Soft photons
and
power corrections

Domenico Bonocore

Technische Universität München

Based on:
2112.08329 DB, Kulesza
2312.11386 Balsach, DB, Kulesza
2406.17959 Bailhache et al.

Heidelberg, 26 May 2025

Outline

Introduction

LBK theorem: the traditional formulation

LBK theorem with shifted kinematics

Results for $e^+e^- \to \mu^+\mu^-\gamma$ and $pp \to \mu^+\mu^-\gamma$

Loop corrections to LBK theorem

Introduction

Power corrections

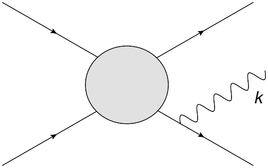
For many kinematic configurations, **perturbative** calculations are the cornerstone of theoretical predictions in hep

E.g. expansion in α: **LO, NLO, NNLO, ...**

Large separation of scales $E/\mu \ll 1$
$\rightarrow$ expansion in other small parameters : **LP, NLP, NNLP, ...**

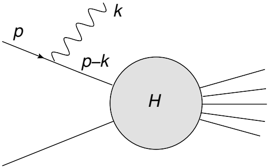
E.g. pert. $(M/\mu)^n$ or non.pert. $(\Lambda/\mu)^n$ VS α^n

In this talk perturbative power corrections: $k \ll p$ (**soft** expansion)

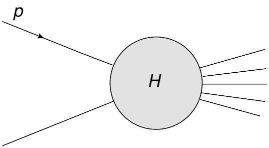


Soft factorization

$\mathcal{A} =$



$\mathcal{H} =$



$$\underbrace{\mathcal{A}(p_1,\ldots,p_n,k)}_{\text{radiative}} = \underbrace{\mathcal{S}(k)}_{\text{soft factor}} \times \underbrace{\mathcal{H}(p_1,\ldots,p_n)}_{\text{non-radiative}}$$

$$\begin{aligned}\mathcal{S} \quad = \quad & \overrightarrow{\mathcal{S}_{\rm LP}^{(0)} + \mathcal{S}_{\rm LP}^{(1)}} + \mathcal{S}_{\rm LP}^{(2)} + \ldots \\ \big\downarrow \quad & + \mathcal{S}_{\rm NLP}^{(0)} + \mathcal{S}_{\rm NLP}^{(1)} + \mathcal{S}_{\rm NLP}^{(2)} + \ldots \\ & + \mathcal{S}_{\rm NNLP}^{(0)} + \mathcal{S}_{\rm NNLP}^{(1)} + \mathcal{S}_{\rm NNLP}^{(2)} + \ldots \\ & + \ldots\end{aligned}$$

More interesting structure for n emissions: $\mathcal{S} = \exp(\mathcal{W})$

History extends to present days (both at LP and NLP)

Soft Factorization has a long history

- LP (QED) [Bethe-Heitler 1934, Bloch-Nordsieck 1937, Yenni-Fratuushi-Suura 1961]
- LP (gravity) [Weinberg 1965]
- NLP (QED-tree) [Low 1958, Burnett-Kroll 1967].

up to more recent times

- LP (QCD) tree [Berends-Giele 1988]
  1-loop[Catani-Grazzini 2000]
  2-loop[Duhr-Gehrmann 2013]
  3-loop[Herzog-Ma-Mistlberger-Suresh 2023, Chen-Luo-ManYan-Zhou 2023]
- LP (massless QED) 3-loop [Ma-Sterman-Venkata 2023]
- NLP (QCD, massless QED) [DelDuca 1990, Casali 2014, Bern-Davies-Nohle2014, Larkoski-Neill-Stewart 2014, DB-Laenen-Magnea-Melville-Vernazza2015, Beneke-Broggio-Jaskiewicz-Vernazza 2019, +many others]
- NLP (QED) 1-loop [Engel,Signer,Ulrich 2021] all-orders [Engel 2023]
- NLP, NNLP gravity [White 2011, Cachazo-Strominger 2014]

Methods

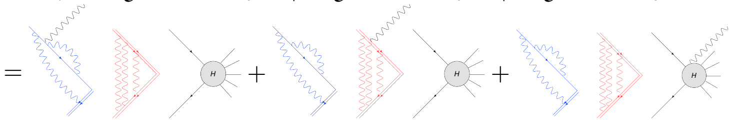
- Effective field theories [Bauer et al 2001, Beneke,Feldmann 2002, ...]

$$\mathcal{L}_{\text{SCET}} = \mathcal{L}_{\text{LP}} + \mathcal{L}_{\text{NLP}}$$

$$\sigma_{\text{DY-LP}} = H \times S\,, \qquad \sigma_{\text{DY-NLP}} = H \otimes J \otimes \bar{J} \otimes \tilde{S}$$

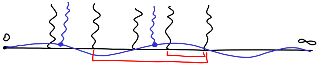
- Diagrammatic factorization [Collins,Soper,Sterman,Libby,..., Catani,Trentadue,...]

$$\mathcal{A}^\mu = \mathcal{J}^\mu \times \mathcal{S} \times \mathcal{H} \quad + \quad \mathcal{J} \times \mathcal{S}^\mu \times \mathcal{H} \quad + \quad \mathcal{J} \times \mathcal{S} \times \mathcal{H}^\mu$$



- Worldline formalism [Feynman 1955, Bern-Kosower 1992, Strassler 1992, Schubert, ...]

$$\langle \frac{\hat{\not{p}} + \not{A}(\hat{x})}{(\hat{\not{p}} - \not{A}(\hat{x}))^2 - m^2} \rangle \sim \int dT \int d\theta \int \mathcal{D}\psi \int \mathcal{D}x\mathcal{D}p\, e^{-i\int dt\, \left(p\cdot\dot{x} + \frac{i}{2}\psi\cdot\dot{\psi} - \text{Den} - \frac{\theta}{T}\text{Num}\right)}\,.$$

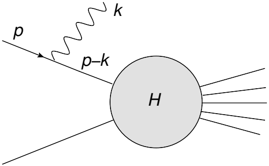


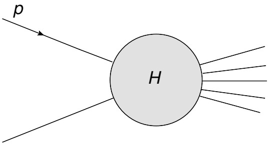
- On-shell methods, celestial methods, ...

LBK theorem: the traditional formulation

DIAGRAMMATICS (LP)

Goal: $\underbrace{\mathcal{A}(p_1,\ldots,p_n,k)}_{\text{radiative}} = \underbrace{\mathcal{S}(k)}_{\text{soft factor}} \times \underbrace{\mathcal{H}(p_1,\ldots,p_n)}_{\text{non-radiative}}$

$\mathcal{A} =$ 

$\mathcal{H} =$ 

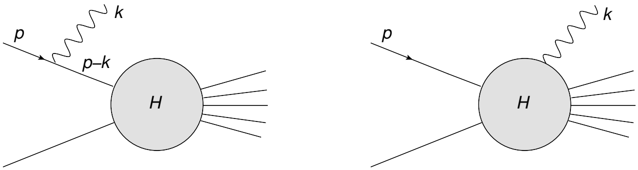
$$\mathcal{A} = H(p-k)\,\frac{(\not{p}-\not{k})}{(p-k)^2}\,(Q\,\epsilon^*(k)\cdot\gamma)\,u(p) \qquad\qquad \mathcal{H} = H\;u(p)$$

At **LP**, take the leading term for $k \to 0$ (**eikonal** approximation):

$$\mathcal{A} = \mathcal{S}_{LP}\,\mathcal{H}\,, \qquad \mathcal{S}_{LP} = \sum_{i=1}^{n} Q_i \eta_i \frac{\epsilon^*(k)\cdot p_i^{\mu}}{p_i\cdot k}$$

- insensitive to spin of hard emitter
- hard particles do not recoil ($k \to 0$)
- insensitive to the short distance physics i.e. non radiative amplitude $\mathcal{H}$

Diagrammatics (NLP)



- External emission: expand up to $\mathcal{O}(k)$

$$
\begin{aligned}
\mathcal{A}^{\mu}_{\text{ext}}(p) &= H(p-k)\,\frac{(\not{p}-\not{k})}{(p-k)^2}\,(Q\,\gamma^{\mu})u(p) \\
&= Q\,H(p)\left(\frac{p^{\mu}}{p\cdot k} + \frac{k^{\mu}}{2p\cdot k} - \frac{k^2 p^{\mu}}{2(p\cdot k)^2} - \frac{i k_{\nu}\sigma^{\mu\nu}}{p\cdot k}\right)u(p) \\
&\quad + \; Q\,\frac{p^{\mu}}{p\cdot k}k^{\nu}\,\underbrace{\left.\frac{\partial H(p-k)}{\partial k_{\nu}}\right|_{k=0}}_{-\frac{\partial H(p)}{\partial p_{\nu}}} u(p) \; + \; \mathcal{O}(k)
\end{aligned}
$$

- Note: expansion performed before imposing momentum conservation. Alternative: do not leave mom-cons. surface $\sum_i p_i = k$ and parametrize $p_i(k)$ (note derivatives on spinors). Parametrization $p_i(k)$ not unique.

Diagrammatics (NLP)

- Internal emission: use Ward identity $k_\mu(\mathcal{A}^\mu_{\text{ext}} + \mathcal{A}^\mu_{\text{int}}) = 0$

$$\mathcal{A}^\mu_{\text{int}} = \sum_i Q_i \frac{\partial H(p^i)}{\partial p^i_\mu} u(p^i) + \underbrace{\Delta^\mu}_{\mathcal{O}(k)}$$

- Adding $\mathcal{A}^\mu_{\text{ext}}$ and $\mathcal{A}^\mu_{\text{int}}$:

$$\begin{aligned}\mathcal{A}^\mu = & \sum_i Q_i \frac{p_i^\mu}{p_i \cdot k} \mathcal{H}(p_1 \ldots p_n) \\ & + \sum_i Q_i \left(\frac{k^\mu}{2p \cdot k} - \frac{k^2 p^\mu}{2(p \cdot k)^2} - \frac{ik_\nu \sigma^{\mu\nu}}{p \cdot k} \right) \mathcal{H}(p_1 \ldots p_n) \\ & + \sum_i Q_i \underbrace{\left(-\frac{p_i^\mu k^\nu}{p_i \cdot k} \frac{\partial}{\partial p_i^\nu} + \frac{\partial}{\partial p_i^\mu} \right)}_{-\frac{k_\nu}{p_i \cdot k} \underbrace{\left(p_i^\mu \frac{\partial}{\partial p_i^\nu} - p_i^\nu \frac{\partial}{\partial p_i^\mu} \right)}_{\equiv L^{\mu\nu}}} \mathcal{H}(p_1 \ldots p_n)\end{aligned}$$

$L^{\mu\nu}$ is the angular momentum generator of the Lorentz group

LBK THEOREM (NLP)

This is the **sub-leading** soft theorem, known as **Low-Burnett-Kroll theorem**:
[Low 1958 (scalar emitters), Burnett-Kroll 1968 (spin $\frac{1}{2}$ emitters, conjecture for generic spin), Bell-VanRoyen 1969 (generic spin), Cachazo-Strominger 2014]

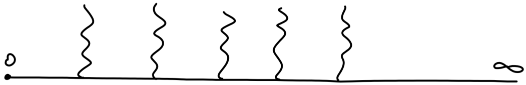
$$\mathcal{A}(p_1,\ldots,p_n,k) = (\mathcal{S}_{LP} + \mathcal{S}_{NLP})\,\mathcal{H}(p_1,\ldots,p_n)\,,$$

$$\mathcal{S}_{LP} = \sum_{i=1}^{n} \eta_i Q_i \frac{\epsilon^*(k)\cdot p_i}{p_i\cdot k}\,, \quad \mathcal{S}_{NLP} = \sum_{i=1}^{n} \eta_i Q_i \frac{\epsilon^*_\mu(k) k_\nu (\sigma^{\mu\nu} + L^{\mu\nu})}{p_i \cdot k}$$

- corrections to the strict limit $k \to 0$: **small recoil** of the emitter taken into account
- sensitive to the **spin** of the emitter (e.g. $\sigma^{\mu\nu} = 0$ for scalars, $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$for spin 1/2, etc.)
- orbital angular momentum $L^{\mu\nu}$ is sensitive to the short distance interactions in $\mathcal{H}$ (hard lines do not start from a pointlike vertex)
- NLP corrections here are valid only at the tree-level

Intermezzo: from 1 photon to many photons

Many emissions exponentiate!

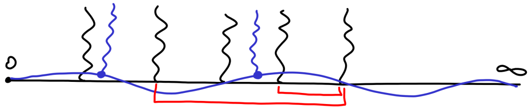


- **LP** emissions on a straight trajectory are **classical** (soft photon spectrum agrees with classical calculation [Jackson 1962]).

$$\exp\left(i\int\frac{d^4k}{(2\pi)^4}\,\frac{p^\mu}{p\cdot k}\,\tilde{A}_\mu(k)\right)=\exp\left(i\int_0^\infty d\lambda\,p^\mu A_\mu(\lambda x)\right)$$

INTERMEZZO: FROM 1 PHOTON TO MANY PHOTONS

Many emissions exponentiate!



- **LP** emissions on a straight trajectory are **classical** (soft photon spectrum agrees with classical calculation [Jackson 1962]).

$$\exp\left(i\int\frac{d^4k}{(2\pi)^4}\,\frac{p^\mu}{p\cdot k}\,\tilde{A}_\mu(k)\right)=\exp\left(i\int_0^\infty d\lambda\,p^\mu A_\mu(\lambda x)\right)$$

- at **NLP** both **classical** (correlations) and **quantum** (recoil) [Laenen, Stavenga, White 2008, DB 2020, DB,Kulesza,Pirsch 2021]

$$\exp\Bigg[\int\frac{d^dk}{(2\pi)^d}\tilde{A}_\mu(k)\left(-\frac{p^\mu}{p\cdot k}+\frac{k^\mu}{2p\cdot k}-k^2\frac{p^\mu}{2(p\cdot k)^2}-\frac{\mathrm{i}k_\nu\sigma^{\nu\mu}}{p\cdot k}\right)$$
$$+\int\frac{d^dk}{(2\pi)^d}\int\frac{d^dl}{(2\pi)^d}\tilde{A}_\mu(k)\tilde{A}_\nu(l)\left(\frac{\eta^{\mu\nu}}{2p\cdot(k+l)}-\frac{p^\nu l^\mu p\cdot k+p^\mu k^\nu p\cdot l}{2(p\cdot l)(p\cdot k)\,[p\cdot(k+l)]}\right.$$
$$\left.+\frac{(k\cdot l)p^\mu p^\nu}{2(p\cdot l)(p\cdot k)\,[p\cdot(k+l)]}-\frac{\mathrm{i}\sigma^{\mu\nu}}{p\cdot(k+l)}\right)\Bigg]$$

From amplitudes to Cross-sections

At amplitude level two NLP contributions:

- Spin $\sigma^{\mu\nu}$
- Orbital $L^{\mu\nu}$ i.e. derivatives

Squaring and **summing over polarizations**, spin contribution becomes also a derivative. Crucial identity e.g. for leg p_1 (neglecting $\sim k^\mu$):

$$\frac{\not{k}\gamma^\mu}{p_1 \cdot k}(\not{p}_1 + m) + (\not{p}_1 + m)\frac{\gamma^\mu \not{k}}{p_1 \cdot k} = -\gamma^\mu + \frac{p_1^\mu}{p_1 \cdot k}\not{k} = G_1^{\mu\nu}\frac{\partial}{\partial p_1^\nu}(\not{p}_1 + m)$$

Then, **traditional LBK with derivatives** reads

$$\overline{|\mathcal{A}(p_1,\ldots,p_n,k)|}^2 = \sum_{ij}(-\eta_i\eta_j Q_i Q_j)\frac{p_i \cdot p_j}{p_i \cdot k\, p_j \cdot k}\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 \quad \to \textbf{LP}$$

$$+ \sum_{ij}(-\eta_i\eta_j Q_i Q_j)\frac{p_\mu^i}{p_i \cdot k}G_j^{\mu\nu}\frac{\partial}{\partial p_j^\nu}\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 \quad \to \textbf{NLP}$$

Ambiguities at NLP

Problem: momentum conservation
l.h.s. $\sum_i p_i = k$ VS $\sum_i p_i = 0$ on the r.h.s. $\rightarrow$ difference for finite $k \neq 0$

Exercise: let us replace in $\mathcal{H}(p_1, \ldots, p_n)$

$$p_i \rightarrow \tilde{\mathbf{p}}_{\mathbf{i}}(\mathbf{k}) = p_i + \mathbf{c_i} k + \mathcal{O}(k^2)$$

$\mathbf{c_i}$ are **arbitrary** coefficients $\Longrightarrow$ Is LBK invariant at NLP?

$$\overline{|\mathcal{A}(p_1, \ldots, p_n, k)|}^2 = \sum_{ij=1}^{n} (-\eta_i \eta_j Q_i Q_j) \frac{p_i \cdot p_j}{p_i \cdot k \, p_j \cdot k} \overline{|\mathcal{H}(\tilde{\mathbf{p}}_{\mathbf{1}}, \ldots, \tilde{\mathbf{p}}_{\mathbf{n}})|}^2$$
$$+ \sum_{ij=1}^{n} (-\eta_i \eta_j Q_i Q_j) \frac{p^i_\mu}{p_i \cdot k} \xi_j \left(\eta^{\mu\nu} - \frac{p_j^\mu k^\nu}{p_j \cdot k} \right) \frac{d}{dp_j^\nu} \overline{|\mathcal{H}(\tilde{\mathbf{p}}_{\mathbf{1}}, \ldots, \tilde{\mathbf{p}}_{\mathbf{n}})|}^2$$

LBK theorem is invariant if $\mathbf{c_i}$ dependence **cancels up to NNLP** corrections.

Ambiguities at NLP

First Taylor expand in k

$$\overline{|\mathcal{H}(\tilde{\mathbf{p}}_1,\ldots,\tilde{\mathbf{p}}_n)|}^2 = \overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 + k^\mu \sum_i \mathbf{c_i} \frac{\partial}{\partial p^i_\mu} \overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 + \mathcal{O}(k^2)$$

Then impose momentum conservation $k = \sum_i p_i$

$$\frac{d}{dp_j^\nu} \overline{|\mathcal{H}(\tilde{\mathbf{p}}_1,\ldots,\tilde{\mathbf{p}}_n)|}^2 = \frac{\partial}{\partial p_j^\nu} \overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 + \eta^{\mu\nu}\xi_j \sum_i \mathbf{c_i} \frac{\partial}{\partial p^i_\mu} \overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 + \mathcal{O}(k)$$

Plug this into LBK with $\tilde{p}_i$
$\rightarrow$ we get original LBK (with p_i) + remainder term that depends on c_i

$$\begin{aligned} R(\mathbf{c_i}) &= \sum_{ij=1}^n (-\eta_i\eta_j Q_i Q_j) \frac{p_i \cdot p_j}{p_i \cdot k\, p_j \cdot k} k^\mu \sum_m \mathbf{c_m} \frac{\partial}{\partial p^m_\mu} \overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 \\ &+ \sum_{ij=1}^n (-\eta_i\eta_j Q_i Q_j) \frac{p^i_\mu}{p_i \cdot k} \xi_j \left(\eta^{\mu\nu} - \frac{p_j^\mu k^\nu}{p_j \cdot k}\right) \xi_j \sum_m \mathbf{c_m} \frac{\partial}{\partial p^\nu_m} \overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 + \mathcal{O}(1) \\ &= \mathbf{0} + \mathcal{O}(1) = \mathbf{NNLP} \end{aligned}$$

$\Longrightarrow$ traditional LBK (i.e. with derivatives) is invariant at NLP under momenta transformation

Ambiguities at NLP

Two cases in particular are relevant:

- $\mathbf{c_i = 0}$ (i.e. **unphysical** momenta, as in original LBK)
- $\sum_i \mathbf{c_i = -1} \implies \sum_i p_i = k, \sum_i \tilde{p}_i = 0$ (i.e. **physical** conserved momenta)

By virtue of the invariance, the two cases are equivalent at NLP. Hence, traditional LBK with unphysical momenta is consistent.

More generally, there are an infinite number of (formally equivalent at NLP) versions of the theorem, that differ by NNLP terms.

Here invariance shown under momenta transformation. The more general invariance of LBK under

$$\mathcal{H} \to \mathcal{H} + \Delta \quad \text{with} \quad \Delta(p_i)\delta(p_i) = 0$$

can be proven [Balsach, DB, Kulesza 2023] which holds also e.g. for constant amplitudes.

Key aspect: **functional dependence** of $\mathcal{H}$ crucial to select the version of the theorem. E.g. $\mathcal{H}(s_L, t)$ VS $\mathcal{H}(s'_L, t_2)$ [Lebiedowicz,Nachmann,Szczurek 2021-23]

Ambiguities at NLP

- Traditional LBK (i.e. the one with derivatives) is consistent at NLP
- Many forms of traditional LBK (all equivalent up to NNLP)
- equivalent≠ identical: different forms differ by NNLP contributions, which might be sizable in photon spectra where, by definition, $k \neq 0$
- consistent ≠ efficient. Some form of the theorem might be more versatile for numerical implementations
- in particular, is there a form where the non-radiative process can be computed with manifestly physical momenta?

LBK theorem with shifted kinematics

From derivatives to shifts

$$\overline{|\mathcal{A}(p_1,\ldots,p_n,k)|}^2 = \sum_{ij}(-\eta_i\eta_j Q_i Q_j)\frac{p_i\cdot p_j}{p_i\cdot k\, p_j\cdot k}\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 \quad \to \textbf{LP}$$

$$+\sum_{ij}(-\eta_i\eta_j Q_i Q_j)\frac{p^i_\mu}{p_i\cdot k}G_j^{\mu\nu}\frac{\partial}{\partial p_j^\nu}\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 \quad \to \textbf{NLP}$$

Exploit the fact that derivatives are generators of translations:

$$f(x+\epsilon) = f(x) + \epsilon\,\frac{d}{dx}f(x)$$

$\to$ convert derivatives into **shifted momenta**

[Burnett-Kroll 1967, DelDuca-Laenen-Magnea-Vernazza-White 2017, vanBeekveld-Beenakker-Laenen-White 2019, DB-Kulesza 2021, Pal-Seth 2024, ...]

$$\overline{|\mathcal{A}(p_1,\ldots,p_n,k)|}^2 = \sum_{i,j=1}^{n} -\eta_i\eta_j\frac{p_i\cdot p_j}{p_i\cdot k\, p_j\cdot k}\left(1-\sum_j \delta p_j^\nu\frac{\partial}{\partial p_j^\nu}\right)\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2$$

From derivatives to shifts

LBK with shifted kinematics: [DB,Kulesza 2021]

$$\overline{|\mathcal{A}(p_1,\ldots,p_n,k)|}^2 = \underbrace{\sum_{i,j=1}^{n} -\eta_i\eta_j Q_i Q_j \frac{p_i\cdot p_j}{p_i\cdot k\, p_j\cdot k}}_{\textbf{LP}\text{ factor!}} \overline{|\mathcal{H}(p_1+\delta p_1,\ldots,p_n+\delta p_n)|}^2$$

$$\delta p_j^\nu = \eta_j \xi_j Q_j \left(\sum_{k,l} \eta_k\eta_l Q_k Q_l \frac{p_k\cdot p_l}{(p_k\cdot k)(p_l\cdot k)}\right)^{-1} \sum_i \left(\frac{\eta_i Q_i p_{i\mu}}{k\cdot p_i}\right)\left(\eta^{\mu\nu} - \frac{p_j^\mu k^\nu}{p_j\cdot k}\right)$$

Note that

$$\delta p_i = \mathcal{O}(k) \qquad \sum_i \delta p_i = -k \qquad p_i\cdot\delta p_i = 0$$

From derivatives to shifts

Simple case: **2 charged particles** [DelDuca-Laenen-Magnea-Vernazza-White 2017, vanBeekveld-Beenakker-Laenen-White 2019]

$$|\mathcal{A}(p_1, p_2, k)|^2 = \left(\sum_{i,j=1}^{2} -\eta_i \eta_j Q_i Q_j \frac{p_i \cdot p_j}{p_i \cdot k \, p_j \cdot k} \right) |\mathcal{H}(p_1 + \delta p_1, p_2 + \delta p_2)|^2 \qquad (1)$$

where

$$\delta p_1^\mu = \frac{1}{2} \left(-\frac{p_2 \cdot k}{p_1 \cdot p_2} p_1^\mu + \frac{p_1 \cdot k}{p_1 \cdot p_2} p_2^\mu - k^\mu \right)$$
$$\delta p_2^\mu = \frac{1}{2} \left(\frac{p_2 \cdot k}{p_1 \cdot p_2} p_1^\mu - \frac{p_1 \cdot k}{p_1 \cdot p_2} p_2^\mu - k^\mu \right)$$

Immediate to see

$$\delta p_i = \mathcal{O}(k) \quad \rightarrow \quad \text{LBK is NLP}$$
$$\delta p_1 + \delta p_2 = -k \quad \rightarrow \quad \text{momentum is conserved}$$
$$p_i \cdot \delta p_i = 0 \quad \rightarrow \quad \text{on shell?}$$

From derivatives to shifts

$$p_i \cdot \delta p_i = 0 \implies (p_i + \delta p_i)^2 = m^2 + \mathcal{O}(k^2) = m^2 + \text{NNLP}$$

Momenta are on-shell at NLP, hence theorem consistent at NLP

However, **masses do get shifted** by a NNLP amount!

$$(\delta p_j)^2 = Q_j^2 \left(\sum_{k,l} \eta_k \eta_l Q_k Q_l \frac{p_k \cdot p_l}{(p_k \cdot k)(p_l \cdot k)} \right)^{-1} \neq 0 \,,$$

i.e. with shifts we recovered momentum conservation, but momenta are off-shell at NNLP $\rightarrow$ problem for numerical implementations, where $k \neq 0$

Is there a LBK formulation fulfilling both **momentum conservation** AND **on-shell** condition exactly (i.e. not just at NLP)?

Modified shifts

Consider the ansatz

$$\delta p_i^\mu = \sum_j A_{ij}^{\mu\nu} p_{j\nu} + B_i^{\mu\nu} k_\nu \ ,$$

and determine coefficients $A_{ij}^{\mu\nu}$ and B_i by imposing conditions (i)-(iii).

$\rightarrow$ conditions not too constraining, many solutions for δp_i. But we seek a single solution!

- restrict our ansatz

$$\delta p_i^\mu = \sum_j A \eta_i \xi_i Q_i \frac{\eta_j Q_j}{k \cdot p_j} p_{j\nu} + B_i^{\mu\nu} k_\nu$$

- impose $p_{j\nu}$ and k_ν to be linear independent
- verify that solution has correct behaviour for $k \rightarrow 0$

MODIFIED SHIFTS

Result: [Balsach, DB, Kulesza 2023]

$$\delta p_i^\mu = A\eta_i\xi_i Q_i \sum_j \frac{\eta_j Q_j}{k\cdot p_j} p_{j\nu} G_i^{\nu\mu} - \frac{1}{2}\frac{A^2 Q_i^2 |\mathcal{S}_{\text{LP}}|^2}{p_i\cdot k} k^\mu \,,$$

with

$$A = \frac{1}{\chi}\left(\sqrt{1+\frac{2\chi}{|\mathcal{S}_{\text{LP}}|^2}} - 1\right) \qquad \chi = \sum_i \frac{\xi_i Q_i^2}{p_i\cdot k}\,.$$

$$|\mathcal{S}_{\text{LP}}|^2 = \sum_{i,j} \eta_i\eta_j Q_i Q_j \frac{p_i\cdot p_j}{(p_i\cdot k)(p_j\cdot k)}$$

- Momentum is **conserved** (exactly)
- Momenta are **on-shell** (exaclty)
- Shifts are $\mathcal{O}(k)$ $\implies$ equivalent to traditional LBK at NLP

$\implies$ This form of LBK allow computation of non-radiative process $\mathcal{H}$ with most general-purpose **event generators**

$\implies$ Numerical stability in **differential** distributions at NNLO in QED

[Banerjee,Engel,Signer,Ulrich 2020, Gogniat,Hoferichter,Ulrich 2025]

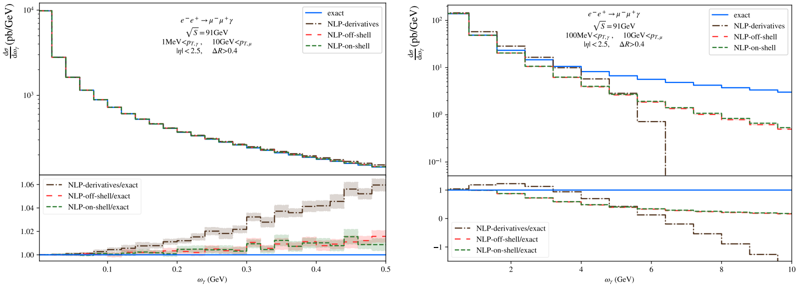
Three versions of (tree-level) LBK

- Three versions (derivatives, off-shell shifts, on-shell shifts) all theoretically consistent at NLP
- NNLP ambiguities contained in all three versions ("scheme" dependence)
- When spectra are computed numerically, NNLP effects are visible
- Which version is more efficient and versatile? Which has more predictive power?
- Once we select the best NLP method, what is resolution in momentum we need for NLP to be measurable?

Results for $e^+e^- \to \mu^+\mu^-\gamma$ and $pp \to \mu^+\mu^-\gamma$

Three versions of (tree-level) LBK

Results for $e^+e^- \to \mu^+\mu^-\gamma$ [Balsach, DB, Kulesza]



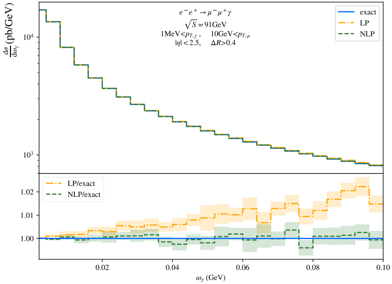
Note

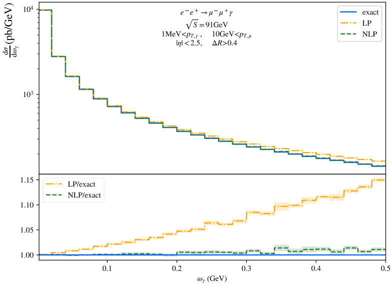
- non-radiative amplitude can be computed analytically (used here for derivatives and off-shell shifts)
- exact means tree-level with no soft expansion
- estimation of NNLP effects

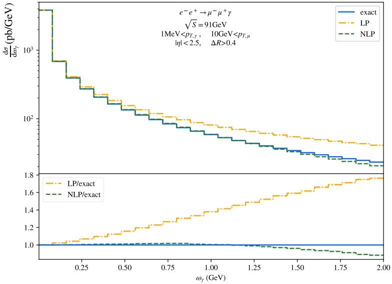
On-shell shifts work better. Used later as NLP

LP VS NLP

$e^+e^- \to \mu^+\mu^-\gamma$: (c.m.) ω distributions [Balsach, DB, Kulesza]

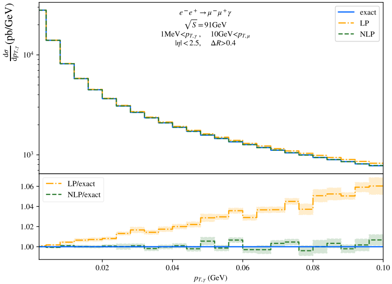


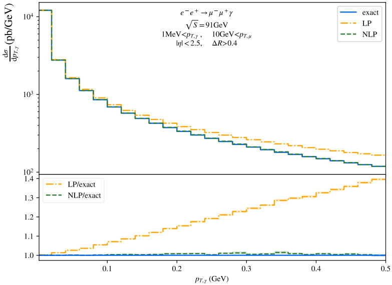


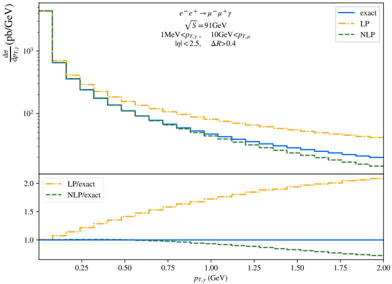


LP VS NLP

$e^+e^- \to \mu^+\mu^-\gamma$: p_t distributions [Balsach, DB, Kulesza]

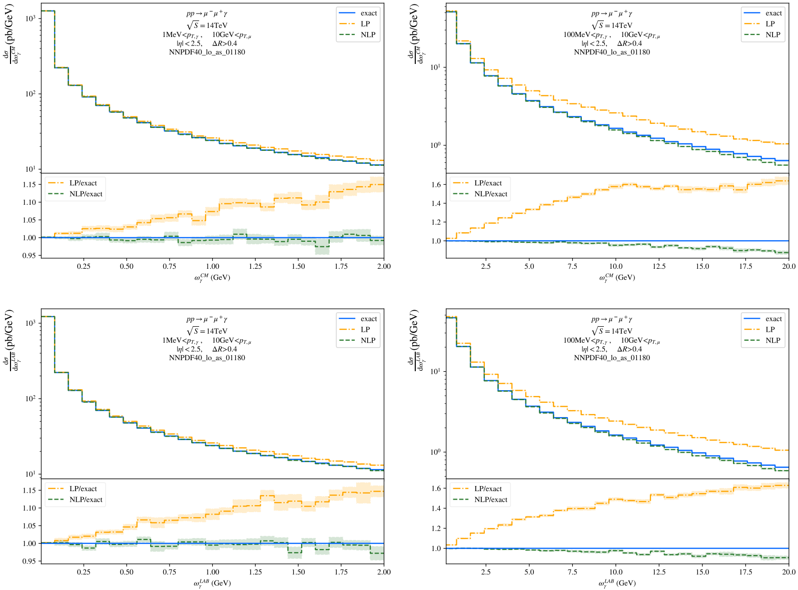






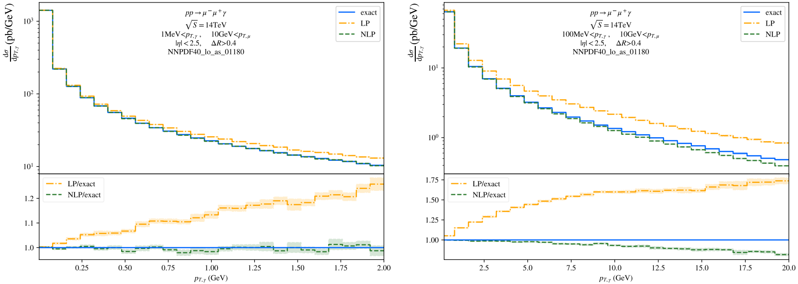
LP VS NLP

$pp \to \mu^+\mu^-\gamma$: (c.m. and lab) ω distributions [Balsach, DB, Kulesza]



LP VS NLP

$pp \to \mu^+\mu^-\gamma$: (c.m.) ω distributions [Balsach, DB, Kulesza]



Loop corrections to LBK theorem

NLP bremsstrahlung with QCD corrections

- LP soft photon theorem **does not** receive corrections at **one-loop**.

$$\epsilon^*_\mu(k)\mathcal{A}^\mu = \mathcal{S}_{LP}\ \mathcal{A}_n\ , \qquad \mathcal{A}_n = \mathcal{A}_n^{(0)}, \mathcal{A}_n^{(1)}, \mathcal{A}_n^{(2)}, \ldots$$

$$\mathcal{S}_{LP} = \sum_{i=1}^{n} Q_i \frac{\epsilon^*(k)\cdot p_i}{p_i\cdot k}\ ,$$

- at NLP, soft theorems **do** receive **one-loop corrections**.[Bern,Davies,Nohle 2014, He,Huang,Wen 2014, Larkoski,Neill, Stewart 2014, DB,Laenen,Magnea,Vernazza,White 2014]

$$\epsilon^*_\mu(k)\mathcal{A}^{\mu(0)} = (\mathcal{S}_{LP} + \mathcal{S}_{NLP-tree})\ \mathcal{A}_n^{(0)}\ ,$$

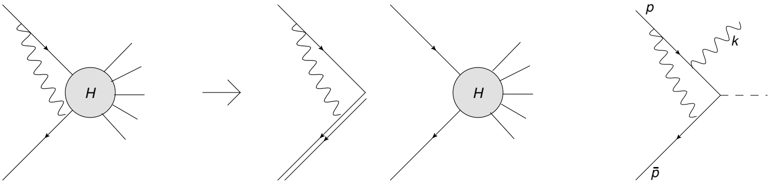
$$\epsilon^*_\mu(k)\mathcal{A}^{\mu(1)} = (\mathcal{S}_{LP} + \mathcal{S}_{NLP-tree})\ \mathcal{A}_n^{(1)} + \ ?\ ,$$

$$\mathcal{S}_{LP} = \sum_{i=1}^{n} Q_i \frac{\epsilon^*(k)\cdot p_i}{p_i\cdot k}\ , \qquad \mathcal{S}_{NLP-tree} = \sum_{i=1}^{n} Q_i \frac{\epsilon^*_\mu(k)k_\nu(\sigma^{\mu\nu} + L^{\mu\nu})}{p_i\cdot k}$$

Various sources of correction. E.g. soft region in the massive case [Engel,Signer,Ulrich 2021]. In the high energy limit, it is interesting to look at the **massless limit** (crucial for the massless parton model) and the **collinear region**

NLP BREMSSTRAHLUNG WITH QCD CORRECTIONS

Virtual collinear effects are captured by **radiative jet functions** J^μ [DelDuca 1990, DB, Laenen, Magnea, Vernazza, White 2014, Gervais 2017, Beneke, Garny, Szafron, Wang 2018, Laenen, Damste, Vernazza, Waalewijn, Zoppi 2020, Liu, Neubert, Schnubel, Wang 2021].



In particular, the one-loop quark radiative jet function in dimensional regularization (with $d = 4 - 2\epsilon$ and $\bar{\mu}$ the $\overline{\text{MS}}$ scale) reads [DB,Laenen,Magnea,Melville,Vernazza,White,2015]

$$J^{\mu(1)} = \left(\frac{\bar{\mu}^2}{2p\cdot k}\right)^{\epsilon} \Bigg[\left(\frac{2}{\epsilon} + 4 + 8\epsilon\right)\left(\frac{n\cdot k}{p\cdot k}\frac{p^\mu}{p\cdot n} - \frac{n^\mu}{p\cdot n}\right) - (1+2\epsilon)\frac{ik_\alpha S^{\alpha\mu}}{p\cdot k}$$

$$+ \left(\frac{1}{\epsilon} - \frac{1}{2} - 3\epsilon\right)\frac{k^\mu}{p\cdot k} + (1+3\epsilon)\left(\frac{\gamma^\mu \not{n}}{p\cdot n} - \frac{p^\mu}{p\cdot k}\frac{\not{k}\not{n}}{p\cdot n}\right)\Bigg] + \mathcal{O}(\epsilon^2, k)$$

NLP BREMSSTRAHLUNG WITH QCD CORRECTIONS

Thus, the next-to-soft theorem (i.e. LBK theorem) receives a **logarithmic correction**:

$$\epsilon^*_\mu(k)\mathcal{A}^{\mu(0)} = \left(\mathcal{S}_{LP} + \mathcal{S}_{NLP-tree}\right)\mathcal{A}_n^{(0)} \,,$$

$$\epsilon^*_\mu(k)\mathcal{A}^{\mu(1)} = \left(\mathcal{S}_{LP} + \mathcal{S}_{NLP-tree}\right)\mathcal{A}_n^{(1)} + \left(\sum_i \epsilon^*_\mu(k)\, q_i\, J_i^{\mu(1)}\right)\mathcal{A}_n^{(0)} \,,$$

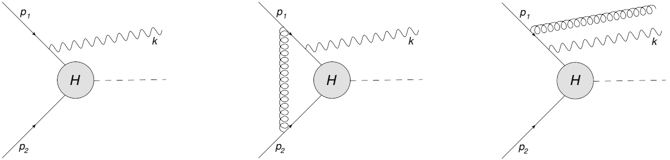
$$\mathcal{S}_{LP} = \sum_{i=1}^{n} Q_i \frac{\epsilon^*(k)\cdot p_i}{p_i\cdot k} \,, \qquad \mathcal{S}_{NLP-tree} = \sum_{i=1}^{n} Q_i \frac{\epsilon^*_\mu(k) k_\nu (\sigma^{\mu\nu} + L^{\mu\nu})}{p_i\cdot k}$$

$$\left(\sum_i \epsilon^*_\mu(k)\, q_i\, J_i^{\mu(1)}\right)\mathcal{A}_n^{(0)} = \frac{2}{p_1\cdot p_2}\left[\sum_{ij}\left(\frac{1}{\epsilon} + \log\left(\frac{\bar{\mu}^2}{2p_i\cdot k}\right)\right) q_j\, p_i\cdot k\, \frac{p_j\cdot\epsilon}{p_j\cdot k}\right]\mathcal{A}_n^{(0)}$$

- Note that amplitude is IR divergent $\epsilon \to 0$
- $\log(\omega_k)$ corrections to soft theorems in QED also discussed (mainly classically) by Laddha-Sahoo-Sen. Here however more standard approach (i.e. dim.reg.) to regularization of soft and collinear divergences, which allows implementation in the massless limit required in QCD partonic calculations.

NLP BREMSSTRAHLUNG WITH QCD CORRECTIONS

IR divergences ($1/\epsilon$) cancel by adding real emission diagram:



The **soft photon** emission from the loop with a **collinear gluon** is captured by the radiative jet function J^μ (note here the **mixed QED-QCD** effect)
The corresponding contribution is what is needed for a process with a single quark-antiquark pair in the **massless limit** such as

- $e^+e^- \to q\,\bar{q}\,\gamma$
- $p\,p \to \mu^+\mu^-\gamma$
- ...

For processes with more than two colored particles situation more subtle (but structure is similar)

NLP bremsstrahlung with QCD corrections

The soft photon bremsstrahlung at $\mathcal{O}(\alpha_s)$ becomes

$$\frac{d\sigma_{\text{NLP}}}{d^3k} = \frac{d\sigma_{\text{LP+(NLP-tree)}}}{d^3k} + \frac{\alpha_s}{4\pi}\frac{d\sigma_{\text{NLP-}J}}{d^3k} \,,$$

where

$$\frac{d\sigma_{\text{NLP-}J}}{d^3k} \propto \frac{\alpha}{\omega_k}\int d^3p_3 \ldots d^3p_n \left(\sum_{i=1}^{2}\eta_i \frac{\log\left(\frac{\bar{\mu}^2}{2p_i\cdot k}\right)}{p_i\cdot k}\right) d\sigma_H(p_1, ..., p_n)$$

- **Correction of order** $\boxed{\alpha_s \log\left(\frac{\bar{\mu}^2}{2p_i\cdot k}\right)}$ **to LP spectrum** $\frac{d\sigma}{d\omega_k}$
  hence particularly enhanced for small ω_k **and small** k_t
- **expecially relevant for hadrons (since for leptons -** $\alpha \ll \alpha_s$**,** $m \to 0$**)**

CONCLUSIONS

- Different formulations of (tree-level) LBK theorem all theoretically consistent at NLP
- Different formulations correspond to reshuffling of NNLP effects, which might be numerically relevant (scheme choice) $\Longrightarrow$ not all formulations equally efficient
- New LBK formulation with on-shell shifted kinematics allows standard event generation for non-radiative process
- Numerical results show resolution in energy/momentum for NLP effects to be visible