Cluster virial expansion and generalized Beth-Uhlenbeck formula from a Φ-derivable approach

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based on: arXiv:2512.03876
(with Gordon Baym and Gerd Röpke)

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Cluster virial expansion & generalized Beth-Uhlenbeck

Aspects of the quantum statistical description of many-particle systems with bound and scattering states

low density limit	*high density modification (medium effects)*
(1) elementary particles	
Ideal Fermi gas:	*Quasiparticle quantum liquid:*
neutrons, protons	mean-field approximation
(electrons, neutrinos,. . .)	Skyrme, Gogny, RMF
(2) bound state formation	
Nuclear statistical equilibrium:	*Chemical equilibrium of quasiparticle clusters:*
ideal mixture of all bound states	medium modified bound state energies
chemical equilibrium, mass action law	self-energy and Pauli blocking
(3) continuum contributions	
Second virial coefficient:	*Generalized Beth-Uhlenbeck formula:*
account of continuum correlations ($A = 2$)	medium modified binding energies,
scattering phase shifts, Beth-Uhlenbeck Eq.	medium modified scattering phase shifts
(4) chemical & physical picture	
Cluster virial approach:	*Correlated medium:*
all bound states (clusters)	phase space occupation by all bound states
scattering phase shifts of all pairs	in-medium correlations, quantum condensates

From N. Bastian et al. arXiv:1804.10178 [nucl-th]

Cluster virial expansion & generalized Beth-Uhlenbeck

Cluster virial expansion of the density
We have in the non-degenerate case $\left[\sum_P \rightarrow V/(2\pi)^3 \int d^3P\right]$

$$
\begin{aligned}
n_{A,Z}(T,\mu_p,\mu_n) &= \frac{1}{V}\sum_{\nu,P} f_{A,Z}[E_{A,\nu}(P)] \\
&= \sum_c e^{(N\mu_n+Z\mu_p)/T}\int\frac{d^3P}{(2\pi)^3}\sum_{\nu_c} g_{A,\nu_c}\, e^{-E_{A,\nu_c}(P)/T} \\
&= \sum_c \int \frac{d^3P}{(2\pi)^3} z_{A,c}(P)
\end{aligned}
$$

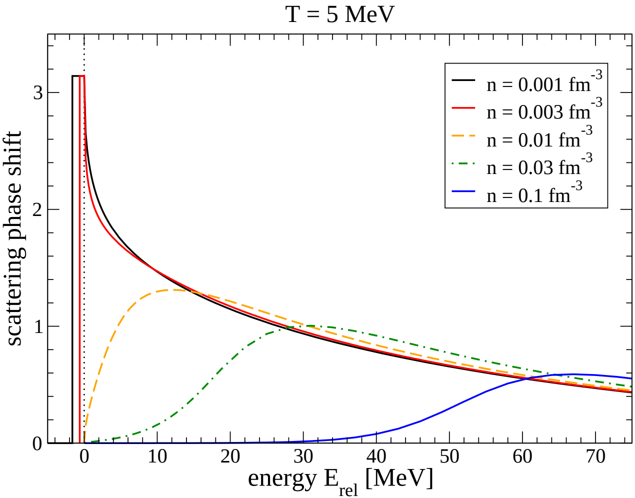
with $g_{A,c} = 2s_{A,c}+1$ the degeneration factor in the channel c. The partial density of the channel c at $\mathbf{P}$ contains the intrinsic partition function

$$
z_{A,c}(P;T,\mu_n,\mu_p) = \int_{-\infty}^{\infty}\frac{dE}{2\pi}\; e^{\left(-E-P^2/(2M_A)+N\mu_n+Z\mu_p\right)/T}\, 2\sin^2\delta_c(E)\,\frac{d\delta_c(E)}{dE}\ .
$$

Derivation for $A=2$ (deuterons) within thermodynamic Green's function approach, see M. Schmidt, G. Röpke, H. Schulz, Ann. Phys. (1990)

Cluster virial expansion & generalized Beth-Uhlenbeck

Example: deuteron channel in nuclear matter



Clustering in a model field theory for quark matter

Partition functionfor the basic NJL model

$$\mathcal{Z} = \int DqD\bar{q} \exp\left[\int \mathrm{d}^4x_E \ \mathcal{L}\right] , \quad \mathcal{L} = \bar{q}(\partial\!\!\!/ - m_0 + \mu\gamma_0)q + G_\mathrm{S}\left[(\bar{q}q)^2 + (\bar{q}\gamma_5\vec{\tau}q)^2\right] ,$$

After bosonization (Hubbard-Stratonovich trick),

$$Z = \int D\sigma\mathcal{D}\vec{\pi}\, \mathrm{e}^{-\int \mathrm{d}^4x_E \ \left\{\frac{\sigma^2+\vec{\pi}^2}{4G_\mathrm{S}}\right\} + \frac{1}{2}\ln\det\left(\beta S^{-1}\right)} .$$

Thermodynamic potential in mean field approximation

$$\begin{aligned}\Omega_\mathrm{MF} &= -\frac{T}{V}\ln Z \\ &= \frac{\sigma^2_\mathrm{MF}}{4G_\mathrm{S}} - 2N_f N_c \int \frac{\mathrm{d}^3p}{(2\pi)^3} \left[E_\mathbf{p} + T\ln\left(1+\mathrm{e}^{-(E_\mathbf{p}-\mu)/T}\right) + \text{antiparticle}\right] .\end{aligned}$$

From D. Blaschke et al. arXiv:1305.3907 [hep-ph]

Clustering in a model field theory for quark matter

Fluctuations in Gaussian approximation

$$\Omega_{\rm Gauß} = -\frac{T}{V}\ln Z_{\rm Gauß} = \Omega_{\rm MF} + \Omega^{(2)} = \Omega_{\rm cond} + \Omega_{\rm Q} + \sum_{\rm X}\Omega_{\rm X} \ ,$$

$$\Omega_{\rm X}(T,\mu) = \frac{d_X}{2}\frac{T}{V}\ln\left[\beta^2 S_{\rm X}^{-1}(z_n,\mathbf{q})\right] \ , \quad S_{\rm X}^{-1}(z_n,\mathbf{q}) = \frac{1}{G_{\rm X}} - \Pi_{\rm X}(z_n,\mathbf{q}) \ .$$

Polar representation $\longrightarrow$ Phase shift

$$S_{\rm X} = |S_{\rm X}|^{\delta_{\rm X}} \ , \quad \delta_{\rm X}(\omega,\mathbf{q}) = -\ln\left[\beta^2 S_{\rm X}^{-1}(\omega-\mu_X+\eta,\mathbf{q})\right]$$

$$\Omega_{\rm X}(T,\mu) = -d_{\rm X}\int\frac{{\rm d}^3q}{(2\pi)^3}\int_0^\infty\frac{{\rm d}\omega}{2\pi}\,[1+n_X^-(\omega)+n_X^+(\omega)]\delta_{\rm X}(\omega,\mathbf{q})$$

Beth-Uhlenbeck formula $\longrightarrow$ where is the factor $\sin^2\delta_X$??

$$p_X(T,\mu) = -d_{\rm X}T\int\frac{{\rm d}^3q}{(2\pi)^3}\int_0^\infty\frac{{\rm d}\omega}{2\pi}\left\{\ln\left(1-{\rm e}^{-(\omega-\mu_{\rm X})/T}\right)+{\rm antip.}\right\}\frac{d\delta_{\rm X}(\omega,\mathbf{q})}{d\omega}$$

From D. Blaschke et al. arXiv:1305.3907 [hep-ph]

Φ-derivable approach (conserving approximation scheme)

For QED, make ansatz [Vanderheyden & Baym, J. Stat. Phys. (1998)]

$$\beta\Omega = \Phi\left[G, D\right] - \mathrm{Tr}\Sigma G + \mathrm{Tr}\ln\left(-\gamma_0 G\right) + \frac{1}{2}\mathrm{Tr}\Pi D - \frac{1}{2}\mathrm{Tr}\ln\left(-D\right)$$

Stationarity of the thermodynamic potential,

$$\frac{\delta\Omega}{\delta G} = \frac{\delta\Omega}{\delta D} = 0 \ ,$$

is equivalent to fulfillment of Dyson equations for propagators of electrons G and photons D,

$$G^{-1} \quad = \quad G_0^{-1} - \Sigma \ , \quad D^{-1} = D_0^{-1} - \Pi \ ,$$

where G_0 and D_0 are the bare electron and photon propagators, and Σ and Π are calculated as functional derivatives,

$$\Sigma = \frac{\delta\Phi}{\delta G} \ , \quad \Pi = 2\frac{\delta\Phi}{\delta D} \ .$$

From Blaschke, Röpke, Baym, arXiv:2512.03876 [nucl-th]

Φ-derivable approach (conserving approximation scheme)

Using spectral representations, the frequency sums can be evaluated

$$\Omega = T\Phi[G,D] + \sum_{\mathbf{p}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f(\omega) \mathrm{Tr}_{\mathrm{Dirac}} \mathrm{Im}\left[\Sigma G + \ln\left(-\gamma_0 G^{-1}\right)\right]$$
$$+ \sum_{\mathbf{q}} \int \frac{d\omega}{2\pi} n(\omega) \sum_{l=L,T} g_l \mathrm{Im}\left[\Pi_l D_l + \ln\left(-D_l^{-1}\right)\right],$$

Let us consider the entropy as the derivative with respect to the temperature,

$$S = -\left.\frac{\partial \Omega}{\partial T}\right|_{\mu,V,A,B} = S_f + S_b + S',$$

$$S_b \equiv -\sum_{\mathbf{q},l} g_l \int \frac{d\omega}{2\pi} \frac{\partial n(\omega)}{\partial T} \left[\mathrm{Im}\Pi_l \,\mathrm{Re} D_l + \mathrm{Im}\ln\left(-D_l^{-1}\right)\right],$$

The remaining term vanishes in the two-loop approximation for Φ

$$S' \equiv -\left.\frac{\partial (T\Phi)}{\partial T}\right|_B - \sum_{\mathbf{q},l} g_l \int \frac{d\omega}{2\pi} \frac{\partial n(\omega)}{\partial T} \mathrm{Im} D_l \,\mathrm{Re}\Pi_l = 0$$

From Blaschke, Röpke, Baym, arXiv:2512.03876 [nucl-th]

Generalized Beth-Uhlenbeck formula from Φ-derivable ...

With the polar representation of the propagator $D = |D| \exp(i\delta)$ follow the relations

$$D\, \mathrm{Im}\Pi = \sin\delta \mathrm{e}^{i\delta} \;, \quad D^{*\prime}\, \mathrm{Im}\Pi = -i\delta' \sin\delta \mathrm{e}^{-i\delta} \;,$$

which allow to rewrite

$$-\left[\mathrm{Re}D\, \mathrm{Im}\Pi + \mathrm{Im}\ln(-D^{-1})\right]' = 2\delta' \sin^2\delta$$

and obtain the entropy in the form of a generalized Beth-Uhlenbeck formula,

$$S_b = 2V \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \int_0^\infty \frac{\mathrm{d}\omega}{\pi}\, \sigma_b(\omega)\, \sin^2\delta(\omega, \mathbf{q}) \frac{\partial \delta(\omega, \mathbf{q})}{\partial \omega} \;.$$

Lorentz-profile $\longrightarrow$ "squared Lorentzian"

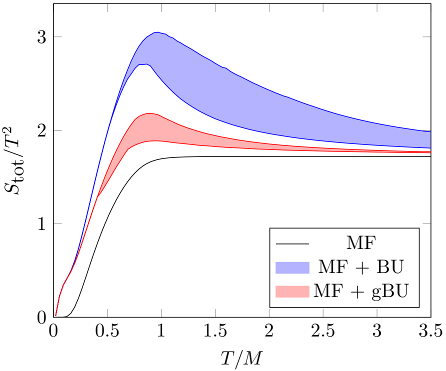
$$\frac{\partial\delta(\omega)}{\partial\omega} = -\frac{\Gamma}{(\omega-\omega_R)^2 + \Gamma^2} \quad \longrightarrow \quad \sin^2\delta(\omega) \frac{\partial\delta(\omega)}{\partial\omega} = \frac{\Gamma^3}{[(\omega-\omega_R)^2 + \Gamma^2]^2} \;.$$

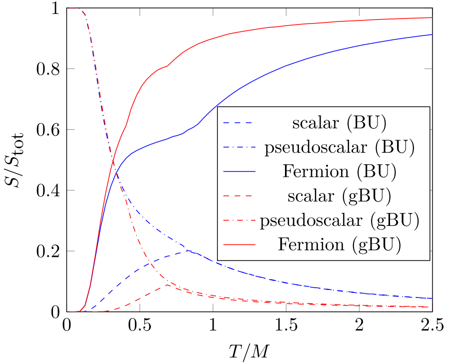
From Blaschke, Röpke, Baym, arXiv:2512.03876 [nucl-th]

Application: quark-meson plasma with backreaction

Generalized Beth-Uhlenbeck formula for 2+1D GN model, with correction $\sin(2\delta)/2$

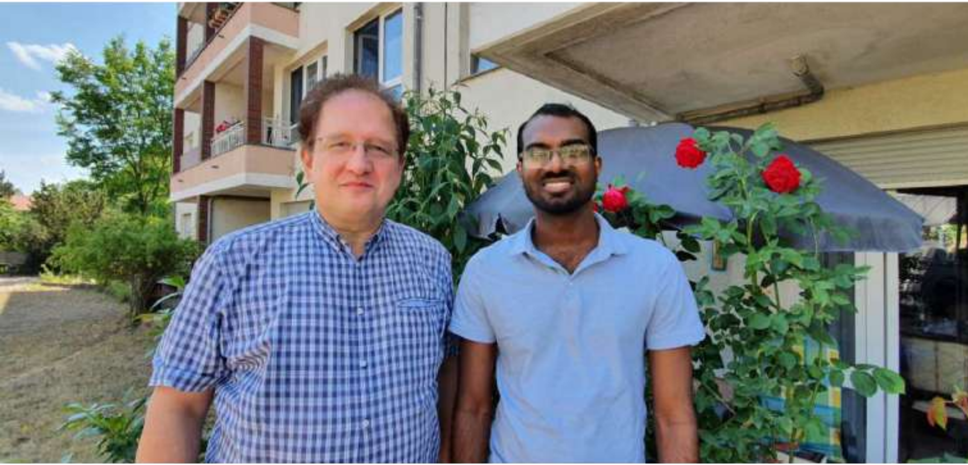
$$S_{fl,gBU} = \sum_{i=\pi,\sigma} \int \frac{d^2q}{(2\pi)^2} \int_0^\infty \frac{d\omega}{2\pi} \frac{dg(\omega)}{dT} \{\delta_i(\omega,q) - \sin[2\delta_i(\omega,q)]/2)\}$$





Figures from B. Mahato & D. Blaschke, arXiv:2604.02246

Application: quark-meson plasma with backreaction



Further applications are found in the PhD Thesis of Biplab Mahato (Uni Wroclaw)

University of Wroclaw, Oratorium Marianum, CPOD 2016