Measurement of a dynamic field by rotating coil

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Outline

- Introduction
  - Plane circular multipoles
  - Measurement of a static field
- Measurement of a dynamic field
  - "Fast coil" case
  - Case study for the "fast coil" case
  - "Fast magnet" case
  - The signal processing in a nutshell
- Remarks
- Conclusions

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Introduction: Plane circular multipoles

For most practical purposes the magnet field analysis for accelerator applications can be reduced to a two-dimensional problem and the field can be represented by the power series expansion:

$$\boldsymbol{B}(z) = B_y + iB_x = \sum_{n=1}^{\infty} c_n \left(\frac{z}{r_{ref}} \right)^{n-1} \qquad (1)$$

where B_x and B_y are the x- and y-components of the magnetic flux density, $z = x + iy$ and r_{ref} is a reference radius. The complex expansion coefficients $c_n = b_n + ia_n$ in commonly accepted jargon are called “harmonics” or “multipoles”.

Introduction: Measurement of a static field

The flux $\Phi(\phi)$ through an arbitrary coil array has the form:

$$\begin{aligned} \Phi(\phi) &= \operatorname{Re} \sum_{n=1}^{\infty} \underbrace{s_n c_n}_{\gamma_n} e^{in\phi} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \gamma_n e^{in\phi} + (\mathbf{c.c.}) \end{aligned} \tag{2}$$

In eqn. (2) s_n are complex "sensitivity factors", ϕ is the angle of rotation, "**(c.c.)**" means complex conjugate. We introduced variables $\gamma_n = s_n c_n$ in order to make further calculations less cumbersome.

Introduction: Measurement of a static field (cont.)

Flux as a function of the rotation angle is evaluated through the time integral of induced voltage $V(t)$, readings of the voltage integrator are triggered by an angular encoder. In terms of time as a function of the rotation angle $t(\phi)$

$$-\int\limits_{t(0)}^{t(\phi)} V(t)dt = \int\limits_{t(0)}^{t(\phi)} \frac{d\Phi}{dt}dt = \int\limits_{t(0)}^{t(\phi)} d\Phi = \Phi(\phi) - \Phi(0) \qquad (3)$$

Introduction: Measurement of a static field (cont.)

As the $\Phi(\phi)$ is evaluated in the measurement, γ_m and consequently c_m can be found by the Fourier transform:

$$\frac{1}{\pi}\int_0^{2\pi} \Phi e^{-im\phi}\,d\phi = \frac{1}{2\pi}\sum_{n=1}^{\infty}\gamma_n \underbrace{\int_0^{2\pi} e^{in\phi}e^{-im\phi}\,d\phi}_{2\pi\delta_{nm}} + \\ + \overline{\gamma_n}\underbrace{\int_0^{2\pi} e^{-in\phi}e^{-im\phi}\,d\phi}_{0} \\ = \gamma_m = s_m c_m \tag{4}$$

Measurement of a dynamic field, “fast coil”

We assume the measuring coil rotates uniformly making an integer number of turns M during the magnet cycling period T. Let us use $\frac{T}{2\pi}$ as the time unit, consequently the dependence of the γ_n on time can be represented by a Fourier series:

$$\gamma_n(t) = \sum_{k=0}^{\infty} \sigma_{nk} e^{ikt} \tag{5}$$

and the flux through the coil vs. time is:

$$\Phi(t) = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \sigma_{nk} e^{i(k+Mn)t} + \textbf{(c.c.)} \tag{6}$$

The Fourier transform of (6) yields

$$\frac{1}{\pi}\int_0^{2\pi} \Phi\, e^{-i\widehat{m}t}\, dt = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \delta_{\widehat{m},(k+Mn)}\sigma_{nk} \tag{7}$$

Let us consider a partial sum of (6) with $n \leq N$, $k < K$ and map the σ_{nk} to a vector $\mathbf{z}$ of length NK as follows:

$$z_{k+(n-1)K} = \sigma_{nk} \quad \Leftrightarrow \quad z_j = \sigma_{\mathrm{floor}\left(\frac{j}{K}\right)+1,\, j \bmod K}$$
$$0 \leq j < NK \tag{8}$$

In the above equation $\mathrm{floor}(j/K) \equiv (j - (j \bmod K))/K$ is the integer division.

The equation $k + nM = \widehat{m}$ in terms of the linear index j defined in (8) can be derived as

$$(j \bmod K)(K - M) + jM = (\widehat{m} - M)K \qquad (9)$$

The smallest solution of the eqn. (9) is found for $\widehat{m} = M$, $j = 0$. Let us denote $m = \widehat{m} - M$. Eqn. (7) can then be rewritten in the form $\mathbf{Az} = \mathbf{b}$ where the matrix $\mathbf{A}$ and vector $\mathbf{b}$ are defined as:

$$\begin{aligned} A_{mj} &= \delta_{(j \bmod K)(K-M)+jM,\, mK} \\ b_m &= \frac{1}{\pi} \int_0^{2\pi} \Phi \, e^{-i(m+M)t} \, dt \end{aligned} \qquad (10)$$

The case $K = M$ is a trivial one, because $\mathbf{A}$ turns to the identity matrix, yielding $z_m = b_m$.

In a case $K > M$, which is of more practical importance, the structure of matrix $\mathbf{A}$ is more complicated. Let us consider as a case study $\mathbf{A}$ for $K = 3$, $M = 1$ and $N = 3$:

$$\mathbf{A}\begin{pmatrix} K = 3, \\ M = 1, \\ N = 3 \end{pmatrix} = \begin{bmatrix} 1 & & & & & & & & \\ & 1 & & 1 & & & & & \\ & & 1 & & 1 & & 1 & & \\ & & & & & 1 & & 1 & \\ & & & & & & & & 1 \end{bmatrix} \tag{11}$$

The matrix contains $KN - (K - M)(N - 1) = K + M(N - 1)$ rows. Each unknown enters only into one equation, thus all equations are decoupled, but the number of equations is less than the number of unknowns. The system must be supplemented by more equations to have a unique solution.

Additional equations can be obtained using the translation properties of σ_{nk} and b_m.

Each equation with $L > 1$ terms has, after mapping back from z_j to σ_{nk}, the following form:

$$\sum_{l=0}^{L-1} z_{j_{\min}+l(K-M)} = b_m \;\Leftrightarrow\; \sum_{l=0}^{L-1} \sigma_{n_{\min}+l,\, k_{\max}-Ml} = b_m \qquad (12)$$

In equation (12) $j_{\min}$ is the minimal index with z-mapping of unknowns (8), $n_{\min}$ and $k_{\max}$ are the minimal value of index n and the maximal value of index k in the corresponding σ_{nk} set, respectively.

Measurement of a dynamic field, “fast coil” (cont.)

If the measurement clock is shifted by τ, from (5) and (10) it follows that σ_{nk} and b_m are transformed as:

$$\begin{aligned} \sigma_{nk} &\rightarrow \tilde{\sigma}_{nk} = e^{ik\tau}\sigma_{nk} \\ b_m &\rightarrow \tilde{b}_m = e^{i(m+M)\tau} b_m \end{aligned} \tag{13}$$

In terms of the equation for z_j, from the original equation (12) we can construct another one:

$$\begin{aligned} &\sum_{l=0}^{L-1} z_{j_{\min}+l(K-M)} = b_m \quad \rightarrow \\ &\rightarrow \sum_{l=0}^{L-1} e^{-ilM\tau} \cdot z_{j_{\min}+l(K-M)} = e^{i[m+M-(j_{\min} \bmod K)]\tau} b_m \end{aligned} \tag{14}$$

Measurement of a dynamic field, "fast coil" (cont.)

Let us construct $L-1$ additional equations as in (14) with different time shifts τ_l. Denoting $p_l = e^{-iM\tau_l}$ we see that the matrix $\mathbf{V}$ constructed from row of coefficients of the equation (12) is a square Vandermonde matrix with a known determinant:

$$\mathbf{V} = \begin{bmatrix} 1 & p_0 & p_0^2 & \cdots & p_0^{L-1} \\ 1 & p_1 & p_1^2 & \cdots & p_1^{L-1} \\ 1 & p_2 & p_2^2 & \cdots & p_2^{L-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & p_{L-1} & p_{L-1}^2 & \cdots & p_{L-1}^{L-1} \end{bmatrix} \tag{15}$$

$$|\mathbf{V}| = \prod_{0 \le i < j < L} (p_j - p_i)$$

Measurement of a dynamic field, "fast coil" (cont.)

If all p_l are different (a natural choice: $p_l = e^{-2\pi i \frac{l}{L}}$), $\mathbf{V}$ has a non-zero determinant, therefore it is invertable.
Applying the above technique to the case study ($K = 3$, $M = 1$, $N = 3$) we obtain the following set of equations:

$$
\begin{gathered}
z_0 = b_0 \\
\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ -b_1 \end{bmatrix} \\
\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-i\frac{2\pi}{3}} & e^{-i\frac{4\pi}{3}} \\ 1 & e^{-i\frac{4\pi}{3}} & e^{-i\frac{8\pi}{3}} \end{bmatrix} \begin{bmatrix} z_2 \\ z_4 \\ z_6 \end{bmatrix} = \begin{bmatrix} b_2 \\ e^{i\frac{2\pi}{3}} b_2 \\ e^{i\frac{4\pi}{3}} b_2 \end{bmatrix} \\
\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} z_5 \\ z_7 \end{bmatrix} = \begin{bmatrix} b_3 \\ b_3 \end{bmatrix} \\
z_8 = b_4
\end{gathered}
\tag{16}
$$

Measurement of a dynamic field, "fast coil" (cont.)

As all matrices in (16) are invertible, the case study is solved. This technique can be applied for any N, M and $K > M$. Instead of inverting the $NK \times NK$ matrix, the problem is reduced to inverting of a number of matrices of smaller dimensions. It can be shown for the general case, that the maximum matrix dimension does not exceed $(K - M + 1) \times (K - M + 1)$.

Measurement of a dynamic field, "fast magnet"

Let us assume the measuring coil rotates uniformly and the magnet is cycled M times during one turn of the coil. In this case we denote as T the coil rotating period and measure time in units of $\frac{T}{2\pi}$. For the purpose of a partial reuse of results obtained for the "fast magnet" case let us swap the meaning of indices: now k stands for the multipole number and n for term number in the Fourier expansion of the multipole:

$$\gamma_k(t) = \sum_{n=0}^{\infty} \sigma_{kn} e^{iMnt} \tag{17}$$

The flux through the coil vs. time is then given by equation which is similar to (7), except for the low limits of sums:

$$\Phi(t) = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\sigma_{kn}e^{i(k+Mn)t} + \textbf{(c.c.)} \tag{18}$$

The Fourier transform of (18) yields

$$\frac{1}{\pi}\int\limits_{0}^{2\pi}\Phi\, e^{-i\widehat{m}t}\,dt = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\delta_{\widehat{m},(k+Mn)}\sigma_{kn} \tag{19}$$

Let us consider a partial sum of (18) with $n < N$, $k \leq K$ and map the σ_{nk} to a vector $\mathbf{z}$ of length NK as follows:

$$z_{k-1+nK} = \sigma_{kn} \quad \Leftrightarrow \quad z_j = \sigma_{j \bmod K+1,\, \mathrm{floor}\left(\frac{j}{K}\right)} \qquad 0 \leq j < NK \tag{20}$$

The equation $k + nM = \widehat{m}$ in terms of the linear index j defined in (20) can be derived as

$$(j \bmod K)(K - M) + jM = (\widehat{m} - 1)K \tag{21}$$

The smallest solution of the eqn. (21) is found for $\widehat{m} = 1$, $j = 0$.

Let us denote $m = \widehat{m} - 1$, then (19) can be rewritten in the form $\mathbf{Az} = \mathbf{b}$ where the matrix $\mathbf{A}$ and vector $\mathbf{b}$ are defined as:

$$\begin{aligned} A_{mj} &= \delta_{(j \bmod K)(K-M)+jM,\, mK} \\ b_m &= \frac{1}{\pi}\int_0^{2\pi} \Phi\, e^{-i(m+1)t}\, dt \end{aligned} \tag{22}$$

Comparing (22) and (10) one can see that the same matrix $\mathbf{A}$ is obtained (albeit after swapping meanings of k and n). Therefore the consideration of the “fast magnet” case is very similar to the one of “fast coil”.

Measurement of a dynamic field, “fast magnet” (cont.)

The case $K = M$ is a trivial one, because $\mathbf{A}$ turns to the identity matrix, yielding $z_m = b_m$.
For $K > M$ each equation with $L > 1$ terms has, after mapping from z_j to σ_{kn}, the following form:

$$\sum_{l=0}^{L-1} z_{j_{\min}+l(K-M)} = b_m \quad \Leftrightarrow \quad \sum_{l=0}^{L-1} \sigma_{k_{\max}-Ml,\, n_{\min}+l} = b_m \qquad (23)$$

If the measurement clock is shifted by τ, σ_{kn} and b_m are transformed as:

$$\begin{aligned} \sigma_{kn} \;&\rightarrow\; \tilde{\sigma}_{kn} = e^{inM\tau}\sigma_{kn} \\ b_m \;&\rightarrow\; \tilde{b}_m = e^{i(m+1)\tau} b_m \end{aligned} \qquad (24)$$

Measurement of a dynamic field, “fast magnet” (cont.)

In terms of equation for z_j, from the original equation (23) we can construct another one:

$$\sum_{l=0}^{L-1} z_{j_{\min}+l(K-M)} = b_m \quad \rightarrow$$
$$\rightarrow \sum_{l=0}^{L-1} e^{ilM\tau} \cdot z_{j_{\min}+l(K-M)} = e^{i\left[m+1-M\cdot\text{floor}\left(\frac{j_{\min}}{K}\right)\right]\tau} \cdot b_m \tag{25}$$

Consequently, exactly as for the “fast coil” case, from each equation with $L > 1$ terms we construct $L - 1$ additional equations using the σ_{kn} and b_m time translation properties, obtaining a linear system with a Vandermonde matrix, which has a non-zero determinant.

The signal processing in a nutshell

The evaluation of harmonics of a time-periodic field comprises the following steps:

- Presentation of time-dependent harmonics as Fourier series
- Choice of a natural time unit
- Obtaing of (incomplete) set of equations by Fourier transform
- Use of the Fourier coefficients translation properties to obtain missing equations

Remarks

For the "fast magnet" case a good synchronization of the coil rotation and the magnet cycling is essential. As a possible implementation one can consider the magnet power supply driven by a waveform generator which is clocked by the angular encoder pulses.

Much looser requirements can be applied for the "fast coil" case. The magnet cycle starts from the injection plateau and comes back to it. If multipoles stay constant at the injection plateau then expulsion of a part of the plateau, followed by "gluing" of remaining parts preserves continuity of each multipoles as a function of time.

Remarks (cont.)

Therefore in the “fast coil” case the measurement can be started at any point shortly before the ramp up, and ended at any point at the plateau after ramp down when the coil comes back to the start angle. In other words, the magnet cycle is not required to be a multiple of the coil rotation period. This option, to choose a “fictitious” magnet cycle, may be exploited. It may be shown for a cycle with the same ramp up and ramp down rates, that the closer the “fictitious” injection plateau duration is to the duration of the cycle flat-top the lesser are higher harmonics in the Fourier expansion (5).
The developed data processing technique operates with truncated expansions (partial sums) to represent the field and multipoles. To check if upper limits of sums are large enough to provide the precision required they may be varied and the effect of variations on obtained results must be analyzed.

Conclusions

- The data processing technique has been developed which allows the use of the rotating coil method for the measurement of a dynamic time-periodic field.
- This technique allows to obtain time-dependent multipoles in a single measurement which lasts one power cycle of the magnet or the coil rotation period, whichever is longer.