

TMD factorization in diffractive heavy quark production in $\gamma^* A$

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Outline

Production of heavy quarks in diffraction

1) Gluon diffractive TMD

a. Introduction

b. Color Glass Condensate theory

c. Light-cone perturbation theory (LCPT)

d. TMD factorization

2) Quark diffractive TMD

3) Summary

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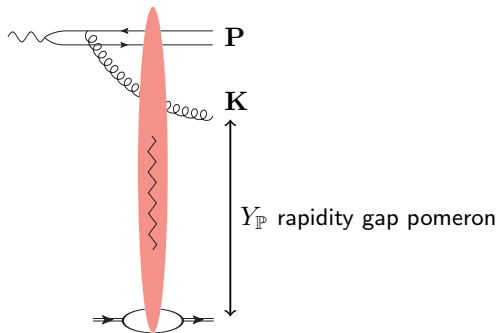
d. TMD factorization

2) Quark diffractive TMD

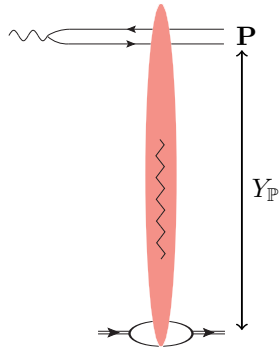
3) Summary

Introduction

Motivation



$$\text{Diffractive jets} \sim \frac{1}{P_{\perp}^4}$$



$$\text{Exclusive jets} \sim \frac{1}{P_{\perp}^6} \rightarrow \text{More suppressed!}$$

- New experimental data on diffractive J/ψ , D , Υ production

Introduction

General picture

Correlation limit (back-to-back)

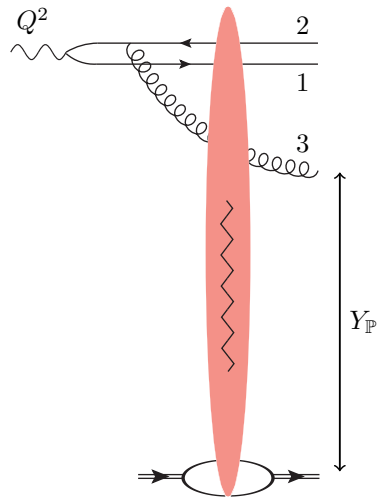
hard jets: $m, k_{1\perp} \simeq k_{2\perp} \sim Q \gg Q_s$

semi-hard jet: $k_{3\perp} \sim Q_s$

Goal $\rightarrow$ TMD factorization from dipole picture

$$\frac{d\sigma}{dz_1 dz_2 d^2\mathbf{K} d^2\mathbf{P} dY_{\mathbb{P}}} = \underbrace{H(z_1, z_2, P_{\perp}, m)}_{\text{"hard" factor}} \underbrace{\frac{dx\mathcal{G}_{\mathbb{P}}(x, x_{\mathbb{P}}, K_{\perp})}{d^2\mathbf{K}}}_{\text{gluon diffractive TMD}}$$

m	quark mass
$Q_s \sim \text{GeV}$	saturation scale
$\mathbf{P} \simeq \mathbf{k}_1 \simeq -\mathbf{k}_2$	relative momentum
$\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2 = -\mathbf{k}_3$	imbalance



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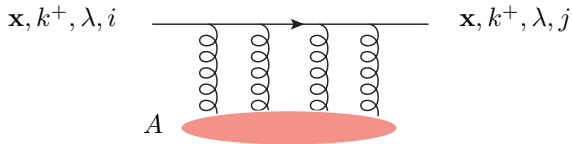
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Color Glass Condensate

Eikonal interaction

- At small x many gluons in the target $\rightarrow$ described as a classical gluon field A
- Eikonal interaction probe with color field in mixed space $(\mathbf{x}, k^+)$:



λ : helicity, i, j : color

Wilson line

Path ordered exponential $\rightarrow$ resums multiple scatterings (not perturbative)

$$V_{F,A}(\mathbf{x}) = \mathcal{P} \exp \left(-ig \int dx^+ A_a^-(x^+, \mathbf{x}) t^a \right) \quad t^a : SU(3) \text{ color matrix}$$

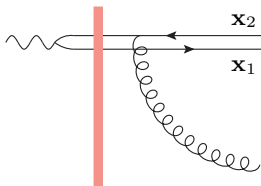
Color Glass Condensate

Eikonal interaction

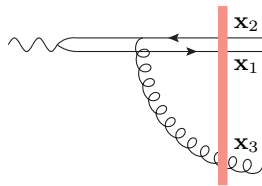
- scattering operator $\hat{S}$ on states $\Rightarrow$ color rotation ($i \rightarrow j$) in mixed space $\Rightarrow$ Wilson line

$$\hat{S} |q(k^+, \mathbf{x}, i)\rangle = \sum_j V_F(\mathbf{x})_{ji} |q(k^+, \mathbf{x}, j)\rangle$$

Wilson
line



$$S_{q\bar{q}}(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{N_c} \langle \text{Tr} [V_F(\mathbf{x}_1) V_F^\dagger(\mathbf{x}_2)] \rangle$$



$$S_{q\bar{q}g}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \frac{1}{N_c C_F} \langle V_A^{ba}(\mathbf{x}_3) \text{Tr} [t^b V_F(\mathbf{x}_1) t^a V_F^\dagger(\mathbf{x}_2)] \rangle$$

Hard quarks

- Change of variables

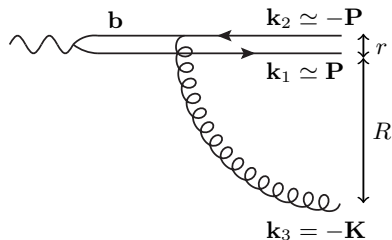
relative momentum: $\mathbf{P} \equiv z_2 \mathbf{k}_1 - z_1 \mathbf{k}_2$

total momentum: $\mathbf{K} \equiv \mathbf{k}_1 + \mathbf{k}_2$

center of mass: $\mathbf{b} \equiv z_1 \mathbf{x}_1 + z_2 \mathbf{x}_2$

$q\bar{q}$ size: $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$

separation $g - q\bar{q}$: $\mathbf{R} = \mathbf{x}_3 - \mathbf{b}$



- Correlation limit: $\mathbf{P}, m \sim Q \gg K_{\perp} = k_{3\perp} \sim Q_s$

- Exponent of the Fourier transform has $\mathbf{P} \cdot \mathbf{r}$

- $\Psi^{\gamma^* \rightarrow q\bar{q}g}$ from LCPT has Bessel function $K_n(r\sqrt{Q^2 z_1 z_2 + m^2})$

$\Rightarrow r \ll R$

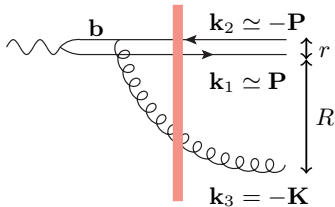
Color Glass Condensate

Effective gg dipole

- correlation limit $\Rightarrow r \ll R$ in coordinate space

$$\mathcal{S}_{q\bar{q}g}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \simeq \mathcal{S}_{q\bar{q}g}(\mathbf{b}, \mathbf{b}, \mathbf{R} + \mathbf{b}) \sim \langle U^{ba}(\mathbf{R} + \mathbf{b}) \text{Tr} [t^b U^{\dagger ab}(\mathbf{b}) t^b] \rangle \sim \langle \text{Tr} [U(\mathbf{R} + \mathbf{b}) U^\dagger(\mathbf{b})] \rangle \sim \mathcal{S}_g(\mathbf{R}).$$

$S_g(\mathbf{R})$: Effective gluon-gluon dipole of transverse separation $\mathbf{R}$



$$\mathcal{S}_{q\bar{q}g}(\mathbf{b}, \mathbf{b}, \mathbf{R} + \mathbf{b}) - \underbrace{\mathcal{S}_{q\bar{q}}(\mathbf{b}, \mathbf{b})}_{\text{color transparency}} \simeq \mathcal{S}_g(\mathbf{R}) - 1 = -\mathcal{N}_g(\mathbf{R})$$

independent of $\mathbf{r}$

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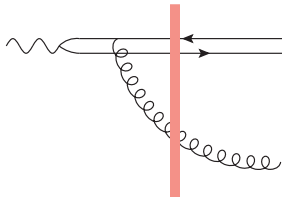
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Light Cone Perturbation Theory

Invariant amplitude

- Fock state expansion projectile (dipole frame)



$$|\gamma^*\rangle^{\text{in}} \sim |\gamma^*\rangle_0 + \Psi^{\gamma^* \rightarrow q\bar{q}} |q\bar{q}\rangle_0 + \Psi^{\gamma^* \rightarrow q\bar{q}g} |q\bar{q}g\rangle_0 + \dots$$

$$|q\bar{q}g\rangle_s^{\text{out}} \sim |q\bar{q}g\rangle_0 + \Psi^{q\bar{q}g \rightarrow q\bar{q}} |q\bar{q}\rangle_0 + \dots$$

color singlet
($\mathbb{P}$ exchange)

$\Psi(\mathbf{k}, z)$: perturbative light cone wavefunctions

- Invariant amplitude from scattering matrix element:

$$\mathcal{M}^{\gamma^* \rightarrow q\bar{q}g} \sim {}^{\text{out}}_s \langle q\bar{q}g | 1 - \hat{S} | \gamma^* \rangle^{\text{in}}$$

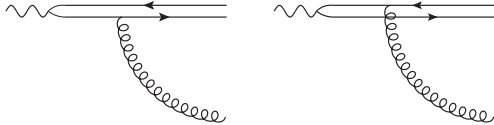
$\hat{S}$: scattering operator

z : fraction of q^+ carried by final partons

Light Cone Perturbation Theory

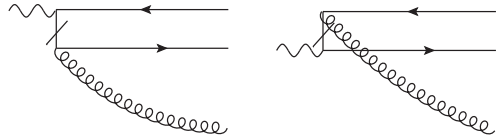
Light cone wavefunction

- $\Psi^{\gamma^* \rightarrow q\bar{q}g} = \Psi_{\text{regular}}^{\gamma^* \rightarrow q\bar{q}g} + \Psi_{\text{instantaneous}}^{\gamma^* \rightarrow q\bar{q}g}$



Regular contribution

+



Instantaneous contribution

Light Cone Perturbation Theory

Light cone wavefunction

$$\begin{aligned}\Psi^{\gamma^* \rightarrow q\bar{q}g} &= \Psi_{\text{regular}}^{\gamma^* \rightarrow q\bar{q}g} + \Psi_{\text{instantaneous}}^{\gamma^* \rightarrow q\bar{q}g} = \varepsilon_{\lambda}^i \varepsilon_{\sigma}^{*m} H^{in}(z_1, z_2, \mathbf{P}, m, Q) \frac{k_3^n k_3^m + \delta^{nm} \Omega^2 / 2}{k_{3\perp}^2 + \Omega^2} \\ &= \varepsilon_{\lambda}^i \varepsilon_{\sigma}^{*m} \left(H^{in}(z_1, z_2, \mathbf{P}, m, Q) \frac{k_3^n k_3^m - \delta^{nm} k_{3\perp}^2 / 2}{k_{3\perp}^2 + \Omega^2} + \frac{H^{in}(z_1, z_2, \mathbf{P}, m, Q)}{2} \right) \\ &\quad \text{traceless 2D tensor } (\mathbb{P}) \qquad \text{diagonal term}\end{aligned}$$

○ H^{in} : hard factor

$$H^{in}(z_1, z_2, \mathbf{P}, m, Q) = \frac{\varphi^{ij}(z_1, h_1) \delta^{jn} \delta_{h_1-h_2}}{P_{\perp}^2 + \tilde{Q}^2} - \frac{\varphi^{ij}(z_1, h_1) 2P^j P^n \delta_{h_1-h_2} - 2\sqrt{2}m\varepsilon_{h_1}^{i*} P^n \delta_{h_1 h_2}}{(P_{\perp}^2 + \tilde{Q}^2)^2}$$

$$\tilde{Q}^2 = Q^2 z_1 z_2 + m^2 \quad \Omega^2 = \frac{z_3}{z_1 z_2} (P_{\perp}^2 + \tilde{Q}^2)$$

Light Cone Perturbation Theory

Light cone wavefunction

- Fourier Transform to mixed space $(z, \mathbf{x})$

(!) small dipole $\mathbf{r}$ (color transparency) $\rightarrow$ neglect the change due to the scattering of $\mathbf{P}$.

$$\begin{aligned}\Psi^{\gamma^* \rightarrow q\bar{q}g}(z_1, z_2, z_3, \mathbf{P}, \mathbf{R}) &= \int \frac{d^2\mathbf{k}_3}{(2\pi)^2} e^{i\mathbf{k}_3 \cdot \mathbf{R}} \left[\Psi_{\mathbb{P}}^{im}(z_1, z_2, z_3, \mathbf{P}, \mathbf{k}_3) + \cancel{\Psi_{\text{diag}}^{im}(z_1, z_2, \mathbf{P})} \right] \\ &= H^{in}(z_1, z_2, \mathbf{P}, m, Q) \frac{1}{2\pi} \left[\frac{\delta^{nm}}{2} - \left(\frac{R^n R^m}{R^2} \right) \right] \Omega^2 K_2(\Omega R) .\end{aligned}$$

Light Cone Perturbation theory

Invariant amplitude

$$\mathcal{M}^{\gamma^* \rightarrow q\bar{q}g} \sim \varepsilon_\lambda^i \varepsilon_\sigma^{*m} H^{in}(z_1, z_2, \mathbf{P}, m, Q) G^{nm}(\mathbf{K}, \Omega, Y_{\mathbb{P}})$$

where

$$G^{nm}(\mathbf{K}, \Omega, Y_{\mathbb{P}}) = \frac{1}{2\pi} \int d^2\mathbf{R} e^{i\mathbf{K}\mathbf{R}} \left(\delta^{nm} - \frac{2R^n R^m}{R^2} \right) \Omega^2 K_2(\Omega R) \mathcal{N}_g(R) = \left(\frac{K^n K^m}{K_\perp^2} - \frac{\delta^{nm}}{2} \right) \mathcal{G}(K_\perp, \Omega, Y_{\mathbb{P}})$$

and

$$\mathcal{G}(K_\perp, \Omega, Y_{\mathbb{P}}) = 2(2\pi)\Omega^2 \int_0^\infty dR R J_2(K_\perp R) K_2(\Omega R) \mathcal{N}_g(R, Y_{\mathbb{P}})$$

- In collinear factorization: semi-hard gluon TMD is a target distribution. But here (!)

$$\Omega^2 \simeq \frac{z_3}{z_1 z_2} (P_\perp^2 + \tilde{Q}^2)$$

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Strategy

dipole picture (LCPT)

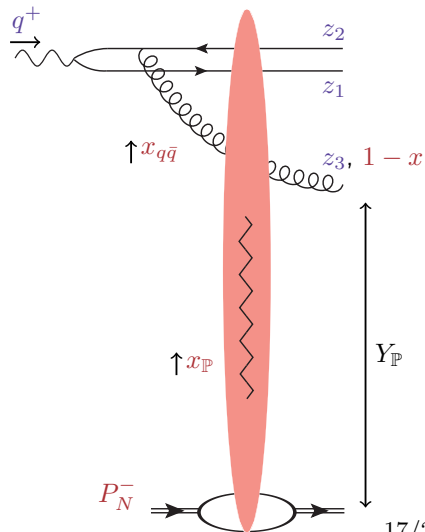
q^+ longitudinal momentum photon
 z fraction of q^+ carried by final partons



target variables

P_N^- longitudinal momentum target
 $x_{\mathbb{P}}$ fraction of P_N^- carried by the $\mathbb{P}$
 $x_{q\bar{q}}$ fraction of P_N^- transferred to $q\bar{q}$
 $x = x_{q\bar{q}}/x_{\mathbb{P}}$ fraction of $\mathbb{P}P_N^-$ carried by the gluon

gluon DTMD $\rightarrow$ target distribution $\frac{dx\mathcal{G}_{\mathbb{P}}(x, x_{\mathbb{P}}, K_{\perp})}{d^2\mathbf{K}}$



TMD factorization

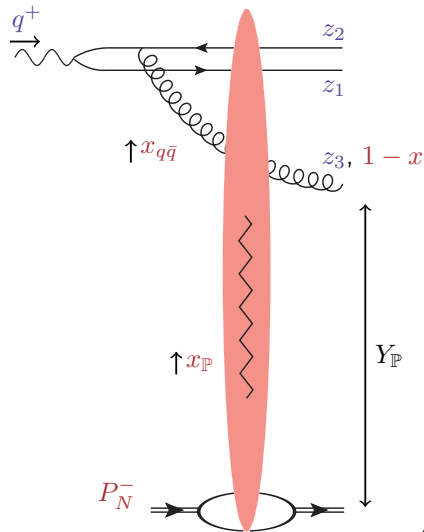
Change of variables

$$\Omega^2 \simeq \frac{z_3}{z_1 z_2} (P_\perp^2 + \tilde{Q}^2) \quad \rightarrow \quad \Omega^2 = \frac{x}{1-x} K_\perp^2$$

and

$$\frac{dz_3}{z_3} = \frac{dx_{\mathbb{P}}}{x_{\mathbb{P}} - x_{q\bar{q}}} = \frac{dx_{\mathbb{P}}}{x_{\mathbb{P}}} \frac{1}{1-x}$$

→ Gluon is seen as emitted from the $\mathbb{P}$.



TMD factorization

Cross section

$$\frac{d\sigma_D^{\gamma_T^* A \rightarrow q\bar{q}gA}}{dz_1 dz_2 d^2\mathbf{K} d^2\mathbf{P} dY_{\mathbb{P}}} = \delta(1-z_1-z_2) \alpha_{\text{e.m.}} \alpha_s e_f^2 \overbrace{\left[(z_1^2 + z_2^2) \frac{P_{\perp}^4 + \tilde{Q}^4}{[P_{\perp}^2 + \tilde{Q}^2]^4} + 2m^2 \frac{P_{\perp}^2}{[P_{\perp}^2 + \tilde{Q}^2]^4} \right]}^{\text{hard factor}} \overbrace{\frac{dx G_{\mathbb{P}}(x, K_{\perp}, Y_{\mathbb{P}})}{d^2\mathbf{K}}}^{\text{gluon DTMD}}$$

Gluon DTMD

density of gluons with $K_{\perp}$ and x in the $\mathbb{P}$ wavefunction.

$$\frac{dx G_{\mathbb{P}}(x, K_{\perp}, Y_{\mathbb{P}})}{d^2\mathbf{K}} \equiv \frac{S_{\perp}(N_c^2 - 1)}{8\pi^4(1-x)} \left[\Omega^2 \int_0^{\infty} dR R J_2(K_{\perp} R) K_2(\Omega R) \mathcal{N}_g(R, Y_{\mathbb{P}}) \right]^2$$

- > Same hard factor as in inclusive DIS (Dominguez, Marquet, Xiao, Yuan (2011))
- > Generalizes result of Iancu, Mueller, Triantafyllopoulos, Wei (2022) to heavy quarks

$$\tilde{Q}^2 = z_1 z_2 Q^2 + m^2 \quad \Omega^2 = K_{\perp}^2 x / (1-x)$$

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General picture

Correlation limit (back-to-back)

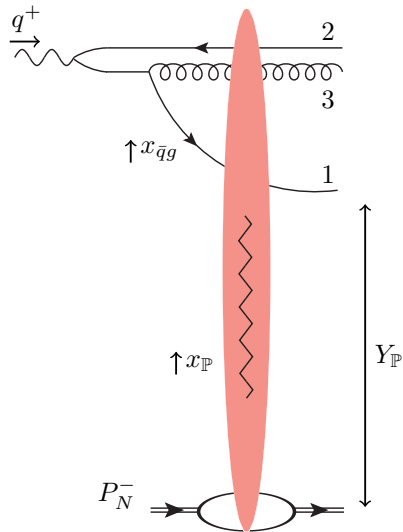
$$k_{2\perp}, k_{3\perp} \sim Q \gg k_{1\perp}, m \sim Q_s$$

Goal $\rightarrow$ TMD factorization from dipole frame

$$\frac{d\sigma}{dz_2 dz_3 d^2\mathbf{K} d^2\mathbf{P} dY_{\mathbb{P}}} = \underbrace{H(z_2, z_3, P_{\perp}, Q)}_{\text{"hard" factor}} \underbrace{\frac{dx \mathcal{G}_{\mathbb{P}}(x, x_{\mathbb{P}}, K_{\perp}, m)}{d^2\mathbf{K}}}_{\text{quark diffractive TMD}}$$

relative momentum $\mathbf{P} \simeq \mathbf{k}_2 \simeq -\mathbf{k}_3$

imbalance $\mathbf{K} = \mathbf{k}_2 + \mathbf{k}_3 = -\mathbf{k}_1$

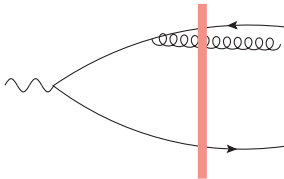
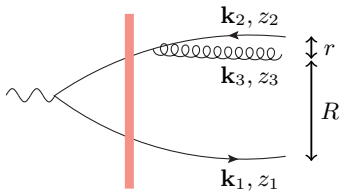


Light Cone Perturbation theory

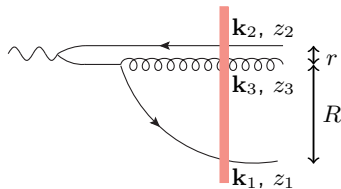
Invariant amplitude

- Diagrams that contribute to the amplitude

> Gluon emission by antiquark before and after shockwave



> Gluon emission by quark



- Calculation in LCPT + change of variables:

longitudinal momentum fraction z_1 of the projectile $\rightarrow$ target longitudinal momentum x .

TMD factorization

Cross section

$$\frac{d\sigma^{\gamma^* \rightarrow q\bar{q}g}}{dz_2 dz_3 d^2\mathbf{K} d^2\mathbf{P} dY_{\mathbb{P}}} = H(z_2, z_3, P_{\perp}) \frac{dx q_{\mathbb{P}}(x, x_{\mathbb{P}}, K_{\perp}, m)}{d^2\mathbf{K}}$$

> Hard factor:

$$H(z_2, z_3, P_{\perp}, Q) = \delta(1 - z_2 - z_3) \frac{\alpha_s C_F}{2\pi^2} \frac{1}{z_3} \frac{\hat{Q}^2 [(P_{\perp}^2 + \hat{Q}^2)^2 + P_{\perp}^4 z_3^2 + \hat{Q}^4 z_2^2]}{P_{\perp}^2 (P_{\perp}^2 + \hat{Q}^2)^3}$$

Quark diffractive TMD

density of quarks with $K_{\perp}$, x and m in the $\mathbb{P}$ wavefunction

$$\frac{dx q_{\mathbb{P}}(x, x_{\mathbb{P}}, K_{\perp}, m)}{d^2\mathbf{K}} = \frac{S_{\perp} N_c}{4\pi^3} \frac{1}{2\pi} \frac{x}{1-x} \frac{K_{\perp}^2 + m^2}{(xK_{\perp}^2 + m^2)} \left(Q_1^2(x, x_{\mathbb{P}}, K_{\perp}) + \frac{(1-x)m^2}{xK_{\perp}^2 + m^2} Q_0^2(x, x_{\mathbb{P}}, K_{\perp}) \right)$$

$$Q_1(x, x_{\mathbb{P}}, K_{\perp}) \equiv \omega^2 \int_0^{\infty} dR R J_1(KR) K_1(\omega R) \mathcal{N}(R), \quad \hat{Q}^2 = z_2 z_3 Q^2$$

$$Q_0(x, x_{\mathbb{P}}, K_{\perp}) = \omega^2 \int_0^{\infty} dR R J_0(KR) K_0(\omega R) \mathcal{N}(R), \quad \omega^2 = (m^2 + xK_{\perp}^2)/(1-x)$$

Summary

- TMD factorization for diffractive production of a pair of heavy quarks in the CGC formalism
 - Hard $\bar{q}q$, semi-hard g
 - known massive hard factor inclusive DIS (Dominguez, Marquet, Xiao, Yuan (2011)) 
 - generalized result to heavy quarks of Iancu, Mueller, Triantafyllopoulos, Wei (2022) 
 - mass independent gluon DTMD
 - Hard $\bar{q}g$, semi-hard q
 - new mass dependent quark DTMD
- Outlook
 - quarkonium, open charm predictions for UPCs at LHC, diffraction at EIC

Backup

- Invariant mass

$$M_{q\bar{q}}^2 = (k_1 + k_2)^2 = \left(\frac{k_{1\perp}^2}{z_1} + \frac{k_{2\perp}^2}{z_2} + \frac{m^2}{z_1 z_2} - K_{\perp}^2 \right)$$

$$M_{q\bar{q}g}^2 = (k_1 + k_2 + k_3)^2 = \left(\frac{k_{1\perp}^2}{z_1} + \frac{k_{2\perp}^2}{z_2} + \frac{k_{3\perp}^2}{z_3} + \frac{m^2}{z_1 z_2} \right) = M_{q\bar{q}}^2 + K_{\perp}^2 + \frac{k_3^2}{z_3}$$

- β variable

$$\beta \equiv \frac{Q^2}{Q^2 + M_{q\bar{q}g}^2}$$

- Target longitudinal momentum fraction

$$x_{q\bar{q}} = \frac{Q^2 + M_{q\bar{q}}^2 + K_{\perp}^2}{2P_N \cdot q} \quad x_{\mathbb{P}} = \frac{Q^2 + M_{q\bar{q}g}^2}{2q \cdot P_N}$$

$$x \equiv \frac{x_{q\bar{q}}}{x_{\mathbb{P}}} = \frac{Q^2 + M_{q\bar{q}}^2 + K_{\perp}^2}{Q^2 + M_{q\bar{q}}^2 + K_{\perp}^2 + k_{3\perp}^2/z_3},$$

- Mixed space wavefunction and scattering

- Fourier transform Pomeron term (traceless 2D tensor):

small dipole $\mathbf{r}$ (color transparency) $\rightarrow$ neglect the change due to the scattering of $\mathbf{P}$.

$$\begin{aligned}\Psi_{\mathbb{P}}^{im}(z_1, z_2, z_3, \mathbf{P}, \mathbf{R}) &= \int \frac{d^2\mathbf{k}_3}{(2\pi)^2} e^{i\mathbf{k}_3 \cdot \mathbf{R}} \Psi_{\mathbb{P}}^{im}(z_1, z_2, z_3, \mathbf{P}, \mathbf{k}_3) \\ &= H^{in}(z_1, z_2, \mathbf{P}, m) \int \frac{d^2\mathbf{k}_3}{(2\pi)^2} e^{i\mathbf{k}_3 \cdot \mathbf{R}} \frac{k_3^n k_3^m - \delta^{nm} \frac{k_{3\perp}^2}{2}}{k_{3\perp}^2 + \Omega^2} \\ &= H^{in}(z_1, z_2, \mathbf{P}, m) \frac{1}{2\pi} \left[\frac{\delta^{nm}}{2} - \left(\frac{R^n R^m}{R^2} \right) \right] \Omega^2 K_2(\Omega R).\end{aligned}$$

- The Fourier transform of $\Psi_{\text{diag}}(\mathbf{P})$ gives a $\delta^2(\mathbf{R})$.

$\delta^2(\mathbf{R})\mathcal{N}(\mathbf{R}) \rightarrow 0$ due to color transparency

$\rightarrow$ We keep only the $\Psi_{\mathbb{P}}$ term.

- Change of variables: $z_3 \rightarrow x$ or $x_{\mathbb{P}}$ using

$$\Omega^2 \simeq \frac{z_3}{z_1 z_2} (P_{\perp}^2 + \tilde{Q}^2) \qquad x \simeq \frac{P_{\perp}^2 + \tilde{Q}^2}{P_{\perp}^2 + \tilde{Q}^2 + z_1 z_2 \frac{K_{\perp}^2}{z_3}}$$

we get

$$\boxed{\Omega^2 = \frac{x}{1-x} K_{\perp}^2}$$

Using also $x_{\mathbb{P}}$:

$$x_{\mathbb{P}} = \frac{Q^2 + M_{q\bar{q}g}^2}{2q \cdot P_N} \qquad x_{q\bar{q}} = \frac{Q^2 + M_{q\bar{q}}^2 + K_{\perp}^2}{2P_N \cdot q} \qquad x \equiv \frac{x_{q\bar{q}}}{x_{\mathbb{P}}} \simeq \frac{Q^2 + M_{q\bar{q}}^2 + K_{\perp}^2}{Q^2 + M_{q\bar{q}}^2 + K_{\perp}^2 + k_{3\perp}^2/z_3}$$

we have the change of variables:

$$\frac{x_{\mathbb{P}} - x_{q\bar{q}}}{x_{q\bar{q}}} = \frac{x_{\mathbb{P}}}{x_{q\bar{q}}} - 1 = \frac{Q^2 + M_{q\bar{q}}^2 + K_{\perp}^2 + k_{3\perp}^2/z_3}{Q^2 + M_{q\bar{q}}^2 + K_{\perp}^2} - 1 = \frac{k_{3\perp}^2/z_3}{Q^2 + M_{q\bar{q}}^2 + K_{\perp}^2}$$

Taking the logarithmic differential:

$$\boxed{\frac{dz_3}{z_3} = \frac{dx_{\mathbb{P}}}{x_{\mathbb{P}} - x_{q\bar{q}}} = \frac{dx_{\mathbb{P}}}{x_{\mathbb{P}}} \frac{1}{1-x}}$$