Different versions of soft-photon theorems

O. Nachtmann

Inst. für Theoretische Physik, Univ. Heidelberg

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1 Introduction

My interest in soft-photon production in hadronic collisions goes back more than 40 years. At that time we speculated that soft photons could have an "anomalous" source in the non-trivial structure of the QCD vacuum. Think of the "gluon condensate", of "instantons", and the like. Maybe I can say a little more on this during a discussion session. We published a number of papers on this subject.

- O.N. and A. Reiter, "The Vacuum Structure in QCD and Hadron - Hadron Scattering", Z. Phys. C 24, 283 (1984)
- G.W. Botz, P. Haberl, O.N., "Soft photons in hadron-hadron collisions: synchrotron radiation from the QCD vacuum?" Z. Phys. C 67, 143 (1995).
- O.N. "Spin correlations in the Drell-Yan process, parton entanglement, and other unconventional QCD effects", Ann. Phys. 350 (2014) 347.
- O.N. "The QCD vacuum structure and its manifestations" in "On Confinement Physics", Eds. S. D. Bass and P. A. M. Guichon, Editions Frontières, 1996.

Some years ago Johanna Stachel, Peter Braun-Munzinger and collaborators started a seminar series devoted to soft-photon problems and asked if we would be interested to contribute. By that time Carlo Ewerz, Antoni Szczurek, Piotr Lebiedowicz and myself were heavily involved in the theory of high-energy hadronic soft diffractive reactions. That is, we studied pomeron, odderon, Reggeon physics. It was then natural for us to study soft-photon production in soft diffractive reactions. These reactions are, in a sense "clean" processes both from the theoretical and experimental sides. We could publish a number of papers on this subject where we used our tensor-pomeron model. Let me emphasize at this point that the experimental study of soft photons in diffractive reactions would profit enormously if the outgoing protons in reactions like

$$p + p \longrightarrow p + \gamma + p$$

could be measured.

Piotr Lebiedowicz, O. N., Antoni Szczurek,

results within the tensor-pomeron model:

- "High-energy $\pi\pi$ scattering without and with photon radiation," PRD 105, 014022 (2022); 109, 099901(E) (2024).

- "Soft-photon radiation in high-energy proton-proton collisions within the tensor-pomeron approach: Bremsstrahlung," PRD 106, 034023 (2022).

- "Central exclusive diffractive production of a single photon in high-energy proton-proton collisions within the tensor-pomeron approach," PRD 107, 074014 (2023).

- "Exclusive diffractive Bremsstrahlung of one and two photons at forward rapidities: Possibilities for experimental studies in pp collisions at the LHC", PL B 843, 138053 (2023).

When studying – as the simplest case – soft-photon production in the reaction

$$\pi^- + \pi^0 \longrightarrow \pi^- + \pi^0 + \gamma$$

we looked in the framework of the tensor-pomeron model at the leading and next-to-leading terms in the photon energy ω for $\omega \to 0$. We found surprises when we compared our results to those in the famous paper of 1958 by F. E. Low. These surprises led us to study soft-photon theorems in a rigorous QFT framework.

Piotr Lebiedowicz, O. N., Antoni Szczurek,
exact results within QFT:

- "High-energy $\pi\pi$ scattering without and with photon radiation"
PR D 105, 014022 (2022), Erratum: PRD 109, 099901(E) (2024).

- "Different versions of soft-photon theorems exemplified at leading and next-to-leading terms for pion-pion and pion-proton scattering" PRD 109, 094042 (2024).

- "Soft-photon theorem for pion-proton elastic scattering revisited", PRD 110, 094028 (2024).

My talk today is based on these papers. I shall mainly discuss the following reactions:

$$\pi^-(p_a) + \pi^0(p_b) \longrightarrow \pi^-(p_1) + \pi^0(p_2), \tag{1.1}$$

$$\pi^-(p_a) + \pi^0(p_b) \longrightarrow \pi^-(p_1') + \pi^0(p_2') + \gamma(k, \varepsilon). \tag{1.2}$$

Let $\omega = k^0$ be the photon energy in the c.m. system. We are interested in the limit $\omega \to 0$. The classical papers dealing with this soft-photon limit are

F.E. Low, "Bremsstrahlung of Very Low Energy Quanta in Elementary Particle Collisions", P.R. <u>110</u>, 974 (1958),

S. Weinberg, "Infrared Photons and Gravitons", P.R. <u>140</u>, B516 (1965)

I hope to convince you in my talk that Low and Weinberg present quite different versions of soft-photon theorems. I shall also show you their relation.

Framework

We consider the reactions (1.1) and (1.2) in QCD plus leading order in electromagnetism. We use only exact QFT methods in this framework.

- energy - momentum conservation,
- gauge invariance,
- invariance under parity (P), charge conjugation (C), and time reversal (T),
- the generalised Ward identity for the pion fields, which in QCD are composite local fields,
- analyticity properties of amplitudes, in particular the Landau conditions.

2 Kinematics and phase space

We start with the elastic reaction

$$\pi^-(p_a) + \pi^0(p_b) \longrightarrow \pi^-(p_1) + \pi^0(p_2),$$

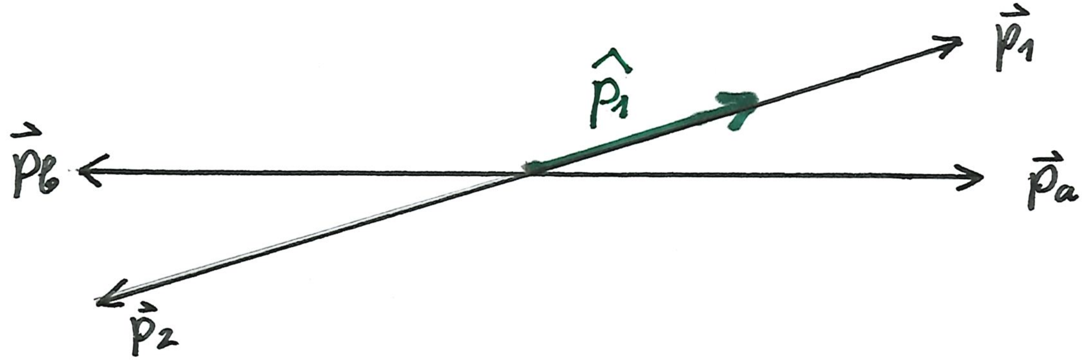
$$p_a + p_b = p_1 + p_2. \tag{2.1}$$

We set as usual

$$s = (p_a + p_b)^2 = (p_1 + p_2)^2,$$

$$t = (p_a - p_1)^2 = (p_b - p_2)^2. \tag{2.2}$$

We look at the reaction (2.1) in the c.m. system and consider a fixed value of the c.m. energy squared s. Then the energies and absolute values of the momenta are fixed. For a given initial configuration we can only vary $\hat{\mathbf{p}}_1 = \vec{p}_1/|\vec{p}_1|$, the unit vector in direction of $\vec{\mathbf{p}}_1$. The phase space is the unit sphere.



$$p_a^0 = p_b^0 = p_1^0 = p_2^0 = \frac{1}{2}\sqrt{s}\,,$$

$$|\vec{p}_a| = |\vec{p}_b| = |\vec{p}_1| = |\vec{p}_2| = \sqrt{\frac{s}{4} - m_\pi^2}\,,$$

$$\hat{p}_1 = \vec{p}_1 / |\vec{p}_1|\,. \tag{2.3}$$

Following Low we use as energy variable

$$\nu = s - 2m_\pi^2 = p_a \cdot p_b + p_1 \cdot p_2\,.$$

We denote the on-shell amplitude by

$$\mathcal{M}^{(on)}(\nu, t) = \langle \pi^-(p_1), \pi^0(p_2) | T | \pi^-(p_a), \pi^0(p_b) \rangle .$$

Now we go to the reaction with photon radiation

$$\pi^-(p_a) + \pi^0(p_b) \longrightarrow \pi^-(p_1') + \pi^0(p_2') + \gamma(k, \varepsilon),$$

$$p_a + p_b = p_1' + p_2' + k. \tag{2.4}$$

Here we set

$$\begin{aligned} s &= (p_a + p_b)^2 = (p_1' + p_2' + k)^2, \\ t_1 &= (p_a - p_1')^2 = (p_b - p_2' - k)^2, \\ t_2 &= (p_b - p_2')^2 = (p_a - p_1' - k)^2. \end{aligned} \tag{2.5}$$

We shall consider real and virtual photon emission and require

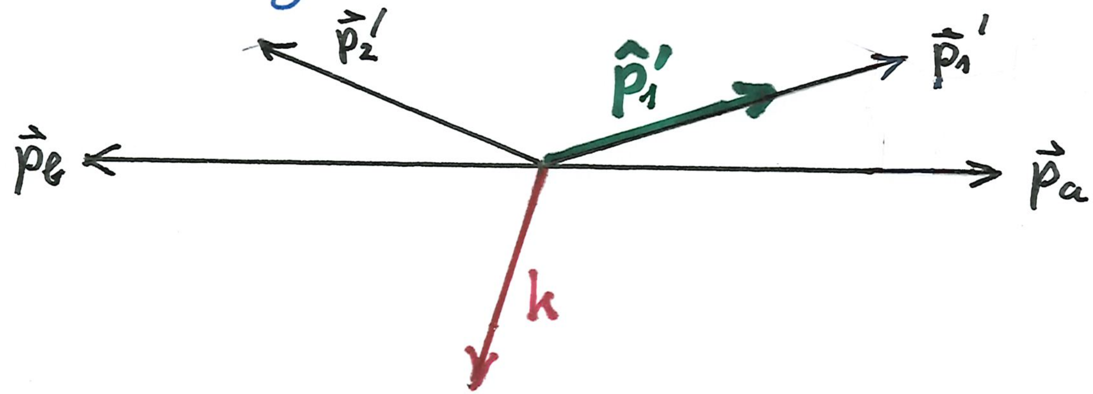
$$k^2 \geq 0, \quad k^0 \geq 0, \tag{2.6}$$

For a given value of s what are the free parameters of the reaction (2.4) for small k, say $|k^\mu| \ll \sqrt{s - 4m_\pi^2}$ $(\mu = 0,1,2,3)$, in the c. m. system?

A convenient set of such parameters is given by the four-vector k plus the unit vector $\hat{p}_1' = \vec{p}_1{}'/|\vec{p}_1{}'|$.

$$\text{Phase space of } (2.4) = \{(k, \hat{p}_1{}'); \; k \in \text{part of } R_4, \; |\hat{p}_1{}'| = 1\}. \tag{2.7}$$

We see this easily by considering (2.4) for given k in the rest system of $p_a + p_b - k = p_1' + p_2'$. In this system $\vec{p}_1{}'$ and $\vec{p}_2{}'$ are back to back with fixed $|\vec{p}_1{}'| = |\vec{p}_2{}'|$. The only freedom left is to vary $\vec{p}_1{}'$ in any direction. The same is then also true in the c.m. system if k is small enough.



Here we denote the amplitude by

$$\mathcal{M}_\lambda \left(p_a, p_b, p_1', p_2', k \right). \tag{2.8}$$

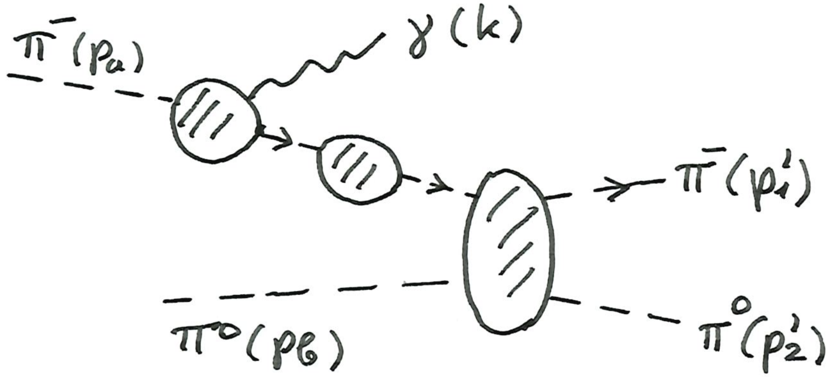
We consider real and timelike virtual photon emission

$$k^2 \geqslant 0 \quad , \quad k^0 \geqslant 0 ,$$

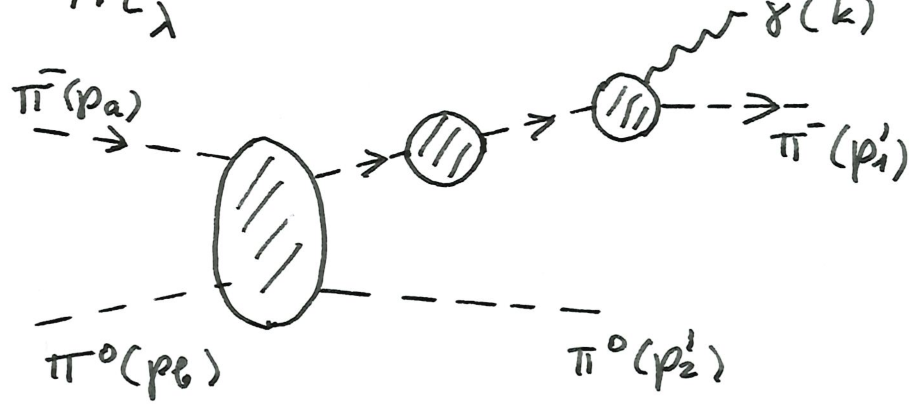
as we mentioned already.

3 QFT analysis of $\pi\pi \to \pi\pi$ and $\pi\pi \to \pi\pi\gamma$

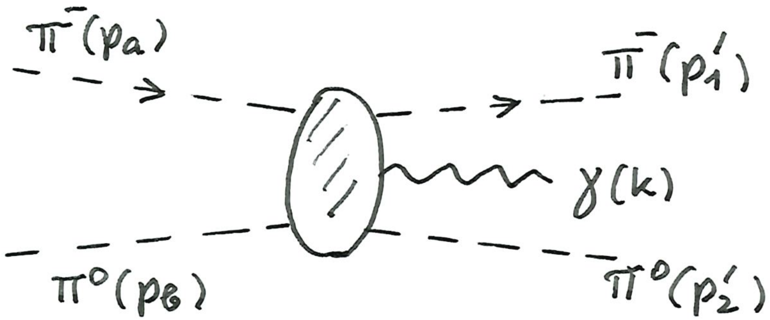
We have three diagrams for $\mathcal{M}_\lambda$



(a)



(b)



(c)

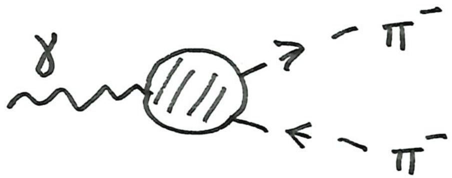
(a), (b) 1 particle reducible

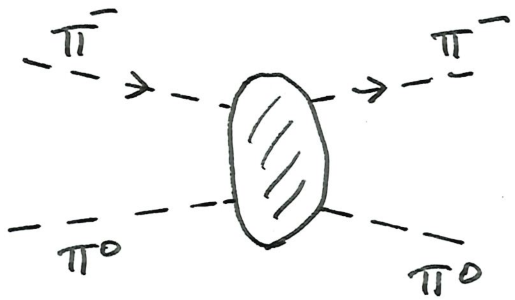
(c) 1 particle irreducible

In our QFT analysis we use

the full pion propagator 

,

the full pion-photon vertex function 

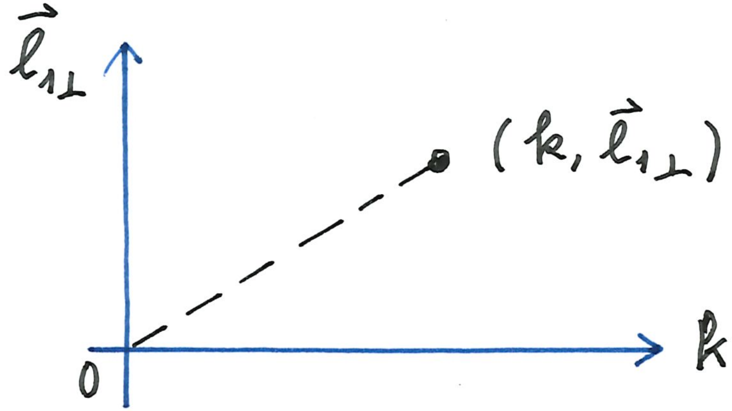
the off-shell pion-pion scattering amplitude 

4 Soft photon theorem I

In this section we shall give the expansion of the amplitude $\mathcal{M}_\lambda$ around the phase-space point $(k=0,\ \hat{p}_1' = \hat{p}_1)$. In a small neighbourhood of this phase-space point we set:

$$\hat{p}_1' = \hat{p}_1 - \vec{\ell}_{1\perp}/|\vec{p}_1| \,, \quad \vec{\ell}_{1\perp} \cdot \hat{p}_1 = 0 \,, \quad k, \vec{\ell}_{1\perp} = \mathcal{O}(\omega) \,. \tag{4.1}$$

This neighbourhood has 6 dimensions. Schematically we represent it as follows:



Below we shall study $\mathcal{M}_\lambda$ on such a ray starting from the origin.

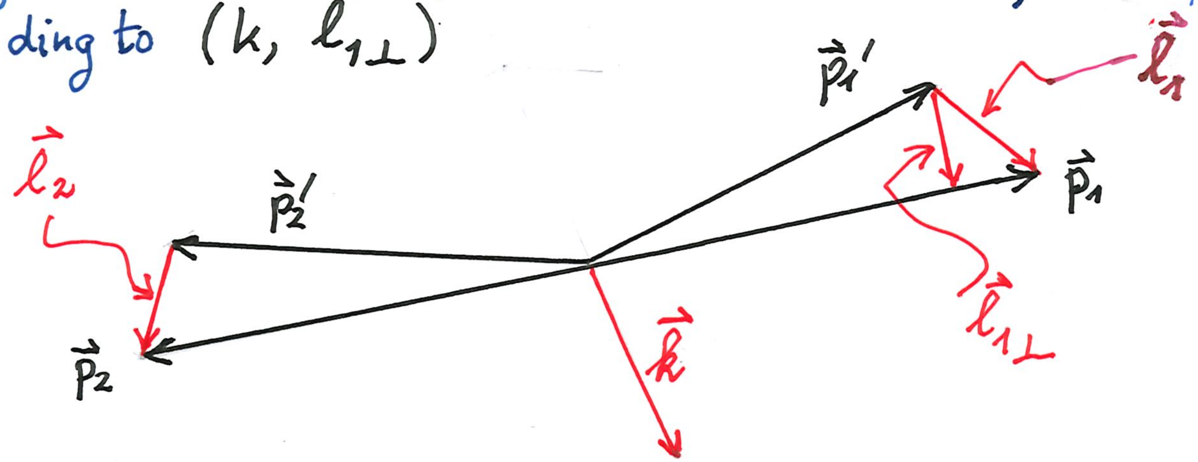
For $(k=0, \vec{\ell}_{1\perp}=0)$ we have the kinematics of the reaction without radiation, $\pi^-(p_a)+\pi^0(p_b) \to \pi^-(p_1)+\pi^0(p_2)$. For $(k, \vec{\ell}_{1\perp})$ we have the kinematics of photon radiation with

$$p_1' = p_1 - \ell_1 \,, \quad p_2' = p_2 - \ell_2 \,, \qquad (4.2)$$
$$\ell_1 + \ell_2 = k \,,$$

as required by energy-momentum conservation

$$p_a + p_b = p_1' + p_2' + k = p_1 + p_2 \,.$$

Working in the c.m. system we find easily the four vectors $\ell_{1,2}$ corresponding to $(k, \vec{\ell}_{1\perp})$



$$p_1 = \begin{pmatrix} p_1^0 \\ \\ |\vec{p}_1|\,\hat{p}_1 \end{pmatrix} , \qquad p_2 = \begin{pmatrix} p_1^0 \\ \\ -|\vec{p}_1|\,\hat{p}_1 \end{pmatrix}$$

$$s = (p_1 + p_2)^2 , \quad p_1^0 = \frac{1}{2}\sqrt{s} , \quad |\vec{p}_1| = \sqrt{\frac{s}{4} - m_\pi^2} , \quad |\hat{p}_1| = 1 , \tag{4.3}$$

$$l_1 = \begin{pmatrix} \dfrac{p_2 \cdot k}{\sqrt{s}} \\ \\ \dfrac{p_1^0}{|\vec{p}_1|\sqrt{s}}(p_2 \cdot k)\,\hat{p}_1 + \vec{l}_{1\perp} \end{pmatrix} + \mathcal{O}(\omega^2) , \quad l_2 = \begin{pmatrix} \dfrac{p_1 \cdot k}{\sqrt{s}} \\ \\ \vec{k} - \dfrac{p_1^0\,\hat{p}_1}{|\vec{p}_1|\sqrt{s}}(p_2 \cdot k) - \vec{l}_{1\perp} \end{pmatrix} + \mathcal{O}(\omega^2)$$

$$p_1 \cdot l_1 = 0 + \mathcal{O}(\omega^2) , \qquad p_2 \cdot l_2 = 0 + \mathcal{O}(\omega^2) . \tag{4.4}$$

Now we come to the expansion of the amplitude $\mathcal{M}_\lambda$ for $\omega \to 0$.
To be precise we set, in the c.m. system with $\omega \geq 0$:

$$k = \omega \begin{pmatrix} 1 \\ \vec{\tilde{k}} \end{pmatrix}, \quad |\vec{\tilde{k}}| \leq 1, \quad \vec{l}_{1\perp} = \omega\, \vec{\tilde{l}}_{1\perp}, \quad |\vec{\tilde{l}}_{1\perp}| = \mathcal{O}(1). \tag{4.5}$$

We keep $\vec{\tilde{k}}$ and $\vec{\tilde{l}}_{1\perp}$ fixed and consider the expansion of the radiative amplitude for $\omega \to 0$. That is, we consider $\mathcal{M}_\lambda$ on a ray starting at the origin in the phase space $\{(k, \vec{l}_{1\perp})\}$.
Of course, we shall get a Laurent expansion for $\mathcal{M}_\lambda$.
We illustrate this for the term $\mathcal{M}_\lambda^{(a)}$.

This gives us our final result for real photon emission

$$\mathcal{M}_\lambda(p_a, p_b, p_1', p_2', k) = e\left[\frac{p_{a\lambda}}{p_a\cdot k} - \frac{p_{1\lambda}}{p_1\cdot k}\right]\mathcal{M}^{(on)}(\nu, t) \qquad \Big|\ \omega^{-1}$$

$$- e\,\frac{1}{(p_1\cdot k)^2}\left[p_{1\lambda}(l_1\cdot k) - l_{1\lambda}(p_1\cdot k)\right]\mathcal{M}^{(on)}(\nu, t)$$

$$- 2e\left[p_{a\lambda}\frac{p_b\cdot k}{p_a\cdot k} - p_{b\lambda}\right]\frac{\partial}{\partial\nu}\mathcal{M}^{(on)}(\nu, t) \qquad \Big|\ \omega^{0}$$

$$- 2e\left[(p_a - p_1, k) - (p_a\cdot l_1)\right]\left[\frac{p_{a\lambda}}{p_a\cdot k} - \frac{p_{1\lambda}}{p_1\cdot k}\right]\frac{\partial}{\partial t}\mathcal{M}^{(on)}(\nu, t)$$

$$+ \mathcal{O}(\omega). \qquad (4.11)$$

$$\nu = s - 2m_\pi^2\,, \quad t = (p_a - p_1)^2 = (p_b - p_2)^2.$$

With (4.11) we have given the Laurent expansion of $\mathcal{M}_\lambda$ to the orders ω^{-1} and ω^0 around the phase space point $(k=0,\ \vec{l}_{1\perp}=0)$ corresponding to $k=0, p_1'=p_1, p_2'=p_2$.

The amplitude $\mathcal{M}^{(on)}(\nu, t)$ corresponds to the basic process

$$\pi^-(p_a) + \pi^0(p_b) \rightarrow \pi^-(p_1) + \pi^0(p_2).$$

The pole term $\propto \omega^{-1}$ in (4.11) is exactly Weinberg's soft-photon term. He writes

"Hence the effect of attaching one soft-photon line to an arbitrary diagram is simply to supply an extra factor,

$$\sum_n e_n \eta_n p_n{}^\mu / [p_n \cdot q - i\eta_n \varepsilon], \tag{4.12}$$

the sum running over all external lines in the original diagram."

Here $\eta_n = +1\ (-1)$ for an outgoing (incoming) charged particle.

In our work we have given the next to leading term, $\mathcal{O}(\omega^0)$, to Weinberg's pole term.

5 Soft photon theorem II

Now we want to discuss Low's version of soft-photon theorem.

Low only discusses $\mathcal{M}_\lambda(p_a, p_b, p_1', p_2', k)$ at a given phase-space point p_1', p_2', k. He gives an approximate expression for $\mathcal{M}_\lambda$ at this point:

$$\mathcal{M}_\lambda(p_a, p_b, p_1', p_2', k) = e\left[\frac{p_{a\lambda}}{p_a \cdot k} - \frac{p'_{1\lambda}}{p_1' \cdot k}\right] \mathcal{M}^{(on)}(\nu_L, t_2)$$

$$- e\left[p_{a\lambda}\frac{p_b \cdot k}{p_a \cdot k} + p'_{1\lambda}\frac{p_2' \cdot k}{p_1' \cdot k} - p_{b\lambda} - p'_{2\lambda}\right] \frac{\partial}{\partial \nu_L}\mathcal{M}^{(on)}(\nu_L, t_2) + \mathcal{O}(k). \tag{5.1}$$

$$\nu_L = p_a \cdot p_b + p_1' \cdot p_2' = s - 2m_\pi^2 - (p_a + p_b \cdot k),$$

$$t_2 = (p_b - p_2')^2 = (p_a - p_1' - k)^2. \tag{5.2}$$

We emphasize that Low's result (5.1) is not an expansion of $\mathcal{M}_\lambda(p_a, p_b, p_1', p_2', k)$ around a phase-space point of no radiation. Low's result (5.1) gives an approximate expression for $\mathcal{M}_\lambda$ at a given phase-space point p_1', p_2', k. There k is fixed by energy-momentum conservation

$$p_a + p_b = p_1' + p_2' + k$$

for given p_1', p_2'. Varying only k in (5.1) violates energy-momentum conservation.

Comparing (4.11) and (5.1) we see clearly that Low's and Weinberg's versions of soft-photon theorems are different. To see their relation we expand Low's approximate expression around the phase space point $p_1' = p_1$, $p_2' = p_2$, $k = 0$ of no radiation. We get then complete agreement with our result (4.11).

6 Conclusions

We have discussed the reactions

$$\pi^-(p_a) + \pi^0(p_b) \longrightarrow \pi^-(p_1) + \pi^0(p_2), \tag{6.1}$$

and

$$\pi^-(p_a) + \pi^0(p_b) \longrightarrow \pi^-(p_1') + \pi^0(p_2') + \gamma(k, \varepsilon) \tag{6.2}$$

for $k \to 0$. For small enough k the phase-space of (6.2) is given by $\{(\mathbf{k}, \hat{p}_1')\}$, where we consider the c.m. system and $\hat{p}_1' = \vec{p}_1'/|\vec{p}_1'|$ varies over the unit sphere. We set $k^0 = \omega$.

(i) In Sec. 4 we have given the expansion of the amplitude for (6.2), $\mathcal{M}_\lambda$, around the phase-space point $(k = 0, \hat{p}_1)$ corresponding to the basic elastic process (6.1). This expansion is a Laurent series. The pole term $\propto \omega^{-1}$ is given by Weinberg's version of the soft-photon theorem. We have calculated the next to leading term $\propto \omega^0$ of this series.

(ii) Low's version of the soft-photon theorem is not an expansion of this kind but gives an approximate expression for $\mathcal{M}_\lambda$ at a given phase-space point.

(iii) What is frequently called Low's theorem is in reality the soft-photon theorem in the version of Weinberg; see (5.1) and (4.11).

(iv) We have shown how we get from Low's formula, giving an approximate expression for $\mathcal{M}_\lambda$ at a given phase-space point, to the Laurent expansion of $\mathcal{M}_\lambda$ around a phase-space point of no radiation.

(v) We have discussed in our papers also the expansion in ω of the cross section for $\pi\pi \to \pi\pi\gamma$. In the same spirit as summarised in (i) we have treated the reactions

$\pi p \to \pi p$ and $\pi p \to \pi p \gamma$.

(vi) We hope that with this investigation we could clarify the meaning of Low's and Weinberg's versions of soft-photon theorems and their relation, in particular, concerning the next to leading terms.

Thank you for
your attention!