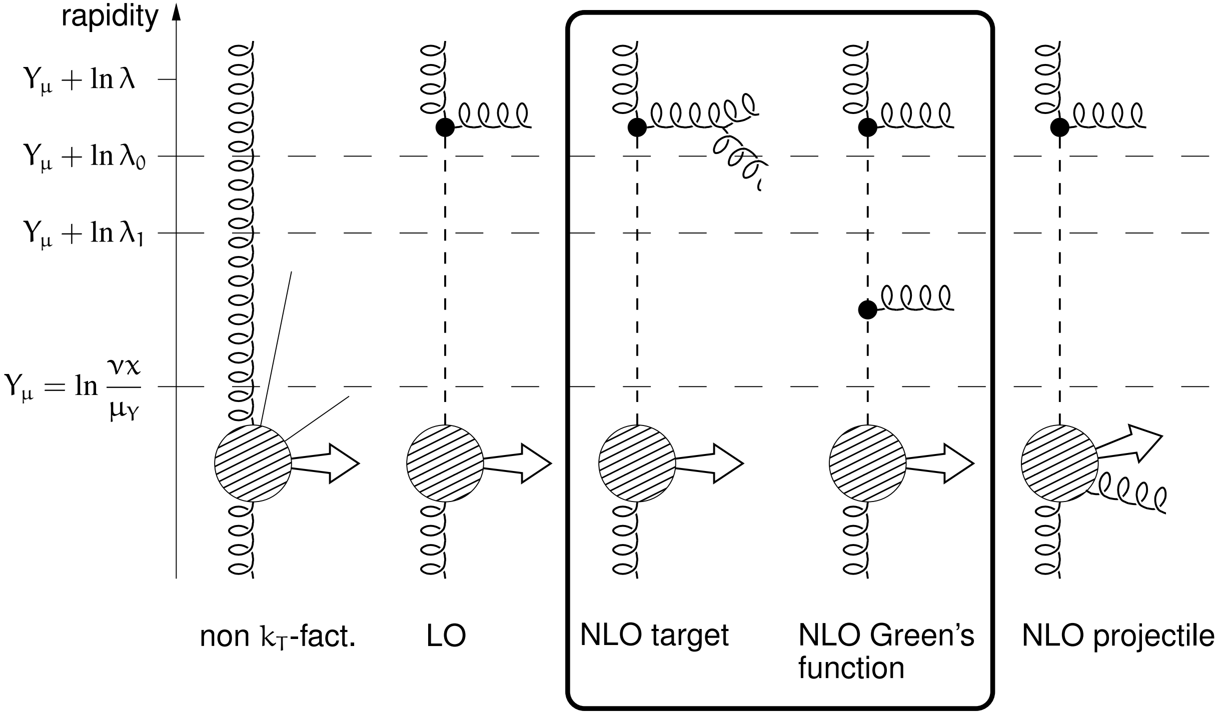
- Closely related to the LO Collins-Soper-Sterman equation
- Very similar to the equation from Born-Oppenheimer renormalization group by Duan, Kovner, Lublinsky 2024.

# Real target and Green's function contribution



# Real target and Green's function contribution



# Target collinear counter term

We need to include the collinear counter term for the target-side collinear PDF

$$\frac{d\sigma_{\lambda}^{\mathsf{CF,B}}(\{p\}_n)}{dx\,d^2k_{\perp}} = \sum_{i,\bar{\imath}} \boxed{\int_0^1 dX\, f_i(X)\,\theta(X > \delta_0)} \int_0^1 d\bar{x}\, f_{\bar{\imath}}(\bar{x})\, \frac{d\hat{\sigma}^{\mathsf{B}}_{i\bar{\imath}}(\lambda X, \bar{x}\,;\{p\}_n)}{dx\,d^2k_{\perp}}$$

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$$\boxed{\int_0^1 dX\, f_i(X)\,\theta(X>\delta_0) + \int_0^1 dX\,\frac{a_\epsilon}{\epsilon}\left(\frac{\mu^2}{\mu_F^2}\right)^{\epsilon}\sum_{i'}\big[\mathcal{P}_{ii'}\otimes f_{i'}\theta_{>\delta_0}\big](X)\,\theta(X>\delta_1)}$$

$$\frac{\lambda_0 x|k_\perp|}{\lambda\mu_Y} = \delta_0 \succ \delta_1 = \frac{\lambda_1 x|k_\perp|}{\lambda\mu_Y}$$

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$$\frac{\lambda_0 x|k_\perp|}{\lambda\mu_Y} = \delta_0 \succ \delta_1 = \frac{\lambda_1 x|k_\perp|}{\lambda\mu_Y}$$

# Complete finite target impact factor contribution

after proper coupling constant renormalization

$$[\mathcal{V}+\mathcal{R}-\mathcal{C}]_i^{\mathrm{targ}}(\Lambda,\lambda_1,\mu_Y,\mu_F;x,k_\perp) = \left[\mathcal{J}_i^{(0)} - 2N_c\ln\frac{\Lambda\mu_Y}{\lambda_1 x|k_\perp|}\right]\ln\frac{\mu_F^2}{|k_\perp|^2} + 2\gamma_g\ln\frac{\mu^2}{|k_\perp|^2} + 2\mathcal{K} + \mathcal{J}_i^{(1)} - \frac{N_c}{2}\ln^2\frac{\mu_Y^2}{|k_\perp|^2}$$

$$\mathcal{J}_{q/\bar{q}} = \frac{3N_c}{2} + \frac{N_c}{2}\epsilon \quad , \quad \mathcal{J}_g = \frac{11N_c}{6} + \frac{n_f}{3N_c^2} - \frac{n_f}{6N_c^2}\epsilon$$

Ciafaloni, Colferai 1999

# Unintegrated PDF

$$\frac{d\sigma^{\mathrm{CF,B+NLO}}_{\lambda\to\infty,\mathrm{targ+Green}}(\{p\}_n)}{dx\,d^2k_\perp} = \boxed{F^{\mathrm{LO+NLO}}(\cdot)} \sum_{\bar\imath}\int_0^1 d\bar x\, f_{\bar\imath}(\bar x)\, dB_{\star\bar\imath}(x,k_\perp,\bar x\,;\{p\}_n)$$

$$\boxed{\sum_i \frac{\alpha_s C_i}{2\pi^2|k_\perp|^2}\int_{\delta_0}^1 dX\, f_i(X,\mu_F)\Big[1 + a_\epsilon[\mathcal{V}+\mathcal{R}-\mathcal{C}]_i^{\mathrm{targ}}(\lambda X,\lambda_1,\mu_Y,\mu_F\,;x,k_\perp) + a_\epsilon[\mathcal{V}+\mathcal{R}-\mathcal{C}]^{\mathrm{Green}}(\lambda_1,\mu_F\,;k_\perp)\Big]}$$

# Unintegrated PDF

$$\frac{d\sigma^{\mathrm{CF,B+NLO}}_{\lambda\to\infty,\mathrm{targ+Green}}(\{p\}_n)}{dx\,d^2k_\perp} = \boxed{F^{\mathrm{LO+NLO}}(\cdot)}\sum_{\bar{\imath}}\int_0^1 d\bar{x}\, f_{\bar{\imath}}(\bar{x})\, dB_{\star\bar{\imath}}(x,k_\perp,\bar{x};\{p\}_n)$$

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$\lambda\to\infty$ for $x$-fixed, is equivalent to $x\to 0$ for $\lambda=1$
set $\lambda_1 = (X\mu_Y)/(x|k_\perp|)$
set $\mu_Y = |k_\perp|$

$$\boxed{F(x,k_\perp,\mu_Y=|k_\perp|) = \sum_i\int_x^1 dX\, f_i(X,\mu_F)\int d^{2-2\epsilon}k'_\perp\, I_i(k'_\perp,\mu_F)\, G\left(k'_\perp,k_\perp,\frac{X}{x},\mu_F\right)}$$

# Unintegrated PDF

$$\frac{d\sigma^{\mathrm{CF,B+NLO}}_{\lambda\to\infty,\mathrm{targ+Green}}(\{p\}_n)}{dx\,d^2k_\perp} = \boxed{F^{\mathrm{LO+NLO}}(\cdot)}\sum_{\bar{\imath}}\int_0^1 d\bar{x}\,f_{\bar{\imath}}(\bar{x})\,dB_{\star\bar{\imath}}(x,k_\perp,\bar{x};\{p\}_n)$$

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$$I_i(k_\perp,\mu_F) = \frac{\alpha_s C_i}{2\pi^2|k_\perp|^2}\left\{1 + \frac{\alpha_s}{2\pi}\left[\mathcal{J}_i^{(0)}\ln\frac{\mu_F^2}{|k_\perp|^2} + 2\gamma_g\ln\frac{\mu^2}{|k_\perp|^2} + 2\mathcal{K} + \mathcal{J}_i^{(1)}\right] + \mathcal{O}(\alpha_s^2)\right\}$$

$$\mathcal{J}_{q/\bar{q}} = \frac{3N_c}{2} + \frac{N_c}{2}\epsilon$$

$$\mathcal{J}_g = \frac{11N_c}{6} + \frac{n_f}{3N_c^2} - \frac{n_f}{6N_c^2}\epsilon$$

# Unintegrated PDF

$$\frac{d\sigma^{\mathrm{CF,B+NLO}}_{\lambda\to\infty,\mathrm{targ+Green}}(\{p\}_n)}{dx\,d^2k_\perp} = \boxed{F^{\mathrm{LO+NLO}}(\cdot)}\sum_{\bar{\imath}}\int_0^1 d\bar{x}\, f_{\bar{\imath}}(\bar{x})\, dB_{\star\bar{\imath}}(x,k_\perp,\bar{x};\{p\}_n)$$

$$\boxed{\sum_i \frac{\alpha_s C_i}{2\pi^2|k_\perp|^2}\int_{\delta_0}^1 dX\, f_i(X,\mu_F)\Big[1 + a_\epsilon[\mathcal{V}+\mathcal{R}-\mathcal{C}]_i^{\mathrm{targ}}(\lambda X,\lambda_1,\mu_Y,\mu_F;x,k_\perp) + a_\epsilon[\mathcal{V}+\mathcal{R}-\mathcal{C}]^{\mathrm{Green}}(\lambda_1,\mu_F;k_\perp)\Big]}$$

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$$G(k'_\perp,k_\perp,y,\mu_F) = \delta^{(2-2\epsilon)}(k'_\perp-k_\perp)+a_\epsilon\int_y^1\frac{dz}{z}\int d^{2-2\epsilon}q_\perp\Big[K_{\mathrm{BFKL}}(k'_\perp,q_\perp)-\theta(\mu_F^2-|k'_\perp|^2)\,K_{\mathrm{BFKL}}(0,q_\perp)\Big]G\left(q_\perp,k_\perp,\frac{y}{z},\mu_F\right)$$

$$K_{\mathrm{BFKL}}(k'_\perp,q_\perp) = 4N_c\frac{\mu^{2\epsilon}}{2\pi_\epsilon}\left[\frac{1}{|k'_\perp-q_\perp|^2}+\delta^{(2-2\epsilon)}(k'_\perp-q_\perp)\frac{\pi_\epsilon}{\epsilon}|q_\perp^2|^{-\epsilon}\right]$$

# Unintegrated PDF

$$\frac{d\sigma^{\mathrm{CF,B+NLO}}_{\lambda\to\infty,\mathrm{targ+Green}}(\{p\}_n)}{dx\,d^2k_\perp} = \boxed{F^{\mathrm{LO+NLO}}(\cdot)} \sum_{\bar{\imath}} \int_0^1 d\bar{x}\, f_{\bar{\imath}}(\bar{x})\, dB_{\star\bar{\imath}}(x,k_\perp,\bar{x};\{p\}_n)$$

$$\boxed{\sum_i \frac{\alpha_s C_i}{2\pi^2|k_\perp|^2}\int_{\delta_0}^1 dX\, f_i(X,\mu_F)\Big[1 + a_\epsilon[\mathcal{V}+\mathcal{R}-\mathcal{C}]_i^{\mathrm{targ}}(\lambda X,\lambda_1,\mu_Y,\mu_F\,;x,k_\perp) + a_\epsilon[\mathcal{V}+\mathcal{R}-\mathcal{C}]^{\mathrm{Green}}(\lambda_1,\mu_F\,;k_\perp)\Big]}$$

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$$G(k'_\perp,k_\perp,y,\mu_F) = \delta^{(2-2\epsilon)}(k'_\perp-k_\perp) + a_\epsilon\int_y^1\frac{dz}{z}\int d^{2-2\epsilon}q_\perp\Big[K_{\mathrm{BFKL}}(k'_\perp,q_\perp) - \theta(\mu_F^2-|k'_\perp|^2)\,K_{\mathrm{BFKL}}(0,q_\perp)\Big]G\Big(q_\perp,k_\perp,\frac{y}{z},\mu_F\Big)$$

The initial condition $F(x,x_\perp,\mu_Y=|k_\perp|)$ resums $\ln(X/x)$,
while the $\mu_Y$-evolution resums $\ln(\mu_Y/|k_\perp|)$.

# Conclusions

- ITMD factorization provides a momentum-space formula suitable to study saturation in QCD within an automatable Monte Carlo approach
- collinear embedding is a powerfull strategy to extract NLO formulas for cross sections and evolution equations within $k_T$-type factorization