Multiplicity distribution of gluons produced in deep inelastic dispersion

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In collaboration with
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Outline

- Introduction
- Multiplicity distribution and AGK cutting rules
- Multiplicity distribution Equation
- BK and Homotopic approach
- Solution and Applications
- Summary and Outlook

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Phys. Rev. D 111 (2026) 9, 096025

Measurement of particle multiplicity distributions in DIS at HERA and its implication to entanglement entropy of partons

H1 Collaboration

arXiv:2011.01812v1

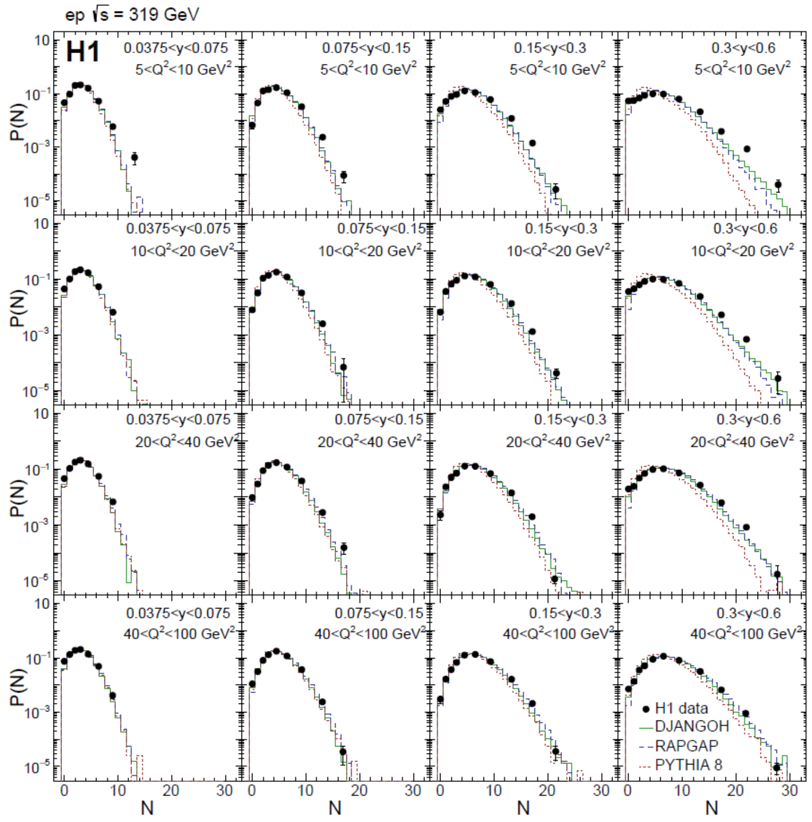
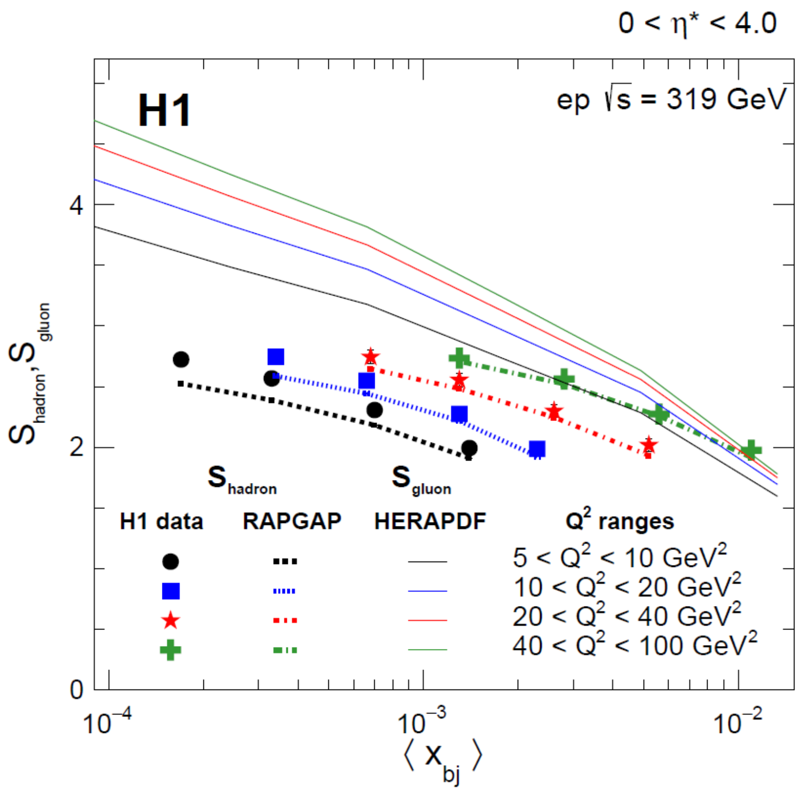


Figure 2: Charged particle multiplicity distributions $P(N)$ as a function of the number of particles N at $\sqrt{s} = 319\,\mathrm{GeV}$ ep collisions. Different panels correspond to different Q^2 and y bins, as indicated by the text in the figure. The phase space restrictions are given in Table 1. Predictions from DJANGOH, RAPGAP and PYTHIA 8 are also shown. The total uncertainty is denoted by the error bars.

$$P_n(z)$$



$$S_E(z) = -\sum_n P_n(z) \ln(P_n(z))$$

$$Q^2 = 4E_eE_e'\cos\frac{\theta_e}{2}^2, \quad y = 2E_e\frac{\Sigma}{[\Sigma + E_e'(1-\cos\theta_e)]^2}, \quad x_{\mathrm{bj}} = \frac{Q^2}{sy}.$$

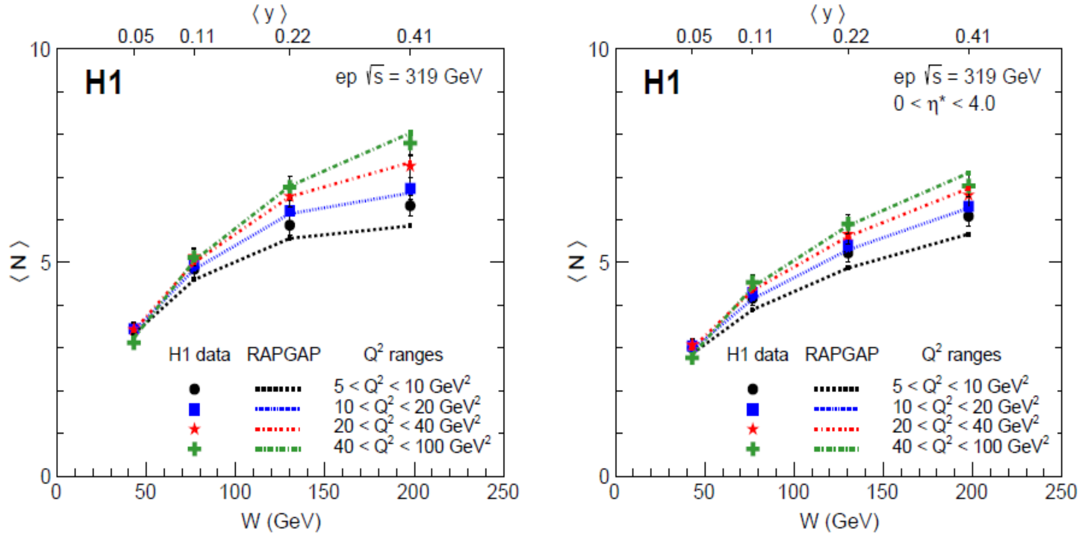


Figure 8: Mean multiplicity $\langle N \rangle$ as a function of W measured at $\sqrt{s} = 319$ GeV ep collisions (left) and with an additional restriction to the current hemisphere $0 < \eta^* < 4$ (right). Further phase space restrictions are given in Table 1. The corresponding $\langle y \rangle$ is indicated by the top axis for each measured W. Predictions from the RAPGAP model are shown by dashed lines. The total uncertainty is represented by the error bar.

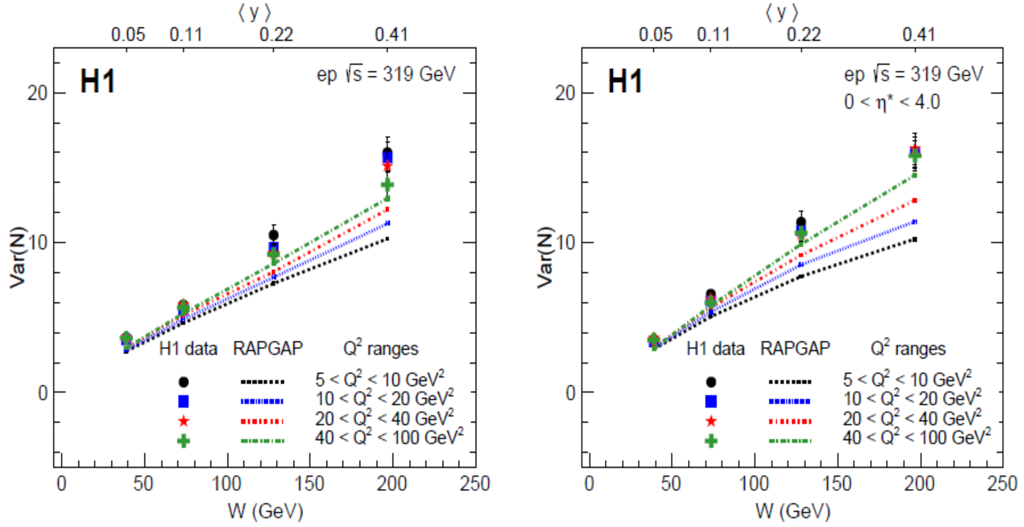


Figure 9: Second moment (variance) of the multiplicity distributions $Var(N)$ as a function of W measured at $\sqrt{s} = 319$ GeV ep collisions (left) and with an additional restriction to the current hemisphere $0 < \eta^* < 4$ (right). Further phase space restrictions are given in Table 1. The corresponding $\langle y \rangle$ is indicated by the top axis for each measured W. Predictions from the RAPGAP model are shown by dashed lines. The total uncertainty is represented by the error bar.

$$W = \sqrt{sy - Q^2 + M_p^2},$$

$$\mathit{Momentum\ of\ the\ multiplicity}$$
$$< N > = \sum n\, P_n(z)$$

AGK Motivation:

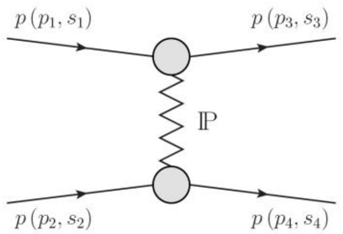
Charged–particle multiplicities in proton–proton collisions at $\sqrt{s}$ = 0.9 to 8 TeV ALICE Collaboration **arXiv:1509.07541v2**

Abramovsky, Gribov, and Kancheli in their pioneering paper have pointed out that, for high energy hadron hadron scattering, the multiple exchange of Pomerons leads to observable effects in multiparticle final states.

V. A. Abramovsky, V. N. Gribov and O. V. Kancheli, Yad. Fiz. 18 (1973) 595 [Sov. J. Nucl. Phys. 18 (1974) 308]

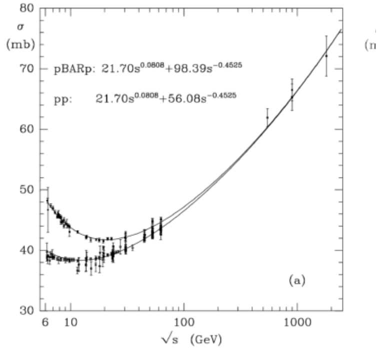
In the Regge theory one of the most successful models for describing hadronic interactions, the asymptotic behaviour of cross-sections for elastic scattering and multiple production of hadrons is determined by the properties of the Pomeron in the t-channel (soft Pomeron)

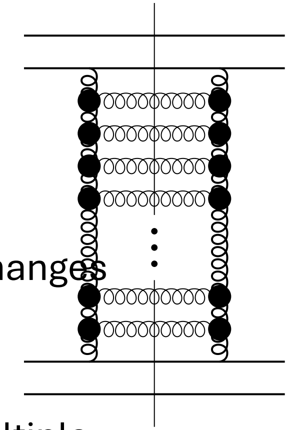
Regge theory: (Gribov 1986) - total Cross section, considering the optical theorem is given by: $\sigma_T = A_i s^{\alpha_i(0)-1}$



Soft Pomeron: $\mu = \alpha_0 - 1 = 0.08$ **intercept and slope** $\alpha' = 0.25\ GeV^{-2}$

A. Donnachie and Landshoff : 1992





In QCD, the BFKL (Balitsky, Fadin, Kuraev, Lipatov 1977-1978) Pomeron has vacuum quantum numbers, is related to gluonic exchanges in the t-channel. Pomeron is described by the sum of BFKL ladder diagrams and described by the BFKL Evolution equation

In connection with saturation and the color glass condensate, both in DIS at HERA and in heavy ion collisions at RHIC also the multiple exchange of BFKL Pomerons has to be consider; a convenient framework for this is the nonlinear Balitsky-Kovchegov (BK) equation

AGK Motivation:

Charged–particle multiplicities in proton–proton collisions at $\sqrt{s}$ = 0.9 to 8 TeV ALICE Collaboration
arXiv:1509.07541v2

AGK analysis in the framework of perturbative QCD can be found in **J. Bartels, M. Salvadore and G. P. Vacca arxiv 0503049v1**

In this presentation we use the equivalent formulation of the AGK cutting rules within the dipole approach to high-energy QCD.

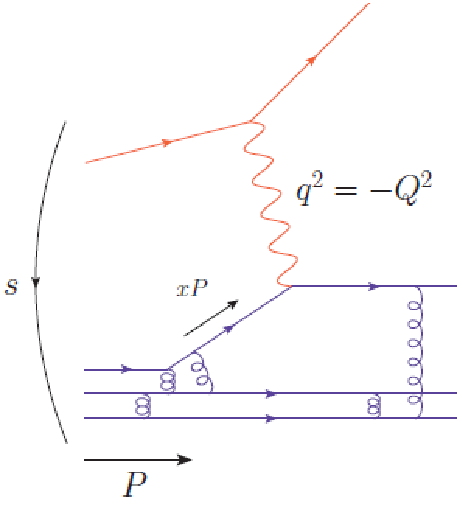
We study the evolution equation for σ_n whose solution gives the full AGK cross sections.

We suggest the homotopy approach for the solution of the cross section of the n particle.

Deep Inelastic Scatering DIS

Mueller Dipole Approximation

(A. H. Mueller in ’93-’94)



The total DIS cross section is expressed in terms of the quark Dipole-Hadron amplitude $N(\vec{x}, \vec{b}, Y)$

At small x, the dominant contribution to DIS structure functions comes from the dipole picture of DIS, where the virtual photon splits into a quark-antiquark pair, which then interacts with the target.

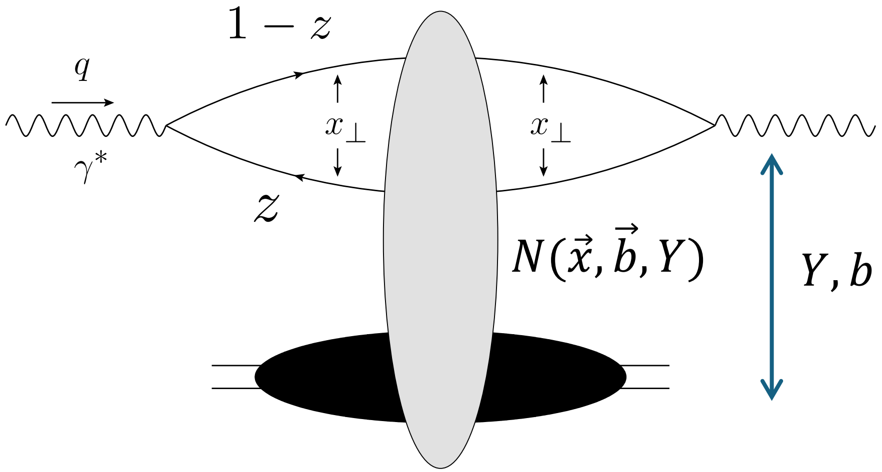
$$\sigma_{tot}^{\gamma^* A} = \int \frac{d^2 x_\perp}{2\,\pi}\, d^2 b_\perp \int_0^1 \frac{dz}{z\,(1-z)}\, |\Psi^{\gamma^* \to q\bar{q}}(\vec{x}_\perp, z)|^2\, N(\vec{x}_\perp, \vec{b}_\perp, Y)$$

$Y = \ln\frac{1}{x}$ is the rapidity

b is the impact parameter

$x_\perp$ the dipole transverse size

$\Psi^{\gamma}{}_{q\bar{q}}\,(x_\perp, b, Y)$ Photon wave function in dipole approximation (known to NLO light cone)



Quantum Chromodynamics at high Energy, Kovchegov and Levin Cambridge 2012
Mueller 2002

Nonlinear Evolution

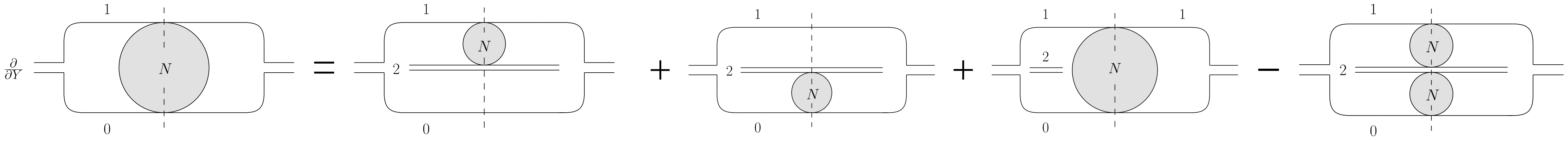
Summing the gluon cascade at large - N_C one can find the Balitsky – Kovchegov evolution equation for the dipole scattering amplitude N

$$Y = \ln \frac{1}{x} \sim \ln s$$

$$\frac{\partial N(x_{10},b,Y)}{\partial Y} = \frac{\bar{\alpha}}{2\,\pi}\int d^2x_2\,\frac{x_{10}^2}{x_{12}^2\,x_{02}^2}\left(N(x_{12},b,Y) + N(x_{20},b,Y) - N(x_{10},b,Y)\ - N(x_{12},b,Y) * N(x_{20},b,Y)\right)$$

Balitsky ‘96, Yu.Kovchegov. ’99

Diagrammatic representation of the BK evolution equation for the forward dipole–nucleus scattering amplitude *N*, denoted by a shaded circle.



At fixed rapidity a colorless dipole with size x_{10} decays into two dipoles with sizes x_{21} and x_{20}.
Either one dipole proceeds to evolve and interact with the target while the other dipole remains a spectator or both dipoles evolve and interact with the target (the nonlinear term in BK).

Beyond the large-N_c limit we have to consider the JIMWLK functional evolution equation (Jalilian-Marian, Iancu, McLerran, Weigert Leonidov and Kovner 1997-2002)

The JIMWLK and BK equation was solved on the lattice and there is excellent agreement between the JIMWLK and BK simulations K. Rummukainen and H. Weigert ’04

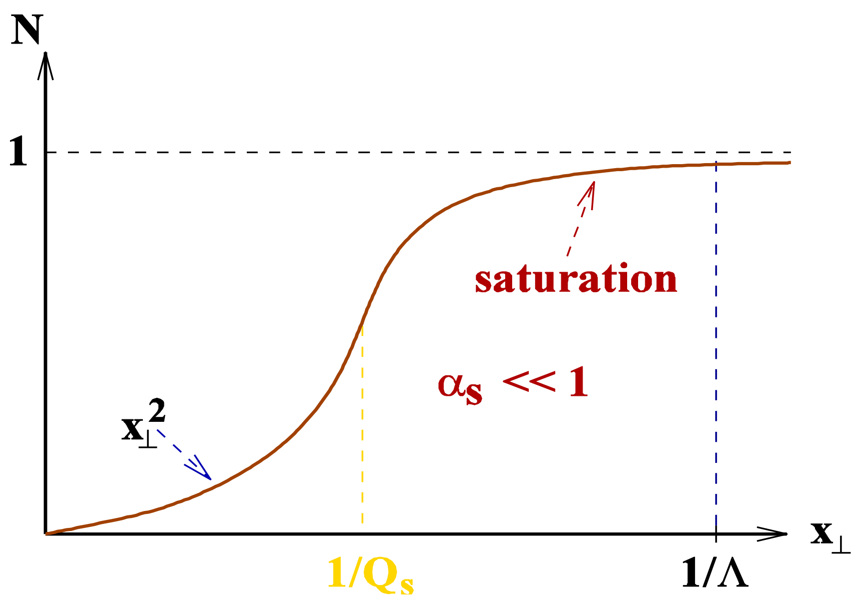
Solution of the BK nonlinear equation

$$\frac{\partial N(x_{10},b,Y)}{\partial Y} = \frac{\bar{\alpha}}{2\,\pi}\int d^2x_2\,\frac{x_{10}^2}{x_{12}^2\,x_{02}^2}\left(N(x_{12},b,Y) + N(x_{20},b,Y) - N(x_{10},b,Y)\; - N(x_{12},b,Y) * N(x_{20},b,Y)\right)$$

Analyzing the behavior of the solution BK equation we can easily see that *N* = 1 and N = 0 are stationary solution of that equation.

The dipole-nucleus amplitude should be

The fact that the forward amplitude *N* goes to zero as $x\perp \to 0$ is based on a fundamental physical principle of **color transparency** in the zero-size dipole the color charges of the quark and the anti-quark cancel each other, leading to disappearance of the interactions with the target.



$N(x_{10},Y)$ grows with $x\perp$, but a very large $x\perp$ saturates to $N \to 1$

Black disk limit,

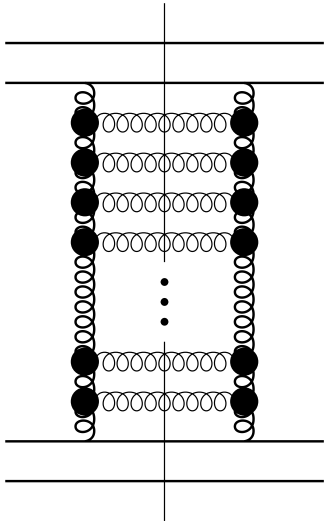
At the moment there is no exact analytical solution of the BK equation

$$\frac{\partial N(x_{10},b,Y)}{\partial Y} = \frac{\bar{\alpha}}{2\,\pi}\int d^2x_2 \frac{x_{10}^2}{x_{12}^2\,x_{02}^2}\left(N(x_{12},b,Y) + N(x_{20},b,Y) - N(x_{10},b,Y) \;-N(x_{12},b,Y) * N(x_{20},b,Y)\right)$$

But for $N(x_{10},Y) < 1$ at small side of the dipole $\mathrm{x}_{\perp} < 1/Q_{so}$

$$\boxed{\frac{\partial N(x_{10},b,Y)}{\partial Y} = \frac{\bar{\alpha}}{2\,\pi}\int d^2x_2 \frac{x_{10}^2}{x_{12}^2\,x_{02}^2}\left(N(x_{12},b,Y) + N(x_{20},b,Y) - N(x_{10},b,Y)\right)}$$

BFKL equation



BFKL equation: Balitsky, Fadin, Kuraev, Lipatov 1977-78.

- One starts with a two-gluon exchange diagram and resum its radiative corrections.
- The leading high-energy contribution can be drawn as a ladder diagram, with the t-channel gluons being the special "reggeized" gluons and the thick dots representing effective Lipatov vertices.
- Each rung of the ladder brings in a power of $\alpha \ln s$

Nonlinear saturation effects become important when N ~ N^2 ⇨ N ~ 1. This happens at $k_T \sim Q_S$ and we define the saturation scale $Q_s(s)$

Typical partons have $k_T \sim Q_S$, so that their characteristic size is of the order $r \sim 1/k_T \sim 1/Q_S$
Typical parton size decreases with energy and $Q_s(y) = Q_{s0}(y)e^{\Delta\, y}$ with the rapidity $y = \ln\frac{1}{x}$

Analytical Solutions to the BFKL equation

$$\frac{\partial N(x_{10},b,Y)}{\partial Y} = \frac{\bar{\alpha}}{2\,\pi}\int d^2x_2\,\frac{x_{10}^2}{x_{12}^2\,x_{02}^2}\left(N(x_{12},b,Y) + N(x_{20},b,Y) - N(x_{10},b,Y)\right)$$

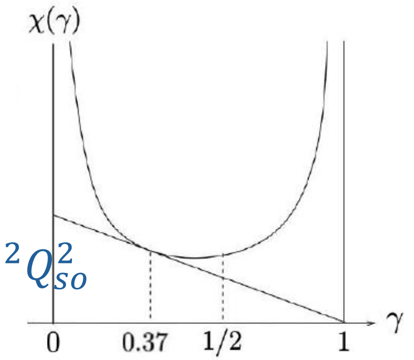
First BFKL Pomeron Lipatov solution (1986)

$$\bar{\alpha}_s \equiv \frac{\alpha_s N_c}{\pi}.$$

We can consider that the solution in Mellin representation is

$N_{BFKL}(r,Y,b) = \int d\nu\, r^{2\gamma}\,\tilde{N}(\nu,Y,b) \;\rightarrow\; \frac{\partial \tilde{N}(\nu,Y,b)}{\partial Y} = \bar{\alpha}\,\chi(\gamma)\,\tilde{N}(\nu,Y,b)$ and $\tilde{N}(\nu,Y,b) = \tilde{N}_0\, e^{\bar{\alpha}\,\chi(\gamma)\,Y}$

where $\chi(\gamma) = 2\psi(1) - \psi(1-\gamma) - \psi(\gamma) = \chi(0,\frac{1}{2} + i\,\nu)$; $\psi(z) = \frac{d\ln\Gamma(z)}{dz}$ Digamma Function $\gamma = \frac{1}{2} + i\,\nu$



Then $N_{BFKL}(r,Y,b) = \int d\nu\, r^{2\gamma}\,\tilde{N}_0\, e^{\bar{\alpha}\,\chi(\gamma)\,Y}$

Dominant contribution to high Y is

$$N_{BFKL}(r,Y,b) = \int \frac{d\gamma}{2\pi}\, r^{2\gamma}\,\tilde{N}_0 e^{\bar{\alpha}\,\chi(\gamma)\,Y} = \int \frac{d\gamma}{2\pi}\,\tilde{N}_0 e^{\bar{\alpha}\,\chi(\gamma)\,Y + \gamma\xi} \approx e^{\bar{\alpha}\,\chi(\gamma_{cr})\,Y + \gamma_{cr}\xi} \quad ; \; \xi = \ln r^2 Q_{so}^2$$

and using Saddle point $N_{BFKL}(r,Y,b) \approx e^{\bar{\alpha}\,\chi(\gamma_{cr})\,Y}(r\,Q_{so})^{2\,\gamma_{cr}}$

Saturation scale $Q_s(s)$ is happens at $r = 1/Q_s$ and we find that the saturation scale $Q_s(s)$ is

$$Q_s^2(Y) = Q_{s0}^2\, e^{\bar{\alpha}\,\frac{\chi(\gamma_{cr})}{\gamma_{cr}}\,Y} = e^{\bar{\alpha}\,\kappa\,Y} \text{ where } \kappa = \frac{\chi(\gamma_{cr})}{\gamma_{cr}}$$

BFKL and BK equation:
Transition to saturation and Geometric Scaling

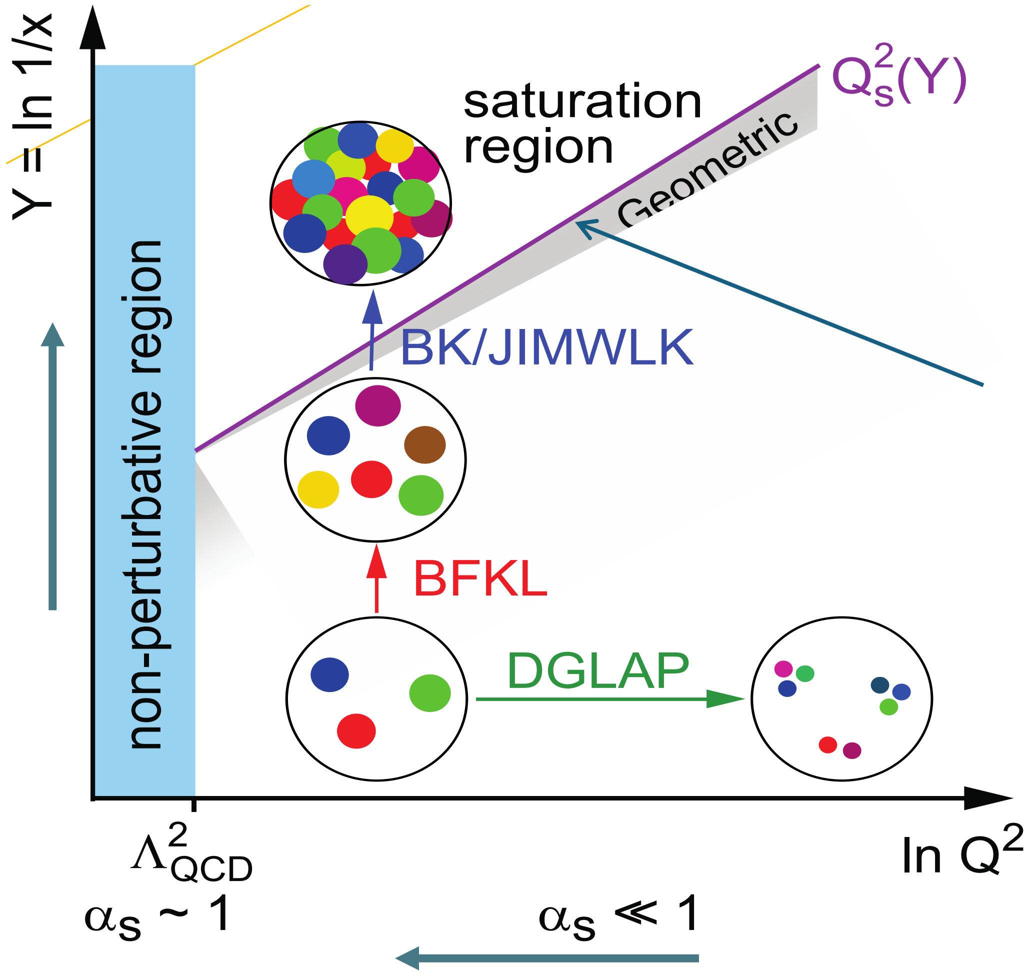
$N_{BFKL}(r,Y,b) \rightarrow N_{BK}(r,Y,b)$

Nonperturbative analysis
BK – nonlinear

energy

for High energy or small x we have a high number of gluons leading a very dense system of gluons (CGC)

Saturation region and the transition from BFKL to BK is to $Q_s(Y)$ and Geometric scaling



There is a huge number gluons at small-x

$$Q_s^2 \sim \left(\frac{1}{x}\right)^{\lambda}$$

Saturation Scale
grows with energy

Perturbative analysis
BFKL, linear equations

Evalution in Q^2:
$DGLAP\ equation$
Better resolution in the partons
Dokshitzer–Gribov–Lipatov–Altarelli–Parisi

BFKL and BK equation: Transition to saturation and Geometric Scaling

Stasto, Golec-Biernat and Kwiecinski

Phys. Rev. Lett. 86 (2001) 596.

have shown that the HERA data on DIS at low x , functions of two independent variables — the photon virtuality Q^2 and the Bjorken variable x , are consistent with scaling in terms of the variable τ

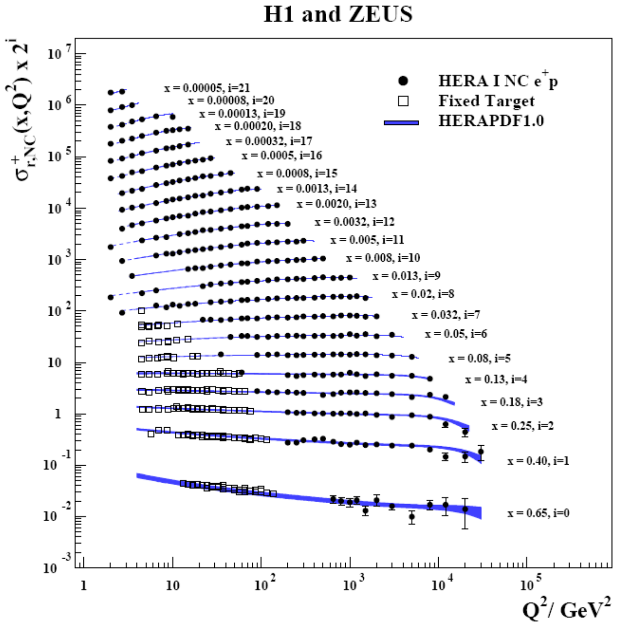
Saturation momentum: $$Q_s^2(Y) = Q_{s0}^2 e^{\lambda Y} = Q_{s0}^2 \left(\frac{x_0}{x}\right)^{\lambda} \rightarrow \tau = r^2 Q_s^2(Y); \;\; \tau = \frac{Q^2}{Q_s^2(Y)} = Q^2/Q_0^2 \left(\frac{x}{x_0}\right)^{\lambda} \quad \lambda = 0.3 - 0.4$$

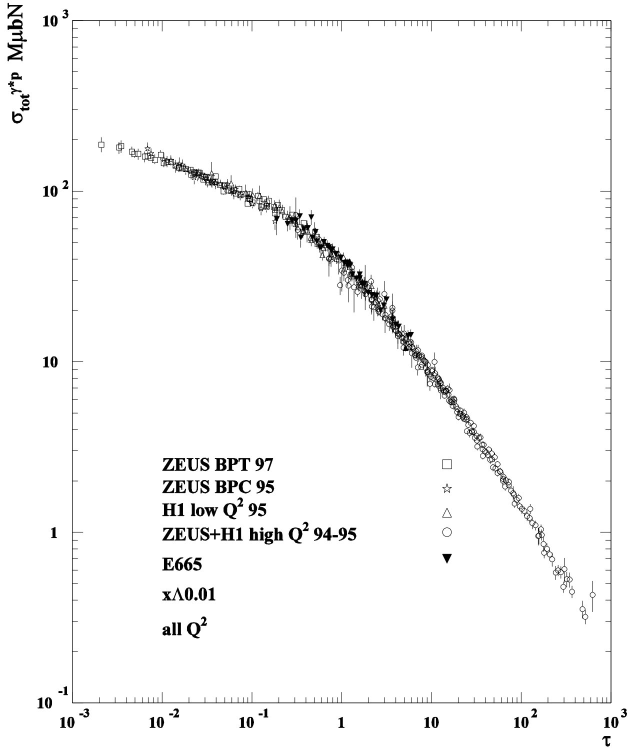
They plot the total DIS cross section, which is a function of 2 variables, Q^2 and x, as a function of just <u>one</u> variable

$$\sigma(x, Q) = \sigma(\tau)$$

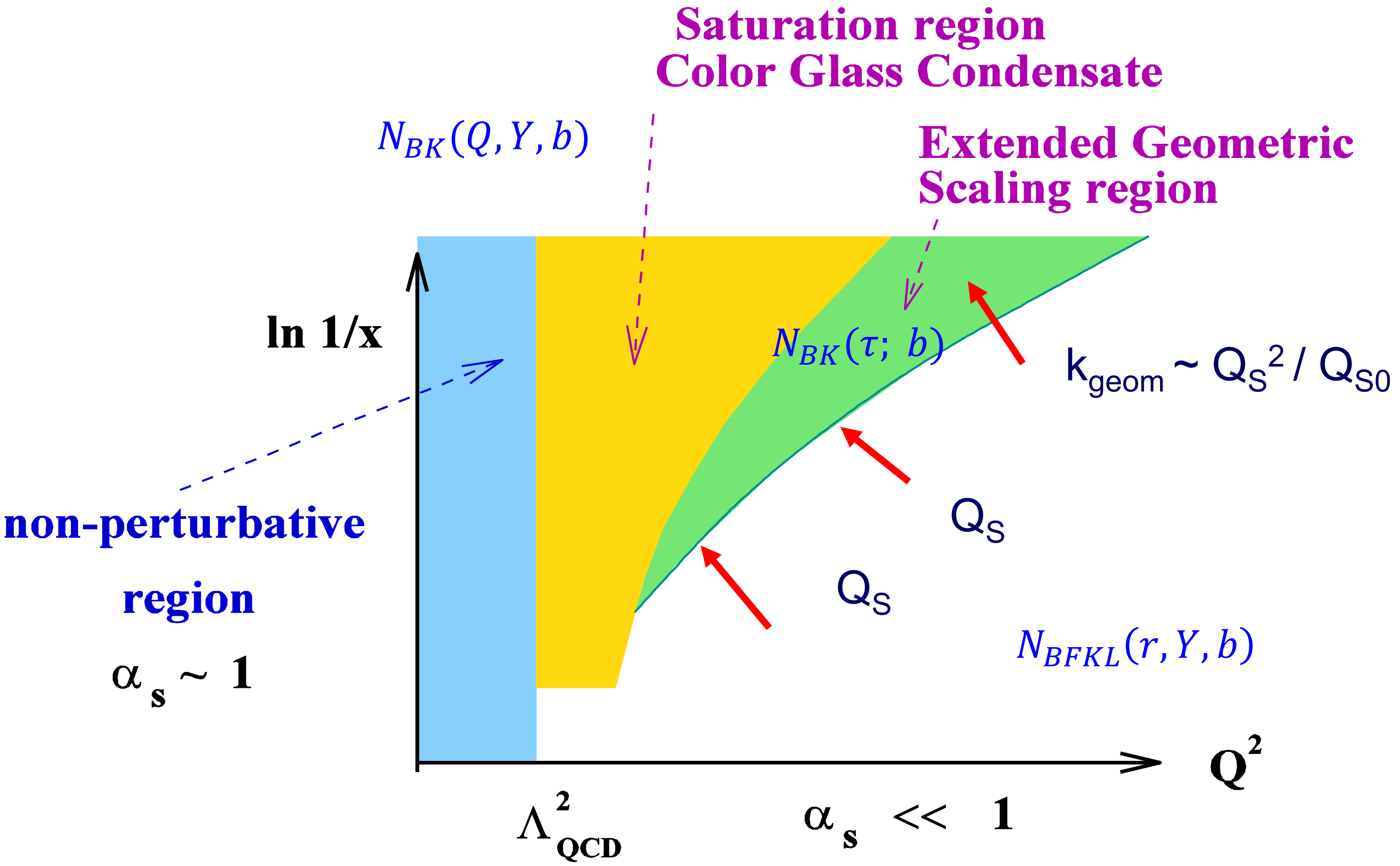
Geometric Scaling

is to show that the scaling region for the various distribution functions is in fact much larger than the saturation region





Balistky Phys. Rev. D75, 014001 (2007)
I. Balitsky, Nucl. Phys. B463, 99 (1996)
Kovchgov, Iancu, Itakura, and Larry McLerran arXiv 0203.137
Jalilian-Marian-Iancu-McLerran-Weigert-Leonidov-Kovner (JIMWLK)



Saturation physics is based on the existence of a large internal transverse momentum scale Q_S which grows with both decreasing Bjorken *x* and with increasing nuclear atomic number A

$$Q_s^2 \sim A^{1/3} \left(\frac{1}{x}\right)^{\lambda}$$

such that

$$\boxed{\alpha_S = \alpha_S(Q_S) \ll 1}$$

and we can use perturbation theory to calculate total cross sections, particle spectra and multiplicities, correlations, etc, from first principles.

Pomeron Calculus

The scattering amplitude can also be calculated as the sum of the “fan” diagrams of the BFKL Pomeron calculus

$$N(Y, \mathbf{x}_{01}, \mathbf{b}) = \sum_{k=1}^{\infty} (-1)^{k+1} \underbrace{C_k(\mathbf{x}_{01}, \mathbf{b}) \left(\mathrm{Im}\, G_{\mathrm{BFKL}}(Y, \mathbf{x}_{01}, \mathbf{b})\right)^k}_{= F_k(Y, \mathbf{x}_{01}, \mathbf{b})}$$

where F_k is the contribution of the exchange of k- Pomeron to the cross section.

And for the Initial condition IC (Mc Lerran Venugopalan)

$$N^{(0)}(Y = Y_A, \mathbf{x}_{01}, \mathbf{b}) = 1 - \exp\left(-\frac{1}{4}\, x_{01}^2\, Q_s^2(Y_A, \mathbf{b})\right)$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k!} \left(\frac{x_{01}^2 Q_s^2(Y_A, \mathbf{b})}{4}\right)^k$$

$$\Rightarrow F_k(Y = Y_A; \mathbf{x}_{01}, \mathbf{b}) = \frac{1}{k!} \left(\frac{x_{01}^2\, Q_{s0}^2}{4}\right)^k$$

K. Golec-Biernat and M. Wuesthoff, Phys. Rev. D 59, 014017 (1998).
L. McLerran and R. Venugopalan, Phys. Rev. D 49, 2233 (1994); 49, 3352 (1994); 50, 2225 (1994)

$Q_s(Y) = Q_{s0}(Y) e^{\Delta\, Y}$
Saturation Scale

The AGK cutting rules

The Abramovsky-Gribov-Kancheli **AGK cutting rules in Regge Theory** state that the total amplitude of a high-energy process can be decomposed into specific integer-weighted combinations of "cut" diagrams (producing real particles) and "uncut" diagrams.

The cross section, σ_n involving the exchange of exactly n cut pomerons can be expressed as follow:

$$\sigma_n^{\mathrm{AGK}}(Y;\mathbf{r},\mathbf{b}) = \sum_{k=n}^{\infty} \sigma_n^k(Y;\mathbf{r},\mathbf{b})$$

A.H. Mueller, G.P. Salam
Nucl.Phys.B475:293-320,1996

where

$$\sigma_n^k(Y;\mathbf{r},\mathbf{b}) = \begin{cases} (-1)^k\left(2^k - 2\right) F_k(Y;\mathbf{r},\mathbf{b}) & \text{for } n=0 \\ (-1)^{k-n}\dfrac{k!}{(k-n)!\,n!}\,2^k\,F_k(Y;\mathbf{r},\mathbf{b}) & \text{for } n\geq 1 \end{cases}$$

REVIEW OF PARTICLE PHYSICS∗ Particle Data Group Phys. Rev D 110, 030001 (2024)

and

$$F_k(Y;\mathbf{x}_{01},\mathbf{b}) = C_k(\mathbf{x}_{01},\mathbf{b})\left(\mathrm{Im}\,G_{\mathrm{BFKL}}(Y;\mathbf{x}_{01},\mathbf{b})\right)^k$$

The AGK cutting rules allows us to calculate the cross sections for production of n-cut Pomerons if we know $\mathbf{F_k}$: the contribution of the exchange of k-Pomerons to the cross section

This series can be calculated for the initial condition.

$$F_k(Y = Y_A; \mathbf{x}_{01}, \mathbf{b}) \;=\; \frac{1}{k!}\left(\frac{x_{01}^2\,Q_{s0}^2}{4}\right)^k$$

$$\sigma_n^{\mathrm{AGK}}(Y = Y_A; \mathbf{x}_{01}, \mathbf{b}) \;=\; \sum_{k=n}^{\infty}(-1)^{k-n}\,\frac{k!}{(k\,-\,n)!\,n!}\,2^k\,\frac{1}{k!}\left(\frac{x_{01}^2\,Q_{s0}^2}{4}\right)^k$$

$$=\; \frac{\left(\frac{1}{2}\,x_{01}^2\,Q_{s0}^2\right)^n}{n!}\,\exp\left\{-\,\frac{1}{2}\,x_{01}^2\,Q_{s0}^2\right\}$$

For $\xi \;=\; \ln\left(x_{01}^2\,Q_{s0}^2\right)$

$$\sigma_n^{\mathrm{AGK}}(Y = Y_A; \mathbf{x}_{01}, \mathbf{b}) \;=\; \frac{\left(\frac{1}{2}\,e^{\xi}\right)^n}{n!}\,\exp\left\{-\,\frac{1}{2}\,e^{\xi}\right\}$$

The multiplicity distribution

The multiplicity distribution of Gluons in the final state is defined by the expression: the convolution of the σ_k^{AGK} *with the distribution of the gluons inside of the k − cut Pomerons*.

$$\sigma_n^{f.s.}(Y;\mathbf{x}_{01},\mathbf{b}) \;=\; \sum_{k=1}^{\infty}\sigma_k^{\mathrm{AGK}}(Y;\mathbf{x}_{01},\mathbf{b})\,\mathcal{P}_n\underbrace{(k\,\Delta^{I\!P}_{\mathrm{BFKL}}Y)}_{\text{Poisson distribution}}$$

We find that the produced gluons in the BFKL follows a Poisson distribution

$$P^{\boldsymbol{P}}_{n-Gluons}(Y,k) = \frac{(\Delta_{BFKL}\,k\,Y)^n}{n!}\,\mathrm{e}^{-(\Delta_{BFKL}\,k\,Y)}$$

E. Levin, Phys. Rev. D 49, 4469 (1994).
Phys. Rev. D 111, 016019 (2025).

The main equation for $\sigma_n^{AGK}(Y, x_{01}, b) = \sigma_n$ has been derived by Kormilizin, Levin and Prygarin Nucl. Phys A 813 1 (2008)

$$\frac{\partial}{\partial Y} \sigma_n (Y; \mathbf{x}_{01}, \mathbf{b}) = \frac{\bar{\alpha}_S}{2\pi} \int d^2 x_2 \frac{x_{01}^2}{x_{02}^2 x_{12}^2}$$

$$\times [\sigma_n (Y; \mathbf{x}_{12}, \mathbf{b}) + \sigma_n (Y; \mathbf{x}_{02}, \mathbf{b}) - \sigma_n (Y; \mathbf{x}_{01}, \mathbf{b})$$

$$+ \sigma_n (Y; \mathbf{x}_{12}, \mathbf{b}) \sigma_{SD} (Y; \mathbf{x}_{02}, \mathbf{b}) + \sigma_n (Y; \mathbf{x}_{02}, \mathbf{b}) \sigma_{SD} (Y; \mathbf{x}_{12}, \mathbf{b})$$

$$+ \sum_{k=1}^{n-1} \sigma_{n-k} (Y; \mathbf{x}_{02}, \mathbf{b}) \sigma_k (Y; \mathbf{x}_{12}, \mathbf{b})$$

$$- 2 \sigma_n (Y; \mathbf{x}_{12}, \mathbf{b}) N (Y; \mathbf{x}_{02}, \mathbf{b}) - 2 \sigma_n (Y; \mathbf{x}_{02}, \mathbf{b}) N (Y; \mathbf{x}_{12}, \mathbf{b})]$$

Where σ_{SD} corresponds to single diffraction scattering, and N is the scattering amplitud of the dipole with size x_{ij}

This is a nonlinear generalized BK equation, and we need it solution.

Equivalent: What is the solution to the BK equation?

BK evolution in coordinate space

$$\frac{\partial N(x_{10},b,Y)}{\partial Y} = \frac{\bar{\alpha}}{2\,\pi}\int d^2x_2 \frac{x_{02}^2}{x_{12}^2\,x_{01}^2}\left(N(x_{12},b,Y) + N(x_{20},b,Y) - N(x_{10},b,Y)\; - N(x_{12},b,Y) * N(x_{20},b,Y)\right)$$

In general $N(x_{10}, b, Y)$:

- The dependence on the impact parameter b which becomes an external parameter or can be absorb in Q_s
- satisfying the initial condition given by the BFKL Pomeron
- satisfying the Geometric Scaling inside the saturation region, Non-perturbative

Initial condition at Y_0 is McLerran -Venugopalan model MV:

$$\mathrm{N}(r_\perp, b, Y = Y_0) = 1 - \exp\left[-\frac{r_\perp^2 Q_{s0}^2}{4}\ln\left(\frac{1}{r_\perp \Lambda} + e\right)\right]$$

A solution of BK equation is helpful to calculate F2 and compare with experimental data.

Kowalski, Lappi, Marquet, Venugopalan, PRC 78 (2008) 045201
Lappi, M¨antysaari, PRD 88 (2013) 114020

Analytical Solutions to the BK equation in coodinate space

Levin – Tuchin (2000) Solutions BK eq. inside the saturation region: $Q_s(Y) \gg Q$

$$\frac{\partial(N(x_{10},b,Y)}{\partial Y} = \frac{\bar{\alpha}}{2\,\pi}\int d^2x_2\,\frac{x_{02}^2}{x_{12}^2\,x_{01}^2}(N(x_{12},b,Y) + N(x_{20},b,Y) - N(x_{10},b,Y)\ - N(x_{12},b,Y) * N(x_{20},b,Y)$$

If N$(r_\perp,b,Y)$ = 1 - $\Delta(r_\perp,Y,b)$ and $r_\perp = |x_{10}| \gg \frac{1}{Q_s(Y)}$, then we can find a solution with the condition that:

$x_{12}^2 \sim x_{10}^2$ *and* $x_{10}^2 > x_{20}^2 \gg 1/Q_s^2$ one of the daughter dipole is smaller

$$\frac{\partial\Delta(r_\perp,Y,b)}{\partial Y} = -\frac{\bar{\alpha}}{2\,\pi}\int d^2x_{20}\,\frac{x_{10}^2}{x_{12}^2\,x_{20}^2}\Delta(x_{10},b,Y) = -\bar{\alpha}\,z\,\Delta(r_\perp,Y,b)$$

where

$$z = \ln r^2 Q_s^2(Y) = \ln\tau = \bar{\alpha}\,\kappa\,Y + \ln r^2 Q_s^2(Y_0) = \bar{\alpha}\,\kappa\,Y + \xi_0 \qquad \tau = r^2 Q_s^2(Y)\,; Q_s^2(Y) = Q_{s0}^2 e^{\lambda Y}$$

$$\Delta_{LT}(r_\perp,Y,b) = C\,e^{-\frac{z^2}{2\kappa}}$$

we can find a solution with Geometric Scaling in the saturation region

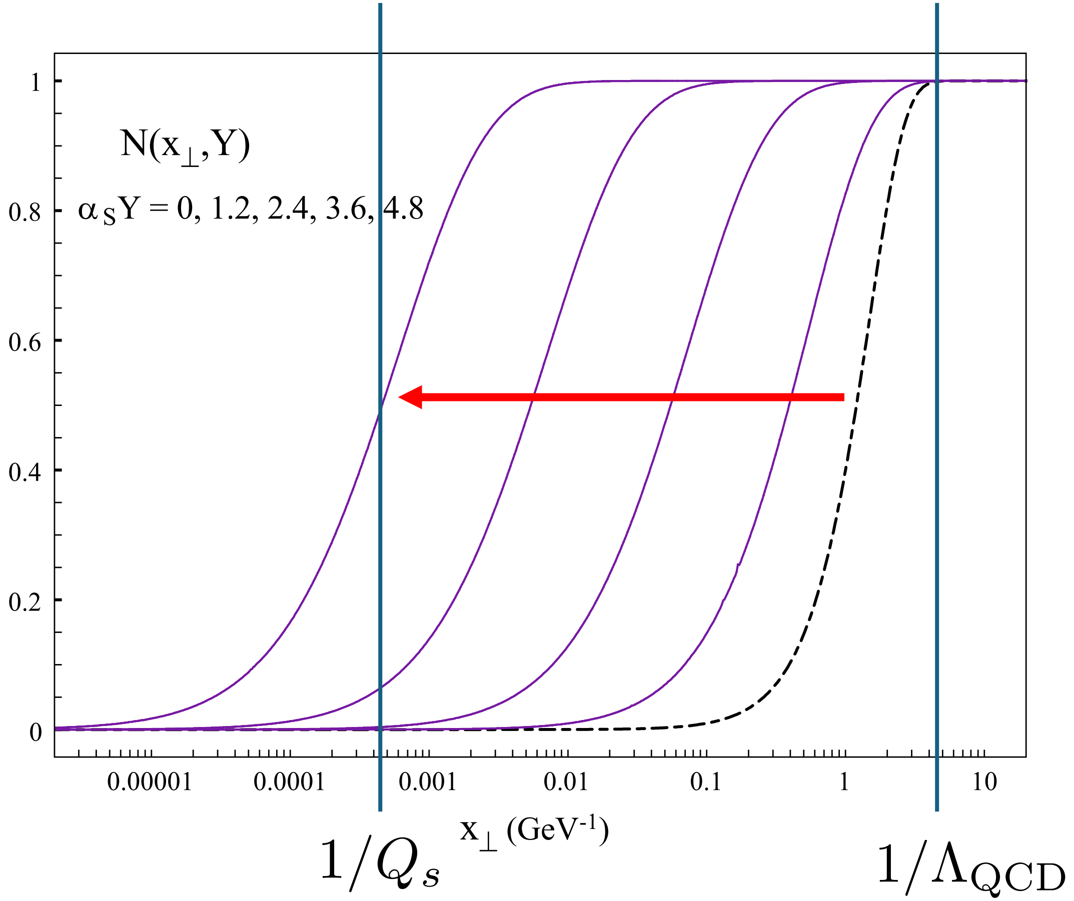
E. Levin and K. Tuchin, Nucl. Phys. B 573, 833 (2000) hep-ph/9908317;
Nucl. Phys. A 691, 779 (2001) [hep-ph/0012167]; 693, 787 (2001) hep-ph/0101275

Numerical Solution of BK equation

We conclude that

$$Q_s^2 \sim \left(\frac{1}{x}\right)^{\lambda}$$

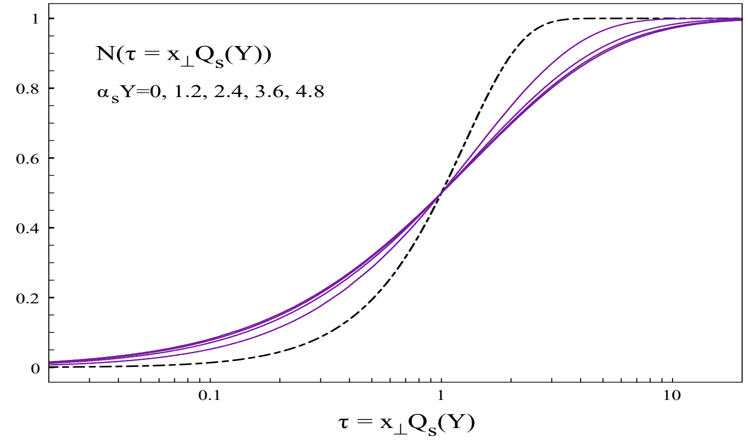
Energy increases → Q_s increases moving further away from Λ_{QCD}



numerical solution
by J. Albacete '03

Albacete and V. Kovchegov 2007
Lappi et all. 2012- 2015

J. Albacete (ca. 2006)



BK solution preserves the black disk limit, always

$$\sigma^{q\bar{q}A} = 2\int d^2b\, N(x_\perp, b_\perp, Y)$$

Our Proposal to find solution in the Saturation Region
Homotopic Aproach

One can see that a numerical solution of the BK equation does not allow us to introduce explicit dependence on Qs(Y).

we need to have the analytical solution in which we can see explicitly the dependence of the scattering amplitude on Qs(Y; b).

we suggest a procedure to find the solution to the BK equation as a sequence of iterations, based on Homotopy approach

Homotopy Method: consider a nonlinear differential equation

$$\mathcal{L}[u] \quad + \quad \mathcal{N}_{\mathcal{L}}[u] = 0 \to \mathcal{K}[p,u] = \mathcal{L}[u] \quad + p\,\mathcal{N}_{\mathcal{L}}[u] = 0$$

$\mathcal{L}[u]$ *linear part* $\mathcal{N}_{\mathcal{L}}[u]$=Nonlinear Integral differential Operator

And $\mathcal{K}[p,u]$ is the Homotopic Function with $p \in [0,1]$ small parameter. The solution is

$$u_p(r,Y;b) = u_0(r,Y;b) + pu_1(r,Y;b) + p^2u_2(r,Y;b) + \cdots$$

$\mathcal{L}[u_0] = 0$ *give de solution from linear part and* u_1 *from the nonlinear part,* u_2 *from recursive equation.*

The convergency of the Homotopic approach has been proved by J. He (1999) , but we need to check to each solution.

Saikia arXiv 2204.10111

Homotopy solution to the BK evolution

Phys.Rev.D 107 (2023) 9, 094030

$$\frac{\partial N(x_{10},b,Y)}{\partial Y} = \int d^2x_2 \frac{r_{10}}{r_{12}r_{20}}\left(N(x_{12},b,Y) + N(x_{20},b,Y) - N(x_{10},b,Y) - N(x_{12},b,Y) * N(x_{20},b,Y)\right)$$

Kinematic region

$x_{10}^2 Q_s^2(Y) < 1 \quad pQCD$

$x_{10}^2 Q_s^2(Y) \approx 1 \ \ Vicinity\ Saturation \rightarrow N_{10'}(z) = C_1\left(x_{10}^2 Q_s^2(Y)\right)^{\bar{\gamma}}$

$x_{10}^2 Q_s^2(Y) > 1 \ \ saturation \ \rightarrow N_{10'LT}(z) = 1 - C\, e^{-\frac{z^2}{2\kappa}}$

Variables:

$(\mathrm{r},\mathrm{Y}) \rightarrow\ \mathrm{z} = \ln\, r^2 Q_s^2(Y,b) = \bar{\alpha}\,\kappa\,(Y - Y_A) + \ln\, r^2 Q_s^2(Y_A,b)$, $\quad \ln(Q_{so}^2(Y_A,b) r^2) = \xi$

$Q_s^2(Y) = Q_{so}^2(Y = Y_A)\ e^{\bar{\alpha}\,\kappa\,(Y - Y_A)}$, and $\kappa = \frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}$ and $\chi(\gamma_{cr})$ is the BFKL characteristic functon $\bar{\gamma} = 0.73$ $\kappa = 4.88$

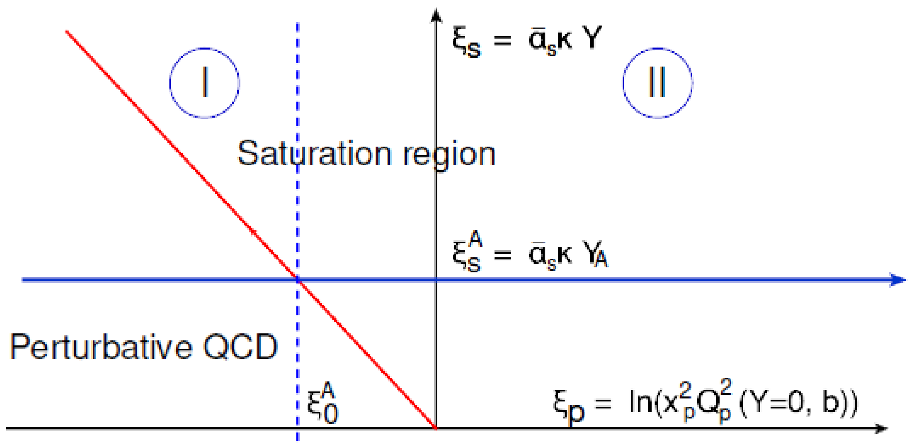
$Y_A = \ln A^{\frac{1}{3}}$ where A is the number of nucleons in a nucleon $\quad N_I(z) \quad N_{II}(z,\xi)$

Initial Condition: at Y_A for DIS with nuclei we consieder the McLerran-Venugopalan

$$N(x_{10},b,Y_A) = 1 - e^{-\frac{r^2 Q_s^2(Y=Y_A)}{4}} = 1 - e^{-\frac{1}{4}e^{\xi}}$$

Boundary Condition: at $x_{10}^2 Q_s^2(Y) = 1$, $z = 0$, we have the following boundary condition

$$\mathrm{N}_{01}(z=0) = N_0 \quad \text{and} \quad \frac{d\mathrm{N}_{01}(z)}{dz}\Big|_{z=0} = \bar{\gamma}\, N_0$$



Summary at first calculation

We developed the homotopy approac for solving the non-linear evolution Balitsky-Kovchegov equation in the saturation region

This solution satisfies the boundary and initial conditions which are given perturbative QCD approach for $r\,Q_s < 1$ and by McLerran-Venugopalan for $Y = Y_A$.

The numerical part of the calculations is expressed through well converged integrals and can be easily estimated.

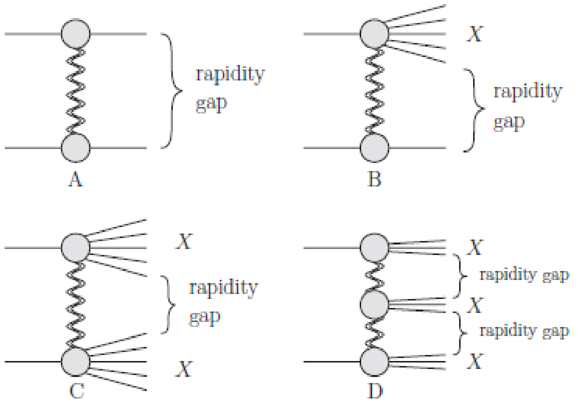
We found a way to study the BK equation in Momentum or coordinated representation, which can be extended to another process.

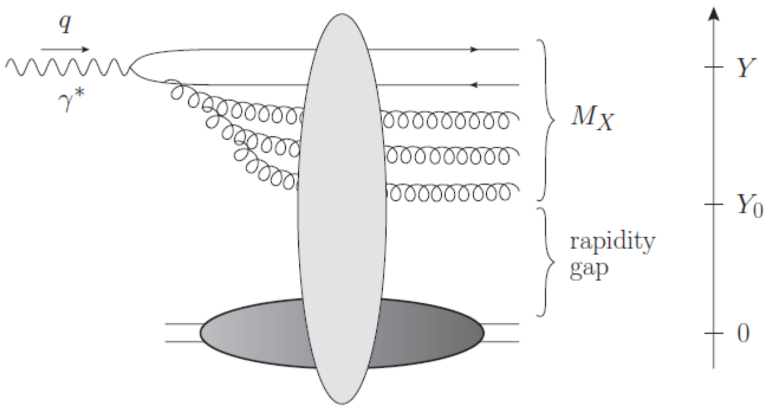
Applications:

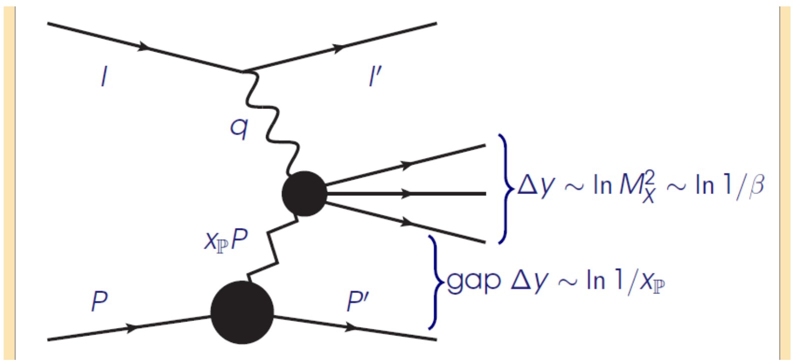
Diffractive Scattering
BK Equation with Running coupling constant

Diffraction at high energy

Single Diffraction







cross section of diffractive production with the rapidity gap larger than Y_0

$$\sigma^{\text{diff}}(Y, Y_0, Q^2) \;=\; \int d^2 r_\perp \int dz \, |\Psi^{\gamma^*}(Q^2; r_\perp, z)|^2 \, \sigma^{diff}_{\text{dipole}}(r_\perp, Y, Y_0)$$

where

$$\sigma^{diff}_{dipole}(r_\perp, Y, Y_o) = \int d^2 b \, N^D(r_\perp, Y, Y_0, b)$$

Mining for gluon saturation at the LHC and the future EIC

Evolution equation for Diffraction at high energy

Kovchegov-Levin (KL) equation (2000)

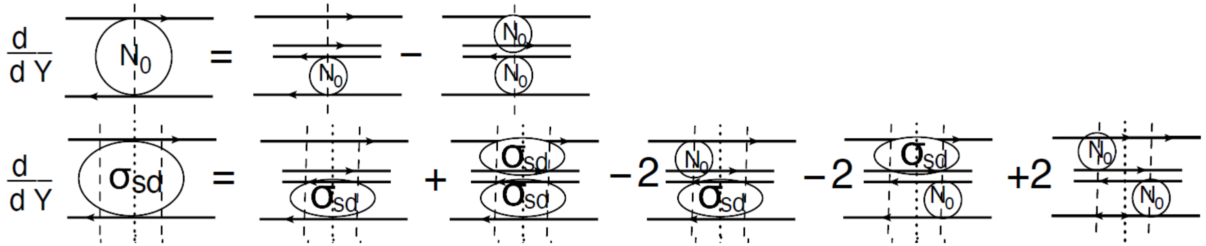
Contreras, Garrido, Levin and Meneses Phys. Rev D. 107 (2023)

$$\frac{\partial N^D\left(Y,Y_0,r_{10};b\right)}{\partial Y_M} =$$
$$\frac{\bar{\alpha}_S}{2\pi}\int d^2 r_2\, K\left(r_{10}|r_{12},r_{02}\right)\left\{N^D\left(Y,Y_0,r_{12};b\right)\,+\,N^D\left(Y,Y_0,r_{20};b\right)\,-\,N^D\left(Y,Y_0,r_{10};b\right)\right.$$
$$+N^D(Y;Y_0,r_{12};b)N^D(Y;Y_0,r_{20};b)\,-\,2\,N^D(Y;Y_0,r_{12};b)\,N(Y;r_{20};b)$$
$$\left.-\,2\,N(Y;r_{12};b)N^D(Y;Y_0,r_{20};b)\,+\,2N(Y;r_{12};b)N(Y;r_{20};b)\right\}$$

N^D the evolution equation has been derived by Kovchegov and Levin in the leading log(1/x) approximation (LLA) of perturbative QCD

- This is a nonlinear evolution equation with the initial condition specified at $Y = Y_0$
- To solve it one has first to solve the BK equation to find the dipole amplitude N
- For this equation: N^D no analytic solution exists.
- We will use the Homotopic approach and the previous solution for N which satisfice the BK equation.

$$\bar{\alpha}_s\,=\,\frac{\alpha_s\,N_c}{\pi}$$



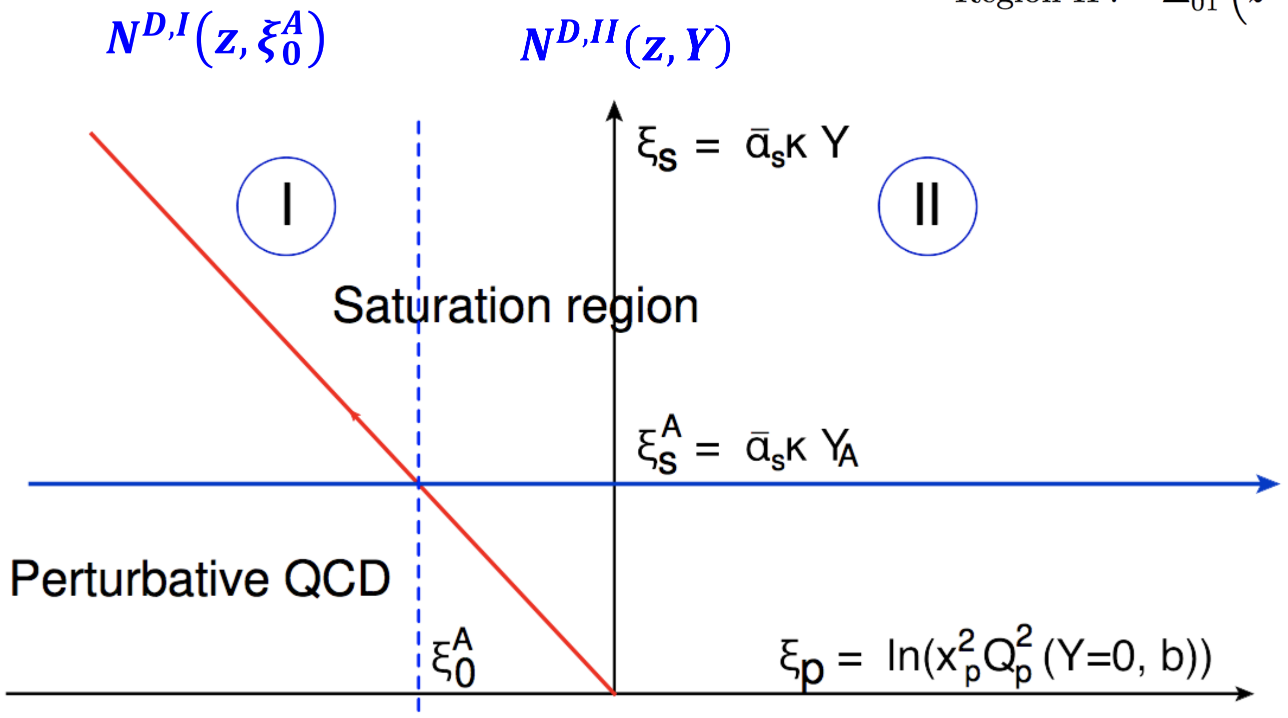
This equation takes the form of the Balitsky-Kovchegov (BK) equation

$$\mathcal{N}(z, \delta\tilde{Y}, \delta Y_0) = 2\,N(z, \delta\tilde{Y}) - N^D(z, \delta\tilde{Y}, \delta Y_0)$$

$$\mathcal{N}\,(z, \delta Y_0) = 1 - \Delta^D\,(z, \delta Y_0)$$

$$\text{Region I}: \quad \Delta^D_{01}\,(z \to z_0, \delta Y_0) = C^2 \exp\left(-\frac{(z_0 - \tilde{z})^2}{\kappa}\right)$$

$$\text{Region II}: \quad \Delta^D_{01}\left(z \to z_0, \delta\tilde{Y} \to \delta Y_0, \delta Y_0\right) = 1 - N_{in} = G^2\,(\xi) \exp\left(-\frac{(z_0 - \tilde{z})^2}{\kappa}\right)$$



Saturation region of QCD for the scattering amplitude. The critical line (z=0) is shown in red.

Contreras, Garrido, Levin and Meneses Phys. Rev D. 107 (2023)

The main equation for $\sigma_n^{AGK}(Y, x_{01}, b) = \sigma_n$ has been derived by Kormilizin, Levin and Prygarin Nucl. Phys A 813 1 (2008)

$$
\begin{aligned}
\frac{\partial}{\partial Y}\sigma_n\left(Y;\mathbf{x}_{01},\mathbf{b}\right) \; = \; & \frac{\bar{\alpha}_S}{2\,\pi}\int d^2x_2\,\frac{x_{01}^2}{x_{02}^2\,x_{12}^2} \\
& \times\left[\sigma_n\left(Y;\mathbf{x}_{12},\mathbf{b}\right) + \sigma_n\left(Y;\mathbf{x}_{02},\mathbf{b}\right) - \sigma_n\left(Y;\mathbf{x}_{01},\mathbf{b}\right)\right. \\
& + \sigma_n\left(Y;\mathbf{x}_{12},\mathbf{b}\right)\sigma_{SD}\left(Y;\mathbf{x}_{02},\mathbf{b}\right) + \sigma_n\left(Y;\mathbf{x}_{02},\mathbf{b}\right)\sigma_{SD}\left(Y;\mathbf{x}_{12},\mathbf{b}\right) \\
& + \sum_{k=1}^{n-1}\sigma_{n-k}\left(Y;\mathbf{x}_{02},\mathbf{b}\right)\sigma_k\left(Y;\mathbf{x}_{12},\mathbf{b}\right) \\
& \left. - 2\,\sigma_n\left(Y;\mathbf{x}_{12},\mathbf{b}\right)N\left(Y;\mathbf{x}_{02},\mathbf{b}\right) - 2\,\sigma_n\left(Y;\mathbf{x}_{02},\mathbf{b}\right)N\left(Y;\mathbf{x}_{12},\mathbf{b}\right)\right]
\end{aligned}
$$

Where σ_{SD} correspond to the single diffraction, and N is the scattering amplitud of the dipole with size x_{ij}

This is a nonlinear generalized BK equation, and we need it solution.

$$\frac{\partial}{\partial Y}\sigma_n\left(Y,\mathbf{x}_{01},\mathbf{b}\right) \;=\; \frac{\bar{\alpha}_S}{2\,\pi}\int d^2x_2\,\frac{x^2_{01}}{x^2_{12}\,x^2_{02}}\times\Big[-\sigma_n\left(Y,\mathbf{x}_{01},\mathbf{b}\right)\;+\;\sigma_n\left(Y,\mathbf{x}_{12},\mathbf{b}\right)\Delta\left(Y,\mathbf{x}_{02},\mathbf{b}\right)$$

$$+\;\Delta\left(Y,\mathbf{x}_{12},\mathbf{b}\right)\,\sigma_n\left(Y,\mathbf{x}_{02},\mathbf{b}\right)\;+\;\sum_{k=1}^{n-1}\sigma_{n-k}\left(Y,\mathbf{x}_{02},\mathbf{b}\right)\,\sigma_k\left(Y,\mathbf{x}_{12},\mathbf{b}\right)\Big]$$

We have a conected Flow equation:

$$\sigma_n(Y,x_{02})(1+\sigma_{SD}(Y,x_{12})\text{- }2N(Y,x_{12}))$$

$$2N(Y,x_{12})-\sigma_{SD}(Y,x_{12})=1-\Delta\left(Y,x_{12}\right)$$

We need $N(r,Y),\sigma_{SD}(r,Y)$

Kinematic restriction:

$x_{12}^2 \sim x_{10}^2$ *and* $x_{10}^2 > x_{20}^2 \gg 1/Q_s^2$

one of the daughter dipole is small

Region I: $N(r,Y)=N(z),\; z=Log[r^2Q_s^2(Y)]$

$$\kappa\,\frac{d\sigma_1\left(z\right)}{dz}\;=\;-\,z\,\sigma_1\left(z\right)\;+\;\sigma_1\left(z\right)\int\limits_{\xi_0^A}^{z}dz'\,\Delta\left(z'\right)\;+\;\Delta\left(z\right)\int\limits_{\xi_0^A}^{z}dz'\,\sigma_1\left(z'\right)$$

$$\kappa \frac{d\sigma_1(z)}{dz} = -z\,\sigma_1(z) + \sigma_1(z)\int_{\xi_0^A}^{z} dz'\,\Delta(z') + \Delta(z)\int_{\xi_0^A}^{z} dz'\,\sigma_1(z')$$

$$= -\underbrace{\left(z - \int_{\xi_0^A}^{z} dz'\,\Delta(z')\right)}_{T(z)}\sigma_1(z) + \Delta(z)\underbrace{\int_{\xi_0^A}^{z} dz'\,\sigma_1(z')}$$

$$\int_{\xi_0^A}^{z} dz'\,\sigma_1(z') = \underbrace{\int_{\xi_0^A}^{\infty} dz'\,\sigma_1(z')}_{=\sigma_{0,1}} - \underbrace{\int_{z}^{\infty} dz'\,\sigma_1(z')}_{=\tilde{\Sigma}_1(z)}$$

Master equation

$$\Rightarrow \underbrace{\kappa \frac{d\sigma_1(z)}{dz} + T(z)\,\sigma_1(z) - \sigma_{0,1}\,\Delta(z)}_{\mathcal{L}[\sigma_1]} + \underbrace{\Delta(z)\,\tilde{\Sigma}_1(z)}_{\mathcal{N}_{\mathcal{L}}[\sigma_1]} = 0$$

We have divided the equation in two parts

$$\mathcal{L}[\sigma_1] \;=\; \kappa\,\frac{d\sigma_1\,(z)}{dz} \;+\; T(z)\,\sigma_1\,(z) \;-\; \sigma_{0,1}\,\Delta\,(z)$$

$$\mathcal{N}_{\mathcal{L}}[\sigma_1] \;=\; \Delta\,(z)\;\tilde{\Sigma}_1\,(z)$$

As a solution, we introduce $\quad \sigma_1 \;=\; \sigma_1^{(0)} \;+\; p\,\sigma_1^{(1)} \;+\; p^2\,\sigma_1^{(2)} \;+\; \cdots$

such that $\quad \mathcal{L}[\sigma_1^{(p)}] \;+\; p\,\mathcal{N}_{\mathcal{L}}[\sigma_1^{(p)}] \;=\; 0$

The 0th iteration is $\quad \mathcal{L}[\sigma_1^{(0)}];\;\; \kappa\,\frac{d\sigma_1^{(0)}(z)}{dz} \;=\; -\,T(z)\,\sigma_1^{(0)}(z) \;+\; \sigma_{0,1}\,\Delta(z)$

The solution is:

$$\sigma_1^{(0,I)}(z) \;=\; \underbrace{\exp\left(-\frac{1}{\kappa}\int_{\xi_0^A}^{z} dz'\, T(z')\right)}_{\tilde{\sigma}_1^{(0)}} \left\{ \underbrace{\frac{\sigma_{0,1}}{\kappa}\int_{\xi_0^A}^{z} dz'\, \Delta(z')\, \exp\left(\frac{1}{\kappa}\int_{\xi_0^A}^{z'} dz''\, T(z'')\right)}_{\tilde{\sigma}_1^{(0,I)}} + C_\phi^{(1)} \right\}$$

where

$$C_\phi^{(1)} \;=\; \frac{1}{2}e^{\xi_0^A}\exp\left(-\frac{1}{2}e^{\xi_0^A}\right)$$

For the next homotopy iteration we need to account for the linear terms in p:

$$\mathcal{L}[\sigma_1^{(0)} + p\,\sigma_1^{(1)}] \;+\; p\,\mathcal{N}_{\mathcal{L}}[\sigma_1^{(0)}] \;=\; 0$$

The equation takes the form

$$\kappa\,\frac{d\sigma_1^{(1,I)}(z)}{dz} \;=\; -T(z)\,\sigma_1^{(1,I)}(z) \;-\; \Delta(z)\,\tilde{\Sigma}_1^{(0)}(z)$$

$$\tilde{\Sigma}_1^{(0)}(z) \;=\; \int_{z}^{\infty} dz'\sigma_1^{(0)}(z')$$

$$\sigma_1^{(1,I)}(z) \;=\; -\frac{1}{\kappa}\exp\left(-\frac{1}{\kappa}\int_{\xi_0^A}^{z} dz'\, T(z')\right)\int_{\xi_0^A}^{z} dz'\Delta(z')\,\tilde{\Sigma}_1^{(0)}(z')\exp\left(\frac{1}{\kappa}\int_{\xi_0^A}^{z'} dz''\, T(z'')\right)$$

In general, the equation for the p-iteration (p≥1) in region I takes the form

$$\kappa\,\frac{d\sigma_1^{(p,I)}(z)}{dz} \;=\; -T(z)\,\sigma_1^{(p,I)}(z) \;-\; \Delta(z)\,\tilde{\Sigma}_1^{(p-1)}(z)$$

$$\sigma_1^{(p,I)}(z) \;=\; -\frac{1}{\kappa}\exp\left(-\frac{1}{\kappa}\int_{\xi_0^A}^{z} dz'\, T(z')\right)\int_{\xi_0^A}^{z} dz'\,\Delta(z')\,\tilde{\Sigma}_1^{(p-1)}(z')\,\exp\left(\frac{1}{\kappa}\int_{\xi_0^A}^{z'} dz''\, T(z'')\right).$$

Region II: $N(r,Y) = N(z\,,Y)\ ;\ \ z = Log[r^2 Q_s^2(Y)]$

$$\frac{\partial\sigma_1\left(\delta\tilde{Y},z\right)}{\partial\delta\tilde{Y}} + \kappa\,\frac{\partial\sigma_1\left(\delta\tilde{Y},z\right)}{\partial z} =$$

$$-\left(\underbrace{z - \int_{\xi_0^A}^{z} dz'\,\Delta\left(\delta\tilde{Y},z'\right)}_{T(\delta\tilde{Y},\,z)}\right)\sigma_1\left(\delta\tilde{Y},z\right) + \Delta\left(\delta\tilde{Y},z\right)\underbrace{\int_{\xi_0^A}^{z} dz'\,\sigma_1\left(\delta\tilde{Y},z'\right)}$$

$$\int_{\xi_0^A}^{z} dz'\,\sigma_1\left(\delta\tilde{Y},z'\right) = \underbrace{\int_{\xi_0}^{\infty} dz'\,\sigma_1\left(z'\right)}_{=\sigma_{0,1}} + \underbrace{\int_{\xi_0^A}^{\infty} dz'\left(\sigma_1\left(\delta\tilde{Y},z'\right) - \sigma_1\left(z'\right)\right)}_{=\delta\Sigma_1\left(\delta\tilde{Y}\right)} - \underbrace{\int_{z}^{\infty} dz'\sigma_1\left(\delta\tilde{Y},z'\right)}_{=\tilde{\Sigma}_1\left(\delta\tilde{Y},z\right)}$$

$$\underbrace{\frac{\partial\sigma_1\left(\delta\tilde{Y},z\right)}{\partial\delta\tilde{Y}} + \kappa\,\frac{\partial\sigma_1\left(\delta\tilde{Y},z\right)}{\partial z} + T\left(\delta\tilde{Y},z\right)\sigma_1\left(\delta\tilde{Y},z\right) - \sigma_{0,1}\,\Delta\left(\delta\tilde{Y},z\right) + \Delta\left(\delta\tilde{Y},z\right)\delta\Sigma_1\left(\delta\tilde{Y}\right)}_{\mathcal{L}[\sigma_1]}$$

$$\underbrace{-\,\Delta\left(\delta\tilde{Y},z\right)\tilde{\Sigma}_1\left(\delta\tilde{Y},z\right)}_{\mathcal{N}_{\mathcal{L}}[\sigma_1]} = 0$$

In this case, the solution is more complex. After some algebra, the general solution takes the form

$$\sigma_1^{(0,II)}\left(\delta\tilde{Y},z\right) \;=\; \Phi_1\left(-\,\kappa\,\delta\tilde{Y}\,+\,z\right)\sigma_1^{0'}\left(\delta\tilde{Y},z\right)\,+\,\sigma_1^{0'}\left(\delta\tilde{Y},z\right)\tilde{\sigma}_1^{0'}\left(\delta\tilde{Y},z\right)$$

where

$$\sigma_1^{0'}\left(\delta\tilde{Y},z\right) \;=\; \exp\left(-\int_0^{\delta\tilde{Y}} d\delta\tilde{Y}'\,T\left(\delta\tilde{Y}',-\,\kappa\left(\delta\tilde{Y}\,-\,\delta\tilde{Y}'\right)+\,z\right)\right)$$

$$\tilde{\sigma}_1^{0'}\left(\delta\tilde{Y},z\right) \;=\; \frac{1}{\kappa}\int_{\xi_0^A}^{z} dz'\,\frac{\sigma_{0,1}}{\sigma_1^{0'}\left(\delta\tilde{Y}\,-\,\frac{z-z'}{\kappa},z'\right)}\,\Delta\left(\delta\tilde{Y}\,-\,\frac{z-z'}{\kappa},z'\right)\,+\,C_{\sigma_1}$$

$$\Phi_1\left(\xi\right) \;=\; \frac{\frac{1}{2}e^{\xi}\exp\left(-\frac{1}{2}e^{\xi}\right)}{\sigma_1^{0'}\left(\delta\tilde{Y}=0,z=\xi\right)}\;-\;\tilde{\sigma}_1^{0'}\left(\delta\tilde{Y}=0,z=\xi\right)$$

σ_n

Now we need to add to our definition of L[σ_1] the non-homogeneous term

$$\mathcal{L}[\sigma_n] \;=\; \kappa\,\frac{d\sigma_n\,(z)}{dz} \;+\; T(z)\,\sigma_n\,(z) \;-\; \sigma_{0,n}\,\Delta\,(z) \underbrace{-\;\sum_{k=1}^{n-1}\Sigma_{n-k}\,(z)\;\sigma_k\,(z)}_{-\,U_n(z)}$$

$$\mathcal{N}_{\mathcal{L}}[\sigma_n] \;=\; \Delta\,(z)\;\tilde{\Sigma}_n\,(z)$$

We solve the equation in a similar way to σ_1, yielding

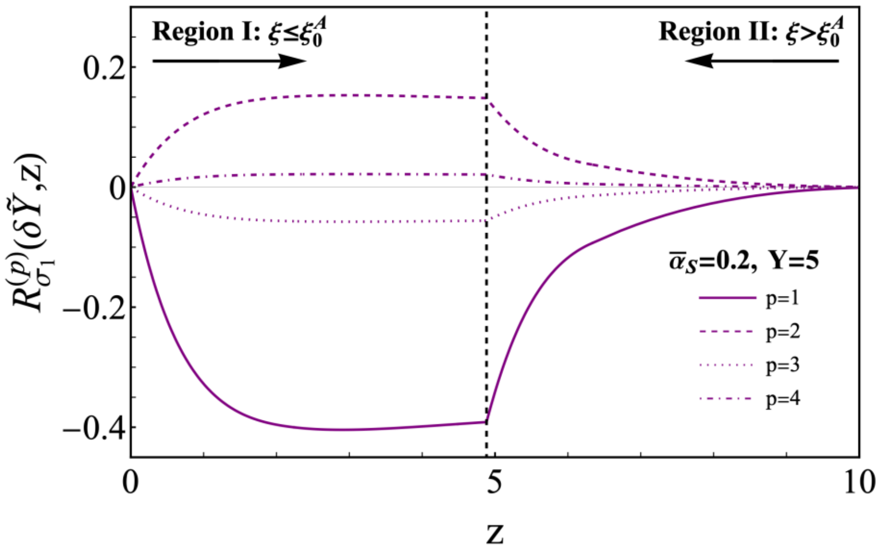
$$\sigma_n^{(0,I)}\,(z) \;=\; \exp\left(-\frac{1}{\kappa}\int_{\xi_0^A}^{z} dz'\,T\,(z')\right)\left\{\frac{1}{\kappa}\int_{\xi_0^A}^{z} dz'\,U_n\,(z')\,\exp\left(\frac{1}{\kappa}\int_{\xi_0^A}^{z'} dz''\,T\,(z'')\right) \;+\; C_\phi^{(n)}\right\}$$

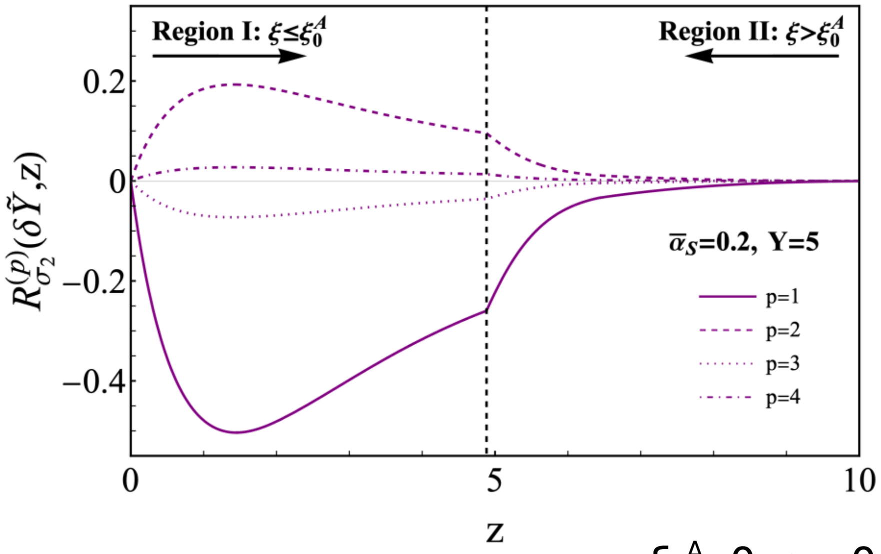
where $$C_\phi^{(n)} \;=\; \frac{\left(\frac{1}{2}e^{\xi_0^A}\right)^n}{n!}e^{-\frac{1}{2}e^{\xi_0^A}}$$

The formula for the p-iteration is

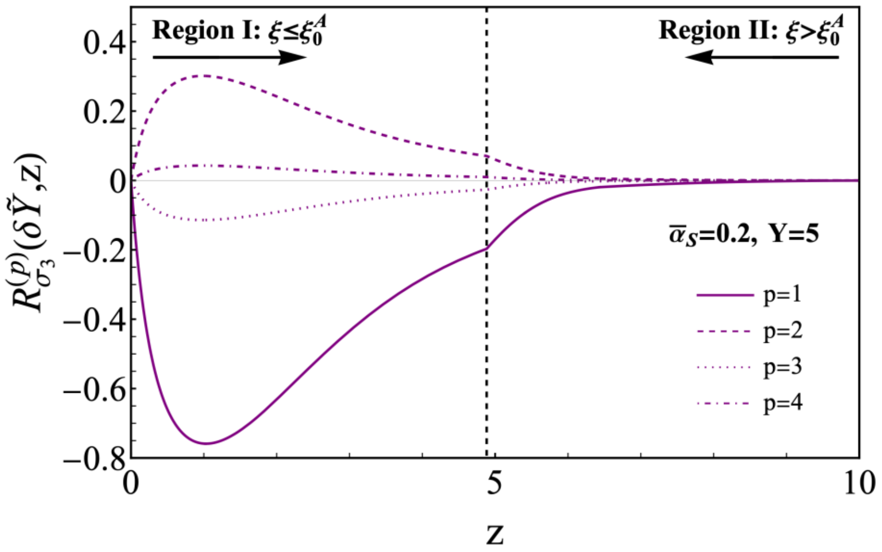
$$\sigma_n^{(p,I)}\,(z) \;=\; \exp\left(-\frac{1}{\kappa}\int_{\xi_0^A}^{z} dz'\,T\,(z')\right)\left\{\frac{1}{\kappa}\int_{\xi_0^A}^{z} dz'\,U_n\,(z')\;\tilde{\Sigma}_n^{(p-1)}\,(z')\,\exp\left(\frac{1}{\kappa}\int_{\xi_0^A}^{z'} dz''\,T\,(z'')\right) \;+\; C_\phi^{(n)}\right\}$$

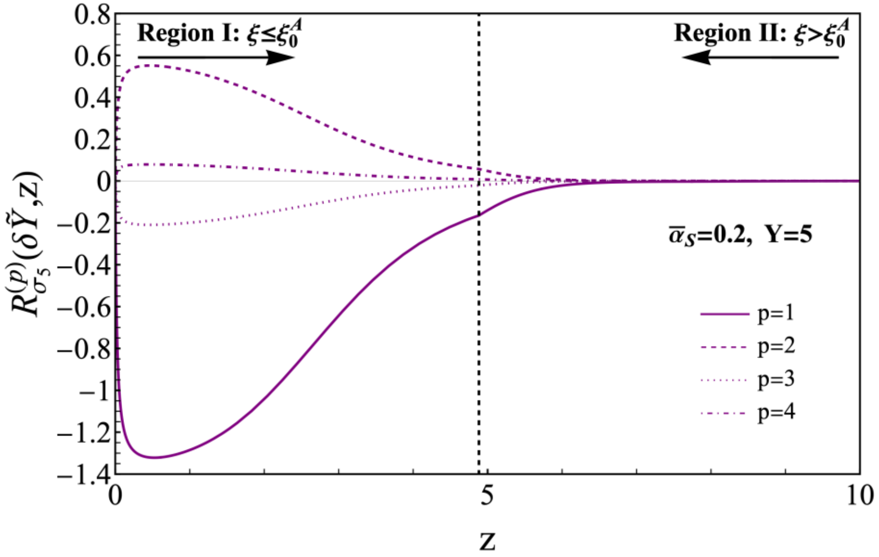
Numerical estimates Ratios of σ_1, σ_2, σ_3 and σ_5



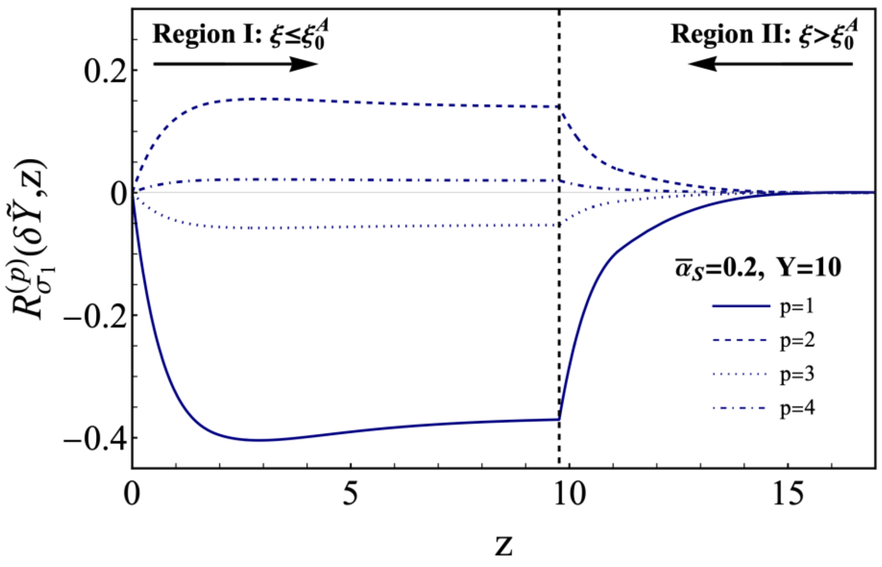


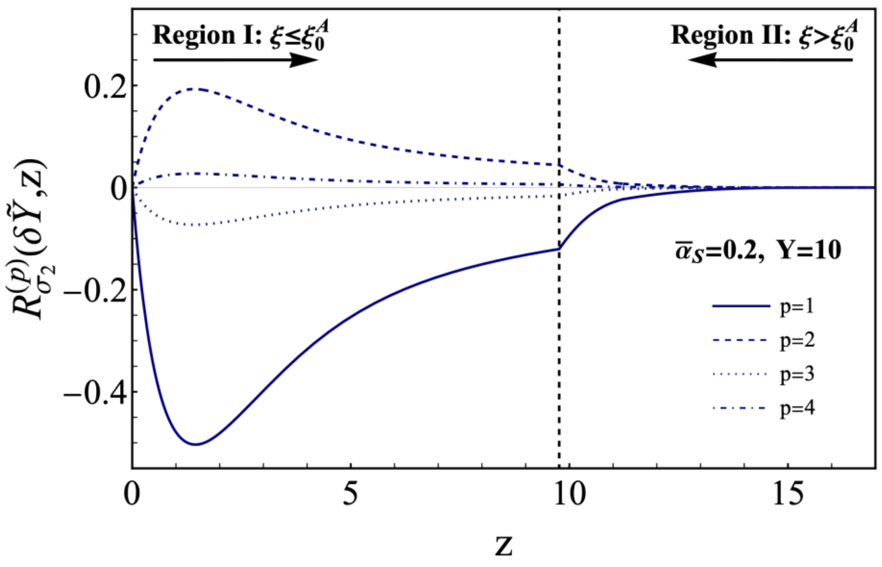
ξ_0^A=0, α_S=0.2, z_0=2

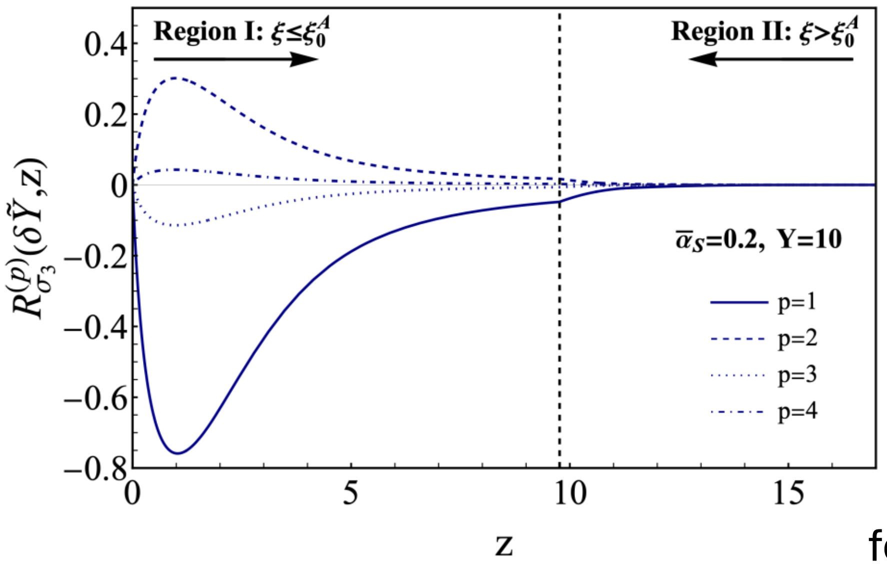


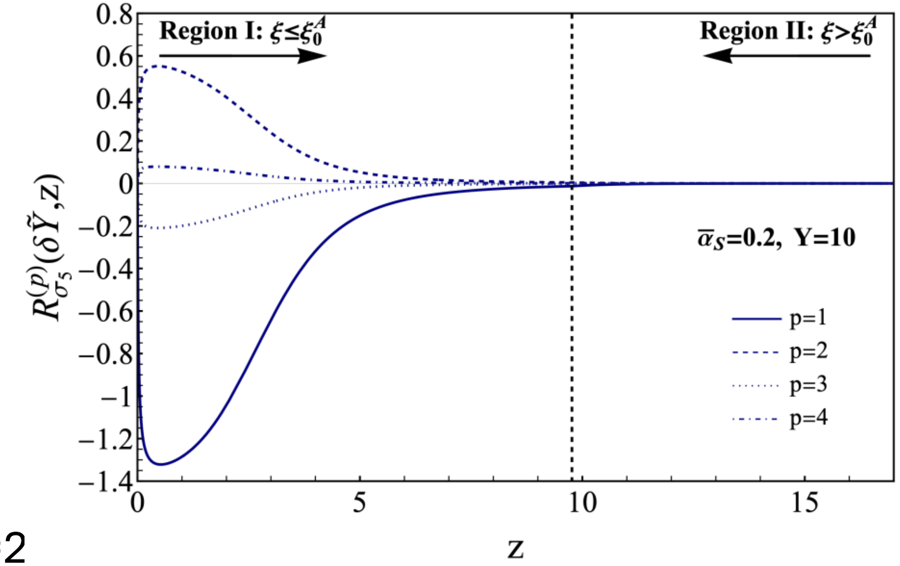


Numerical estimates Ratios of $\sigma_1, \sigma_2, \sigma_3$ and σ_5









for $\xi_0^A=0$, $\alpha_S=0.2$, $z_0=2$

Numerical estimates

One can see that the first iteration turns out to be rather large in the small-z region which corresponds to region I

the fact that after an appropriate number of iterations the corrections become small, the approach may break down for a sufficiently large n.

For this reason, we use this method only for small values of n

In region II, la iteration is convergent

What happends if we solve the equation in the large-n limit.

The large n Solution

- We start with our equation

$$\kappa \frac{d\,\sigma_n(z)}{d\,z} = -T(z)\,\sigma_n(z) + \Delta(z)\,\Sigma_n(z) + \sum_{k=1}^{n-1} \Sigma_{n-k}(z)\,\sigma_k(z)$$

Defining $$\Sigma_n(z) = \int_{\xi_0^A}^{z} dz'\,\sigma_n(z') \qquad \mathcal{S}_n(z) = \int_{\xi_0^A}^{z} dz'\,\Sigma_n(z')$$

$$\kappa \frac{d^2\,\Sigma_n(z)}{d\,z} = -\frac{d}{dz}\left(T(z)\,\Sigma_n(z)\right) + \Sigma_n(z) + \sum_{k=1}^{n-1} \Sigma_{n-k}(z)\,\sigma_k(z)$$

$$\sum_{k=1}^{n-1} \Sigma_{n-k}(z)\,\sigma_k(z) = \frac{1}{2}\frac{d}{d\,z}\sum_{k=1}^{n-1} \Sigma_{n-k}(z)\Sigma_k(z)$$

$$\kappa \frac{d^2\,\Sigma_n(z)}{d\,z} = -\frac{d}{dz}\left(T(z)\,\Sigma_n(z)\right) + \frac{d\,\mathcal{S}_n(z)}{d\,z} + \frac{1}{2}\frac{d}{d\,z}\left[\sum_{k=1}^{n-1} \Sigma_{n-k}(z)\Sigma_k(z)\right]$$

We suggest that the solution has the following form

$$\Sigma_n(z) \;=\; \phi(z)\,\exp\left(-n\,\Phi(z)\right)$$

Functions Φ (z) and ϕ (z) we will find from the equation. Replacing and taking the large z and n limit we find

$$\kappa\frac{d\Sigma_n}{dz} \;=\; -(z\,-\,\Sigma_\Delta(z))\Sigma_n(z)+\mathcal{S}_n(z)\;+\;\frac{n-1}{2}\phi\,(z)\;\Sigma_n\,(z)$$

We can now solve

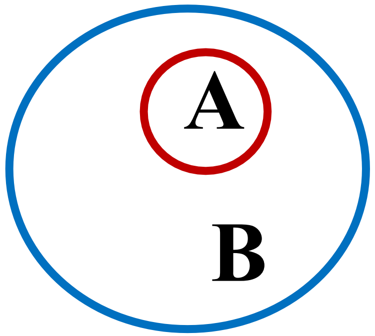
$$\sigma_n\,(z) \;=\; \left(\phi'\,(z)-n\phi\,(z)\,\Phi'\,(z)\right)\exp\left(-n\Phi\,(z)\right)$$

$$\rightarrow\quad \frac{2}{\kappa}\frac{z^2}{\langle n\,(z)\rangle}\Psi\left(\xi\,=\,\frac{n}{\langle n(z)\rangle}\right)$$

where $\Psi\,(\xi)\;=\;\xi\,e^{-\xi}$ **and** $\langle n(z)\rangle\;=\;2\,n_0\,z\,e^{\frac{(z-z_\Delta)^2}{2\,\kappa}}$

Entropy of the produced gluons

DIS probes only a part of the proton's wave function (region A)



$$r^2 \sim 1/Q^2$$

The probability to have n-cut BFKL Pomerons in the final state is equal to

$$P_n^{\rm AGK}\left(z\right) \equiv \frac{\sigma_n^{\rm AGK}\left(z\right)}{\sum_{n=1}^{\infty}\sigma_n^{\rm AGK}\left(z\right)} = \frac{2}{\kappa}\,\frac{z^2}{\langle n(z)\rangle}\,\Psi\left(\frac{n}{\langle n(z)\rangle}\right)$$

where

$$\Psi\left(\xi\right) = \xi\, e^{-\xi} \qquad \text{and} \qquad \langle n(z)\rangle = 2\, n_0\, z\, e^{\frac{(z - z_\Delta)^2}{2\,\kappa}}$$

- The entropy content of multiplicity distribution is given by the von Neumann's formula:

$$S_E\left(z\right) = -\sum_n P_n\left(z\right)\ln\left(P_n\left(z\right)\right) = \ln\langle n(z)\rangle$$

$$\xrightarrow{z \gg 1} \frac{z^2}{2\,\kappa}$$

In agreement with Kharzeev and Levin (2017) predictions

$$S_E = S_{Gluons}$$

Summary and Outlook

Using AGK Rules we define the cross section of the n-cut Pomerons σ_n

Using Homotopy Approach we can find solution for n small and Numerical estimation indicates that the solutions converge in the Saturation region

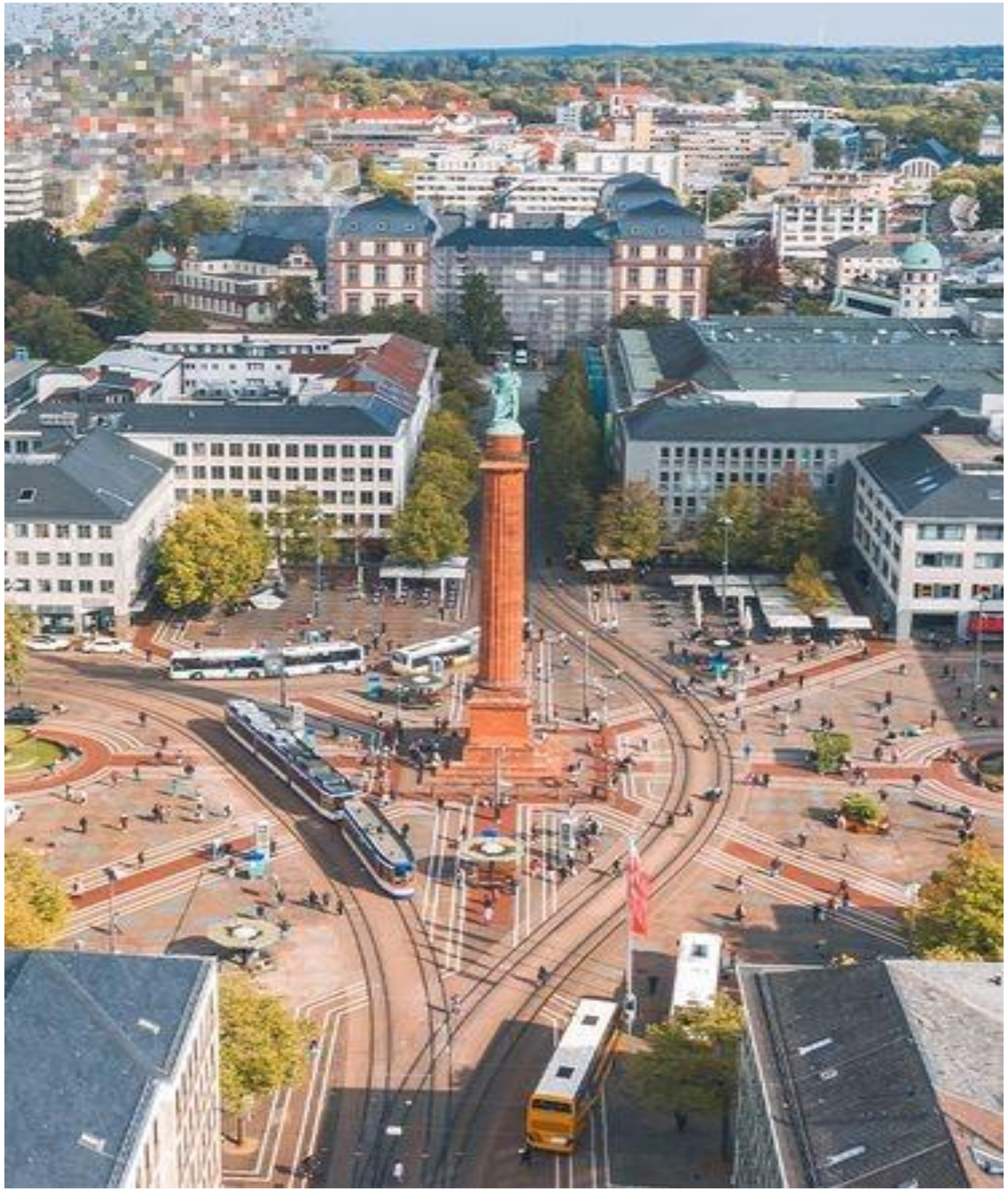
In the large n limit, we obtaine the probability to have n-cut Pomeron , which allow to estimate the entropy, wich is according with

Next steps:

We are in conditions to apply our solutions to investigate whether we can describe the

experimental data inclusive diffractive cross sections measured by H1 and ZEUS collaboration at HERA.

Thank you very much for your attention



BK evolution in momentum space

Unfortunately finding an exact analytical solution it is a very difficult task, this eq. is non–linear and because of a complicated structure of the BFKL kernel $\chi(\lambda)$

First approximation Perturbation theory for the linear part.

Y. Kovchegov arXiv 9905214 Perturbative approach

Analysis of the BK-equation in momentum space, by Fourier transforming N(r, b, Y), into N(q, b, Y), q being the Fourier conjugated of the dipole size.

Motika and Sadzikowski arXiv 230602118 Hight Twist corrections

Numerical solutions in momentum space: traveling wave
In the momentum space, the equation admits travelling-wave solutions.

C. Marquet, G. Soyez arXiv:hep-ph/0504080
C. Marquet, R. Peschanski, G. Soyez arXiv:hep-ph/0509074
S. Munier, R. Peschanski arXiv:hep-ph/0310357 - arXiv:hep-ph/0309177

Result:

The perturbative solution is convergent, and we know the solution for N (k, Y) outside of the saturation region and the high energy asymptotics inside the saturation region

One need to have a better knowledge of the momentum space solution **inside the saturation region**.

We can obtain the solution in coordinate space *N(r , Y)* using the Fourier transformation: To do that one would have to integrate over all values of the transverse momentum k from 0 to ∞.

If $x\perp < 1/Qs$ then the transverse momentum is effectively cut off and then we have to control the region of 0 to $1/x\perp > Qs$

If $x\perp > 1/Qs$ then the transverse momentum is in the saturation region and simply gets cut off by $1/x\perp < Qs$

Homotopic approach to nonlinear Balitsky-Kovchegov (BK) evolution equation with running QCD coupling

The NLO BK: The running-coupling corrections to small-*x* evolution are obtained considering contribution of the quarks loops into de gluon lines

$$\frac{\partial(N(x_{10},b,Y)}{\partial Y} = \int d^2x_2\, K_{rc}\,(r,r_1,r_2)(N(x_{12},b,Y) + N(x_{20},b,Y) - N(x_{10},b,Y) \;-N(x_{12},b,Y) * N(x_{20},b,Y))$$

We have two prescriptions:

- Balistsky (2007) $K_{rc}^{Bal}\,(r,r_1,r_2)$

$$K_{rc}^{Bal}\,(r;r_1,r_2) = \bar{\alpha}_S\,(r^2)\left\{\frac{r^2}{r_1^2\,r_2^2} + \frac{1}{r_1^2}\left(\frac{\bar{\alpha}_S\,(r_1^2)}{\bar{\alpha}_S\,(r_2^2)} - 1\right) + \frac{1}{r_2^2}\left(\frac{\bar{\alpha}_S\,(r_2^2)}{\bar{\alpha}_S\,(r_1^2)} - 1\right)\right\}$$

$Kovchegov\ and\ Weigert\ (2007)\quad K_{rc}^{KW}\,(r,r_1,r_2)$

$$K_{rc}^{KW}\,(r;r_1,r_2) = \bar{\alpha}_S\,(r_1^2)\left\{\frac{1}{r_1^2} - 2\,\frac{\bar{\alpha}_S\,(r_2^2)}{\bar{\alpha}_S\,(R_0^2)}\,\frac{\boldsymbol{r}_1\cdot\boldsymbol{r}_2}{r_1^2 r_2^2} + \frac{1}{r_2^2}\,\frac{\bar{\alpha}_S\,(r_2^2)}{\bar{\alpha}_S\,(r_1^2)}\right\}$$

$$R_0^2\,(r;r_1,r_2) = r_1 r_2\left(\frac{r_2}{r_1}\right)^{\frac{r_1^2+r_2^2}{r_1^2-r_2^2} - 2\frac{r_1^2 r_2^2}{\boldsymbol{r}_1\cdot\boldsymbol{r}_2}\frac{1}{r_1^2-r_2^2}}$$

Both prescriptions neglect different contributions. However, numerical studies indicate that the Balitsky prescription provides a closer result to the full answer, where no approximations are made. Therefore, we adopt the Balitsky prescription in our analysis

Albacete and Y. V. Kovchegov (2007) arXiv:0704.0612

For Numerical calculation with running-coupling corrections to the BK/ JIMWLK evolution equations

See **Lappi and Mäntysaari: JIMWLK equation 1212.4825; 1502.0240**

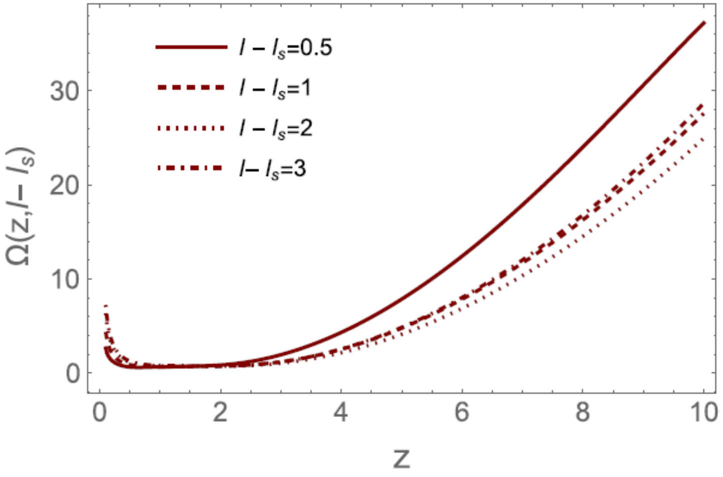
Albacete and Kovchegov (2007) 0704.0612

Analytic solution are calculated using the **homotopy approach** for solving the nonlinear Balitsky-Kovchegov (BK) evolution equation with running QCD coupling. *Phys.Rev.D* 111 (2025) 9, 096025

We found the analytic function for the scattering amplitude, which satisfies the initial and boundary conditions.

We find a good convergency.

The geometric scaling behavior could be in the vicinity of the saturation scale. This Fig. shows an approximate z-scaling behavior in the limited region of z.



$$l - l_s = \int_{1/Q_s^2}^{r^2} dr'^2 \frac{\bar{\alpha}_S(r'^2)}{r'^2} = -\frac{4N_c}{b_0} \ln\left(\frac{-\xi}{\xi_s}\right).$$

We plot in Fig. the first N^0 and the second N^1 iterations in the homotopy approach. One can see that the homotopy approach works quite well for the model

Results