# Loop calculations in light cone perturbation theory

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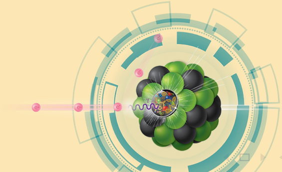
# Outline

## Outline of this talk

- High energy collisions as eikonal scattering
- Light cone perturbation theory
- Diffractive structure function at NLO
  G. Beuf, T. L., H. Mäntysaari, R. Paatelainen, J. Penttala, arXiv:2401.17251 [hep-ph]
  - See talk Kampshoff
  - *$q\bar{q}g$* contribution and the Wüsthoff limit, related to talks by Gimeno-Estivill, Iancu
    G. Beuf, H. Hänninen, T.L., Y. Mulian, H. Mäntysaari, arXiv:2206.13161 [hep-ph]
  - Partial numerical evaluation
    A. Kaushik, H. Mäntysaari and J. Penttala, [arXiv:2510.24171 [hep-ph]]
- Two loops in LCPT T.L., R. Paatelainen and M. Seppälä, [arXiv:2601.10374 [hep-ph]].

## Process of interest

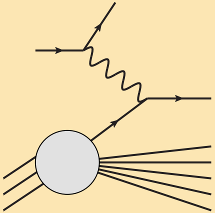
DIS at high energy



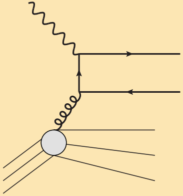
# High energy collisions as eikonal scattering

# Usual parton picture

DIS: electron-proton/nucleus scattering, via $\gamma/W^{\pm}/Z$ exchange
$\Longrightarrow$ For us DIS = $\gamma^*$-proton/nucleus scattering.



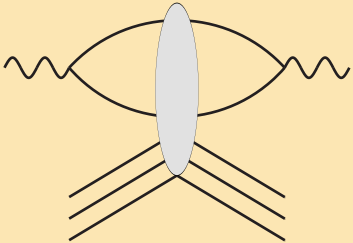
- Proton consists of partons
- Struck by $\gamma^*$: interaction time short
  $\Longrightarrow$ partonic interaction factorizes
- LO $\gamma^* + q \to q$ counts quarks
  $\Longrightarrow$ PDF's
- NLO: also gluon-initiated



Nonperturbative physics parametrized by PDF's: count quarks/gluons

# Inclusive DIS in dipole picture

Different frame: $\gamma^*$ moving fast; not proton



- $\gamma^*$ consists of partons
- Partons struck by target gluon field: interaction time short $\Longrightarrow$ structure of $\gamma^*$ frozen
- LO: $\gamma^*$ is only $q\bar{q}$ dipole
- NLO corrections: also include $\gamma^* \to q\bar{q}g$

**Nonperturbative physics: q/g+p scattering amplitude**

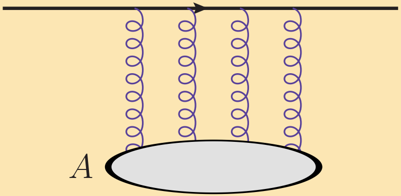
- Do not "count" quarks or gluons
- Both exclusive and inclusive cross sections via **optical theorem**

# What is the target made of?

- So what is the amplitude?
- At high energy: dominantly gluons

**Color Glass Condensate (CGC)**

Many gluons in the target $\Longrightarrow$ classical gluon field $A_\mu$.




Sum all diagrams with $n$ gluon lines

# Eikonal amplitudes

## Quark amplitude: 2-d field of SU($N_c$)-matrices

$$V(\mathbf{x}) \equiv \mathbb{P}\exp\left\{-ig\int_{-\infty}^{\infty} dx^+ A^-(x^+, x^- = 0, \mathbf{x})\right\}$$

— The **Wilson line**

## Dipole amplitude

In DIS incoming state is (color neutral) dipole

$$\mathcal{N}_{q\bar{q}} = 1 - \frac{1}{N_c}\,\mathrm{tr}\, V(\mathbf{x})V^\dagger(\mathbf{y})$$

(Imaginary unit convention $S_{fi} = \langle f|\,\hat{S}\,|f\rangle = 1 + iT_{fi}$ $\sigma_{\mathrm{tot}} = 2\mathrm{Im}T_{ii}$ $\mathcal{N} \equiv \mathrm{Im}T_{ii}$ $S_{ii} = \delta_{ii} - \mathcal{N} + \mathrm{imag}$ )

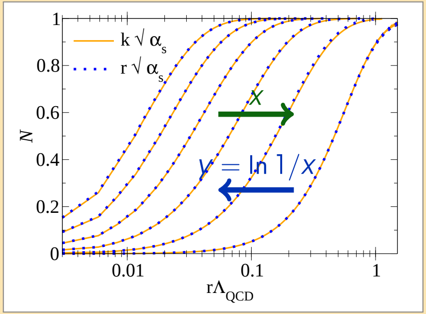
- These amplitudes are in transverse coordinate space
- At high energy gluon field Lorentz-contracted to $x^- = 0$ $\Longrightarrow$ **"shockwave"**

# Saturation of the dipole cross section

Basic features following from

$$\mathcal{N}_{q\bar{q}} = 1 - \frac{1}{N_c} \operatorname{tr} V(\mathbf{x}) V^\dagger(\mathbf{y})$$

- Small dipoles: $\mathcal{N}_{q\bar{q}}(r) \sim r^2 \implies$ perturbative
- Large dipoles $\langle \operatorname{tr} V(\mathbf{x}) V^\dagger(\mathbf{y}) \rangle = 0 \implies \mathcal{N}_{q\bar{q}} \lesssim 1$
- Know that $\mathcal{N}_{q\bar{q}}$ grows at high energy




**Nonperturbative weak coupling unitarization = Saturation**

Saturation scale $Q_s = 1/R_s$ = inverse distance for turnover

# Light Cone Perturbation Theory

# Idea of LCPT calculation

- Know free particle Fock states: $|\gamma^*\rangle_0, \quad |q\bar{q}\rangle_0, \quad |q\bar{q}g\rangle_0$ etc.
- **Interacting** states are superpositions of these:

$$|\gamma^*\rangle = (1+\dots)|\gamma^*\rangle_0 + \psi^{\gamma^*\to q\bar{q}} \otimes |q\bar{q}\rangle_0 + \psi^{\gamma^*\to q\bar{q}g} \otimes |q\bar{q}g\rangle_0 + \dots$$

- Calculate in QM perturbation theory, e.g. ground state $|0\rangle$ wavefunction:

$$\psi^{0\to n} = \sum_n \frac{\langle n|\,\hat{V}\,|0\rangle}{E_n - E_0} + \dots$$

  - Here $1/\Delta E$ is $\sim$ the lifetime of the quantum fluctuation from 0 to $n$
- In "Energy" $E$ is conjugate to "time", LC time is $x^+ \implies$ LC energy $k^-$
- "Old fashioned" perturbation theory: no energy conservation at vertex.

Connection to Feynman perturbation theory

- Matrix elements $\langle n|\,\hat{V}\,|m\rangle$ are vertices in Feynman rules
- LC energy denominators from propagators, integrating over $k^-$ pole

# (D)DIS at NLO: Fock state expansion

$$\left|\gamma^*_\lambda(q^+, \mathbf{q}; Q^2)\right\rangle_D = \sqrt{Z_{\gamma^*_\lambda}} \Bigg\{ \text{Non-QCD Fock states} + \widetilde{\sum_{q_0\bar{q}_1 \text{ F. s.}}} \widetilde{\Psi}_{\gamma^*_\lambda \to q_0\bar{q}_1} \left| q_0\bar{q}_1 \right\rangle + \widetilde{\sum_{q_0\bar{q}_1 g_2 \text{ F. s.}}} \widetilde{\Psi}_{\gamma^*_\lambda \to q_0\bar{q}_1 g_2} \left| q_0\bar{q}_1 g_2 \right\rangle + \cdots \Bigg\},$$

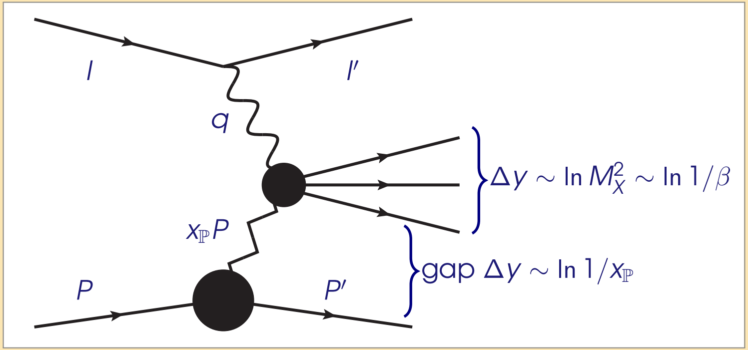
- Non-QCD Fock states: EW states, not needed
- Tree level $\widetilde{\Psi}_{\gamma^*_\lambda \to q_0\bar{q}_1}$: LO
- $\widetilde{\Psi}_{\gamma^*_\lambda \to q_0\bar{q}_1}$: known to 1 loop
- $\widetilde{\Psi}_{\gamma^*_\lambda \to q_0\bar{q}_1 g_2}$: tree level
- Diagram calculations in momentum space, then 2d Fourier-transform

# NLO diffractive DIS

G. Beuf, T. L., H. Mäntysaari, R. Paatelainen, J. Penttala, arXiv:2401.17251 [hep-ph]

# Inclusive diffraction, kinematics

$\gamma^* + A \to X + A$, differential in $M_X$




- Momentum transfer $t = (P - P')^2$
- Gap size $x_{\mathbb{P}}$, target evolution rapidity $\sim \ln 1/x_{\mathbb{P}}$
- Diffractive system mass $M_X^2$, $\beta = Q^2/(Q^2 + M_X^2)$
- Virtuality $Q^2$

$x_{Bj} = x_{\mathbb{P}} \beta$

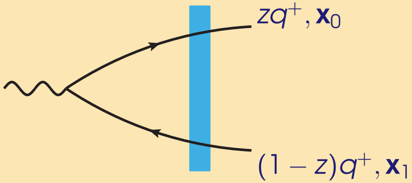
This talk: $x_{\mathbb{P}}$ small, $\beta$ not.

# Diffractive DIS at leading order

- Full kinematics and transverse coordinate dependence
  G. Beuf, H. Hänninen, T.L., Y. Mulian, H. Mäntysaari, `arXiv:2206.13161`

$$\frac{\mathrm{d}\sigma^{\mathrm{D}}_{\lambda,\,q\bar{q}}}{\mathrm{d}M_X^2\,\mathrm{d}|t|} = \frac{N_c}{4\pi}\int_0^1 \mathrm{d}z \int_{\mathbf{x}_0\mathbf{x}_1\bar{\mathbf{x}}_0\bar{\mathbf{x}}_1} \mathcal{I}^{(2)}_{\boldsymbol{\Delta}}\mathcal{I}^{(2)}_{M_X}$$

$$\times \sum_{f,h_0,h_1} \left(\widetilde{\psi}_{\gamma^*_\lambda \to q_{\bar{0}}\bar{q}_{\bar{1}}}\right)^\dagger \left(\widetilde{\psi}_{\gamma^*_\lambda \to q_0\bar{q}_1}\right) \boxed{\mathcal{N}^\dagger_{q\bar{q}}(\bar{\mathbf{x}}_0 - \bar{\mathbf{x}}_1)\mathcal{N}_{q\bar{q}}(\mathbf{x}_0 - \mathbf{x}_1)}$$



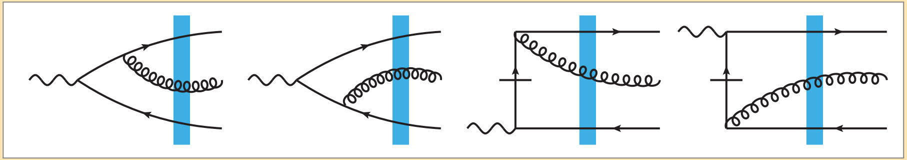
(Blue: shockwave target)

- Fraction of $\gamma^*$ longitudinal momentum $z$
- $q\bar{q}$ crossing shockwave: dipole $\mathcal{N}_{q\bar{q}}$
- "Transfer functions:" relate coordinates at shockwave to:
  - Momentum transfer $t = -\boldsymbol{\Delta}^2$: $\mathcal{I}^{(2)}_{\boldsymbol{\Delta}} = \frac{1}{4\pi} J_0\left(\sqrt{|t|}\,\|z\mathbf{x}_{\bar{0}0} - (1-z)\mathbf{x}_{\bar{1}1}\|\right)$
  - Invariant mass $\mathcal{I}^{(2)}_{M_X} = \frac{1}{4\pi} J_0\left(\sqrt{z(1-z)}M_X\|\bar{\mathbf{r}} - \mathbf{r}\|\right)$

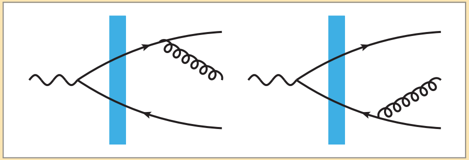
These generalize known results Golec-Biernat, Wüsthoff, Marquet et al

# NLO radiative corrections: amplitudes

- Emission before target

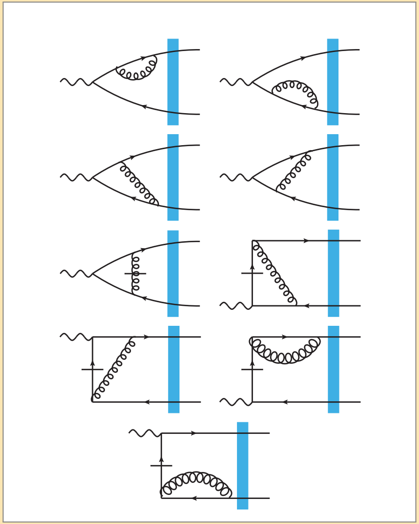


  - Squares already in G. Beuf, H. Hänninen, T.L., Y. Mulian, H. Mäntysaari `arXiv:2206.13161`
  - Contain leading $\ln Q^2$ contribution $\Longrightarrow$ talks by Iancu, Gimeno-Estivill
- Emission after target



- Interferences $\Longrightarrow$ simplify with some of the virtual corrections

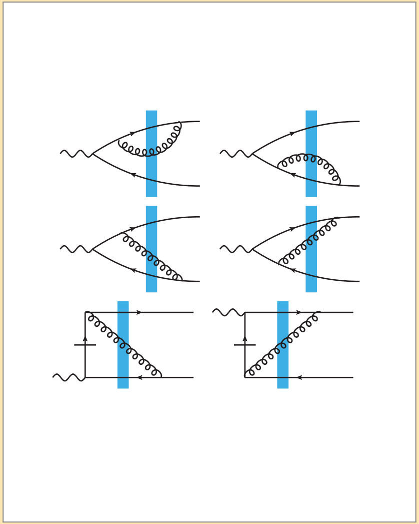
# NLO virtual: amplitudes



- Vertex corrections:
  known 1-loop $\gamma \to q\bar{q}$ wavefunction

See also Boussarie et al 2014: diffractive jets,
also Caucal et al 2021 inclusive

# NLO virtual: amplitudes



- Vertex corrections:
  known 1-loop $\gamma \to q\bar{q}$ wavefunction
- Gluon crosses shockwave, but not the cut:
  - Loop corrections to amplitude,
    tree level wavefunctions
  - 3-point operator of Wilson lines
  - BK/JIMWLK evolution of LO amplitude

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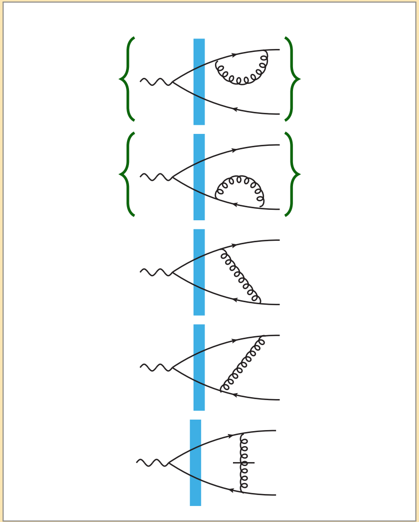
# NLO virtual: amplitudes



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- Gluon crosses shockwave, but not the cut:
  - Loop corrections to amplitude,
    tree level wavefunctions
  - 3-point operator of Wilson lines
  - BK/JIMWLK evolution of LO amplitude
- Final state interactions
  (Propagator corrections $\{\} \to$
  State renormalization, in fact = 0 in dim. reg.)

See also Boussarie et al 2014: diffractive jets,

also Caucal et al 2021 inclusive

# NLO diffractive DIS cross section

# Calculation in 2401.17251 [hep-ph]

Beuf, T.L. , Paatelainen, Mäntysaari, Penttala, see talk Kampshoff

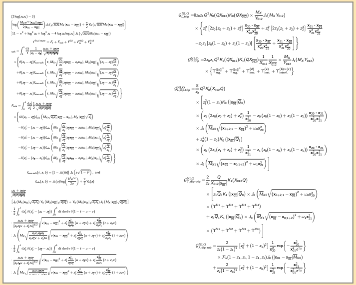
## We have calculated all these contributions

- Diffractive structure function:

  clean IR-safe, [perturbative = experimental] final state definition $M_X$!

  (No fragmentation function, jet definition)

  $\Longrightarrow$ Divergences must cancel



Features of the calculation:

- Divergence structure
- Treatment of energy denominators

(Result: logs, Bessel fn's, polynomials in $z_i$'s)

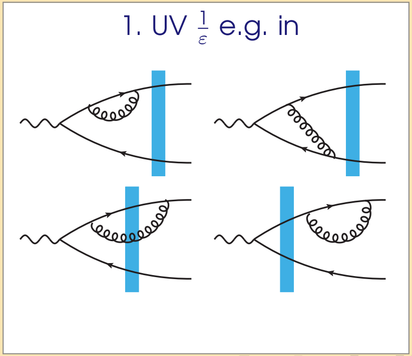
# Regularization and divergences

## Regularization procedure

- Transverse momentum in $2 - 2\varepsilon$ dimensions $\Longrightarrow \frac{1}{\varepsilon}$ divergences, collinear or UV
- Longitudinal $k^+$: cutoff $k^+ > \alpha$, $\alpha \to 0$ $\Longrightarrow 1/\alpha, \ln^2 \alpha, \ln \alpha$ divergences

(Note $k^-$ already integrated with the pole)

1. UV $\frac{1}{\varepsilon}$ and $\frac{1}{\varepsilon} \ln \alpha$ divergences: $\gamma^* \to q\bar{q}$ vertex, gluon crossing shock, wavefunction renormalization



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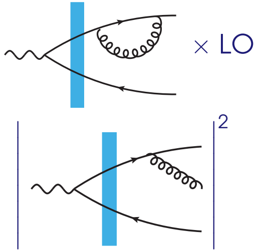
2. Collinear $\frac{1}{\varepsilon}$ :
   wavef. renormalization, final state emission

2. Collinear $\frac{1}{\varepsilon}$ e.g. in



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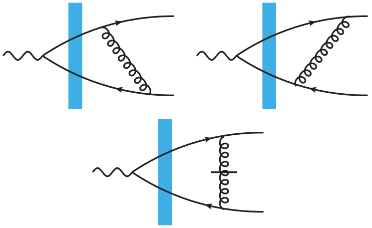

# Regularization and divergences

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- Transverse momentum in $2-2\varepsilon$ dimensions $\Longrightarrow$ $\frac{1}{\varepsilon}$ divergences, collinear or UV
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2. Collinear $\frac{1}{\varepsilon}$ :
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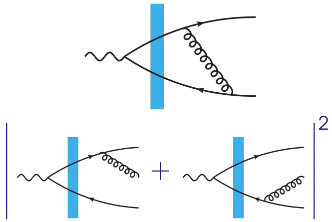
3. $1/\alpha$ cancels between normal and instantaneous exchange
4. $\ln^2\alpha$ from final state exchange and emission
   ($M_X$ restriction matters here!)

4. $\ln^2\alpha$ e.g. in



# Regularization and divergences

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- Transverse momentum in $2-2\varepsilon$ dimensions $\implies \frac{1}{\varepsilon}$ divergences, collinear or UV
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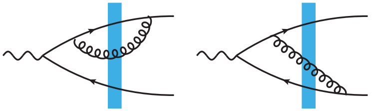
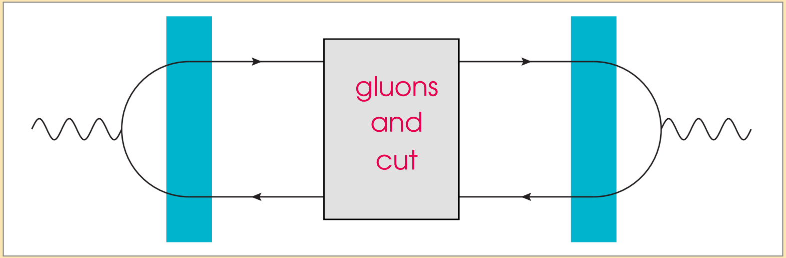
1. UV $\frac{1}{\varepsilon}$ and $\frac{1}{\varepsilon} \ln \alpha$ divergences:
   $\gamma^* \to q\bar{q}$ vertex, gluon crossing shock, wavefunction renormalization
2. Collinear $\frac{1}{\varepsilon}$ :
   wavef. renormalization, final state emission
3. $1/\alpha$ cancels between normal and instantaneous exchange
4. $\ln^2 \alpha$ from final state exchange and emission
   ($M_X$ restriction matters here!)
5. Remaining $\ln \alpha$ absorbed into BK/JIMWLK

5. $\ln \alpha$ e.g. in



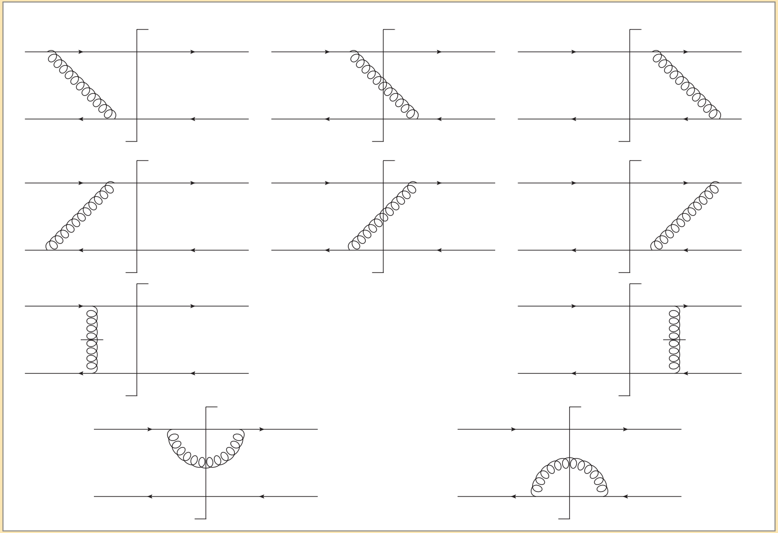
# Purely final state corrections: combine at cross section level




- Only $q\bar{q}$ at shockwave $\Longrightarrow$ dipole $\mathcal{N}_{q\bar{q}}(\mathbf{x}_0 - \mathbf{x}_1)$ and $\mathcal{N}^{\dagger}_{q\bar{q}}(\bar{\mathbf{x}}_0 - \bar{\mathbf{x}}_1)$.
- If fully inclusive, these would cancel, now $M_X$ **restriction**
- Final state=cut can be $|f\rangle = |q\bar{q}\rangle\,, |q\bar{q}g\rangle$ with $M_{q\bar{q}} = M_X, M_{q\bar{q}g} = M_X$
  $\Longrightarrow$ LCWF's $\Psi^*_{q\bar{q}\to q\bar{q}}$ and $\Psi^*_{q\bar{q}g\to q\bar{q}}$ ( LCWF from asymptotic state at the cut = $M_X$)

# How to dig out different types of divergences?

(only drawing part between shockwaves)



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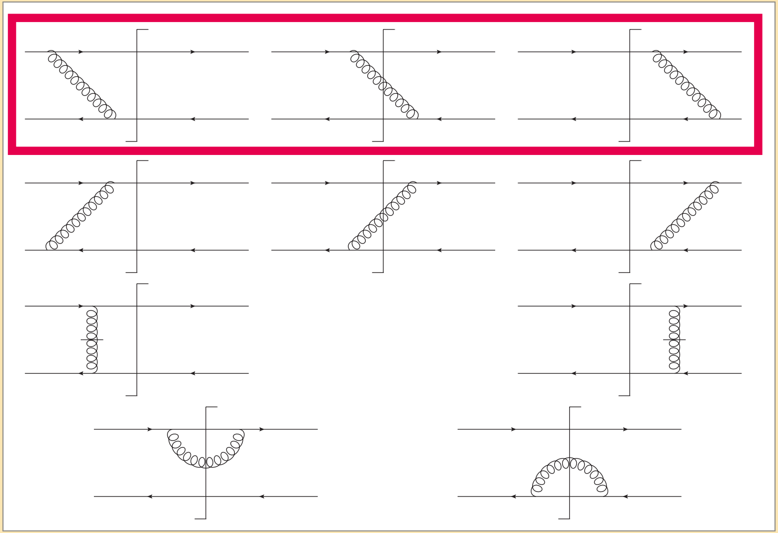
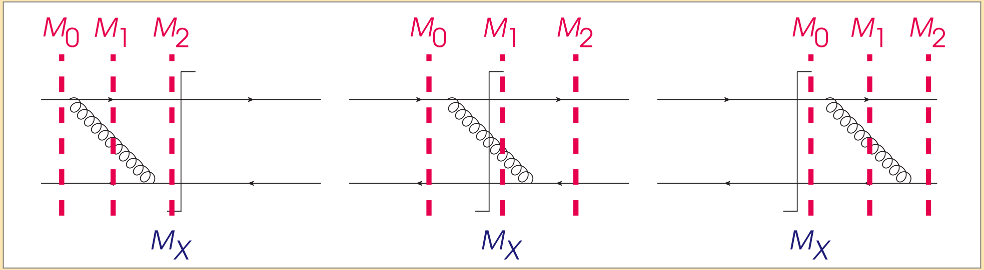


As an example: consider 1st row

# Combine energy denominators

"Beuf trick"= "reverse unitarity": write $M_X$ delta function as imaginary part of "propagator"



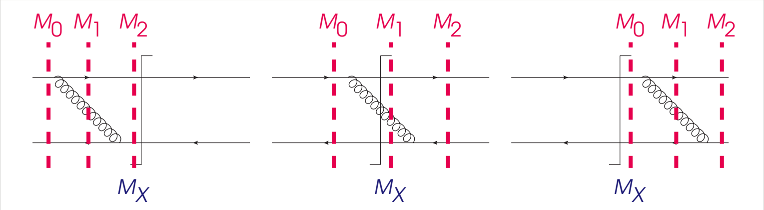

$$\frac{\delta({M_X}^2 - M_2^2)}{(M_2^2 - M_1^2 + i\delta)(M_2^2 - M_0^2 + i\delta)} + \frac{\delta({M_X}^2 - M_1^2)}{(M_1^2 - M_0^2 + i\delta)(M_1^2 - M_2^2 - i\delta)} + \frac{\delta({M_X}^2 - M_0^2)}{(M_0^2 - M_1^2 - i\delta)(M_0^2 - M_2^2 - i\delta)}$$

$$= \frac{1}{2\pi i}\left[\frac{1}{({M_X}^2 - M_0^2 - i\delta)({M_X}^2 - M_1^2 - i\delta)({M_X}^2 - M_2^2 - i\delta)} - \text{c.c.}\right]$$

(Note: sign of $i\delta$ essential)

Looks even more complicated than before, but ...

# Separating divergences




$$\frac{\text{numerator}}{2\pi i}\left[\frac{1}{({M_X}^2 - M_0^2 - i\delta)({M_X}^2 - M_1^2 - i\delta)({M_X}^2 - M_2^2 - i\delta)} - \text{c.c.}\right]$$

$$\text{Numerator} \sim [\ldots]({M_X}^2 - M_0^2) + [\ldots]({M_X}^2 - M_2^2) + [\ldots]({M_X}^2 - M_1^2) + [\ldots]{M_X}^2$$

- Numerator $\sim ({M_X}^2 - M_0^2), \sim ({M_X}^2 - M_2^2)$: divergence $\sim \ln^2 \alpha$
- Numerator $\sim ({M_X}^2 - M_1^2)$: divergence $\sim 1/\alpha$
- Numerator $\sim {M_X}^2$: finite (but most complicated, 3 ED's)

Beuf trick separated divergences —- crucial, specific to fixed $M_X$ final state.

# Two-loop DGLAP splitting function

T.L., R. Paatelainen, M. Seppälä, [arXiv:2601.10374 [hep-ph]]

# Two loops

- Many reasons to go to higher orders:
  - Rederive NLO BK
  - NLO 2-particle cross sections
  - NNLO single inclusive cross sections

    (need to address collinear logs in factorization, though)
  - ...
- At some point need to automatize algebra, but first understand systematics
- Test case: 2-loop calculation of $q \to q$ DGLAP splitting function

  T.L., R. Paatelainen, M. Seppälä, [arXiv:2601.10374 [hep-ph]]

# Setup

- DGLAP evolution: collinear divergence in cross section,
  **UV renormalization** in operator definition of PDF
- Starting point: bare nonsinglet quark distribution, UV divergent

$$f^0_{\mathrm{n.s.}/h}(\mu, x) \equiv \frac{1}{4\pi x}\int \frac{\mathrm{d}^{D-2}\mathbf{k}}{(2\pi)^{D-2}} \frac{\langle h(p^+)|\, a_q^\dagger(xp^+,\mathbf{k}) a_q(xp^+,\mathbf{k}) - a_{\bar q}^\dagger(xp^+,\mathbf{k}) a_{\bar q}(xp^+,\mathbf{k})\,|h(p^+)\rangle}{\langle h(p^+)|h(p^+)\rangle}$$

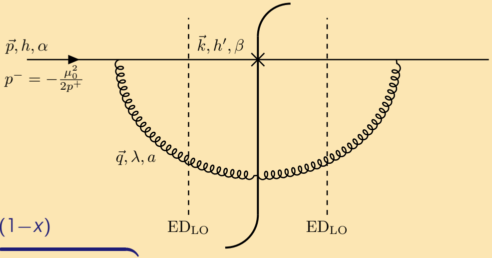
- Take hadron = dressed quark with space-like $p^2 = -\mu_0^2 \implies$ **no IR divergences**

$$|h\rangle = \left|q(\mu_0^2)\right\rangle_{\mathrm{int}} = \sqrt{Z_q(p^+)}\left[\,|q\rangle + \psi^{q\to qg}\,|qg\rangle + \psi^{q\to qgg}\,|qgg\rangle + \dots\right]$$

- At dim. reg scale $\mu = \mu_0$ off-shellness: no logs, identify DGLAP evolution from

$$f^0_{\mathrm{n.s.}/q(\mu_0)}(\mu=\mu_0, x) = \delta(1-x) + \left(\frac{\alpha_s}{2\pi}\right)\frac{1}{\varepsilon}P^{-,(0)}(x) + \frac{1}{2}\left(\frac{\alpha_s}{2\pi}\right)^2\left[\frac{1}{\varepsilon}P^{-,(0)}\right]^2 + \frac{1}{2}\left(\frac{\alpha_s}{2\pi}\right)^2\left[\frac{1}{\varepsilon}P^{-,(1)}(x)\right] + \text{finite, cross terms.}$$

# LO splitting function




Leading order wavefunction:

$$\left| g\psi_{(0)}^{q\to qg} \right|^2 = \overbrace{\frac{g^2 C_F (2p^+)^2 x(1-x)^2 \mathbf{k}^2}{\left[\mathbf{k}^2 + x(1-x)\mu_0^2\right]^2}}^{\to 1/\varepsilon} \overbrace{\left[ D - 3 + \left(1 - \frac{2}{1-x}\right)^2 \right]}^{(1+x^2)/(1-x)},$$

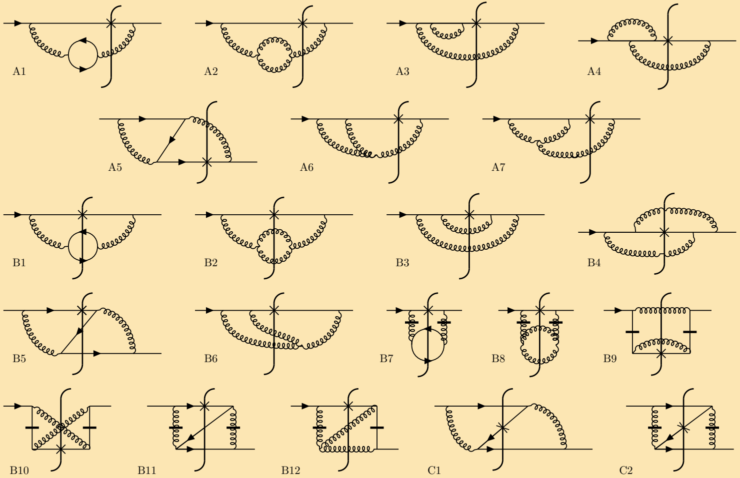
Renormalization constant (Comes with $\delta(1-x)$)

$$Z_q = 1 - \frac{g^2 C_F}{4\pi} \int_0^1 \mathrm{d}z (1-z) \left[ D - 3 + \left(1 - \frac{2}{1-z}\right)^2 \right] \bar{\mu}^{4-D} \overbrace{\int \frac{\mathrm{d}^{D-2}\mathbf{k}}{(2\pi)^{D-2}} \frac{\mathbf{k}^2}{\left[\mathbf{k}^2 + z(1-z)\mu_0^2\right]^2}}^{\sim 1/\varepsilon}$$

$\Longrightarrow$ coefficient of $1/\varepsilon$ combines to LO $q \to q$ splitting function $\sim \frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)$

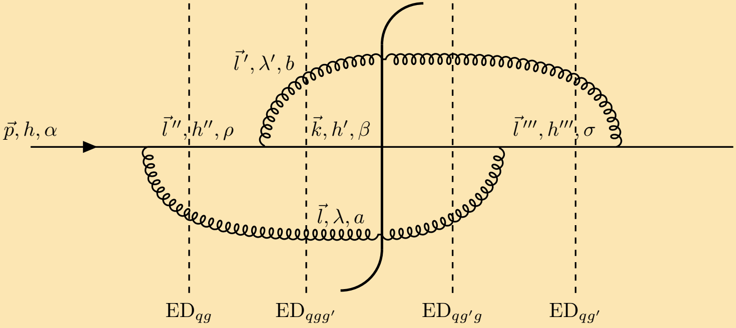
# Diagrams

We calculate these, $\times$ is “measured” quark/antiquark:




Recover standard result

# Example diagram




- 2 $\perp$ loop integrals
- One $k^+$ loop integral (quark $k^+$ is measured)
- 4 ED's, ("propagators") , 4 vertices ($\propto$ loop momentum)

Future: generate and evaluate diagrams for given process automatically.

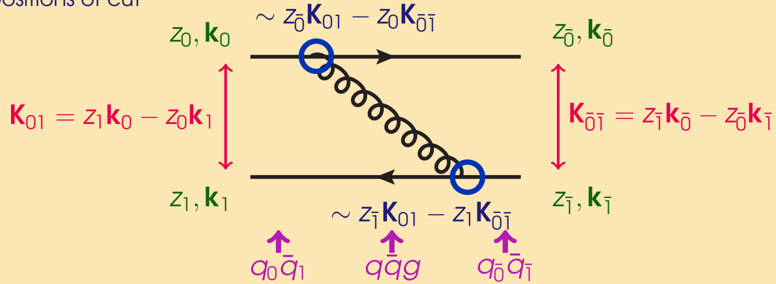
# Conclusions

- High energy scattering of dilute probe off strong color fields:
  - Target: classical color field
  - Probe: virtual photon,
    develop in a Fock state expansion in Light Cone Perturbation Theory
- Inclusive diffraction at one loop: key piece of EIC physics
- Moving to 2 loops, test case NLO $q \to q$ splitting function
- Branchout of splitting function calculation:
  implement collinear factorization with spacelike off-shell external particles?

In prog. with Gimeno-Estivill, Seppälä

# Numerator

Same for all 3 positions of cut




- $\mathbf{K}_{01}, \mathbf{K}_{\bar{0}\bar{1}}$ are conjugate to the dipole size $\Longrightarrow$ integration variables
- Numerator $\sim (z_{\bar{0}}\mathbf{K}_{01} - z_0\mathbf{K}_{\bar{0}\bar{1}}) \cdot (z_{\bar{1}}\mathbf{K}_{01} - z_1\mathbf{K}_{\bar{0}\bar{1}})$
- Invariant masses

$$M^2(q_0\bar{q}_1) = \frac{\mathbf{K}_{01}^2}{z_0 z_1} \quad M^2(q\bar{q}g) = \frac{1}{z_0 - z_{\bar{0}}}\left[\frac{z_{\bar{1}}}{z_1}\mathbf{K}_{01}^2 + \frac{z_0}{z_{\bar{0}}}\mathbf{K}_{\bar{0}\bar{1}}^2 - 2\mathbf{K}_{01}\cdot\mathbf{K}_{\bar{0}\bar{1}}\right] \quad M^2(q_{\bar{0}}\bar{q}_{\bar{1}}) = \frac{\mathbf{K}_{\bar{0}\bar{1}}^2}{z_{\bar{0}} z_{\bar{1}}},$$

- Express numerator as linear combination of invariant masses

# LO, recovering known result

We can recover known results Golec-Biernat, Wüsthoff, Marquet et al :

$$x_{\mathbb{P}} F^{\mathrm{D}}_{L,q\bar{q}}\left(\beta, x_{\mathbb{P}}, Q^2\right) = \frac{N_c Q^4}{2\pi^3 \beta} \sum e_f^2 \int \mathrm{d}^2\mathbf{b} \int_0^1 \mathrm{d}z\, z^3(1-z)^3 Q^2 \Phi_0(z, \beta, Q, \mathbf{b}),$$

$$x_{\mathbb{P}} F^{\mathrm{D}}_{T,q\bar{q}}\left(\beta, x_{\mathbb{P}}, Q^2\right) = \frac{N_c Q^4}{8\pi^3 \beta} \sum e_f^2 \int \mathrm{d}^2\mathbf{b} \int_0^1 \mathrm{d}z\, z^2(1-z)^2 \left(z^2 - (1-z)^2\right) Q^2 \Phi_1(z, \beta, Q, \mathbf{b}),$$

$$\Phi_n(z, \beta, Q, \mathbf{b}) = \left[\int \mathrm{d}r\, r J_n\left(\sqrt{z(1-z)} M_X r\right) K_n\left(\sqrt{z(1-z)} Q\, r\right) (S_{rb} - 1)\right]^2 .$$

Requires approximations

- Dependence on center-of-mass impact parameter $z_0\mathbf{x}_0 + z_1\mathbf{x}_1$ factorizes
- Dipole amplitude does not depend on **b**, **r**-angle:
  $\Longrightarrow$ Bessel function index from angular structure of $\gamma^*_\lambda \to q\bar{q}$ vertex

Now to NLO

# Wüsthoff limit

# Large $Q^2$

G. Beuf, H. Hänninen, T.L., Y. Mulian, H. Mäntysaari, arXiv:2206.13161

Recover "Wüsthoff result" (origin somewhat mysterious)

$$x_{\mathbb{P}} F^{\mathrm{D\,(GBW)}}_{T,q\bar{q}g} = \frac{\alpha_s \beta}{8\pi^4} \sum_f e_f^2 \int_{\mathbf{b}} \int_{\beta}^{1} \mathrm{d}z \left[ \left(1 - \frac{\beta}{z}\right)^2 + \left(\frac{\beta}{z}\right)^2 \right] \int_0^{Q^2} \mathrm{d}k^2 k^4 \ln \frac{Q^2}{k^2}$$

$$\times \left[ \int_0^{\infty} \mathrm{d}r r K_2(\sqrt{z} k r) J_2(\sqrt{1-z} k r) \frac{\mathrm{d}\tilde{\sigma}_{\mathrm{dip}}}{\mathrm{d}^2\mathbf{b}}(\mathbf{b}, \mathbf{r}, x_{\mathbb{P}}) \right]^2 .$$

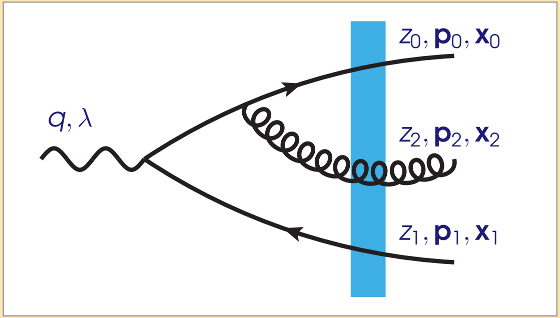
- Explicit $\ln Q^2$
- $g \to q\bar{q}$ DGLAP splitting function: target evolution
- Color-octet small-size $q\bar{q}$ is "effective gluon" $\tilde{g}$ $\Longrightarrow$ adjoint dipole $\Longrightarrow$ diffractive gluon PDF
- $J_2, K_2$ from curious "effective gluon wavefunction"

$$\psi^{\gamma \to g\tilde{g}} \sim k^i k^j - \frac{1}{2}\mathbf{k}^2 \delta^{ij}$$

# Deriving large $Q^2$ limit: aligned jet limit

Leading large $Q^2$ from:

- $z_0 \gg z_1 \gg z_2$
- $\mathbf{p}_0^2 \sim \mathbf{p}_1^2 \gg \mathbf{p}_2^2$
- $p_0^- \sim p_1^- \sim p_2^-$
  $\Longrightarrow$ Wüsthoff momentum fractions

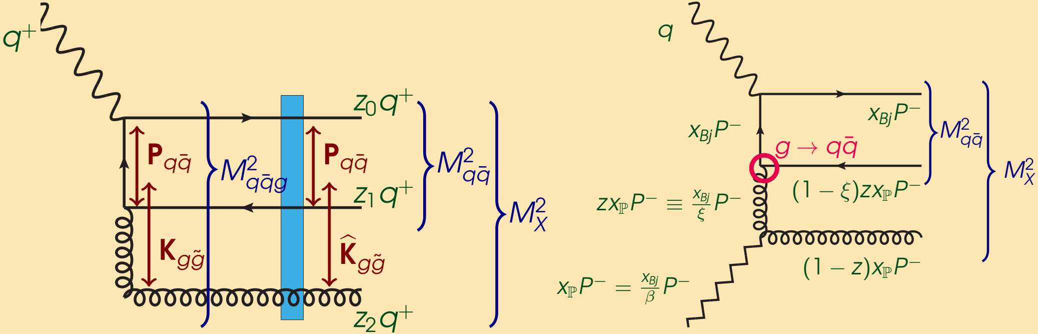



Consistently taking this limit derivation is straightforward:

- $q$ and $\bar{q}$ close: not resolved by target $\Longrightarrow$ point-like "effective gluon" $\tilde{g}$
- Consequently relative momentum of $q\bar{q}$ pair does not change in shockwave
  only relative momentum of $g\tilde{g}$
- Explicit log from $q\bar{q}$ internal dynamics, $g \to q\bar{q}$ (!) splitting function
- Rank-2 tensor for $\gamma^* \to g\tilde{g}$ Edmond and Dionysios had a much more elegant way!

# Deriving large $Q^2$ limit: matching

Identification with collinear variables via invariant masses



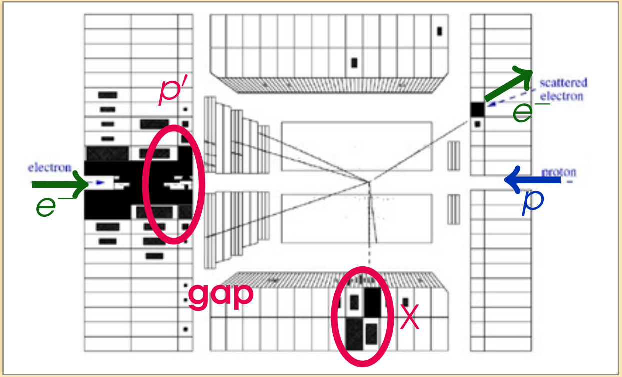

- $\mathbf{K}_{g\tilde{g}}$: before shock, Fourier-transform
- $\widehat{\mathbf{K}}_{g\tilde{g}}$: final state, fixed

$$M^2_{q\bar{q}} \approx \mathbf{P}^2_{q\bar{q}}/z_1 \approx (1/\xi - 1)Q^2$$

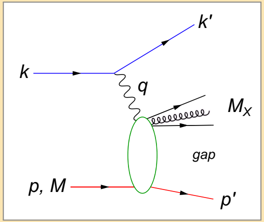
$$M^2_{q\bar{q}g} \approx M^2_{q\bar{q}} + \mathbf{K}^2_{g\tilde{g}}/z_2 \qquad M^2_X = (1/\beta - 1)Q^2 \approx M^2_{q\bar{q}} + \widehat{\mathbf{K}}^2_{g\tilde{g}}/z_2$$

# Color neutral QCD interactions: diffraction

- Around 15% of HERA events **diffractive**.
- Experimental signal:
  rapidity gap.
- Most of the time proton does not break up
  (Even if energy $\gg$ binding energy)




ZEUS detector event display




**Color neutral** QCD interaction ("pomeron") .

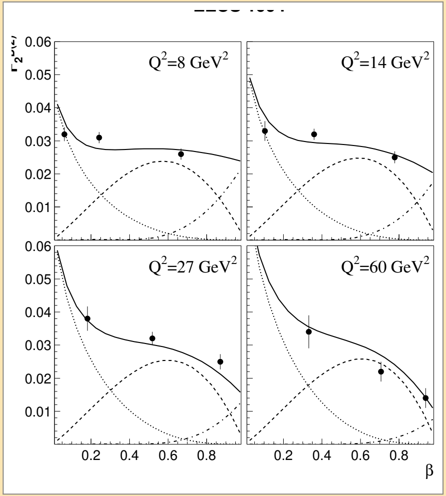
# Dependence on $\beta$, i.e. $M_X$

${M_X}^2$ = photon remnants.

Essential regimes:

LO+NLO
- Large $\beta \to 1$ — small $M_X$: longitudinal $\gamma^* \to q\bar{q}$
- Medium $\beta \sim 0.5$ — ${M_X}^2 \sim Q^2$: transverse $\gamma^* \to q\bar{q}$

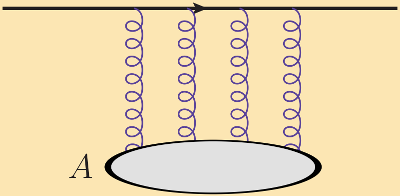
NLO
- Small $\beta \ll 1$ — large ${M_X}^2$: higher Fock states ($q\bar{q}g$ etc.)




LO $q\bar{q}$ and leading ln $Q^2$ $q\bar{q}g$

Golec-Biernat& Wüsthoff hep-ph/9903358

# Scattering off CGC target




Quark in classical color field: Dirac equation

$$(i\partial\!\!\!/ - gA\!\!\!/)\psi(x) = 0$$

(Note: $A\!\!\!/ = A_a^\mu \gamma_\mu t^a$ is $N_c \times N_c \otimes 4 \times 4$-matrix )

High energy: **eikonal** approximation

- Gluon spin 1, couples to a vector: $\sim J^\mu A_\mu$, at high energy $J^\mu \sim p^\mu$
- $p^\mu$ has one large component $\Longrightarrow p^\mu A_\mu \sim p^+ A^-$ $\Longrightarrow$ only need $A^-$

  ($V^\pm = (V^0 \pm V^z)/\sqrt{2}$ )

Ansatz for DE: $\psi(x) = V(x) e^{-ip\cdot x} u(p)$ $\Longrightarrow$ **$N_c \times N_c$-matrix!**

$$\Longrightarrow \quad \partial_+ V(x^+, x^-, \mathbf{x}) = -igA^-(x^+, x^-, \mathbf{x}) V(x^+, x^-, \mathbf{x})$$

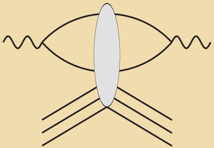
**This is solved by path-ordered exponential**

$$V(x^+, x^-, \mathbf{x}) = \mathbb{P} \exp \left\{ -ig \int^{x^+} dy^+ A^-(y^+, x^-, \mathbf{x}) \right\}$$

# Perturbation theory: total cross section

Limit of small $x$, i.e. high $\gamma^*$-target energy

## Leading order

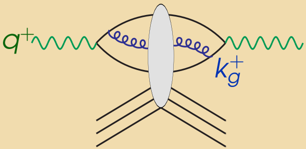


- $\gamma^* \to q\bar{q}$ in vacuum
- $q\bar{q}$ interacts eikonally with target
- $\sigma^{\text{tot}}$ is $2\times$Im-part of amplitude

"Dipole model": Nikolaev, Zakharov 1991 , A. Mueller

Many fits to HERA data, starting with Golec-Biernat, Wüsthoff 1998

## Leading Log: add **soft** gluon




- Soft gluon: large logarithm

$$\int_{x_{Bj}} \frac{dk_g^+}{k_g^+} \sim \ln \frac{1}{x_{Bj}}$$

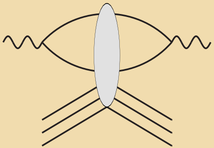
Absorb into renormalization of target:

**BK equation** Balitsky 1995, Kovchegov 1999

# Perturbation theory: total cross section

Limit of small $x$, i.e. high $\gamma^*$-target energy

## Leading order



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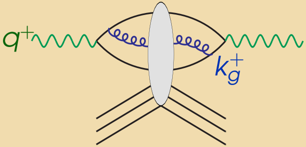


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Absorb into renormalization of target:

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NLO: the same gluon with full kinematics