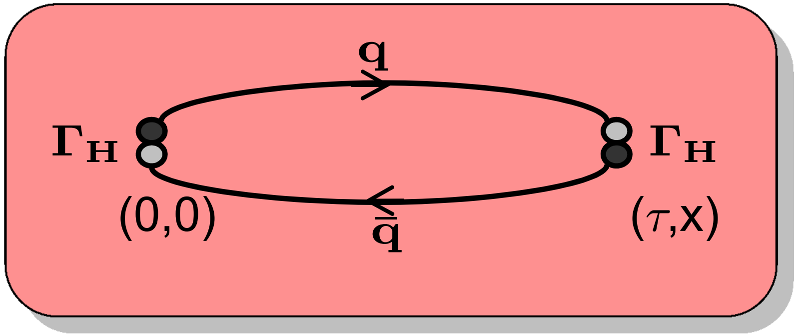
Vector correlation functions at high temperature

$$G(\tau,\vec{p},T) = \int_0^\infty \frac{d\omega}{2\pi} \rho(\omega,\vec{p},T) K(\tau,\omega,T)$$

$$K(\tau,\omega,T) = \frac{\cosh\left(\omega(\tau - \frac{1}{2T})\right)}{\sinh\left(\frac{\omega}{2T}\right)}$$



Lattice observables:

$$G_{\mu\nu}(\tau,\vec{x}) = \langle J_\mu(\tau,\vec{x}) J_\nu^\dagger(0,\vec{0})\rangle$$

$$J_\mu(\tau,\vec{x}) = 2\kappa Z_V \bar{\psi}(\tau,\vec{x})\Gamma_\mu \psi(\tau,\vec{x})$$

← local, non-conserved current, needs to be renormalized

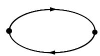
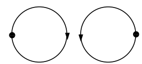
$$G_{\mu\nu}(\tau,\vec{p}) = \sum_{\vec{x}} G_{\mu\nu}(\tau,\vec{x}) e^{i\vec{p}\vec{x}}$$

← only $\vec{p} = 0$ used here

How to extract spectral properties from correlation functions?

The correlators in quenched formulation Eq. (2.81) and Euclidian time $\tau \in [0, 1/T]$ ($\langle \ldots \rangle$ denotes the expectation value at finite T) have the form

$$
\begin{aligned}
G_H(x_f, x_i) &= \langle \mathcal{J}_H(x_f) \mathcal{J}_H^\dagger(x_i) \rangle \\
&= \frac{1}{Z} \int dU d\bar{\psi} d\psi \ \left(\bar{\psi}_A(x_f) \Gamma_H \psi_B(x_f)\right) \left(\bar{\psi}_B(x_i) \Gamma_H^\dagger \psi_A(x_i)\right) \mathrm{e}^{-S} \\
&= \frac{1}{Z} \int dU \ \ \mathrm{Tr} \left(M^{-1}(x_f, x_i) \Gamma_H M^{-1}(x_i, x_f) \Gamma_H^\dagger \right) \\
&\qquad - \mathrm{Tr} \left(\Gamma_H M^{-1}(x_f, x_f) \right) \ \mathrm{Tr} \left(\Gamma_H^\dagger M^{-1}(x_i, x_i) \right) \mathrm{e}^{-S_G(U)} \qquad (2.152)
\end{aligned}
$$

$= x_i$  $x_f - x_i$  x_f, (2.153)

where A and B are the different quark flavors, while Tr implies the trace over color and Dirac indices. The second term describes disconnected diagrams in which each of the quark line starts and ends at the same point. In case of the correlators with $A \neq B$ the contributions from the disconnected diagram vanishes. Taking advantage of Eq. (2.102) we then find our two-point function

$$
G_H(x_f, x_i) = \left\langle \ \mathrm{Tr} \left(M^{-1}(x_f, x_i) \Gamma_H \gamma_5 (M^{-1})^\dagger (x_f, x_i) \gamma_5 \Gamma_H^\dagger \right) \right\rangle_{\mathcal{U}} . \qquad (2.154)
$$

This corresponds to non-singlet (isovector) channels ($I = 1$), which we consider from now

Spectral functions at high temperature

Free theory (massless case):

free non-interacting vector spectral function (infinite temperature):

$$\begin{aligned} \rho_{00}^{free}(\omega) &= 2\pi T^2 \omega \delta(\omega) \\ \rho_{ii}^{free}(\omega) &= 2\pi T^2 \omega \delta(\omega) + \frac{3}{2\pi}\, \omega^2 \tanh(\omega/4T) \end{aligned}$$

δ-functions exactly cancel in $\rho_V(\omega)$=-$\rho_{00}(\omega)$+$\rho_{ii}(\omega)$

With interactions (but without bound states):

while ρ_{00} is protected, the δ-funtion in ρ_{ii} gets smeared:

$\kappa = \frac{\alpha_s}{\pi}$ at leading order

Ansatz:

$$\begin{aligned} \rho_{00}(\omega) &= 2\pi \chi_q \omega \delta(\omega) \\ \rho_{ii}(\omega) &= 2\chi_q c_{BW} \frac{\omega \Gamma/2}{\omega^2 + (\Gamma/2)^2} + \frac{3}{2\pi}(1+\kappa)\, \omega^2 \tanh(\omega/4T) \end{aligned}$$

Ansatz with 3-4 parameters: $(\chi_q), c_{BW}, \Gamma, \kappa$

[“Thermal dilepton rate and electrical conductivity…”, H.T.-Ding, OK et al., PRD83 (2011) 034504]

Electrical Conductivity

Electrical Conductivity ⟷ slope of spectral function at ω=0 (Kubo formula)

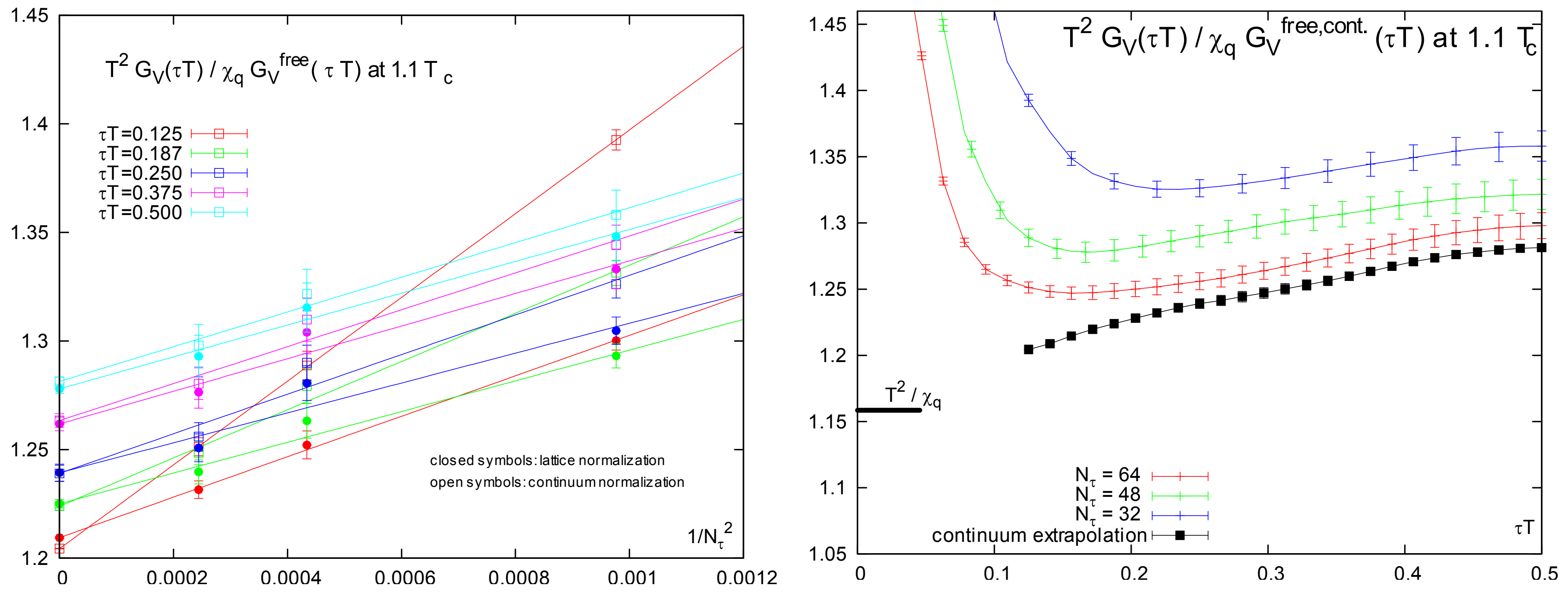
$$\frac{\sigma}{T} = \frac{C_{em}}{6} \lim_{\omega \to 0} \frac{\rho_{ii}(\omega, \vec{p}=0, T)}{\omega T}$$

$$C_{em} = e^2 \sum_{f=1}^{n_f} Q_f^2 = \begin{array}{lll} 5/9\ e^2 & \text{for} & n_f = 2 \\ 6/9\ e^2 & \text{for} & n_f = 3 \end{array}$$

Using our Ansatz for $\rho_{ii}(\omega)$:

$$\frac{\sigma}{T} = \frac{2}{3} \frac{\chi_q}{T^2} \frac{T}{\Gamma} c_{BW}\ C_{em}$$

Continuum extrapolation



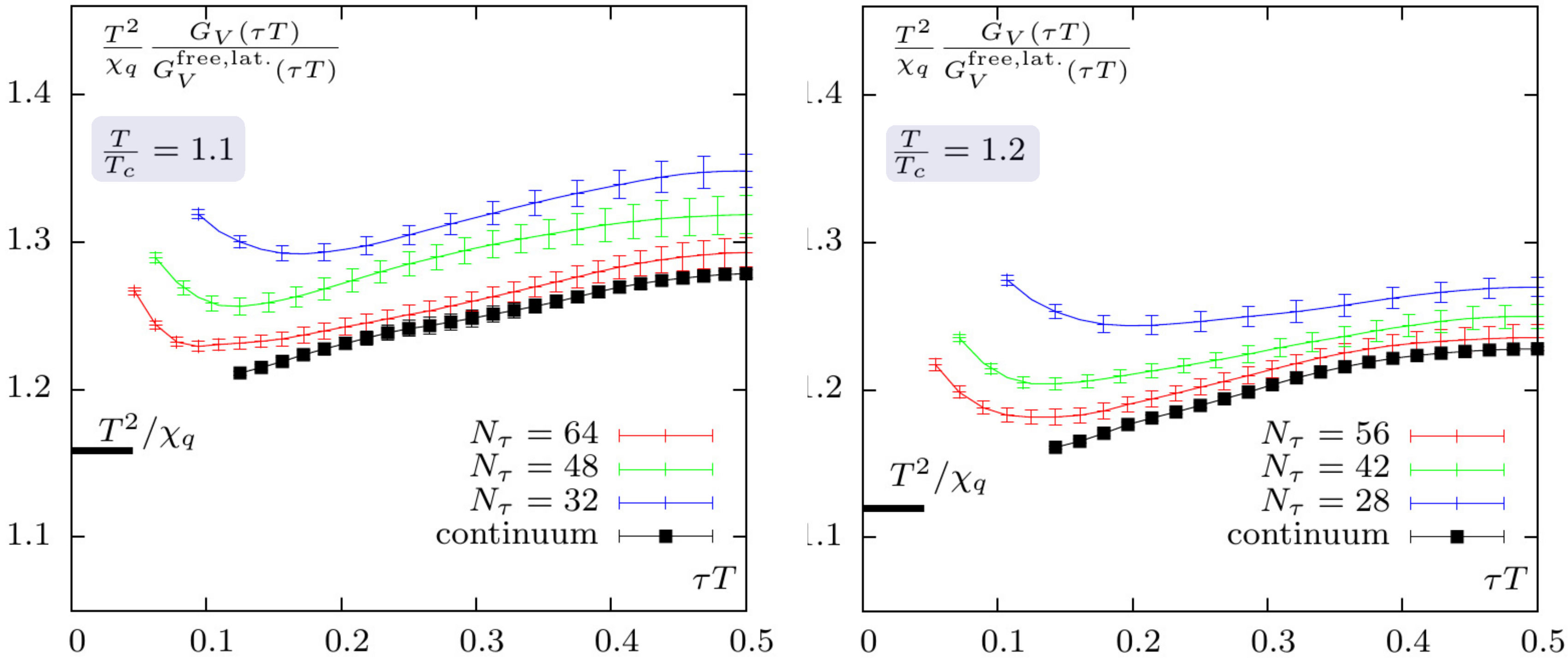
cut-off effects visible at all distances but

well defined continuum limit on the correlator level

well behaved continuum correlator down to small distances

approaching the correct asymptotic limit for $\tau \to 0$

Continuum extrapolation



cut-off effects visible at all distances but

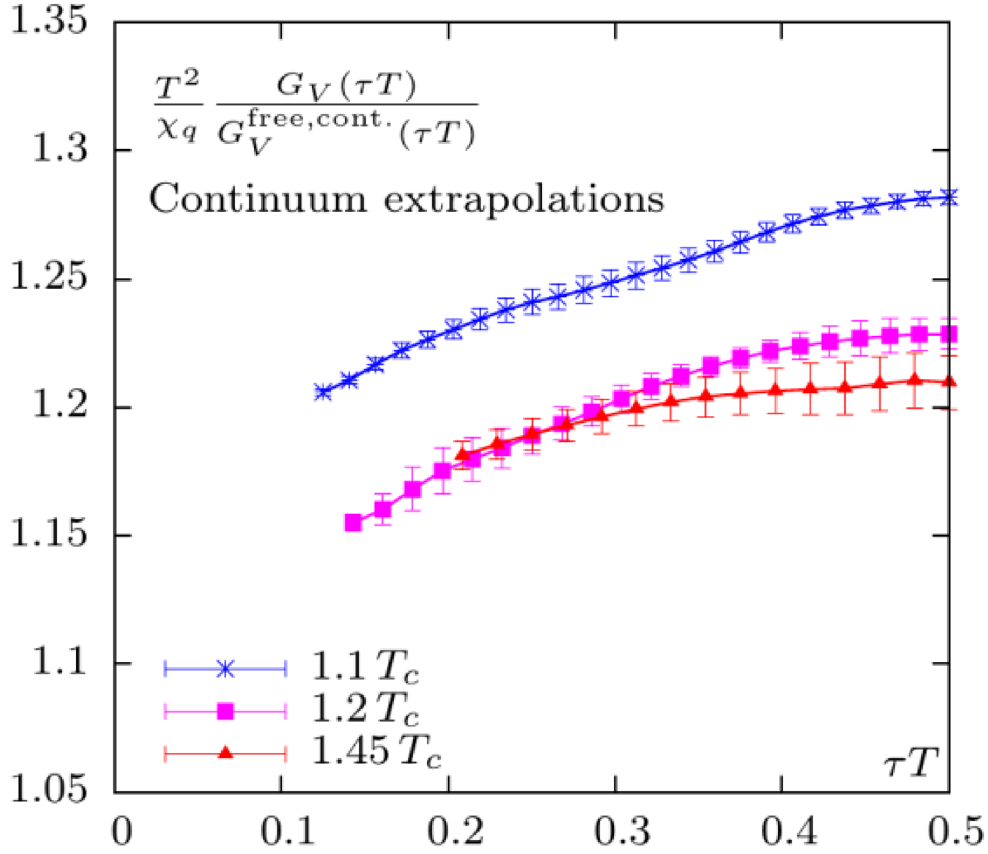
well defined continuum limit on the correlator level

well behaved continuum correlator down to small distances

approaching the correct asymptotic limit for $\tau \to 0$

Continuum extrapolation

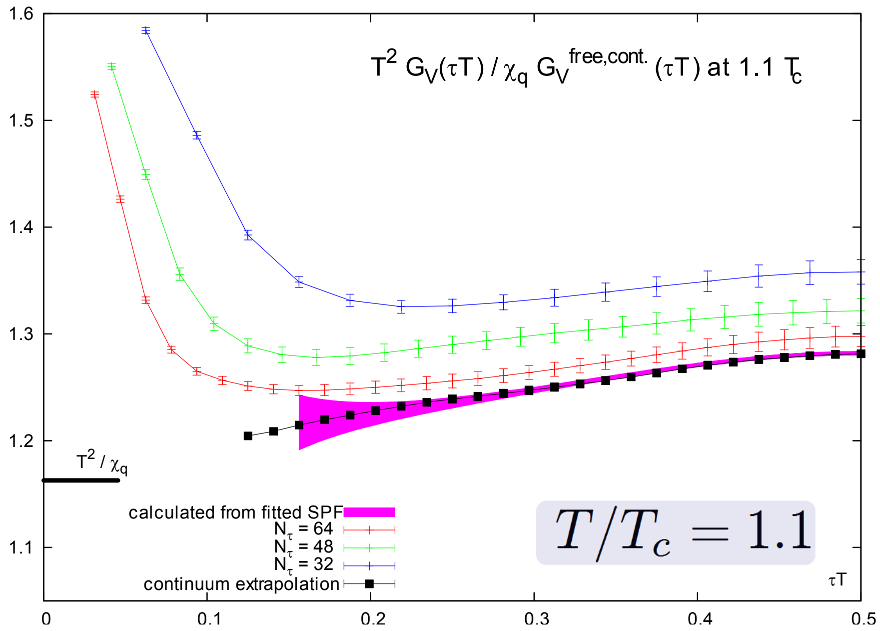
continuum extrapolated correlators

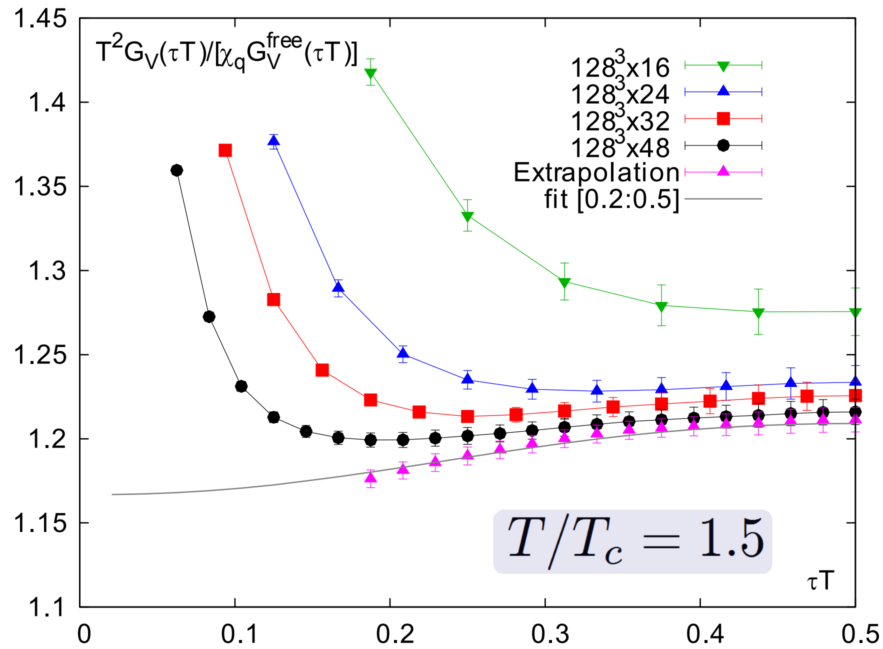


Use our Ansatz for the spectral function

$$\rho_{00}(\omega) = 2\pi\chi_q\omega\delta(\omega)$$
$$\rho_{ii}(\omega) = 2\chi_q c_{BW}\frac{\omega\Gamma/2}{\omega^2+(\Gamma/2)^2} + \frac{3}{2\pi}(1+\kappa)\ \omega^2\ \tanh(\omega/4T)$$

and fit to the continuum extrapolated correlators





[H.T.-Ding, OK et al., PRD83 (2011) 034504]

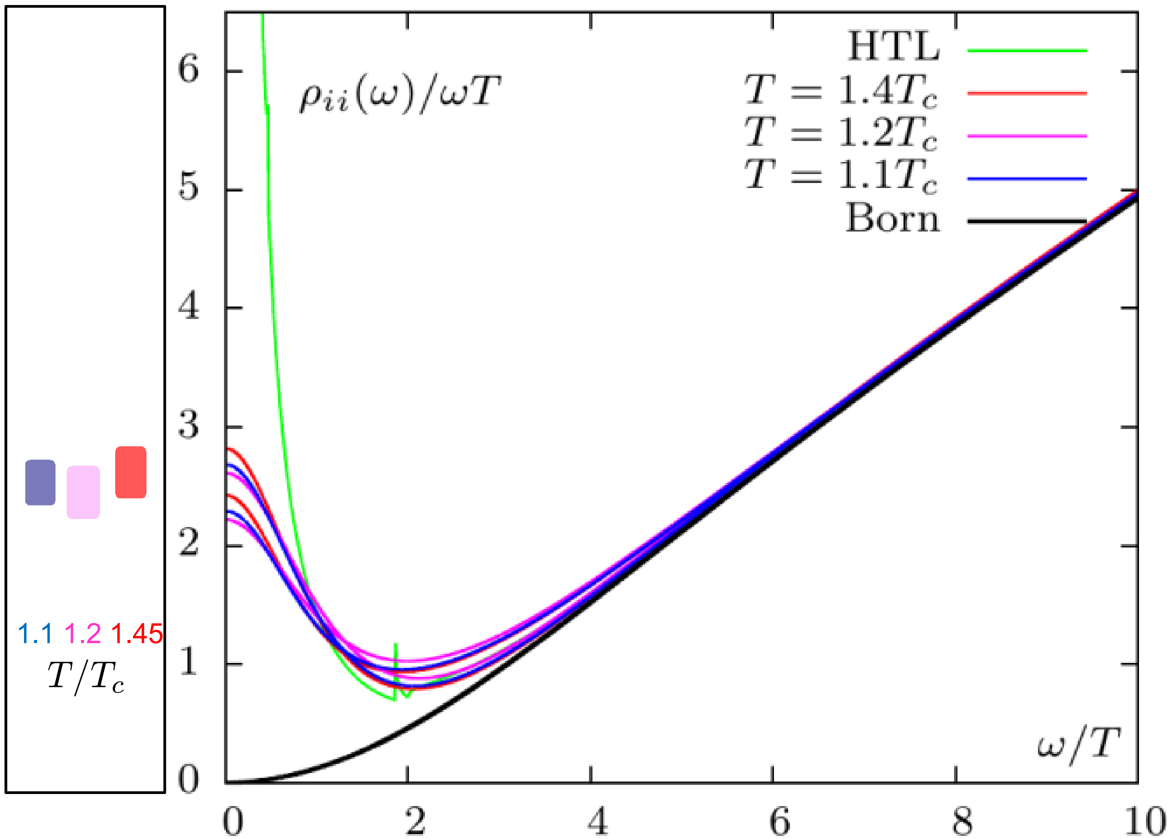
all three temperatures are well described by this rather simple Ansatz

Spectral function and electrical conductivity

Use our Ansatz for the spectral function

$$\rho_{00}(\omega) = 2\pi\chi_q\omega\delta(\omega)$$

$$\rho_{ii}(\omega) = 2\chi_q c_{BW}\frac{\omega\Gamma/2}{\omega^2+(\Gamma/2)^2} + \frac{3}{2\pi}(1+\kappa)\,\omega^2\ \tanh(\omega/4T)\times\Theta(\omega_0,\Delta_\omega)$$



Analysis of the systematic errors

using truncation of the large ω contribution

$$\Theta(\omega_0,\Delta_\omega) = \left(1+e^{(\omega_0^2-\omega^2)/\omega\Delta_\omega}\right)^{-1}$$

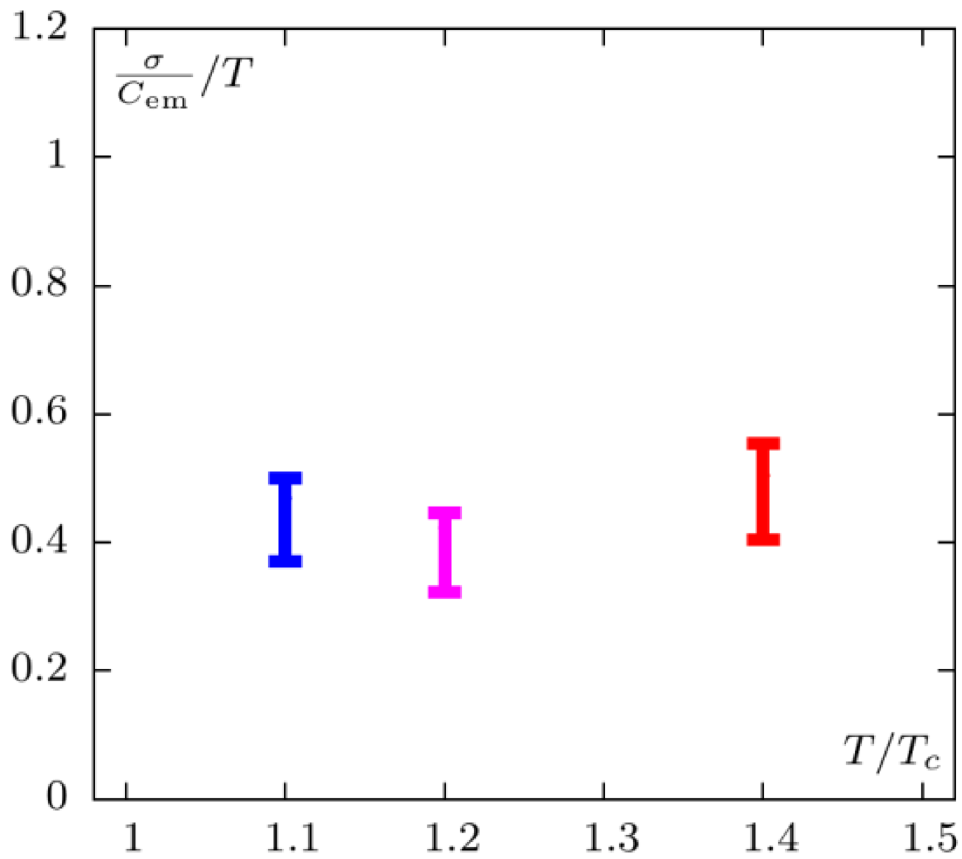
$$\frac{\sigma}{T} = \frac{C_{em}}{6}\lim_{\omega\to 0}\frac{\rho_{ii}(\omega,\vec{p}=0,T)}{\omega T}$$

electrical conductivity

Electrical conductivity

T-dependence of the electrical conductivity:

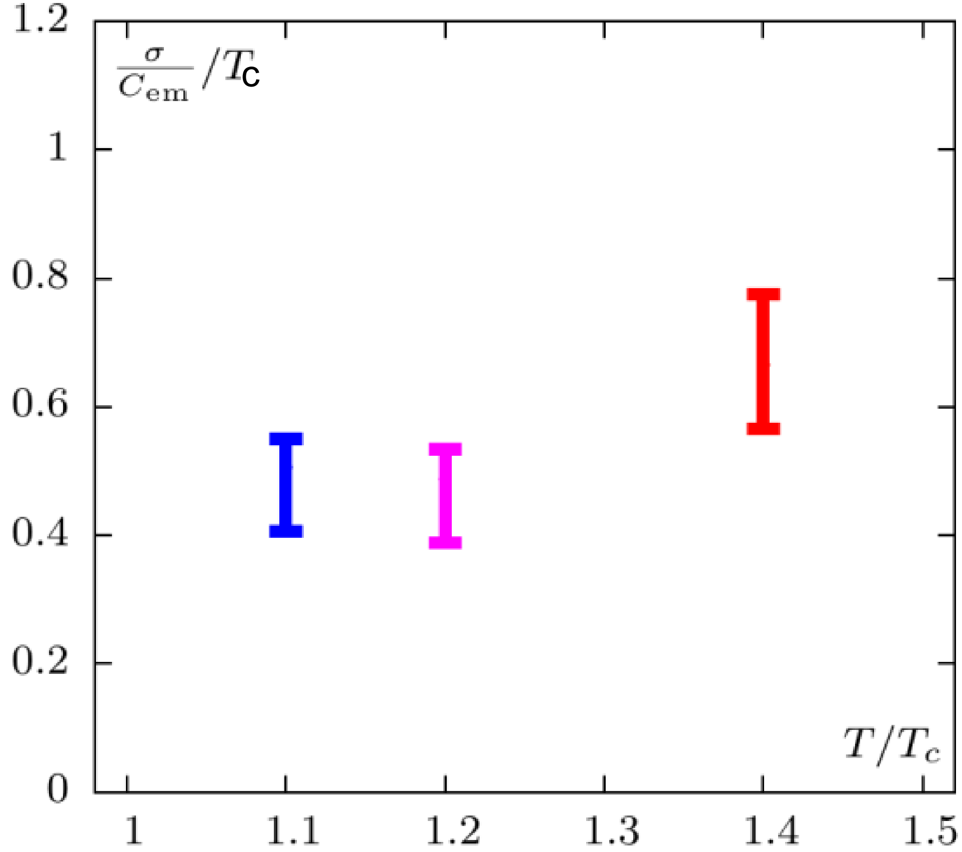
$$\frac{\sigma}{T} = \frac{C_{em}}{6} \lim_{\omega \to 0} \frac{\rho_{ii}(\omega, \vec{p} = 0, T)}{\omega T}$$



Electrical conductivity

T-dependence of the electrical conductivity:

$$\frac{\sigma}{T} = \frac{C_{em}}{6} \lim_{\omega \to 0} \frac{\rho_{ii}(\omega, \vec{p}=0, T)}{\omega T}$$



C. Thermal dilepton and photon rates

The vector spectral function is directly related to the thermal production rate of dilepton pairs with squared invariant mass $\omega^2 - \vec{p}^2$,

$$\frac{\mathrm{d}N_{l^+l^-}}{\mathrm{d}\omega \mathrm{d}^3p} = C_{em}\frac{\alpha_{em}^2}{6\pi^3}\frac{\rho_V(\omega,\vec{p},T)}{(\omega^2-\vec{p}^2)(\mathrm{e}^{\omega/T}-1)} \quad , \tag{II.14}$$

where α_{em} is the electromagnetic fine structure constant.

The vector spectral function at light-like 4-momentum yields the photon emission rate of a thermal medium,

$$\omega\frac{\mathrm{d}R_\gamma}{\mathrm{d}^3p} = C_{em}\frac{\alpha_{em}}{4\pi^2}\frac{\rho_V(\omega=|\vec{p}|,T)}{\mathrm{e}^{\omega/T}-1} \ . \tag{II.15}$$

The emission rate of soft photons, thus can be related to the electrical conductivity,

$$\lim_{\omega\to 0}\omega\frac{\mathrm{d}R_\gamma}{\mathrm{d}^3p} = \frac{3}{2\pi^2}\sigma(T)T\alpha_{em} \ . \tag{II.16}$$

In the limit $\omega \to 0$ the results for $\rho_{ii}(\omega)/\omega$, and thus also for the electrical conductivity, are sensitive to the choice of fit ansatz. Within the class of ansätze used by us a small value of $\rho_{ii}(\omega)/\omega$ seems to be favored. Our current analysis suggests,

$$2 \lesssim \lim_{\omega\to 0} \frac{\rho_{ii}(\omega)}{\omega T} \lesssim 6 \quad \text{at} \quad T \simeq 1.45\, T_c \; . \tag{VI.1}$$

This translates into an estimate for the electrical conductivity

$$1/3 \lesssim \frac{1}{C_{em}} \frac{\sigma}{T} \lesssim 1 \quad \text{at} \quad T \simeq 1.45\, T_c \; . \tag{VI.2}$$

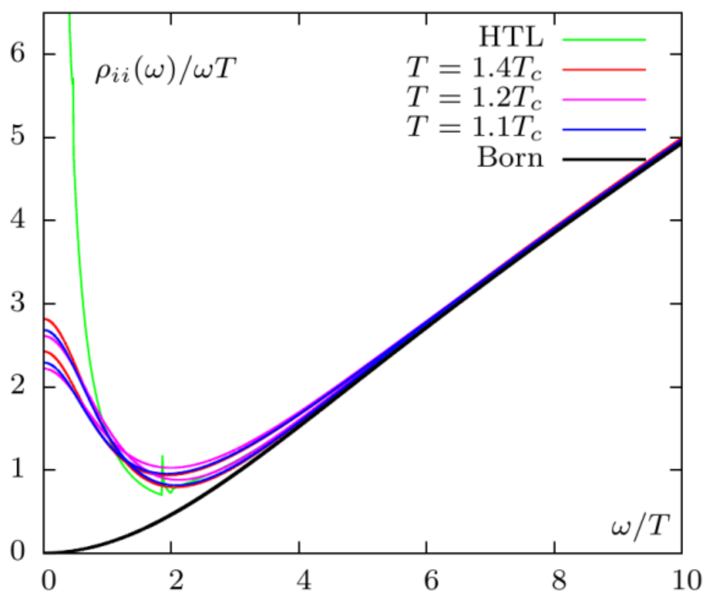
Using Eq. II.15 this yields for the zero energy limit of the thermal photon rate[d],

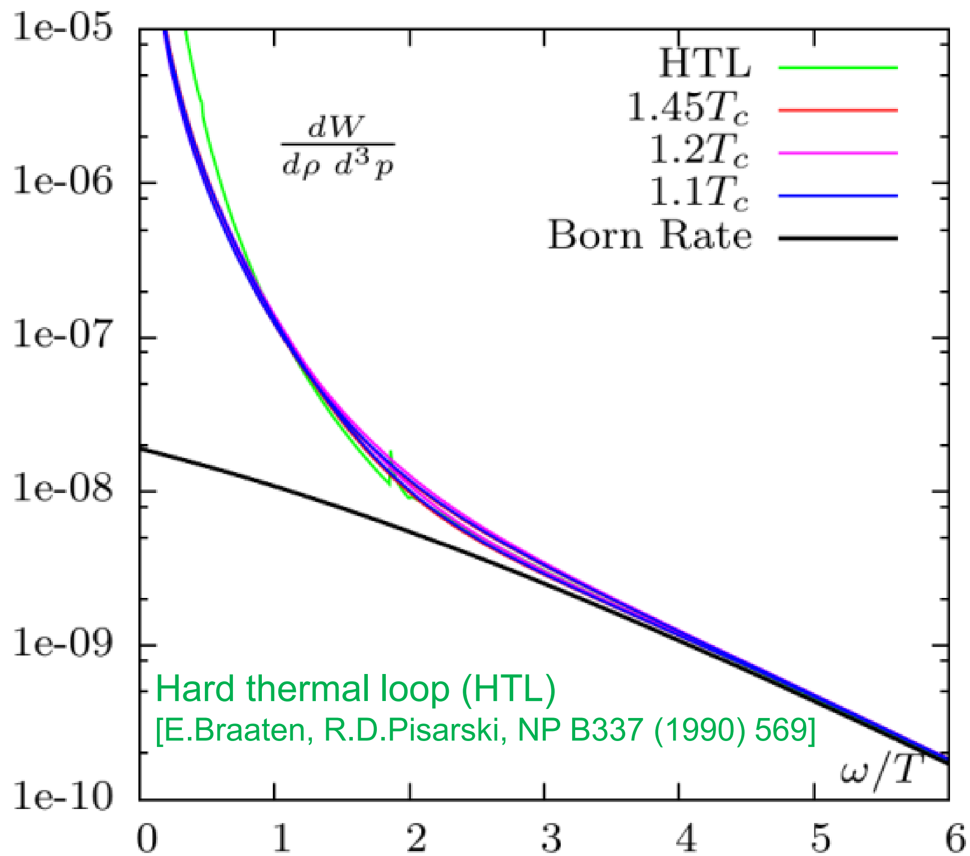
$$\lim_{\omega\to 0} \omega \frac{\mathrm{d}R_\gamma}{\mathrm{d}^3 p} = (0.0004 \; - \; 0.0013)\, T_c^2 \simeq (1-3)\cdot 10^{-5}\ \mathrm{GeV}^2 \quad \text{at} \quad T \simeq 1.45\, T_c \; . \tag{VI.3}$$

Dilepton rates

Dileptonrate directly related to vector spectral function:

$$\frac{\mathrm{d}W}{\mathrm{d}\omega \mathrm{d}^3p} = \frac{5\alpha^2}{54\pi^3} \frac{1}{\omega^2(e^{\omega/T}-1)} \rho_{\mathbf{V}}(\omega, \mathbf{T})$$





Spectral function and electrical conductivity

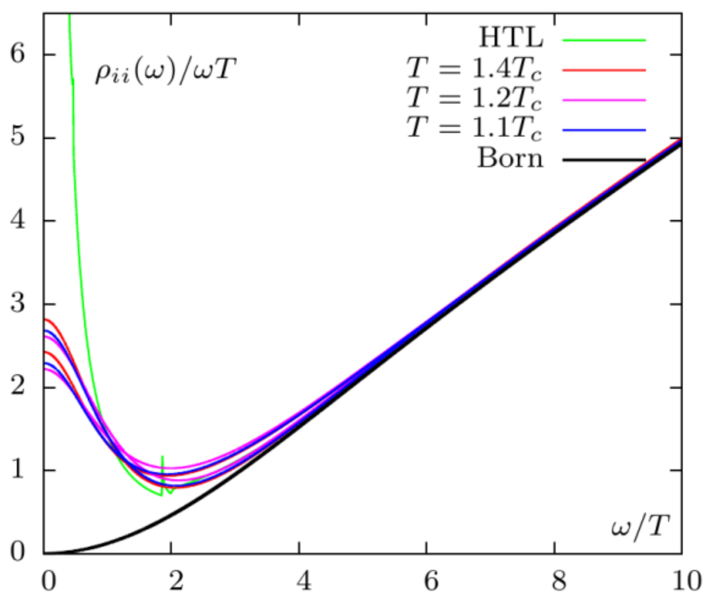
$$\rho_{00}(\omega) = 2\pi\chi_q\omega\delta(\omega)$$

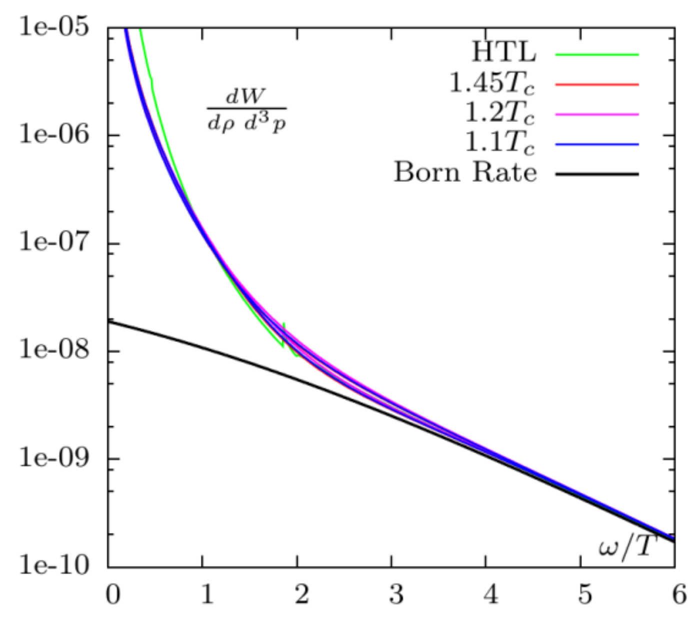
$$\rho_{ii}(\omega) = 2\chi_q c_{BW}\frac{\omega\Gamma/2}{\omega^2+(\Gamma/2)^2} + \frac{3}{2\pi}(1+\kappa)\,\omega^2\ \tanh(\omega/4T)$$

Fitting the continuum extrapolated correlation function $G_{ii}(\tau T)$ in the interval $[0.2:0.5]$ together with the second thermal moment $G_V^{(2)}$ to constrain the fit, we obtain

$$k = 0.0465(30)\ ,\ \bar{\Gamma} = 2.235(75)\ ,\ 2c_{BW}\bar{\chi}_q/\bar{\Gamma} = 1.098(27)\ . \qquad \text{(V.8)}$$

This three parameter fit has a $\chi^2/d.o.f. = 0.06$ for 12 degrees of freedom. The small $\chi^2/d.o.f.$ clearly reflects that even after the continuum extrapolation data at different distances are strongly



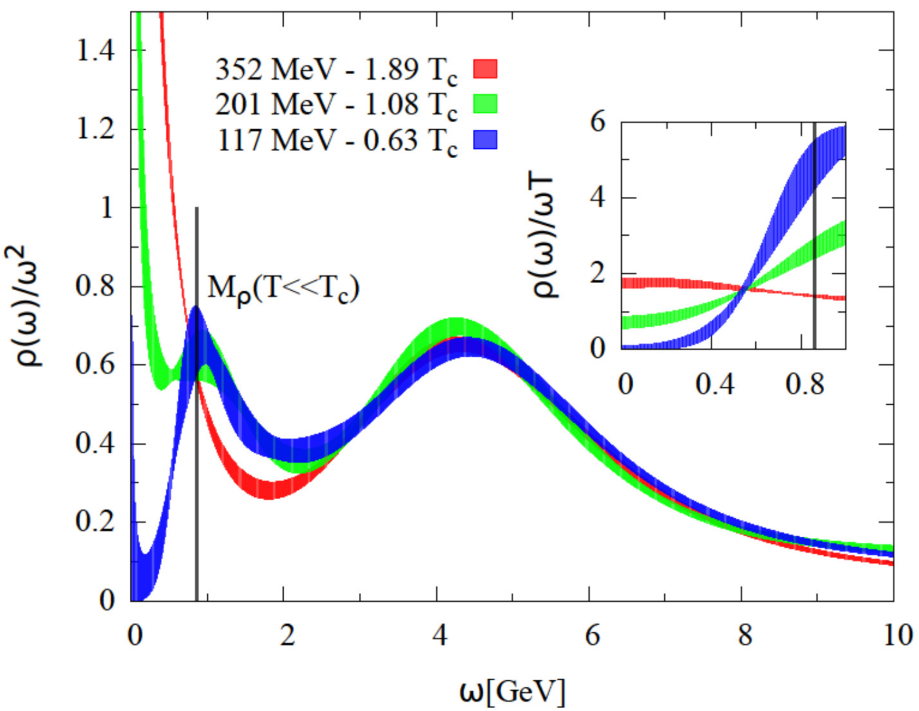


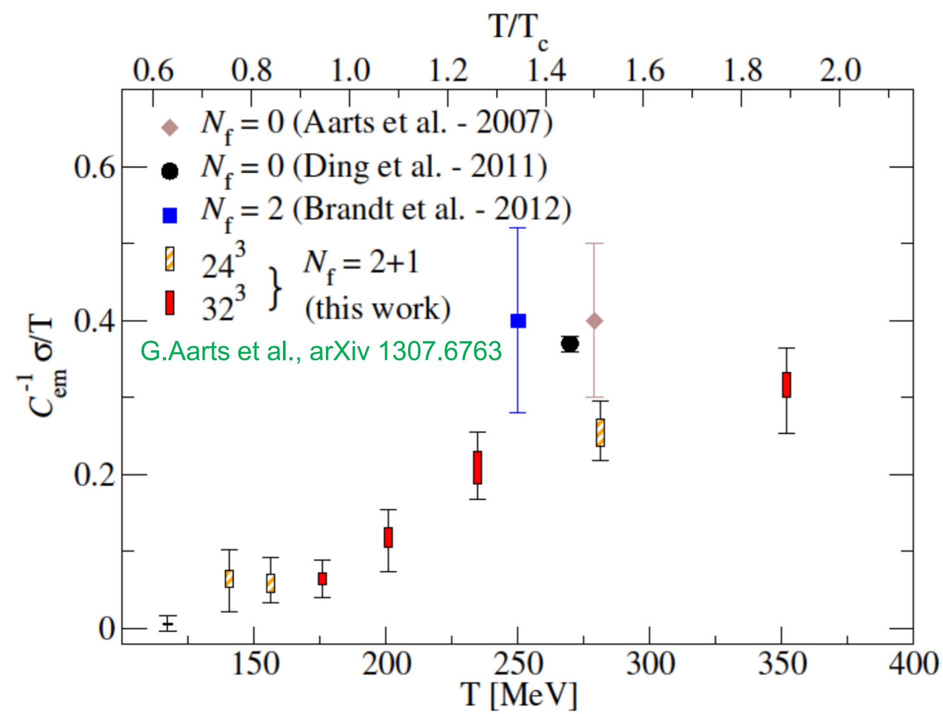
Electrical conductivity

T-dependence of the electrical conductivity:

$$\frac{\sigma}{T} = \frac{C_{em}}{6} \lim_{\omega \to 0} \frac{\rho_{ii}(\omega, \vec{p}=0, T)}{\omega T}$$

G.Aarts et al., arXiv 1307.6763





Ding et al.: **Quenched** on **isotropic lattices** + ***continuum limit***

Aarts et al.: **2+1-flavor dynamical Wilson fermions on anisotropic lattices (Ns=24-32 Nt=16-48)**
(cut-off effects and energy resolution determined by spatial lattice spacing)

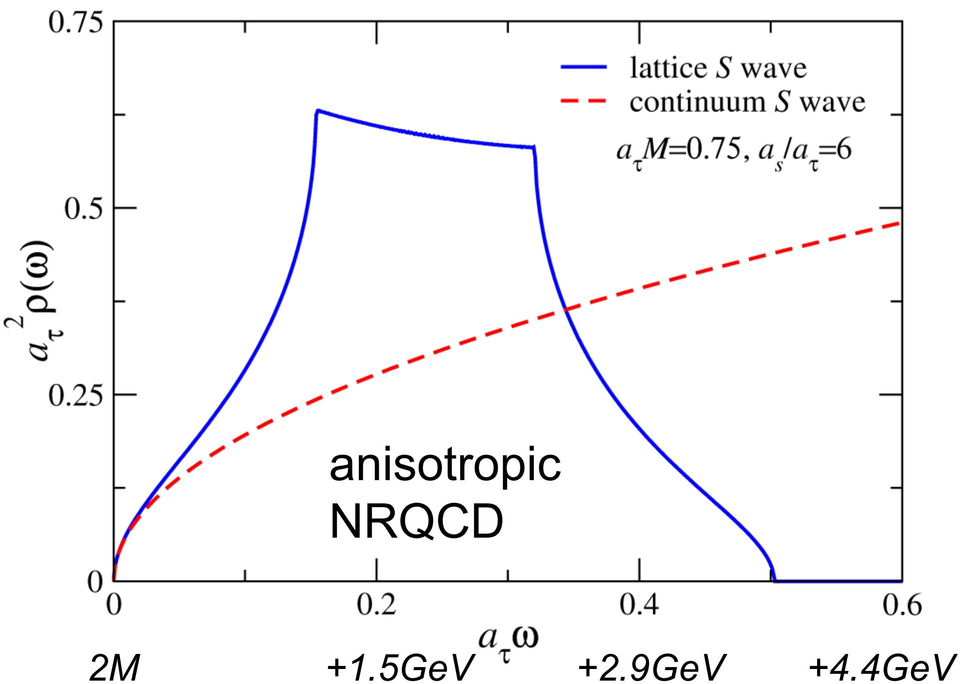
Brandt et al.: **2-flavour dynamical Wilson fermions on isotropic lattices (Ns=64 Nt=16)**

Lattice cut-off effects – free spectral functions

[G.Aarts et al., JHEP11(2011)103]

gauge configurations from n_f=2 dynamical Wilson fermion action

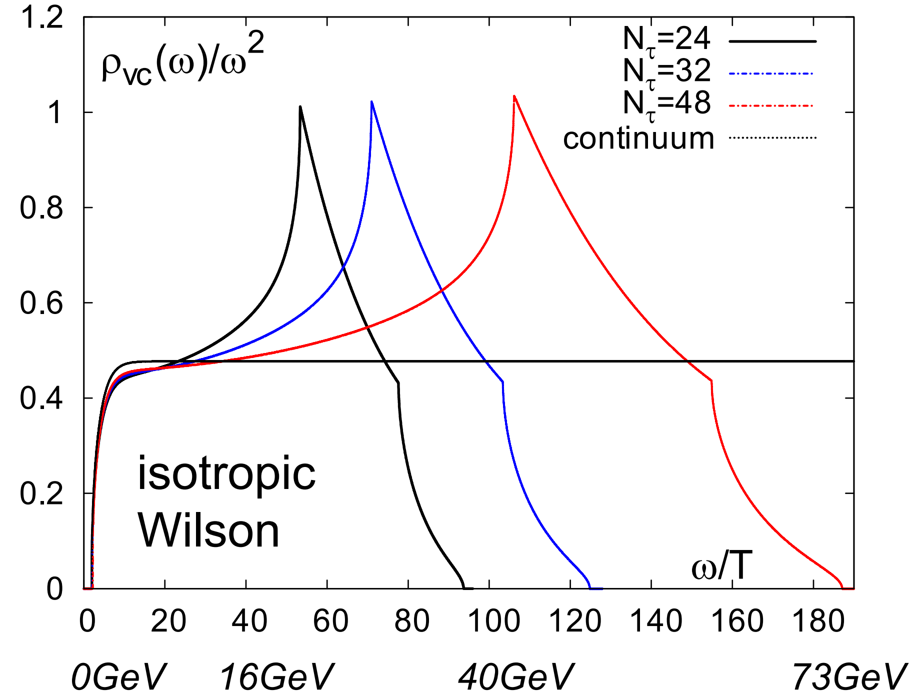
$a_s \simeq 0.162$ fm
$1/a_t \simeq 7.35$ GeV



[H.T.Ding, OK et al., arXiv:1204.4945]

gauge configurations from quenched action

$a \simeq 0.01$ fm
$1/a \simeq 19$ GeV



cut-off effects and energy resolution determined by spatial lattice spacing

no continuum limit in NRQCD, $a_s M \gg 1$

only small energy region accessible

continuum limit straight forward, but expensive

transport properties accessible

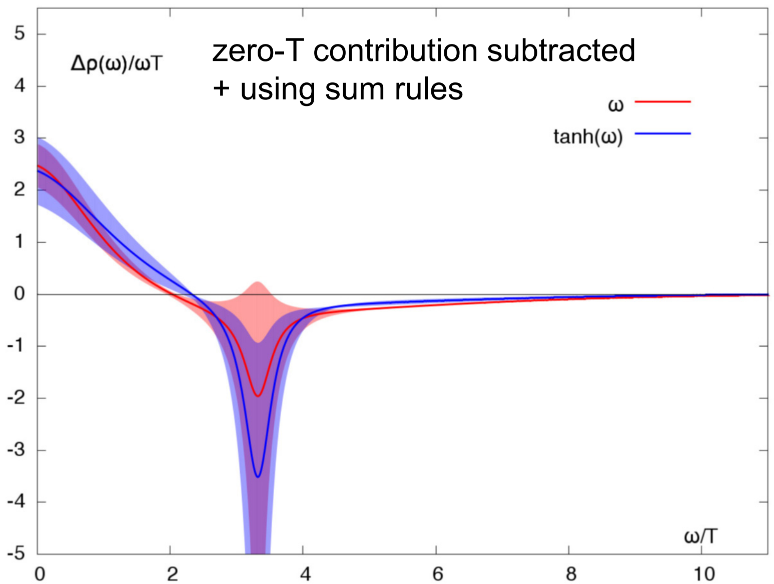
[see also F.Karsch et al., PRD68 (2003) 014504]

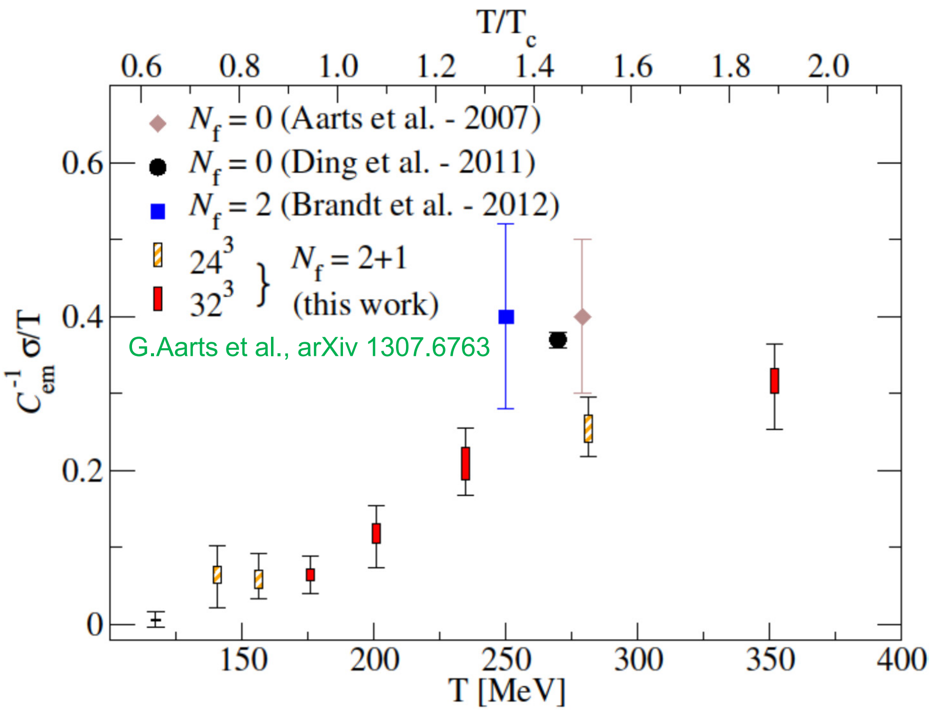
Electrical conductivity

T-dependence of the electrical conductivity:

$$\frac{\sigma}{T} = \frac{C_{em}}{6} \lim_{\omega \to 0} \frac{\rho_{ii}(\omega, \vec{p}=0, T)}{\omega T}$$

Brandt et al., 2012





Ding et al.: **Quenched** on **isotropic lattices** + ***continuum limit***

Aarts et al.: **2+1-flavor dynamical Wilson fermions on anisotropic lattices (Ns=24-32 Nt=16-48)**

Brandt et al.: **2-flavor dynamical Wilson fermions on isotropic lattices (Ns=64 Nt=16)**

Our primary observables are the Euclidean vector current correlators and their spectral representation:

$$G_{\mu\nu}(\tau,T) = \int d^3x\, \langle J_\mu(\tau,\vec{x})\, J_\nu(0)^\dagger \rangle = \int_0^\infty \frac{d\omega}{2\pi}\, \rho_{\mu\nu}(\omega,T)\, \frac{\cosh[\omega(\beta/2-\tau)]}{\sinh(\omega\beta/2)} \tag{2.1}$$

with $J_\mu(x) \equiv \frac{1}{\sqrt{2}}\left(\bar{u}(x)\gamma_\mu u(x) - \bar{d}(x)\gamma_\mu d(x)\right)$ the isospin current. For a given function $\rho(\omega,T)$, the reconstructed correlator is defined as

$$G^{\rm rec}(\tau,T;T') \equiv \int_0^\infty \frac{d\omega}{2\pi}\, \rho(\omega,T')\, \frac{\cosh[\omega(\frac{\beta}{2}-\tau)]}{\sinh(\omega\beta/2)}\,. \tag{2.2}$$

It can be interpreted as the Euclidean correlator that would be realized at temperature T if the spectral function was unchanged between temperature T and T'. For $T'=0$ it can be directly obtained from the zero-temperature Euclidean correlator via [6]

$$G^{\rm rec}(\tau,T) \equiv G^{\rm rec}(\tau,T;0) = \sum_{m\in\mathbb{Z}} G(|\tau+m\beta|,T=0). \tag{2.3}$$

In the thermodynamic limit, the subtracted vector spectral function obeys a sum rule (see [1] sec. 3.2),

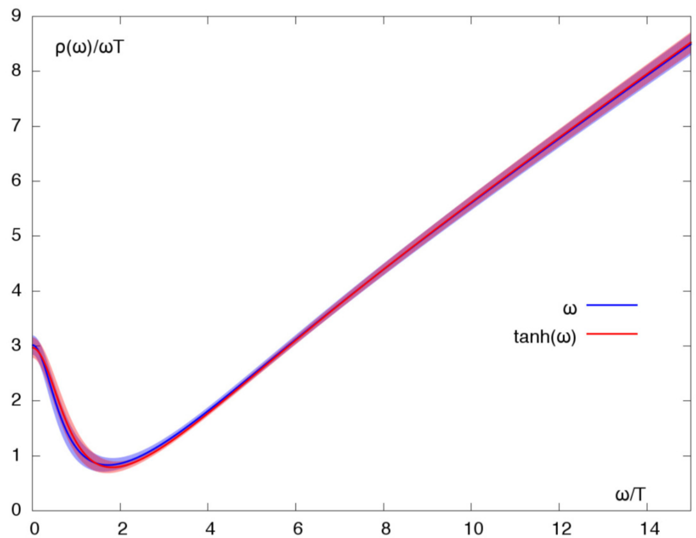
$$\int_{-\infty}^\infty \frac{d\omega}{\omega}\, \Delta\rho(\omega,T) = 0, \qquad\qquad \Delta\rho(\omega,T) \equiv \rho_{ii}(\omega,T) - \rho_{ii}(\omega,0). \tag{2.4}$$

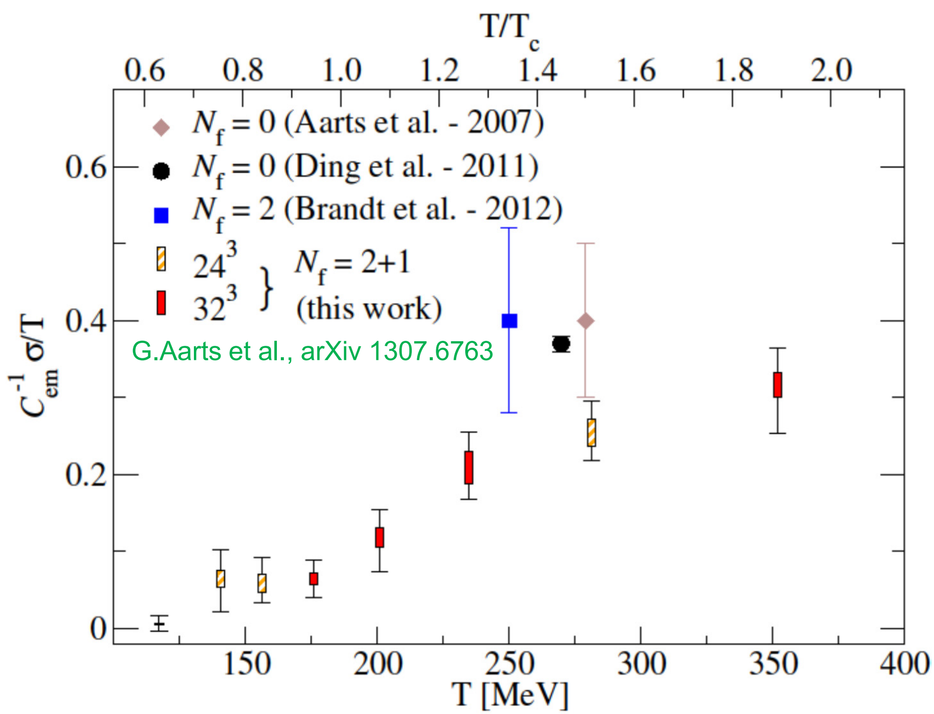
Electrical conductivity

T-dependence of the electrical conductivity:

$$\frac{\sigma}{T} = \frac{C_{em}}{6} \lim_{\omega\to 0} \frac{\rho_{ii}(\omega, \vec{p}=0, T)}{\omega T}$$

Brandt et al., 2012

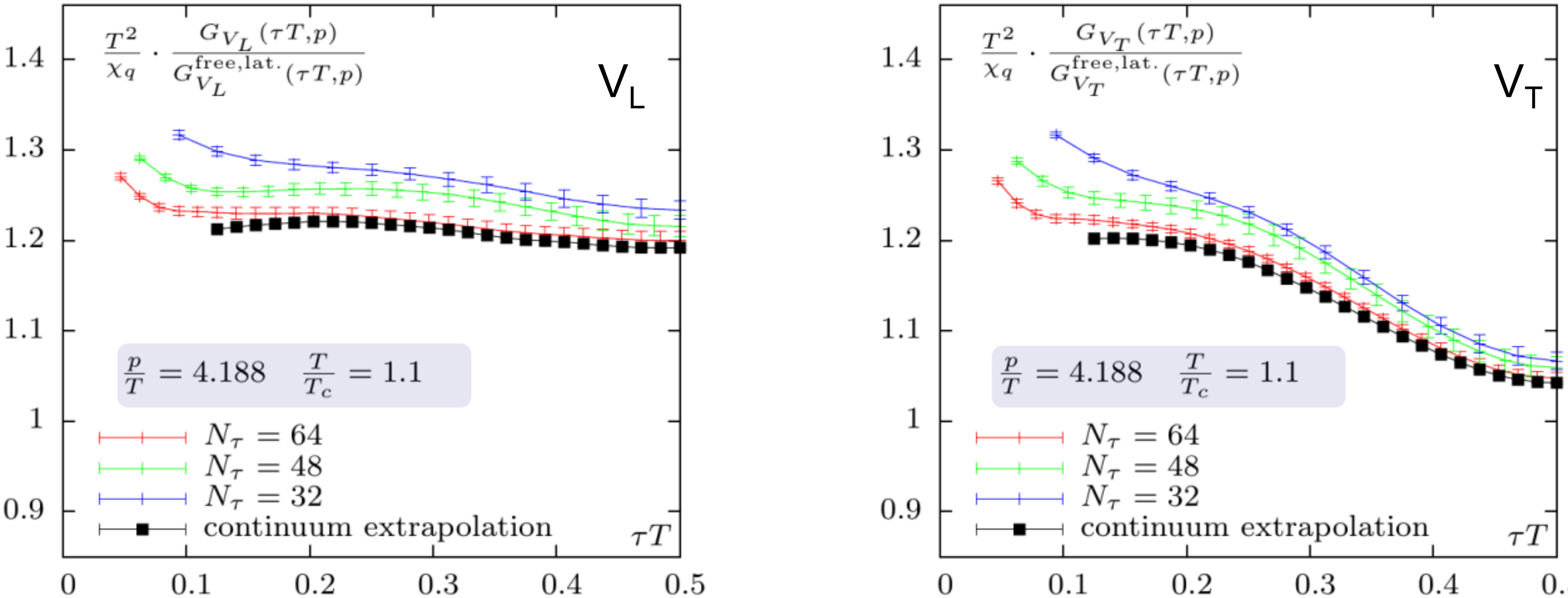




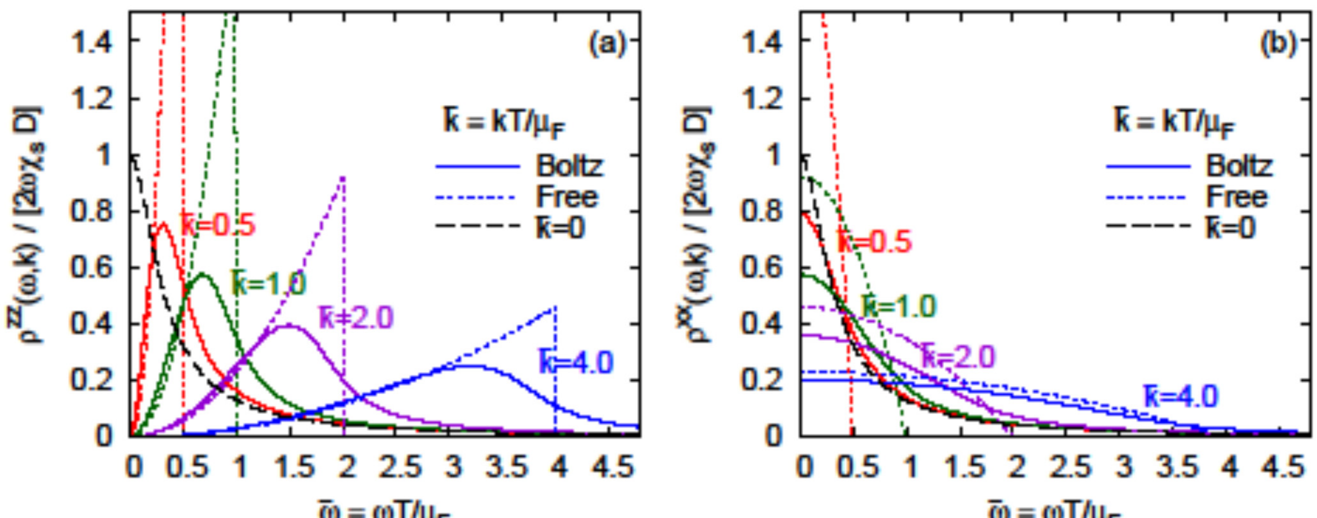
Ding et al.: **Quenched on isotropic lattices + *continuum limit***
Aarts et al.: **2+1-flavor dynamical Wilson fermions on anisotropic lattices (Ns=24-32 Nt=16-48)**
Brandt et al.: **2-flavor dynamical Wilson fermions on isotropic lattices (Ns=64 Nt=16)**

Non-zero momentum

[M.Müller et al., preliminary 2013]



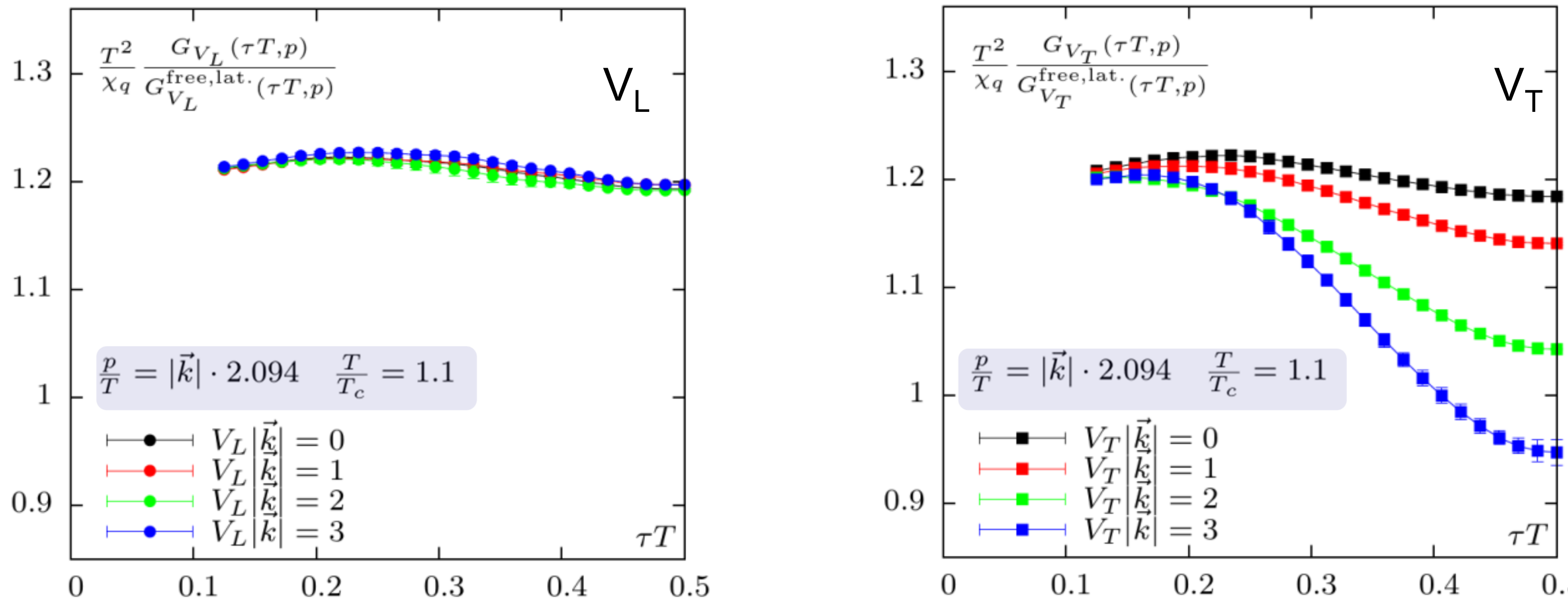
indications for non-trivial behavior of spectral functions at small frequencies:



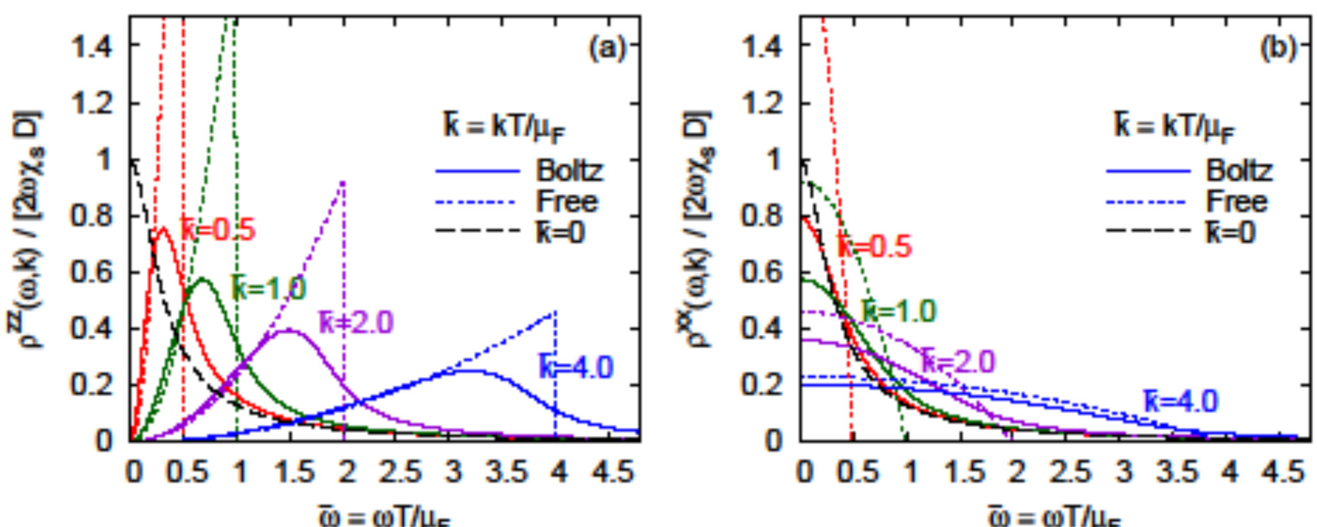
[Hong+Teaney, PRC82 (2010)044908]

Non-zero momentum

[M.Müller et al., preliminary 2013]

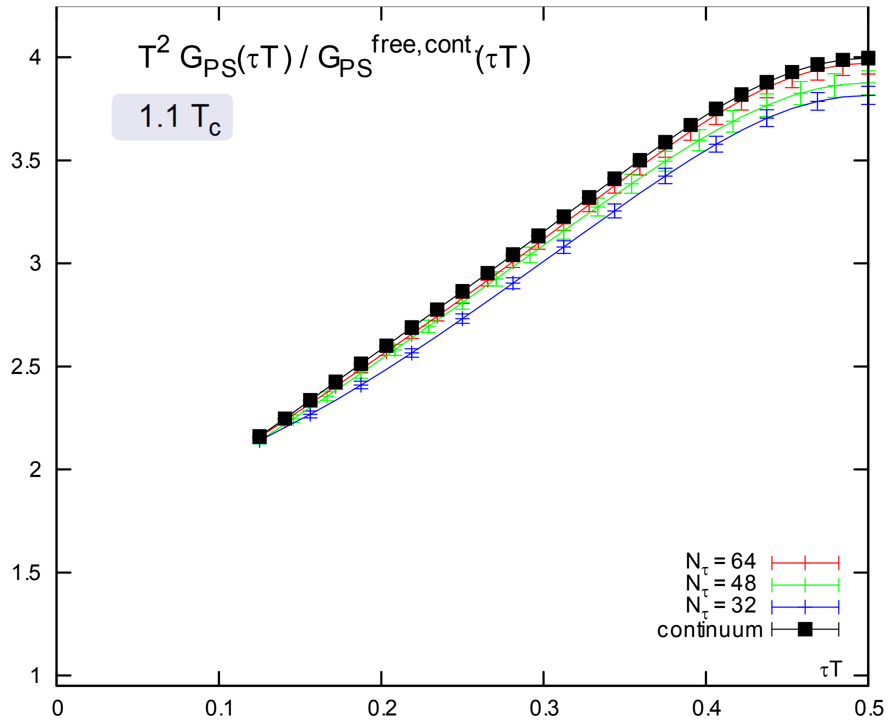


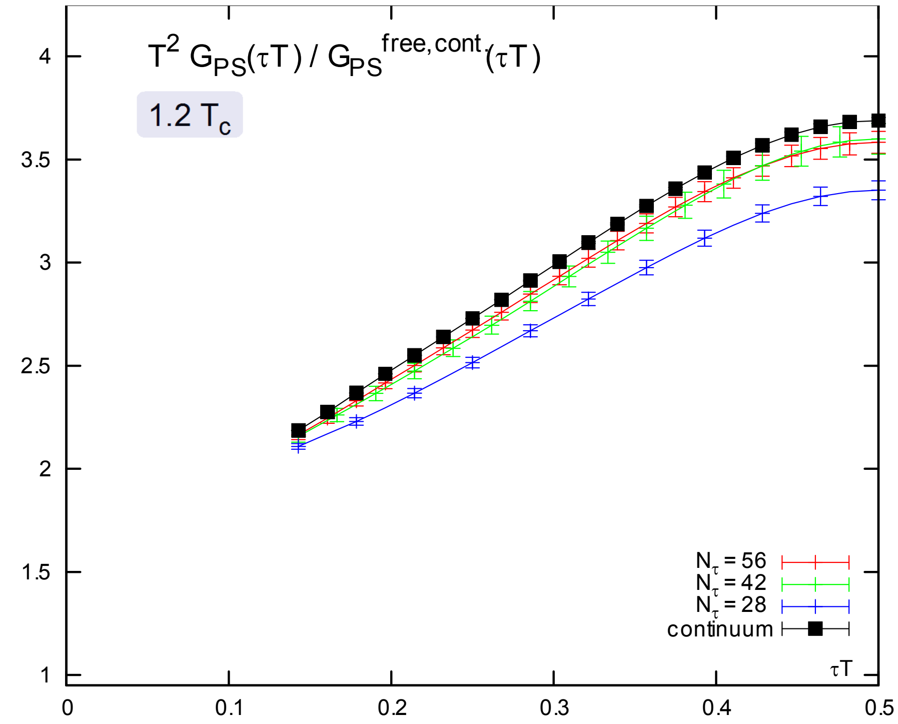
indications for non-trivial behavior of spectral functions at small frequencies:



[Hong+Teaney, PRC82 (2010)044908]

Pseudo-scalar channel





in contrast to the vector channel

no transport peak expected in the pseudo-scalar channel

still strong correlations visible in the pseudo-scalar channel

spectral function still needs to be determined!

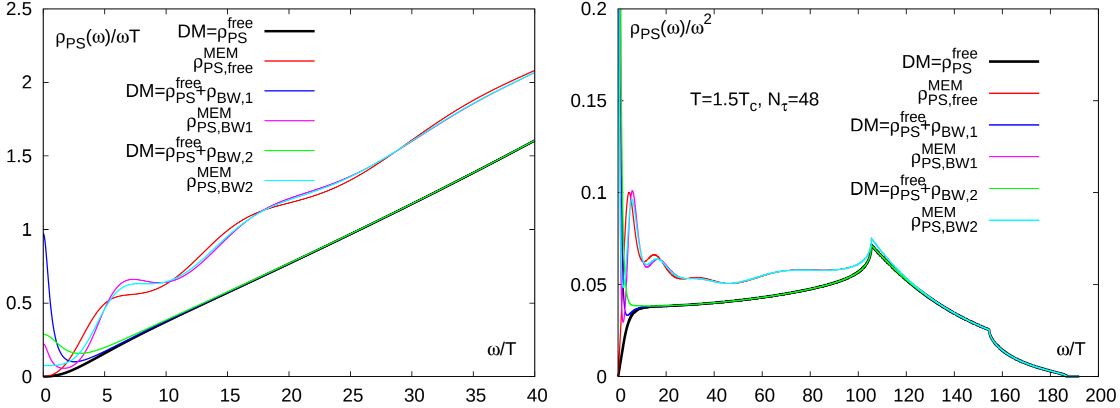


Figure 7.5: *Results of a MEM analysis on the pseudo scalar correlator. Left: The low frequency region of* $\rho(\omega)/\omega T$ *is shown. Right: The full spectral function is given in units* $1/\omega^2$. *Note the index in* $\rho^{MEM}_{PS,index}$ *shows which default model was used as input.*