Scalar size of the pion from lattice QCD

Konstantin Ottnad

Institut für Kernphysik, Johannes Gutenberg-Universität Mainz

EMMI Hadron Physics Seminar @ GSI Darmstadt

November 12, 2025

Introduction

The pion scalar form factor is defined by

$$F_S^{\pi,f}(Q^2) = \langle \pi(p_f)|\mathcal{S}^f|\pi(p_i)\rangle\,,$$

where for $N_f = 2+1$ quark flavors

$$\begin{array}{llll} \mathcal{S}^l = \bar{u}u + \bar{d}d & \rightarrow & F_S^{\pi,l}(Q^2) & \text{(light)} \\ \mathcal{S}^8 = \bar{u}u + \bar{d}d - 2\bar{s}s & \rightarrow & F_S^{\pi,8}(Q^2) & \text{(octet)} \\ \mathcal{S}^0 = \bar{u}u + \bar{d}d + \bar{s}s & \rightarrow & F_S^{\pi,0}(Q^2) & \text{(singlet)} \end{array}$$

The corresponding scalar radii

$$\langle r_S^2\rangle_\pi^f = \frac{-6}{F_S^{\pi,f}(0)} \cdot \frac{dF_S^{\pi,f}(Q^2)}{dQ^2}\bigg|_{Q^2=0}\,,$$

provide insight into the low-energy regime of QCD:

- At NLO in $SU(2)$ χPT $\langle r_S^2\rangle_\pi^l$ is parametrized by a single LEC $\bar{l}_4$. *Annals Phys. 158, 142 (1984)*
- At NLO in $SU(3)$ χPT $\langle r_S^2\rangle_\pi^{0,8,l}$ three LECs, i.e. f_0, L_4 and L_5. *NPB 250 (1985) 517-538*
- $F_S^{\pi,f}(q^2)$ from LQCD **computationally very demanding**:

  $\rightarrow$ Significant quark-disconnected contributions. *JHEP 01 (2012) 007*

  $\rightarrow$ Very few lattice calculations; none with (fully) controlled systematics.

Introduction

In two-flavor χPT: direct relation between $\langle r_S^2 \rangle_\pi^l$ and $\bar{l}_4$:

$$\langle r_S^2 \rangle_\pi^l = \frac{2}{(4\pi f_\pi)^2}\left[-\frac{13}{6} + \left(6\bar{l}_4 + \log\left(\frac{M_{\pi,\mathrm{phys}}^2}{M_\pi^2}\right)\right)\right].$$

Annals Phys. 158, 142 (1984)

However, most lattice studies of $\bar{l}_4$ are based on computations of f_π instead of actually computing $\langle r_S^2 \rangle_\pi^l$:

$$\frac{f_\pi}{\mathring{f}_\pi} = 1 + \frac{1}{6}M_\pi^2 \langle r_S^2 \rangle_\pi^l + \frac{13 M_\pi^2}{96\pi^2 f_\pi^2} + \mathcal{O}(M_\pi^4)\,.$$

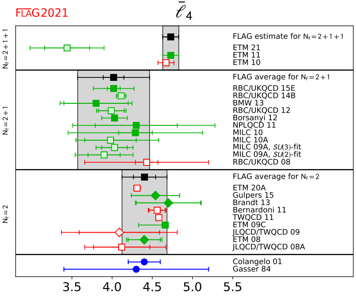


Figure reproduced from FLAG 2021 review.

- **No direct computations of $\bar{l}_4$ from $N_f = 2+1(+1)$ lattice simulations.**
- Few $N_f = 2$ computations of $\langle r_S^2 \rangle_\pi^l$; older than $\gtrsim 10$ years.
- (Very) unphysical pion masses; various other uncontrolled systematics.
- For $N_f = 2+1$ substantial scatter of results; FLAG average $\bar{l}_4 = 4.02(45)$ has $\sim 10\%$ error.

Introduction

In three-flavor χPT:

$$\langle r_S^2\rangle_\pi^0 = \langle r_S^2\rangle_\pi^8 + \langle \delta r_S^2\rangle\,,$$
$$\langle r_S^2\rangle_\pi^l = \langle r_S^2\rangle_\pi^8 + \tfrac{2}{3}\langle \delta r_S^2\rangle\,,$$
$$\langle r_S^2\rangle_\pi^8 = \frac{6}{M_K^2 - M_\pi^2}\left(\frac{f_\pi}{f_K} - 1\right) + \delta_3\,,$$

where

$$\langle \delta r_S^2\rangle = \frac{12}{{f_0}^2}\left[12L_4^r - \frac{3}{64\pi^2}\left(\ln\frac{M_K^2}{\mu^2}+1\right) + \frac{M_\pi^2}{288M_\eta^2}\right]\,,$$
$$\delta_3 = \frac{1}{(4\pi f_0)^2}\left[3\,\frac{2M_K^2 - M_\pi^2}{M_K^2 - M_\pi^2}\ln\frac{M_K^2}{M_\pi^2} + \frac{9}{2}\,\frac{M_\eta^2}{M_K^2 - M_\pi^2}\ln\frac{M_\eta^2}{M_\pi^2} - \left(10 + \frac{1}{3}\,\frac{M_\pi^2}{M_\eta^2}\right)\right]\,.$$

- Only a single $N_f = 2+1$ lattice calculation of $\langle r_S^2\rangle_\pi^{0,8,l}$ exists. *PRD 93 (2016) 5, 054503*
- Very little is known about L_4^r (from older f_π, f_K calculations)

  $L_4^r = -0.02(56)\times 10^{-3}$ $(N_f = 2+1)$ *PRD 88 (2013) 074504*

  $L_4^r = +0.09(34)\times 10^{-3}$ $(N_f = 2+1+1)$ *PoS LAT2010 (2010) 074*
- **No lattice determination of L_4^r available from the pion scalar form factor!**
- L_5^r enters only in $\langle r_S^2\rangle_\pi^8$ through the NLO expression for f_π/f_K.
- FLAG result for f_0 and L_5^r again from (same) single study:

  $f_0 = 114.0(8.5)\,\mathrm{MeV}$ and $L_5^r = 0.95(41)\times 10^{-3}$

Outline

1. Lattice methods and setup
2. Form factor analysis (excited states, Q^2-dependence $\rightarrow$ radii)
3. Physical extrapolations
4. Model averages and final results
5. Summary & outlook

All results in this talk published in:

PRL 135 (2025) 7, 071904
PRD 112 (2025) 3, 034504

How to obtain physical results?

- **Fix bare parameters (a, m_l, m_s,...):**
  - Use known hadronic quantities (e.g. $M_\pi^{\rm phys}$, $M_K^{\rm phys}$, ...)

    $\rightarrow$ Any further observables are predictions.
- **Control discretization effects:**
  - Simulate at different (small) values of a.
  - Perform continuum extrapolation.
  - In modern LQCD calculations lattice artifacts are typically $\propto a^2$.
- **Correct for unphysical quark masses:**
  - Simulate at several light and strange quark masses.
  - Perform chiral extrapolation.
  - State-of-the-art lattice simulations include physical quark masses.
- **Control finite volume effects:**
  - Simulate several physical volumes.
  - Perform infinite volume extrapolation / test for finite volume effects.

Ensembles

ID^{BC}	Traj.	a/fm	T/a	L/a	M_π/MeV	$M_\pi L$	N_{conf}	$N^{\text{3pt}}_{\text{meas}}$	$N^{(2+1)\text{pt}}_{\text{meas}}$
C101^o	tr$[M]$	0.086	96	48	225	4.68	1970	15760	866800
C102^o	m_s		96	48	228	4.75	1467	11736	645480
N101^o	tr$[M]$		128	48	283	5.89	1577	12616	883120
H102^o	tr$[M]$		96	32	358	4.97	2029	16232	1136240
D450^p	tr$[M]$	0.076	128	64	219	5.36	1009	8072	516608
D451^p	m_s		128	64	218	5.35	1482	11856	758784
N451^p	tr$[M]$		128	48	289	5.32	1004	8032	514048
N452^p	tr$[M]$		128	48	354	6.50	998	7984	510976
S400^o	tr$[M]$		128	32	356	4.37	993	7944	460752
E250^p	**both**	0.064	192	96	132	4.07	982	7856	502784
D200^o	tr$[M]$		128	64	204	4.22	1984	15872	1111040
D201^o	m_s		128	64	204	4.21	1062	8496	594720
N200^o	tr$[M]$		128	48	287	4.45	1703	13624	953680
N203^o	tr$[M]$		128	48	349	5.40	1535	12280	859600
E300^o	tr$[M]$	0.049	192	96	176	4.22	1134	9072	521640
J303^o	tr$[M]$		192	64	260	4.17	1068	8544	598080
J304^o	m_s		192	64	261	4.18	1625	13000	910000

- $N_f = 2+1$ flavors of non-perturbatively improved Wilson clover fermions provided by CLS.
  JHEP 1502 (2015) 043 Commun.Math.Phys. 97 (1985) PoS LATTICE2008 (2008) 049
- 13 Ensembles on tr$[M] = 2m_l + m_s = \text{const}$ trajectory, 4 more on $m_s = m_s^{\text{phys}}$ trajcetory.
- Ensembles with open BC ("o") and periodic BC ("p") in time.

Ensembles

ID^{BC}	Traj.	a/fm	T/a	L/a	M_π / MeV	$M_\pi L$	N_{conf}	$N^{\mathrm{3pt}}_{\mathrm{meas}}$	$N^{(2+1)\mathrm{pt}}_{\mathrm{meas}}$
C101^o	tr$[M]$	**0.086**	96	48	225	4.68	1970	15760	866800
C102^o	m_s		96	48	228	4.75	1467	11736	645480
N101^o	tr$[M]$		128	48	283	5.89	1577	12616	883120
H102^o	tr$[M]$		96	32	358	4.97	2029	16232	1136240
D450^p	tr$[M]$	**0.076**	128	64	219	5.36	1009	8072	516608
D451^p	m_s		128	64	218	5.35	1482	11856	758784
N451^p	tr$[M]$		128	48	289	5.32	1004	8032	514048
N452^p	tr$[M]$		128	48	354	6.50	998	7984	510976
S400^o	tr$[M]$		128	32	356	4.37	993	7944	460752
E250^p	both	**0.064**	192	96	132	4.07	982	7856	502784
D200^o	tr$[M]$		128	64	204	4.22	1984	15872	1111040
D201^o	m_s		128	64	204	4.21	1062	8496	594720
N200^o	tr$[M]$		128	48	287	4.45	1703	13624	953680
N203^o	tr$[M]$		128	48	349	5.40	1535	12280	859600
E300^o	tr$[M]$	**0.049**	192	96	176	4.22	1134	9072	521640
J303^o	tr$[M]$		192	64	260	4.17	1068	8544	598080
J304^o	m_s		192	64	261	4.18	1625	13000	910000

- **Ensembles cover four values of the lattice spacing a**
- Pion masses range from $\sim 130\,\mathrm{MeV}$ to $\sim 350\,\mathrm{MeV}$
- Many different physical volumes with $L \approx 2.4, \ldots, 6.1\,\mathrm{fm}$, $M_\pi L > 4$.
- Two very large and fine boxes at (near) physical quark mass and high momentum resolution.

Ensembles

ID^{BC}	Traj.	a/fm	T/a	L/a	M_π / MeV	$M_\pi L$	N_{conf}	$N_{\text{meas}}^{\text{3pt}}$	$N_{\text{meas}}^{(2+1)\text{pt}}$
C101^o	tr$[M]$	0.086	96	48	**225**	4.68	1970	15760	866800
C102^o	m_s		96	48	**228**	4.75	1467	11736	645480
N101^o	tr$[M]$		128	48	**283**	5.89	1577	12616	883120
H102^o	tr$[M]$		96	32	**358**	4.97	2029	16232	1136240
D450^p	tr$[M]$	0.076	128	64	**219**	5.36	1009	8072	516608
D451^p	m_s		128	64	**218**	5.35	1482	11856	758784
N451^p	tr$[M]$		128	48	**289**	5.32	1004	8032	514048
N452^p	tr$[M]$		128	48	**354**	6.50	998	7984	510976
S400^o	tr$[M]$		128	32	**356**	4.37	993	7944	460752
E250^p	both	0.064	192	96	**132**	4.07	982	7856	502784
D200^o	tr$[M]$		128	64	**204**	4.22	1984	15872	1111040
D201^o	m_s		128	64	**204**	4.21	1062	8496	594720
N200^o	tr$[M]$		128	48	**287**	4.45	1703	13624	953680
N203^o	tr$[M]$		128	48	**349**	5.40	1535	12280	859600
E300^o	tr$[M]$	0.049	192	96	**176**	4.22	1134	9072	521640
J303^o	tr$[M]$		192	64	**260**	4.17	1068	8544	598080
J304^o	m_s		192	64	**261**	4.18	1625	13000	910000

- Ensembles cover four values of the lattice spacing a
- **Pion masses range from** $\sim 130\,\text{MeV}$ **to** $\sim 350\,\text{MeV}$
- Many different physical volumes with $L \approx 2.4, \ldots, 6.1\,\text{fm}$, $M_\pi L > 4$.
- Two very large and fine boxes at (near) physical quark mass and high momentum resolution.

Ensembles

ID^{BC}	Traj.	a/fm	T/a	L/a	M_π / MeV	$M_\pi L$	N_conf	N_meas^3pt	$N_\text{meas}^{(2+1)\text{pt}}$
C101^o	tr$[M]$	0.086	96	48	225	**4.68**	1970	15760	866800
C102^o	m_s		96	48	228	**4.75**	1467	11736	645480
N101^o	tr$[M]$		128	48	283	**5.89**	1577	12616	883120
H102^o	tr$[M]$		96	32	358	**4.97**	2029	16232	1136240
D450^p	tr$[M]$	0.076	128	64	219	**5.36**	1009	8072	516608
D451^p	m_s		128	64	218	**5.35**	1482	11856	758784
N451^p	tr$[M]$		128	48	289	**5.32**	1004	8032	514048
N452^p	tr$[M]$		128	48	354	**6.50**	998	7984	510976
S400^o	tr$[M]$		128	32	356	**4.37**	993	7944	460752
E250^p	both	0.064	192	96	132	**4.07**	982	7856	502784
D200^o	tr$[M]$		128	64	204	**4.22**	1984	15872	1111040
D201^o	m_s		128	64	204	**4.21**	1062	8496	594720
N200^o	tr$[M]$		128	48	287	**4.45**	1703	13624	953680
N203^o	tr$[M]$		128	48	349	**5.40**	1535	12280	859600
E300^o	tr$[M]$	0.049	192	96	176	**4.22**	1134	9072	521640
J303^o	tr$[M]$		192	64	260	**4.17**	1068	8544	598080
J304^o	m_s		192	64	261	**4.18**	1625	13000	910000

- Ensembles cover four values of the lattice spacing a
- Pion masses range from $\sim 130\,\text{MeV}$ to $\sim 350\,\text{MeV}$
- **Many different physical volumes with $L \approx 2.4, \ldots, 6.1\,\text{fm}$, $M_\pi L > 4$.**
- Two very large and fine boxes at (near) physical quark mass and high momentum resolution.

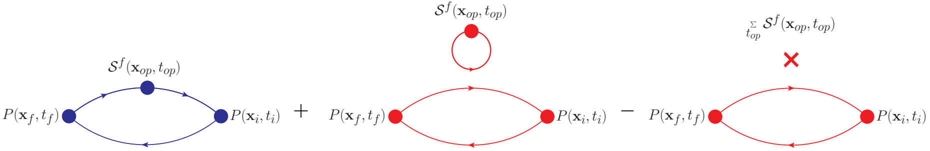
Ensembles

ID^{BC}	Traj.	a/fm	T/a	L/a	M_π / MeV	$M_\pi L$	N_{conf}	$N_{\mathrm{meas}}^{3\mathrm{pt}}$	$N_{\mathrm{meas}}^{(2+1)\mathrm{pt}}$
C101^o	tr$[M]$	0.086	96	48	225	**4.68**	1970	15760	866800
C102^o	m_s		96	48	228	**4.75**	1467	11736	645480
N101^o	tr$[M]$		128	48	283	**5.89**	1577	12616	883120
H102^o	tr$[M]$		96	32	358	**4.97**	2029	16232	1136240
D450^p	tr$[M]$	0.076	128	64	219	**5.36**	1009	8072	516608
D451^p	m_s		128	64	218	**5.35**	1482	11856	758784
N451^p	tr$[M]$		128	48	289	**5.32**	1004	8032	514048
N452^p	tr$[M]$		128	48	354	**6.50**	998	7984	510976
S400^o	tr$[M]$		128	32	356	**4.37**	993	7944	460752
E250^p	both	**0.064**	192	96	**132**	**4.07**	982	7856	502784
D200^o	tr$[M]$		128	64	204	**4.22**	1984	15872	1111040
D201^o	m_s		128	64	204	**4.21**	1062	8496	594720
N200^o	tr$[M]$		128	48	287	**4.45**	1703	13624	953680
N203^o	tr$[M]$		128	48	349	**5.40**	1535	12280	859600
E300^o	tr$[M]$	**0.049**	192	96	**176**	**4.22**	1134	9072	521640
J303^o	tr$[M]$		192	64	260	**4.17**	1068	8544	598080
J304^o	m_s		192	64	261	**4.18**	1625	13000	910000

- Ensembles cover four values of the lattice spacing a
- Pion masses range from $\sim 130\,\mathrm{MeV}$ to $\sim 350\,\mathrm{MeV}$
- Many different physical volumes with $L \approx 2.4, \ldots, 6.1\,\mathrm{fm}$, $M_\pi L > 4$.
- **Two very large and fine boxes at (near) physical quark mass and high momentum resolution.**

Pion scalar form factor in lattice QCD

Study of the pion scalar form factor requires **quark-connected** and **disconnected** three-point functions:



- $P(x) = u(x)\gamma_5 \bar{d}(x)$: interpolating field for the pion.
- Consider all three flavor structures for scalar insertion, i.e.:
  $\mathcal{S}^8 = \bar{u}u + \bar{d}d - 2\bar{s}s$, $\quad \mathcal{S}^0 = \bar{u}u + \bar{d}d + \bar{s}s$ and $\mathcal{S}^l = \bar{u}u + \bar{d}d$
- Basic building blocks to compute are **1pt-**, **2pt-** and **3pt-functions**.
- Last diagram: *vacuum expectation value* (VEV) only contributes at $Q^2 = 0$.
- Evaluation of 2+1 quark-disconnected diagram is very expensive (loops + large two-point statistics)

Quark-connected two- and three-point functions

Setup for the computation of quark-connected two-point and three-point function

$$C_2(\mathbf{p}, \mathbf{x}_i, t_f - t_i) = \sum_{\mathbf{x}_f} e^{i\mathbf{p}\cdot(\mathbf{x}_f - \mathbf{x}_i)} \langle P(\mathbf{x}_f - \mathbf{x}_i, t_f - t_i) P^\dagger(\mathbf{0}, 0)\rangle \,,$$

$$C_3(\mathbf{p}_f, \mathbf{q}, \mathbf{x}_i, t_{\text{sep}}, t_{\text{ins}}) = \sum_{\mathbf{x}_f, \mathbf{x}_{op}} e^{i\mathbf{p}_f\cdot(\mathbf{x}_f - \mathbf{x}_i)} e^{i\mathbf{q}\cdot(\mathbf{x}_{op} - \mathbf{x}_i)} \langle P(\mathbf{x}_f - \mathbf{x}_i, t_{\text{sep}}) \mathcal{S}(\mathbf{x}_{op} - \mathbf{x}_i, t_{\text{ins}}) P^\dagger(\mathbf{0}, 0)\rangle \,.$$

where $t_{\text{sep}} = t_f - t_i$ and $t_{\text{ins}} = t_{op} - t_i$.

- Point sources and *sequential sink method* for three-point functions:
  - 7 or 8 values of source-sink separation t_{sep} with $1\,\text{fm} \lesssim t_{\text{sep}} \lesssim 3.5\,\text{fm}$.
  - Any value of the insertion time $t_{\text{ins}} \in [0, t_{\text{sep}}]$.
  - Two different sink momenta $\mathbf{p}_f \in \{(0,0,0),\ (1,0,0)\}$.
  - Any operator insertion including one-link displaced operators.
- Re-use point-to-all "forward" propagators for two- and three-point functions.
- Truncated solver method gives speedup of 2-5. *Phys.Rev. D91 (2015) 11, 114511*
- For 2+1 disconnected diagrams: ~ 500 (!) additional two-point functions per config.

Quark-connected two- and three-point functions

Setup for the computation of quark-connected two-point and three-point function

$$C_2(\mathbf{p}, \mathbf{x}_i, t_f - t_i) = \sum_{\mathbf{x}_f} e^{i\mathbf{p}\cdot(\mathbf{x}_f - \mathbf{x}_i)} \langle P(\mathbf{x}_f - \mathbf{x}_i, t_f - t_i) P^\dagger(\mathbf{0}, 0)\rangle \,,$$

$$C_3(\mathbf{p}_f, \mathbf{q}, \mathbf{x}_i, t_{\mathrm{sep}}, t_{\mathrm{ins}}) = \sum_{\mathbf{x}_f, \mathbf{x}_{op}} e^{i\mathbf{p}_f\cdot(\mathbf{x}_f - \mathbf{x}_i)} e^{i\mathbf{q}\cdot(\mathbf{x}_{op} - \mathbf{x}_i)} \langle P(\mathbf{x}_f - \mathbf{x}_i, t_{\mathrm{sep}}) \mathcal{S}(\mathbf{x}_{op} - \mathbf{x}_i, t_{\mathrm{ins}}) P^\dagger(\mathbf{0}, 0)\rangle \,.$$

where $t_{\mathrm{sep}} = t_f - t_i$ and $t_{\mathrm{ins}} = t_{op} - t_i$.

Source setup depends on boundary conditions in time:

- **On periodic BC boxes:**
  - Sources randomly distributed on every configuration.
  - 8 source positions x_i for $C_3^{\mathrm{conn}}(\mathbf{p}_f, \mathbf{q}, \mathbf{x}_i, t_{\mathrm{sep}}, t_{\mathrm{ins}})$ and 512 for $C_2(\mathbf{p}, \mathbf{x}_i, t_f - t_i)$.
- **On open BC boxes:**
  - Sources symmetric around $T/2$ $\Rightarrow$ 4+4 sources for $C_3^{\mathrm{conn}}(\mathbf{p}_f, \mathbf{q}, \mathbf{x}_i, t_{\mathrm{sep}}, t_{\mathrm{ins}})$ at each t_{sep}.
  - Additional sources for $C_2(\mathbf{p}, \mathbf{x}_i, t_f - t_i)$ on same t_i + many additional values of t_i.
  - Same number of sources on every t_i (i.e. with random $\mathbf{x}_i$).

Quark-disconnected contribution (I)

Evaluation of quark-disconnected loops for some operator $\mathcal{O}(\mathbf{x}, t)$

$$L \equiv C_1^{\mathcal{O}}(\mathbf{p}, t) = \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} \langle \mathcal{O}_f(\mathbf{x}, t) \rangle_F ,$$

requires **all-to-all propagators** $\rightarrow$ Prohibitively expensive.

Naive stochstastic estimators, e.g.

$$C_1^{\mathcal{O}}(\mathbf{p}, t) = - \lim_{N_s \to \infty} \frac{1}{N_s} \sum_{s=1}^{N_s} \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} \mathrm{tr}\left[\xi_s^\dagger \Gamma_{\mathcal{O}} (D_f(x,x))^{-1} \xi_s \right] , \quad \text{where} \quad \xi_i^\dagger \xi_j = \delta_{ij} + \text{noise} .$$

still way too expensive (besides for $\Gamma = \gamma_5$) as $\sigma^2 \sim \frac{1}{\sqrt{N_s}}$. $\rightarrow$ Need further improvements.

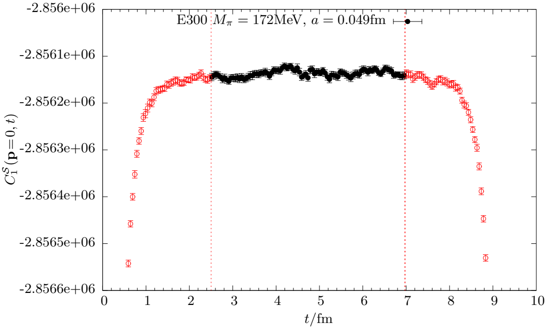
- Use one-end trick

  $$\mathrm{tr}[\Gamma(D_1^{-1} - D_2^{-1})] = (m_2 - m_1)\mathrm{tr}[\Gamma D_1^{-1} D_2^{-1}]$$ *EPJ C58, 261 (2008)* *EPJ C79, 586 (2019)*

  to compute $L_1 - L_2$, ... , $L_{n-1} - L_n$ for quark masses $m_1 < m_2 < \ldots < m_n$ with $\sigma^2 \sim \frac{1}{N_s}$
- Our setup: four quark masses, i.e. $m_l < m_s < m_{\text{extra}} < m_c$.
- $N_s = 512$ for light quark; doubled for each flavor, i.e. $N_s = 4096$ for charm.
- Hopping parameter expansion + hierarchical probing for charm loop. *PRD 89, 094503 (2014)* *SIAM J.Sci.Comput. 35 (2013) 5*

$\rightarrow$ **Reach gauge noise for any local and one-link displaced operator.**

Quark-disconnected contribution (II)



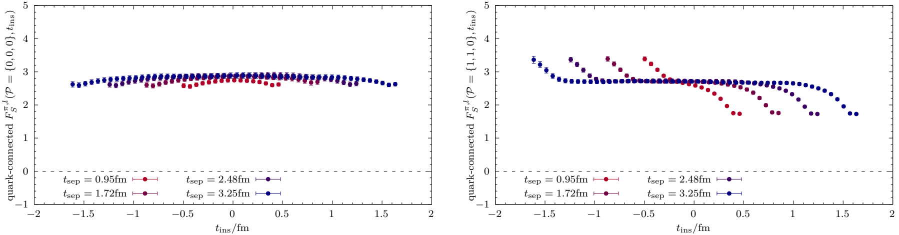
- Statistical precision for $2+1$ diagrams limited only by statistics for $C_2(\mathbf{p}^2, x_i, t)$.
- For periodic BC: forward + backward averaging for any source position $x_i = (t_i, \mathbf{x}_i)$.
- For open BC: **Must stay away from boundary!**

  $\rightarrow$ Exclude timeslices on both sides: $t_{\mathrm{ex}} \lesssim 2.5\,\mathrm{fm}$

  $\rightarrow$ Forward + backward averaging only for $t_i + t_{\mathrm{sep}} < T - t_{\mathrm{ex}}$ and $t_i - t_{\mathrm{sep}} > t_{\mathrm{ex}}$.

  $\Rightarrow$ **Statistics can be severely reduced depending on T, t_{sep} and t_{ex}.**

Ratio method and effective form factor



Light quark-connected effective form factors on E250 ($M_\pi = 132\,\mathrm{MeV}$, $a = 0.064\,\mathrm{fm}$); momentum labels: $\mathcal{P} \equiv (\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2)$

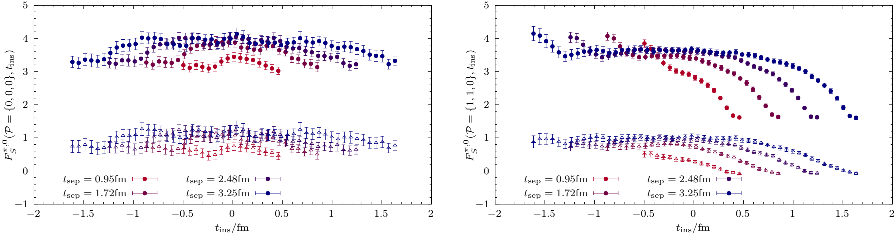
Effective form factors from gauge averages $\langle . \rangle$ via **ratio method**:

$$R(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2, t_{\rm sep}, t_{\rm ins}) = \frac{\langle C_3(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2, t_{\rm sep}, t_{\rm ins})\rangle}{\langle C_2(\mathbf{p}_f^2, t_{\rm sep})\rangle} \sqrt{\frac{\langle C_2(\mathbf{p}_i^2, t_{\rm sep}-t_{\rm ins})\rangle\langle C_2(\mathbf{p}_f^2, t_{\rm ins})\rangle\langle C_2(\mathbf{p}_f^2, t_{\rm sep})\rangle}{\langle C_2(\mathbf{p}_f^2, t_{\rm sep}-t_{\rm ins})\rangle\langle C_2(\mathbf{p}_i^2, t_{\rm ins})\rangle\langle C_2(\mathbf{p}_i^2, t_{\rm sep})\rangle}}.$$

$\Rightarrow$ ground state ME $\langle \pi(\mathbf{p}_f)|S^f(\mathbf{q}^2)|\pi(\mathbf{p}_i)\rangle \sim F_S^{\pi,f}(Q^2)$ for $t_{\rm ins} \to \infty$ and $t_{\rm sep} - t_{\rm ins} \to \infty$.

- Quark-connected data very precise at $Q^2 = 0$ and $Q^2 > 0$.
- Evaluate $R^{\rm conn}(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2, t_{\rm sep}, t_{\rm ins})$ using C_2 and C_3 computed on same set of sources $\{x_i\}$.
- Residual excited state contamination very small; plateau at $t_{\rm sep} \gtrsim 1.5\,\mathrm{fm}$.

Ratio method and effective form factor



Singlet effective form factors on E250 ($M_\pi = 132\,\mathrm{MeV}$, $a = 0.064\,\mathrm{fm}$); momentum labels: $\mathcal{P} \equiv (\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2)$

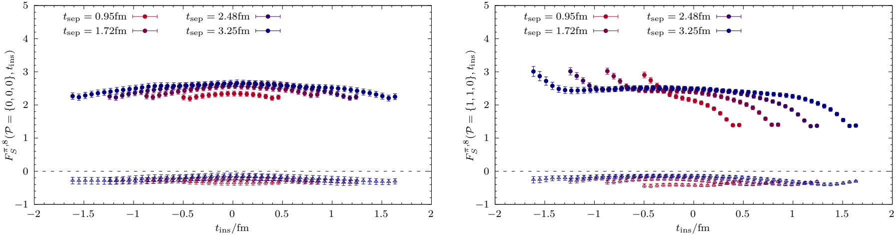
Effective form factors from gauge averages $\langle . \rangle$ via **ratio method**:

$$R(\mathbf{p}_f^2,\mathbf{q}^2,\mathbf{p}_i^2,t_{\mathrm{sep}},t_{\mathrm{ins}}) = \frac{\langle C_3(\mathbf{p}_f^2,\mathbf{q}^2,\mathbf{p}_i^2,t_{\mathrm{sep}},t_{\mathrm{ins}})\rangle}{\langle C_2(\mathbf{p}_f^2,t_{\mathrm{sep}})\rangle}\sqrt{\frac{\langle C_2(\mathbf{p}_i^2,t_{\mathrm{sep}}-t_{\mathrm{ins}})\rangle\langle C_2(\mathbf{p}_f^2,t_{\mathrm{ins}})\rangle\langle C_2(\mathbf{p}_f^2,t_{\mathrm{sep}})\rangle}{\langle C_2(\mathbf{p}_f^2,t_{\mathrm{sep}}-t_{\mathrm{ins}})\rangle\langle C_2(\mathbf{p}_i^2,t_{\mathrm{ins}})\rangle\langle C_2(\mathbf{p}_i^2,t_{\mathrm{sep}})\rangle}}.$$

$\Rightarrow$ ground state ME $\langle \pi(\mathbf{p}_f)|S^f(\mathbf{q}^2)|\pi(\mathbf{p}_i)\rangle \sim F_S^{\pi,f}(Q^2)$ for $t_{\mathrm{ins}} \to \infty$ and $t_{\mathrm{sep}} - t_{\mathrm{ins}} \to \infty$.

- Singlet quark-disconnected contribution very significant $\rightarrow$ up to $\lesssim 100\%$ correction for $a = 0.086\,\mathrm{fm}$.
- Unprecedented data quality with signal for many different values of $Q^2 > 0$.
- Excited state contamination enhanced compared to $R^{\mathrm{conn}}(\mathbf{p}_f^2,\mathbf{q}^2,\mathbf{p}_i^2,t_{\mathrm{sep}},t_{\mathrm{ins}})$.

Ratio method and effective form factor



Octet effective form factors on E250 ($M_\pi = 132\,\mathrm{MeV}$, $a = 0.064\,\mathrm{fm}$); momentum labels: $\mathcal{P} \equiv (\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2)$

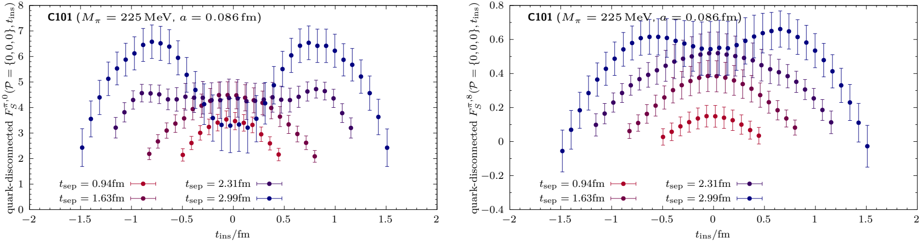
Effective form factors from gauge averages $\langle . \rangle$ via **ratio method**:

$$R(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2, t_{\rm sep}, t_{\rm ins}) = \frac{\langle C_3(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2, t_{\rm sep}, t_{\rm ins})\rangle}{\langle C_2(\mathbf{p}_f^2, t_{\rm sep})\rangle} \sqrt{\frac{\langle C_2(\mathbf{p}_i^2, t_{\rm sep} - t_{\rm ins})\rangle \langle C_2(\mathbf{p}_f^2, t_{\rm ins})\rangle \langle C_2(\mathbf{p}_f^2, t_{\rm sep})\rangle}{\langle C_2(\mathbf{p}_f^2, t_{\rm sep} - t_{\rm ins})\rangle \langle C_2(\mathbf{p}_i^2, t_{\rm ins})\rangle \langle C_2(\mathbf{p}_i^2, t_{\rm sep})\rangle}}.$$

$\Rightarrow$ ground state ME $\langle \pi(\mathbf{p}_f)|S^f(\mathbf{q}^2)|\pi(\mathbf{p}_i)\rangle \sim F_S^{\pi,f}(Q^2)$ for $t_{\rm ins} \to \infty$ and $t_{\rm sep} - t_{\rm ins} \to \infty$.

- Octet quark-disconnected contribution much smaller and opposite sign.
- Statistically very precise; expect $\Delta\langle r_S^2\rangle_\pi^8 < \Delta\langle r_S^2\rangle_\pi^l < \Delta\langle r_S^2\rangle_\pi^0$. (also: $\langle r_S^2\rangle_\pi^8 < \langle r_S^2\rangle_\pi^l < \langle r_S^2\rangle_\pi^0$)
- Still: Error remains entirely dominated by quark-disconnected piece.

VEV subtraction with open boundaries



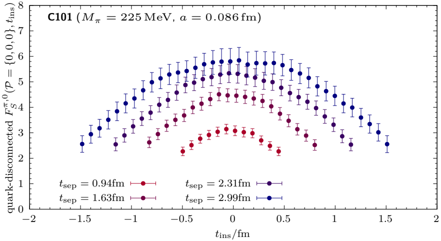
At $Q^2 = 0$: Quark-disconnected signal distorted by significant fluctuations on **open boundary ensembles**:

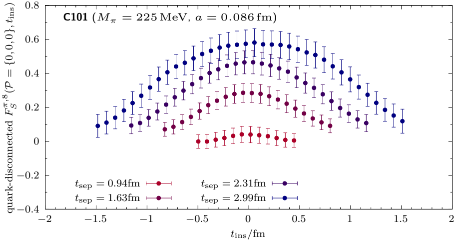
- Correlated fluctuations already visible in the gauge average of one-point functions.
- This "feature" becomes more problematic at larger values of t_{sep}.
- Two-point function source-positions t_i become restricted to a small strip.
- e.g. for C101 with $T = 8.2\,\mathrm{fm}$, $t_{\mathrm{sep}}^{\max} = 3.0\,\mathrm{fm}$, $t_{\mathrm{ex}} = 2.25\,\mathrm{fm}$:

$T - t_{\mathrm{sep}}^{\max} - 2t_{\mathrm{ex}} \approx 0.7\,\mathrm{fm}$ $\Rightarrow$ only $< 10\%$ of the time extent left.

$\Rightarrow$ **Usual method of (global) VEV-subtraction fails.**

VEV subtraction with open boundaries





Solution:

- Subtract VEV **per-timeslice** instead of subtracting bulk-average:

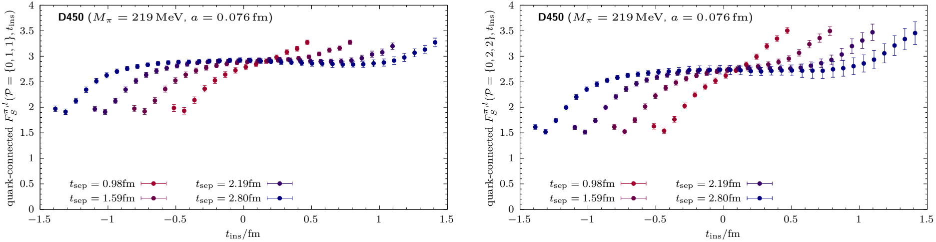
$$\left\langle C_3(t_{\mathrm{sep}}, t_{\mathrm{ins}}, t_i)\right\rangle_{\mathrm{disc}} = \left\langle C_2(t_f - t_i) C_1(t_{op} - t_i)\right\rangle - \left\langle C_2(t_f - t_i)\right\rangle \cdot \left\langle C_1(t_{op} - t_i)\right\rangle .$$

- Compute “fully correlated” ratio:

$$R_{\mathrm{disc}}(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2, t_{\mathrm{sep}}, t_{\mathrm{ins}}) = \frac{1}{N_{t_i}} \sum_{t_i} \frac{\left\langle C_3(t_{\mathrm{sep}}, t_{\mathrm{ins}}, t_i)\right\rangle_{\mathrm{disc}}}{\left\langle C_2(t_{\mathrm{sep}}, t_i)\right\rangle}$$

$\Rightarrow$ **Fluctuations cancel; statistical error significantly reduced.**

Signal-to-noise problem at $Q^2 > 0$ (I)



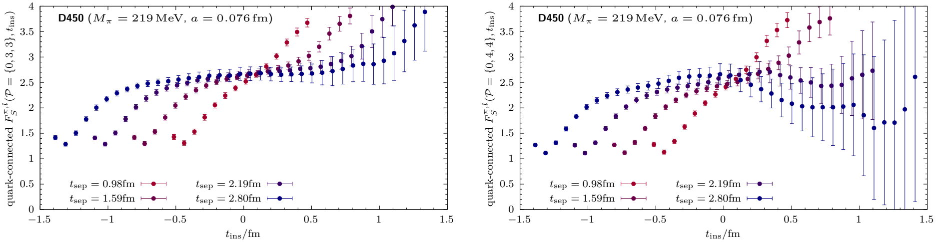
- At $Q^2 = 0$ constant signal-to-noise behavior for eff FF in t_{sep}, t_{ins} (also for $C_2(t)$):

  $\rightarrow$ For pion FF: plateau method (in principle) possible for $t_{\mathrm{sep}} \gtrsim 2.5\,\mathrm{fm}$ **at small** Q^2.

  $\rightarrow$ Different for nucleon calculations where famously $\frac{C_2(\mathbf{p}^2=0,t)}{\Delta C_2(\mathbf{p}^2=0,t)} \sim e^{-m_N - \frac{3}{2}M_\pi}$ *Phys. Rept. 103, 203 (1984)*

  and signal for $R(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2, t_{\mathrm{sep}}, t_{\mathrm{ins}})$ is typically lost for $t_{\mathrm{sep}} \gtrsim 1.5\,\mathrm{fm}$!

Signal-to-noise problem at $Q^2 > 0$ (I)



- At $Q^2 = 0$ constant signal-to-noise behavior for eff FF in t_{sep}, t_{ins} (also for $C_2(t)$):

  → For pion FF: plateau method (in principle) possible for $t_{\mathrm{sep}} \gtrsim 2.5\,\mathrm{fm}$ **at small** Q^2.

  → Different for nucleon calculations where famously $\frac{C_2(\mathbf{p}^2=0,t)}{\Delta C_2(\mathbf{p}^2=0,t)} \sim e^{-m_N - \frac{3}{2}M_\pi}$ *Phys. Rept. 103, 203 (1984)*

  and signal for $R(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2, t_{\mathrm{sep}}, t_{\mathrm{ins}})$ is typically lost for $t_{\mathrm{sep}} \gtrsim 1.5\,\mathrm{fm}$!

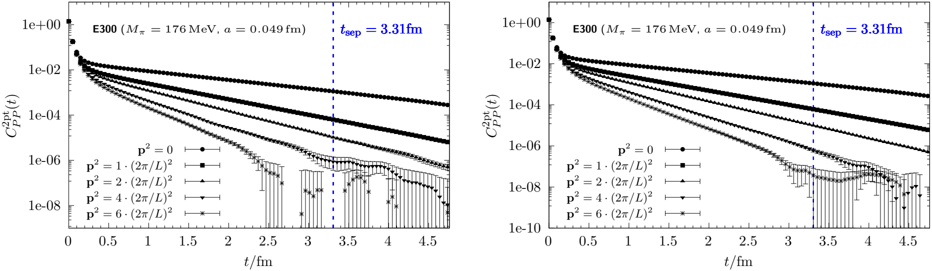
- Injecting momentum $\mathbf{p}_i$ at source: exponential signal-to-noise problem also for pion:

  → Direct extraction of ground state ME problematic for $\mathbf{p}_i^2 \gtrsim 3 \cdot (2\pi/L)^2$.

  → Adding $R^{\mathrm{disc}}(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2, t_{\mathrm{sep}}, t_{\mathrm{ins}})$: even more challenging.

⇒ **Additional effort required to improve signal and convergence ...**

Signal-to-noise problem at $Q^2 > 0$ (II)

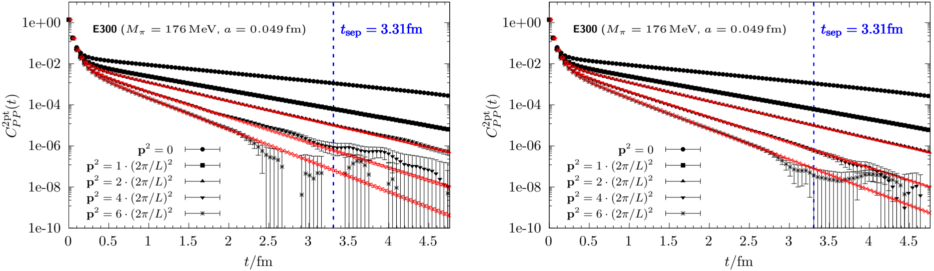


2pt statistics for $R^{\mathrm{conn}}(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2, t_{\mathrm{sep}}, t_{\mathrm{ins}})$ 2pt statistics for $R^{\mathrm{disc}}(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2, t_{\mathrm{sep}}, t_{\mathrm{ins}})$

At non-zero momentum transfer $Q^2 > 0$:

- Two-point and three-point functions develop signal-to-noise problem as $\mathbf{p}_i^2$ ($\mathbf{p}_f^2$) increases.
- Fluctuations for $C_2(\mathbf{p}^2, t)$ more severe $\rightarrow$ square-root in $R(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2, t_{\mathrm{sep}}, t_{\mathrm{ins}})$.
- For $t_{\mathrm{sep}} \gtrsim 2\,\mathrm{fm}$ the signal would be lost at fairly small p_i^2...

Signal-to-noise problem at $Q^2 > 0$ (II)



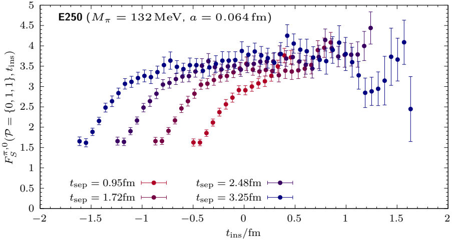
2pt statistics for $R^{\mathrm{conn}}(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2, t_{\mathrm{sep}}, t_{\mathrm{ins}})$ 2pt statistics for $R^{\mathrm{disc}}(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2, t_{\mathrm{sep}}, t_{\mathrm{ins}})$

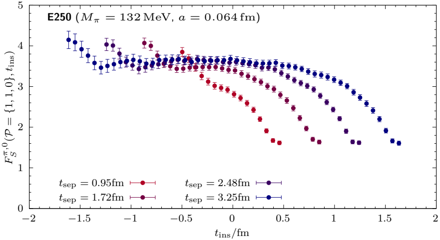
At non-zero momentum transfer $Q^2 > 0$:

- Two-point and three-point functions develop signal-to-noise problem as $\mathbf{p}_i^2$ ($\mathbf{p}_f^2$) increases.
- Fluctuations for $C_2(\mathbf{p}^2, t)$ more severe $\rightarrow$ square-root in $R(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2, t_{\mathrm{sep}}, t_{\mathrm{ins}})$.
- For $t_{\mathrm{sep}} \gtrsim 2\,\mathrm{fm}$ the signal would be lost at fairly small p_i^2...

Solution: Replace two-point functions in ratio by fitted data for $p^2 \geq 2 \cdot (2\pi/L)^2$

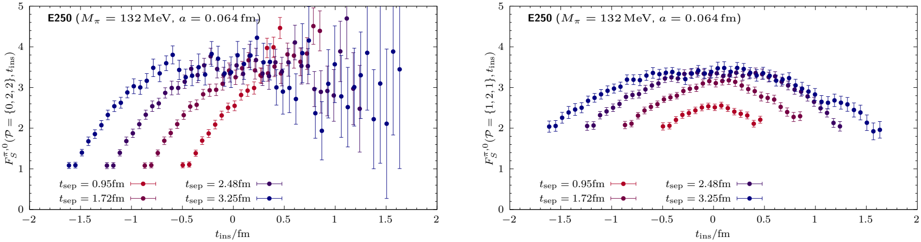
Momentum injection at source and sink





- Can inject momentum at source and / or sink.
- $(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2) = (0,1,1)$ and $(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2) = (1,1,0)$ contribute to same Q^2.
- But statistical precision very different due to use of **point-to-all** propagators.
- Injecting momenta at sink: much better signal, but additional inversions for every value of $\mathbf{p}_f$ contributing in $\langle C_3(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2, t_{\mathrm{sep}}, t_{\mathrm{ins}})\rangle_{\mathrm{conn}}$.
- Even a single, additional value of $\mathbf{p}_f = (1,0,0)$ greatly improves momentum resolution.

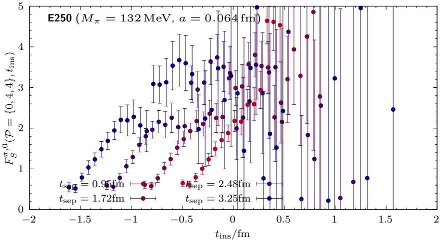
Momentum injection at source and sink

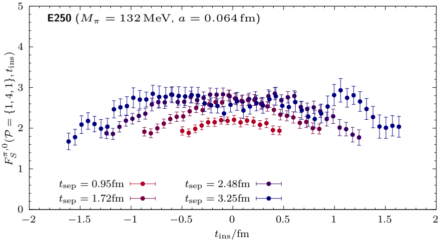


- Can inject momentum at source and / or sink.
- $(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2) = (0,1,1)$ and $(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2) = (1,1,0)$ contribute to same Q^2.
- But statistical precision very different due to use of **point-to-all** propagators.
- Injecting momenta at sink: much better signal, but additional inversions for every value of $\mathbf{p}_f$ contributing in $\langle C_3(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2, t_{\rm sep}, t_{\rm ins})\rangle_{\rm conn}$.
- Even a single, additional value of $\mathbf{p}_f = (1,0,0)$ greatly improves momentum resolution.

Last but not least: Larger $\mathbf{q}^2$ accessible at good stat. precision, e.g. $(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2) = (0,2,2)$ vs. $(1,2,1)$

Momentum injection at source and sink

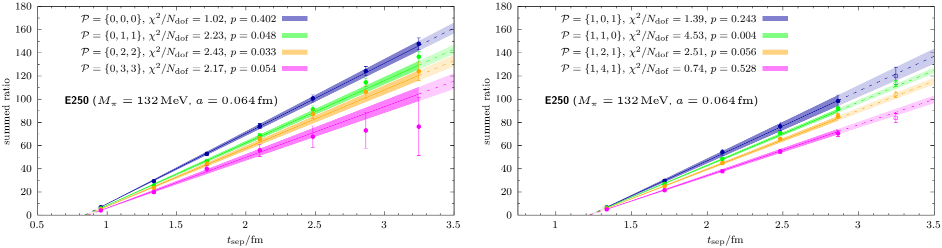




- Can inject momentum at source and / or sink.
- $(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2) = (0,1,1)$ and $(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2) = (1,1,0)$ contribute to same Q^2.
- But statistical precision very different due to use of **point-to-all** propagators.
- Injecting momenta at sink: much better signal, but additional inversions for every value of $\mathbf{p}_f$ contributing in $\langle C_3(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2, t_{\mathrm{sep}}, t_{\mathrm{ins}})\rangle_{\mathrm{conn}}$.
- Even a single, additional value of $\mathbf{p}_f = (1,0,0)$ greatly improves momentum resolution.

Last but not least: Larger $\mathbf{q}^2$ accessible at good stat. precision, e.g. $(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2) = (0,4,4)$ vs. $(1,4,1)$

Summation method

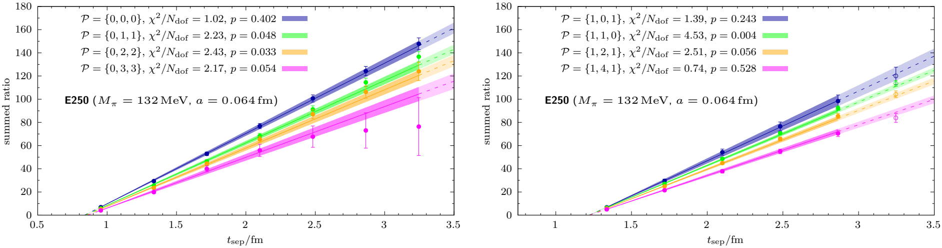


Summation method to extract ground state matrix elements:

$$\sum_{t_{\rm ins}=t_{\rm min}}^{t_{\rm sep}-t_{\rm min}} R(\mathbf{p}_f^2,\mathbf{q}^2,\mathbf{p}_i^2,t_{\rm sep},t_{\rm ins}) = \mathrm{const} + \langle\pi(\mathbf{p}_f)|S(\mathbf{q}^2)|\pi(\mathbf{p}_i)\rangle(t_{\rm sep}-t_0) + \mathcal{O}(e^{-\Delta t_{\rm sep}})\,.$$

- **Favorable excited state suppression $\sim e^{-\Delta t_{\rm sep}}$ vs. $\sim e^{-\Delta t_{\rm sep}/2}$ for plateau (midpoint) method.**
- Less affected by statistical fluctuation than plateau method.
- Linear fits in $t_{\rm sep}$ much more stable than fitting full time-dependence of $R(\mathbf{p}_f^2,\mathbf{q}^2,\mathbf{p}_i^2,t_{\rm sep},t_{\rm ins})$.

Summation method



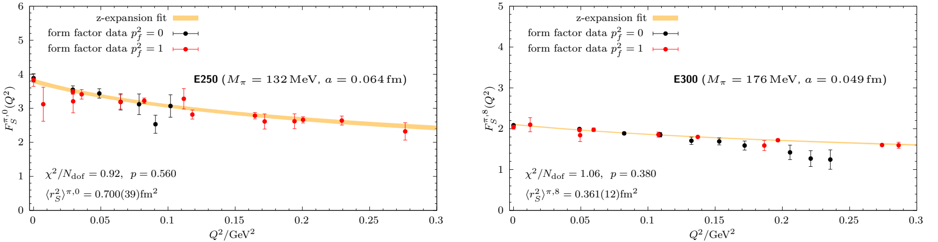
Summation method to extract ground state matrix elements:

$$\sum_{t_{\rm ins}=t_{\rm min}}^{t_{\rm sep}-t_{\rm min}} R(\mathbf{p}_f^2, \mathbf{q}^2, \mathbf{p}_i^2, t_{\rm sep}, t_{\rm ins}) = \mathrm{const} + \langle \pi(\mathbf{p}_f) | S(\mathbf{q}^2) | \pi(\mathbf{p}_i) \rangle (t_{\rm sep} - t_0) + \mathcal{O}(e^{-\Delta t_{\rm sep}}) \,.$$

- Choosing $t_{\rm min} \simeq t_{\rm sep}^{\rm min}/2$ improves signal quality for $Q^2 > 0$.
- Use three values of $t_{\rm sep}^{\rm min} \in [\sim 1.0, \ldots, \sim 1.5]\,\mathrm{fm}$.
- Combined with any $t_{\rm sep}^{\rm max} \in [\sim 2.25, \ldots, \sim 3.25]\,\mathrm{fm}$ s.t. $t_{\rm sep}^{\rm max} - t_{\rm sep}^{\rm min} \geq 1\,\mathrm{fm}$.

Carry out remaining analysis for all variations → final model averages

Form factor parametrization: z-expansion



Data for $Q^2 \leq 0.3\,\mathrm{GeV}^2$, $1.25 \lesssim t_{\mathrm{sep}} \lesssim 3.25\,\mathrm{fm}$.

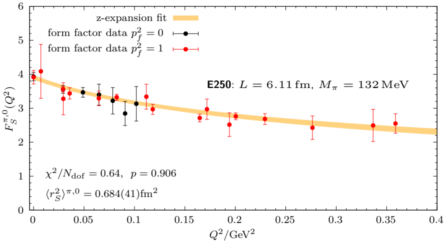
Use z-expansion to extract **radii** from unrenormalized form factor:

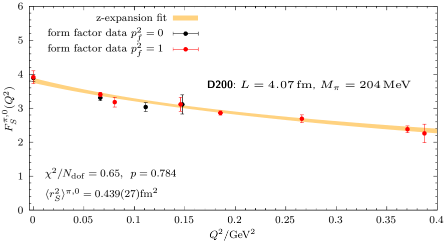
$$F_S^{\pi,f}(Q^2) = \sum_{n=0}^{N_z} a_n z^n\,, \qquad z = \frac{\sqrt{t_{\mathrm{cut}} + Q^2} - \sqrt{t_{\mathrm{cut}} - t_0}}{\sqrt{t_{\mathrm{cut}} + Q^2} + \sqrt{t_{\mathrm{cut}} - t_0}}\,, \qquad a_1 \sim \langle r_S^2\rangle_\pi^f = -\frac{6}{F_S^{\pi,f}(0)} \cdot \left.\frac{dF_S^{\pi,f}(Q^2)}{dQ^2}\right|_{Q^2=0}$$

We use $N_z = 1$, $t_{\mathrm{cut}} = 4M_\pi^2$ and $t_0 = t_0^{\mathrm{opt}} = t_{\mathrm{cut}}\left(1 - \sqrt{1 + Q^2_{\mathrm{max}}/t_{\mathrm{cut}}}\right)$. *PRD 92, 013013 (2015)*

- **Unprecedented Q^2-resolution at (near) physical quark mass.**
- $\mathbf{p}_f = (1,0,0)$ data greatly improves signal quality and Q^2-resolution.
- No renormalization needed for radii.

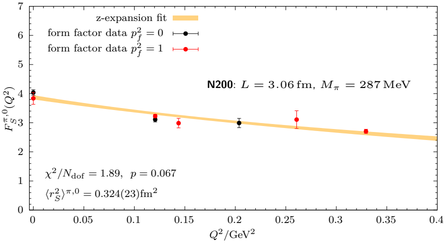
Momentum resolution and Q^2-cuts





- Momentum resolution differs drastically, depending on L.
- For our ensembles: $M_\pi L \gtrsim 4$:
  $\Rightarrow$ Physical L typically smaller at larger M_π.
- Test for systematics by applying data cuts, i.e. $Q^2_{\mathrm{cut}} \in \{0.20, 0.25, 0.30, 0.35, 0.40\}\,\mathrm{GeV}^2$.

Again, run remaining analysis for all variations.



Physical extrapolation

NLO χPT fit ansatz with quark mass proxies $m_l \sim M_\pi^2$ and $m_s \sim 2M_K^2 - M_\pi^2$,

$$\langle r_S^2\rangle_\pi^0 = \frac{1}{(4\pi f_0)^2}\left[768\pi^2\left(3L_4^r + L_5^r\right) - 19 + \frac{m_l}{m_l + 2m_s} - 12\log(m_l) - 6\log\left(\frac{m_l + m_s}{2}\right)\right] + c_0 a^2\,,$$

$$\langle r_S^2\rangle_\pi^l = \frac{1}{(4\pi f_0)^2}\left[768\pi^2\left(2L_4^r + L_5^r\right) - 16 + \frac{m_l}{3(m_l + 2m_s)} - 12\log(m_l) - 3\log\left(\frac{m_l + m_s}{2}\right)\right] + c_l a^2\,,$$

$$\langle r_S^2\rangle_\pi^8 = \frac{1}{(4\pi f_0)^2}\left[768\pi^2 L_5^r - 10 - \frac{m_l}{m_l + 2m_s} - 12\log(m_l) + 3\log\left(\frac{m_l + m_s}{2}\right)\right] + c_8 a^2\,.$$

- Correlated fits are carried out in units of t_0 with $N_b = 1000$ bootstrap samples.
- Scale setting: $\sqrt{t_0} = 0.14464(87)\,\mathrm{fm}$. *Eur. Phys. J. C 82 (2022) 10, 869 (FLAG Review 2021)*
- Radii are fitted individually to compute $\langle r_S^2\rangle_{\pi,\mathrm{phys}}^{0,l,8}$.
- Test systematics of physical extrapolation from (combinations of) data cuts:

  i.e. $M_\pi < \{230, 265, 290\}\,\mathrm{MeV}$, $a < 0.08\,\mathrm{fm}$ and $L > 3.5\,\mathrm{fm}$.
- Fits on every input data set from FF analysis ($\{t_{\mathrm{sep}}^{\min}, t_{\mathrm{sep}}^{\max}\}$-pairs, Q^2-cuts ...) $\to \mathcal{O}(10^7)$ fits in total.

Physical extrapolation

NLO χPT fit ansatz with quark mass proxies $m_l \sim M_\pi^2$ and $m_s \sim 2M_K^2 - M_\pi^2$,

$$\langle r_S^2\rangle_\pi^0 = \frac{1}{(4\pi f_0)^2}\left[768\pi^2\left(3L_4^r + L_5^r\right) - 19 + \frac{m_l}{m_l+2m_s} - 12\log(m_l) - 6\log\left(\frac{m_l+m_s}{2}\right)\right] + c_0 a^2\,,$$

$$\langle r_S^2\rangle_\pi^l = \frac{1}{(4\pi f_0)^2}\left[768\pi^2\left(2L_4^r + L_5^r\right) - 16 + \frac{m_l}{3(m_l+2m_s)} - 12\log(m_l) - 3\log\left(\frac{m_l+m_s}{2}\right)\right] + c_l a^2\,,$$

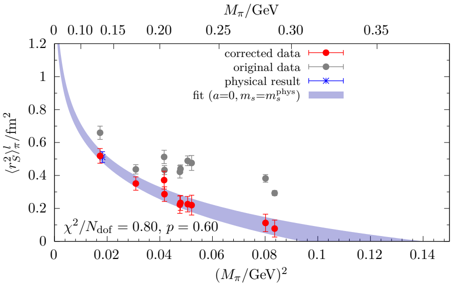
$$\langle r_S^2\rangle_\pi^8 = \frac{1}{(4\pi f_0)^2}\left[768\pi^2 L_5^r - 10 - \frac{m_l}{m_l+2m_s} - 12\log(m_l) + 3\log\left(\frac{m_l+m_s}{2}\right)\right] + c_8 a^2\,.$$

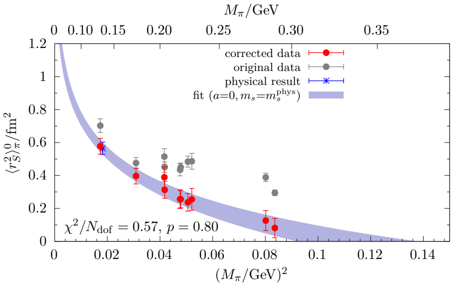
- SU(3) LECs f_0, L_4^r, L_5^r are obtained from fitting the following expressions:

$$\begin{aligned} f_0 &: \quad \langle r_S^2\rangle_\pi^0,\ \langle r_S^2\rangle_\pi^l,\ \langle r_S^2\rangle_\pi^8,\ \langle r_S^2\rangle_\pi^0 - \langle r_S^2\rangle_\pi^l,\ \langle r_S^2\rangle_\pi^0 - \langle r_S^2\rangle_\pi^8,\ \langle r_S^2\rangle_\pi^l - \langle r_S^2\rangle_\pi^8,\\ L_4^r &: \quad \langle r_S^2\rangle_\pi^0 - \langle r_S^2\rangle_\pi^l,\ \langle r_S^2\rangle_\pi^0 - \langle r_S^2\rangle_\pi^8,\ \langle r_S^2\rangle_\pi^l - \langle r_S^2\rangle_\pi^8,\\ L_5^r &: \quad \langle r_S^2\rangle_\pi^8,\ 3\langle r_S^2\rangle_\pi^l - 2\langle r_S^2\rangle_\pi^0 \end{aligned}$$

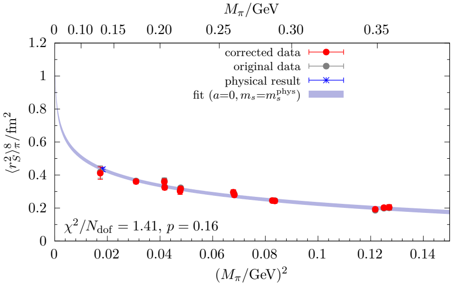
- Extracting L_4^r (and L_5^r) always requires fitting of f_0.
- **Determination of L_4^r more difficult and less precise than $\bar{l}_4$.**
- Using FLAG estimate for f_0 as prior with 50% width to stabilize a small number of fits.

Light chiral extrapolation

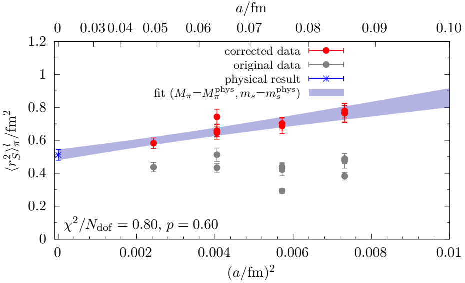


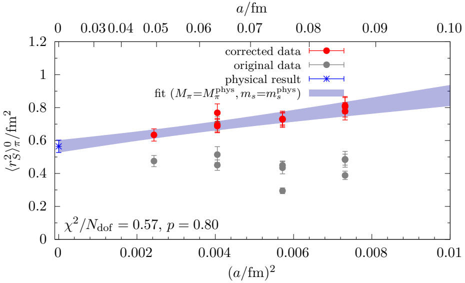


- Shown fits are "representative" of the final result.
- Difference $\langle r_S^2\rangle_\pi^0 - \langle r_S^2\rangle_\pi^l$: **pure strange effect**.
- Errors dominated by quark-disconnected piece.
- Steep slope towards $m_l \to 0$ due to chiral log.
- Physical point in m_l (isospin limit) defined by: $M_\pi^{\mathrm{phys}} = 134.8(3)\,\mathrm{MeV}$ *Eur. Phys. J. C 77 (2017) 2, 112*
- $\langle r_S^2\rangle_\pi^{l,0}$ **data receive large corrections from fit ...**

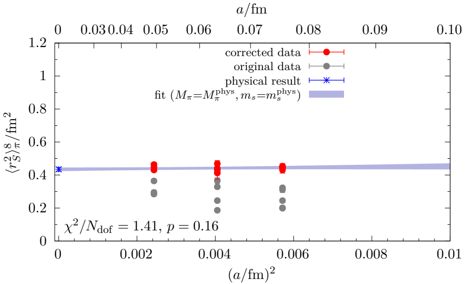


Continuum extrapolation

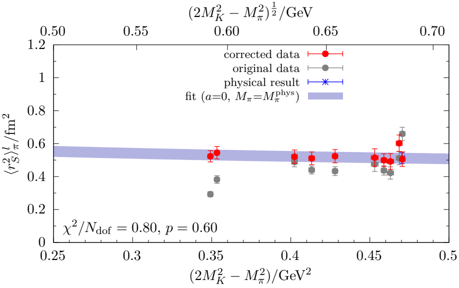


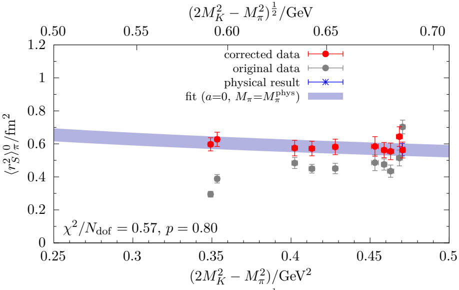


- **Large lattice artifacts in $\langle r_S^2\rangle_\pi^{l,0}$.**
  $\rightarrow$ Corrections up to $\lesssim 40\%$.
- Lattice artifacts mostly due to quark-disconnected contributions.
- Typically rather small corrections for $l-s$.
- Sensitivity to lattice artifacts depends on data cuts
- What about m_s effects?



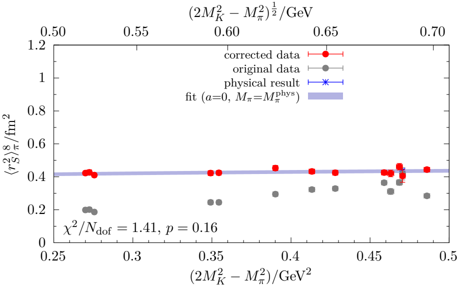
Strange chiral extrapolation



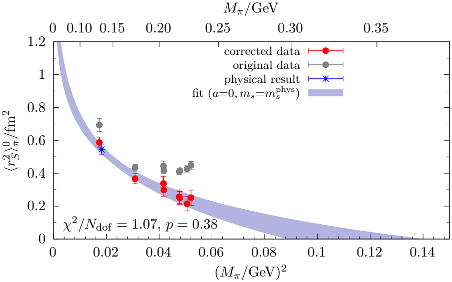


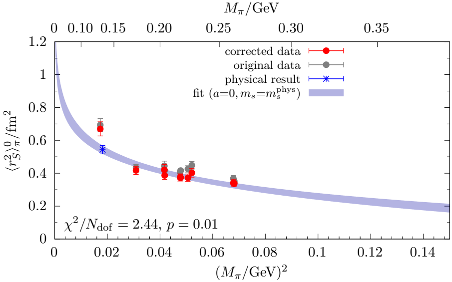
- Strange mass dependence very mild.
- Data cuts have little effect.
- At most percent level corrections.
- Ensembles on $m_s \approx$ phys trajectory cluster around physical point.
- Physical point in $m_s \sim (2M_K^2 - M_\pi^2)$ defined by:

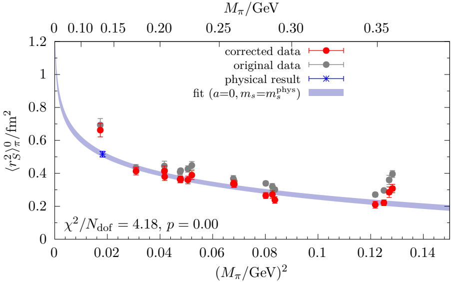
  $M_\pi^{\rm phys} = 134.8(3)\,\mathrm{MeV}$ and
  $M_K^{\rm phys} = 494.2(3)\,\mathrm{MeV}$. *Eur. Phys. J. C 77 (2017) 2, 112*



Data cuts: M_π

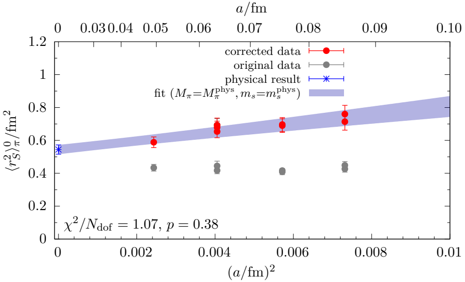


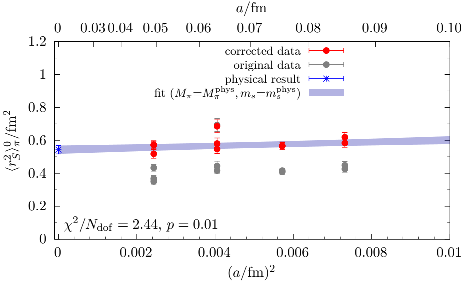




- Example data set for $\langle r_S^2\rangle_\pi^0$ with $1.5\,\mathrm{fm} \lesssim t_{\mathrm{sep}} \lesssim 3.25\,\mathrm{fm}$ and $Q^2 \leq 0.4\,\mathrm{fm}$.
- **Pion mass cuts greatly improve fit quality.**
- Physical values remain compatible.
- E250 falls on fit curve for $M_\pi^{\mathrm{cut}} = 230\,\mathrm{MeV}$.
- Cuts in a (and L) less important.

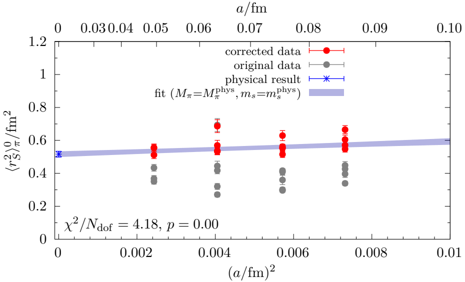
Data cuts: M_π





- Example data set for $\langle r_S^2 \rangle_\pi^0$ with $1.5\,\mathrm{fm} \lesssim t_{\mathrm{sep}} \lesssim 3.25\,\mathrm{fm}$ and $Q^2 \leq 0.4\,\mathrm{fm}$.
- **Pion mass cuts greatly improve fit quality.**
- Physical values remain compatible.
- E250 falls on fit curve for $M_\pi^{\mathrm{cut}} = 230\,\mathrm{MeV}$.
- However: Cuts in M_π affect sensitivity to lattice artifacts

  $\rightarrow$ **Possible higher order effects.**



Model averages

Assign a weight to each model (fit) PRD 103,114502 (2021)

$$w_i \sim \exp\left(-\frac{1}{2}\left[\chi^2 + 2(N_{\text{para}} - N_{\text{prio}}) - 2N_{\text{data}}\right]\right).$$

Central value and total errer for observable y are given by median and 16% and 84% percentiles of the CDF

$$CDF(y,\lambda) = \int_{-\infty}^{y} d\tilde{y} \sum_i w_i N(\tilde{y}, m_i, \sigma_i\sqrt{\lambda}).$$

Separate statistical (σ_{stat}) and systematic (σ_{sys}) errors from solving

$$\lambda\sigma_{\text{stat}}^2 + \sigma_{\text{sys}}^2 = \left(\frac{y_{\text{hi}} - y_{\text{lo}}}{2}\right)^2 \quad \text{where} \quad CDF(y_{\text{hi}},\lambda) = 0.84\,, \quad \text{and} \quad CDF(y_{\text{lo}},\lambda) = 0.16\,,$$

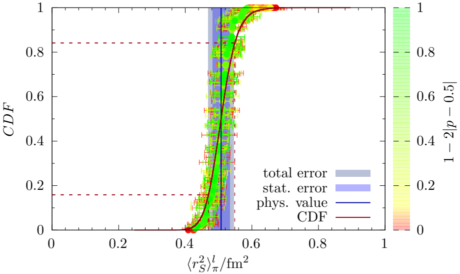
where $\lambda = 2.0$ rescales the statistical errors Nature 593 (2021) 7857, 51-55

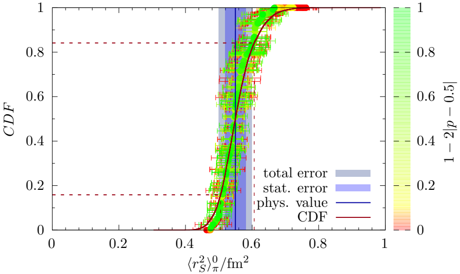
Final set of models: $\underbrace{\left\{(t_{\text{sep}}^{\text{min}}, t_{\text{sep}}^{\text{max}})\text{-pairs}\right\}}_{\text{summation method}} \bigotimes \underbrace{\left\{Q^2\text{-cuts}\right\}}_{\text{z-expansion}} \bigotimes \underbrace{\left\{M_\pi\text{-, } a\text{- and } L\text{-cuts}\right\}}_{\text{physical extrapolation}}$

$\Rightarrow$

observable	$\langle r_S^2\rangle_{\pi,\text{phys}}^{0,l,8}$	f_0	L_4^r	L_5^r
#models	910	5460	2730	1820

Results for radii

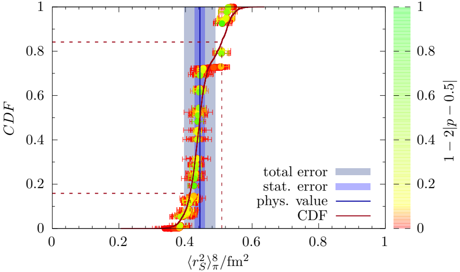




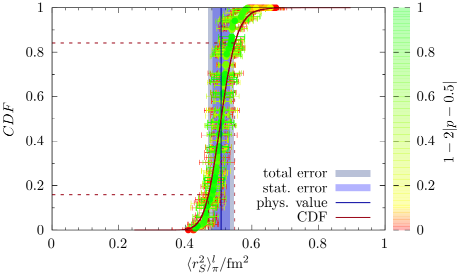
- **Final, physical results for radii:**

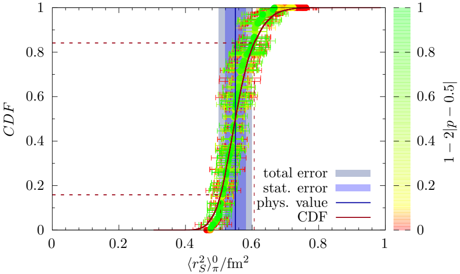
$$\langle r_S^2\rangle_\pi^0 = 0.550(32)_{\mathrm{stat}}(39)_{\mathrm{sys}}[50]_{\mathrm{total}}\ \mathrm{fm}^2\,,$$
$$\langle r_S^2\rangle_\pi^l = 0.508(27)_{\mathrm{stat}}(19)_{\mathrm{sys}}[39]_{\mathrm{total}}\ \mathrm{fm}^2\,,$$
$$\langle r_S^2\rangle_\pi^8 = 0.443(16)_{\mathrm{stat}}(45)_{\mathrm{sys}}[47]_{\mathrm{total}}\ \mathrm{fm}^2\,.$$

- **First lattice calculation with full error budget.**
- Expected hierarchy: $\langle r_S^2\rangle_\pi^8 < \langle r_S^2\rangle_\pi^l < \langle r_S^2\rangle_\pi^0$.
- Error for $\langle r_S^2\rangle_\pi^8$ dominated by systematics.
- Pheno value: $\langle r_S^2\rangle_\pi^l = 0.61(04)\,\mathrm{fm}^2$.



Results for radii



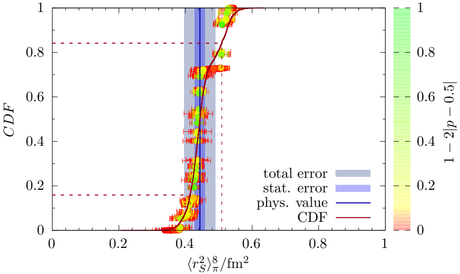


- **Final, physical results for radii:**

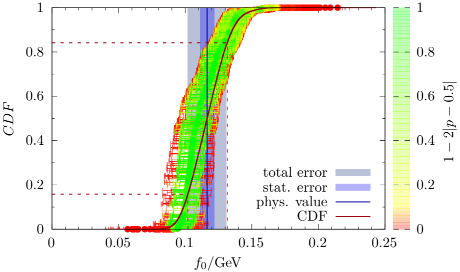
$$\langle r_S^2\rangle_\pi^0 = 0.550(32)_{\text{stat}}(39)_{\text{sys}}[50]_{\text{total}}\ \text{fm}^2\,,$$
$$\langle r_S^2\rangle_\pi^l = 0.508(27)_{\text{stat}}(19)_{\text{sys}}[39]_{\text{total}}\ \text{fm}^2\,,$$
$$\langle r_S^2\rangle_\pi^8 = 0.443(16)_{\text{stat}}(45)_{\text{sys}}[47]_{\text{total}}\ \text{fm}^2\,.$$

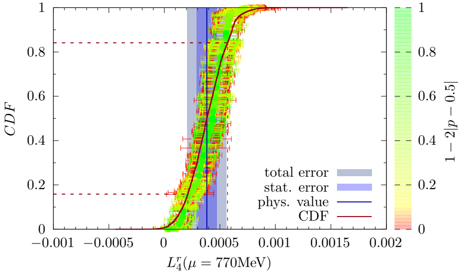
- Compatible with only other LQCD calculation by HPQCD: *PRD 93 (2016) 054503*

$$\langle r_S^2\rangle_\pi^0 = 0.506(38)_{\text{stat}}(53)_{\text{sys}}[65]_{\text{total}}\ \text{fm}^2\,,$$
$$\langle r_S^2\rangle_\pi^l = 0.481(37)_{\text{stat}}(50)_{\text{sys}}[62]_{\text{total}}\ \text{fm}^2\,,$$
$$\langle r_S^2\rangle_\pi^8 = 0.431(38)_{\text{stat}}(46)_{\text{sys}}[60]_{\text{total}}\ \text{fm}^2\,.$$



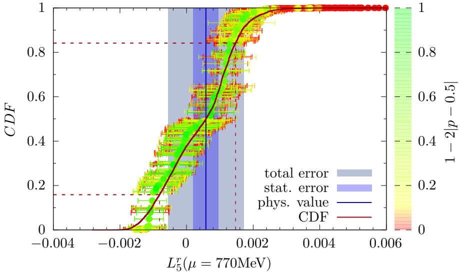
Results for SU(3) LECs





- **Final results for LECs @ $\mu = 770\,\text{MeV}$:**

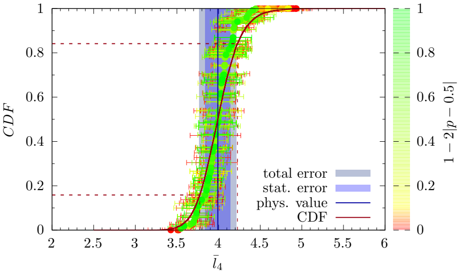
$$f_0 = 116.5(5.5)_{\text{stat}}(13.7)_{\text{sys}}[14.8]_{\text{total}}\ \text{MeV}$$
$$L_4^r(\mu) = +0.38(09)_{\text{stat}}(15)_{\text{sys}}[18]_{\text{total}} \times 10^{-3}$$
$$L_5^r(\mu) = +0.58(38)_{\text{stat}}(1.08)_{\text{sys}}[1.14]_{\text{total}} \times 10^{-3}$$

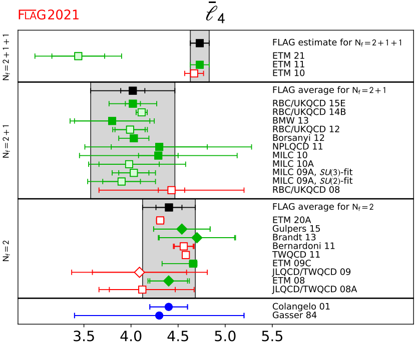


- FLAG 2021 estimates ($N_f = 2+1$):

$$f_0 = 114.0(8.5)\ \text{MeV}$$
$$L_4^r(\mu) = -0.02(56) \times 10^{-3}$$
$$L_5^r(\mu) = +0.95(41) \times 10^{-3}$$

Result for $\bar{l}_4$





- **Final result for $\bar{l}_4$ directly from physical $\langle r_S^2 \rangle_\pi^l$, i.e.** $\bar{l}_4 = \frac{1}{12}\left(13 + (4\pi f_\pi)^2 \langle r_S^2 \rangle_\pi^l\right)$

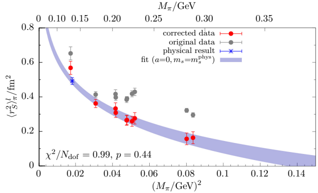
$$\Rightarrow \quad \bar{l}_4 = 3.99(15)_{\mathrm{stat}}(17)_{\mathrm{sys}}[23]_{\mathrm{total}}\,.$$

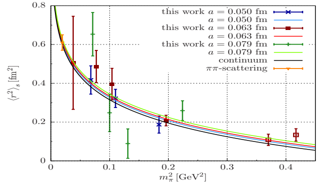
- Very competitive error, compatible with indirect determinations / FLAG average $\bar{l}_4 = 4.02(45)$.
- Pheno value $\bar{l}_4 = 4.39(22)$. *Nucl.Phys.B 603 (2001) 125-179*
- First $N_f = 2+1$ calculation based on scalar FF.
- Much better control over systematics than (older) $N_f = 2$ calculations.

Summary and outlook

Study of the pion scalar radii on 17 CLS $N_f = 2+1$ ensembles:

- Results for radii at the physical point and $SU(3)$ χPT LECs.
- **First calculation with controlled systematics and full error budget, i.e. excited states, Q^2-dependence and physical extrapolation.**
- Most precise existing determination of L_4^r.
- First direct $N_f = 2+1$ lattice calculation of $\bar{l}_4$ with very competitive errors. Good agreement with determinations from f_π and FLAG average.





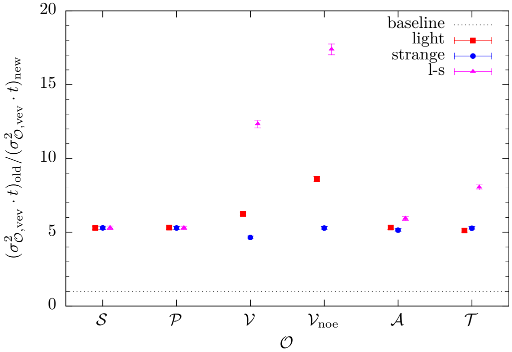
$N_f = 2$ results from Gülpers et al. (*EPJ A (2015) 12, 158*)

Future directions:

- Analyze further form factors, e.g. $F_V^{\pi,K}(Q^2)$.
  $\rightarrow$ data available for all 16 local and one-link displaced operator insertions
- New CLS F300 ensemble ($M_\pi^{\rm phys}$, $a = 0.05\,\text{fm}$, $T/a \times (L/a)^3 = 256 \times 128^3$) currently in production.
  $\rightarrow$ Very promising for form factor calculations.

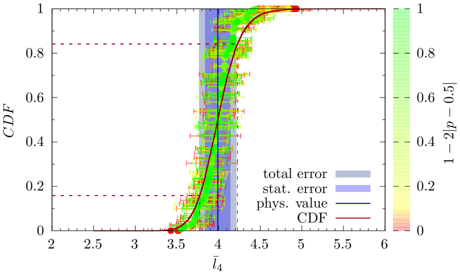
Backup slides

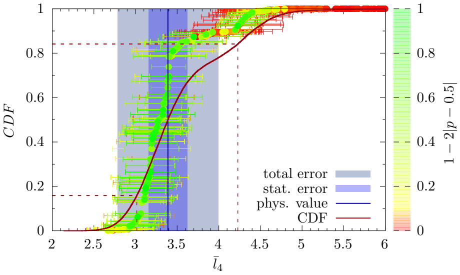
Quark-disconnected contribution (III)



- Several orders of magnitude (!) of speedup $\frac{(\sigma^2 \cdot t)_{\mathrm{naive}}}{(\sigma^2 \cdot t)_{\mathrm{new}}}$ compared to naive stochastic method.
- Also still (at least) factor ~ 5 to ~ 20 compared to plain hierarchical probing ("old" method).
- Quark-disconnected loops available on allmost all CLS ensembles.
- **Re-used in many other projects (e.g. HVP, nucleon structure ...).**
- Cost similar to gauge field generation; several 100MCh in total.

$\bar{l}_4$ – Matching with SU(3)





- Alternative determination via matching condition $\bar{l}_4$: NPB 250 (1985) 517-538

$$\bar{l}_4 = 16\pi^2\left[8L_4^r(\mu^2) + 4L_5^r(\mu^2)\right] - \frac{1}{4}\left[1 + \ln\frac{M_K^2 - \frac{1}{2}M_\pi^2}{\mu^2}\right] - \ln\frac{M_{\pi,\mathrm{phys}}^2}{\mu^2}$$

$\Rightarrow \bar{l}_4 = 3.39(23)_{\mathrm{stat}}(56)_{\mathrm{sys}}[61]_{\mathrm{total}}$

- Final (systematic) error inflated due to L_5^r-dependence.

- Still compatible with $\bar{l}_4 = 3.99(15)_{\mathrm{stat}}(17)_{\mathrm{sys}}[23]_{\mathrm{total}}$.