Effective models of color superconductivity: Challenges and Prospects

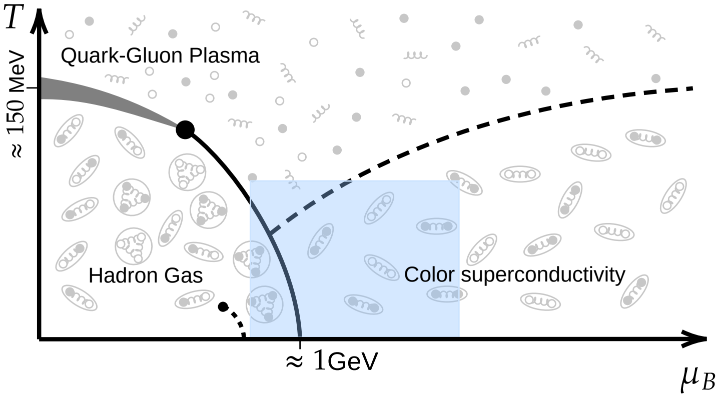
Hosein Gholami (TU Darmstadt)

12/11/2025, EMMI Workshop

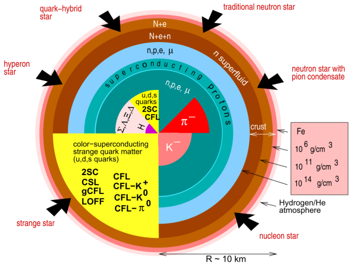
TECHNISCHE UNIVERSITÄT DARMSTADT

CRC-TR 211
Strong-interaction matter under extreme conditions

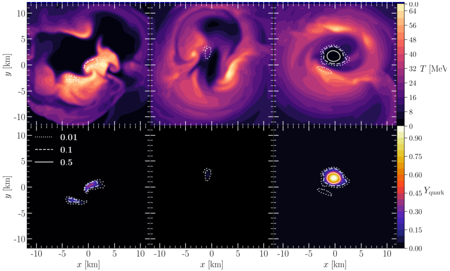
DFG



- QCD phase diagram: where is QCD?
- Phase structure at large (but not asymptotically large) density and moderate T is relevant for neutron stars and neutron star mergers



[Weber (1999)]



[Tootle et al. (2022)]

- Merger simulations: densities produced in merger remnant might be sufficient to produce quark matter
- Expected implications: Modified post-merger frequency spectrum [Elias R. Most et al. (2019)] [Bauswein et al. (2019)]

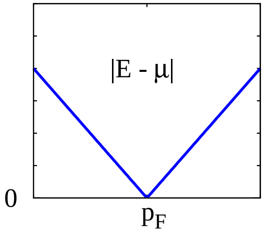
Motivation: signatures of color superconducting phases in neutron star cores or merger remnants?
Method: Use effective models for studying color superconductivity

- This talk: How can we improve this models?

Why (color) Superconductivity

- Noninteracting fermions at $T = 0$
  - Particles at the Fermi surface can be created at the Fermi surface with no free-energy cost.

- Cooper theorem: With a finite attractive interaction between particles
  - Fermi surface becomes unstable against pair creation $\rightarrow$ "Cooper pairs"
  - Bose condensation of the Cooper pairs $\Rightarrow$ Energy gap Δ in exaction spectrum: $\omega = \sqrt{(E-\mu)^2 + \Delta^2}$



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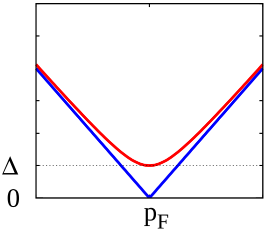

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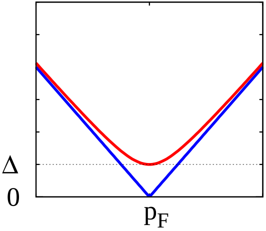
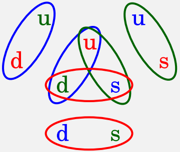


- **Solid-state superconductor:** Effective attractive interaction between electrons through the interaction with lattice $\rightarrow$ phonons
- **Color superconductor:** QCD: attractive quark-quark interaction
  - at higher densities via a single gluon exchange in the $\bar{3}$ color channel
  - at intermediate densities also via instanton exchange [Rapp, Schäfer, Shuryak and Velkovsky 1998]

Color Superconductivity

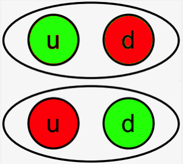
- diquark condensates: $< q_i \mathcal{O}_{ij} q_j >$
- Pauli principle: $\mathcal{O} = \mathcal{O}_{\text{spin}} \otimes \mathcal{O}_{\text{color}} \otimes \mathcal{O}_{\text{flavor}} =$ totally antisymmetric
- most attractive channel: color $\bar{3}$ and spin 0 (antisymmetric) $\Rightarrow$ mixing flavors

Color-flavor-locking (CFL)



Large $\mu \gg M_s$
3 finite gap parameters
$\Delta_{ud}, \Delta_{us}, \Delta_{ds}$

2SC



Intermediate $\mu \lesssim M_s$
1 finite gap parameter
Δ_{ud}

Model

Nambu Jona-Lasinio (NJL)-type model

$$
\begin{aligned}
\mathcal{L} = \quad & \bar{\psi}(i\partial\!\!\!/ - m)\psi && \text{kinetic term} \\
& + G \sum \left[(\bar{\psi}\tau_a\psi)^2 + (\bar{\psi}i\gamma_5\tau_a\psi)^2 \right] && \text{scalar NJL interaction} \\
& - K \left[\det{}_f(\bar{\psi}(\mathbb{1}+\gamma_5)\psi) + \det{}_f(\bar{\psi}(\mathbb{1}-\gamma_5)\psi) \right] && \text{'t Hooft (KMT) interaction} \\
& + G\,\eta_D \sum (\bar{\psi}i\gamma_5\tau_A\lambda_{A'}\psi^c)(\bar{\psi}^c i\gamma_5\tau_A\lambda_{A'}\psi) && \text{diquark interaction} \\
& + G\,\eta_V (\bar{\psi}\gamma^\mu\psi)^2 && \text{vector interaction}
\end{aligned}
$$

Mean field approximation: Linearise theory around condensates

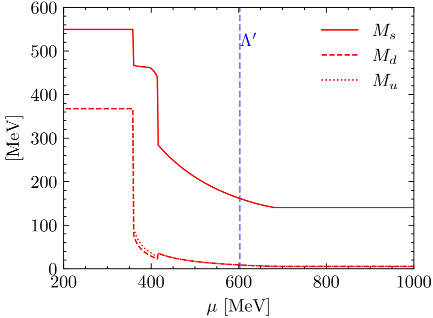
$$
\begin{aligned}
\phi_f =& \langle \bar{\psi}_f \psi_f \rangle && f = u, d, s \\
\Delta_A =& -2G\eta_D \langle \bar{\psi}^c \gamma_5 \tau_A \lambda_A \psi \rangle && A = 2(ud), 5(us), 7(ds)
\end{aligned}
$$

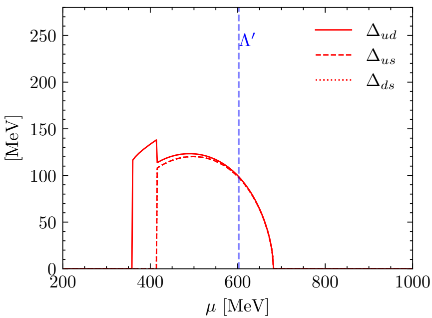
Then minimizing with respect to these condensates and enforce charge and color-neutrality locally

- $\Lambda^{'}, G, K, m$ fitted to vacuum meson spectrum
- Regularization: sharp 3-momentum cutoff $\Lambda^{'} = 602 MeV$

Cutoff Artefacts

- Solution to the gap equations at $T = 0$





- Cutoff artefacts
  - Unexpected behaviour for the gaps: diquark gaps decrease in value at $\mu \sim \Lambda'$ and eventually vanish

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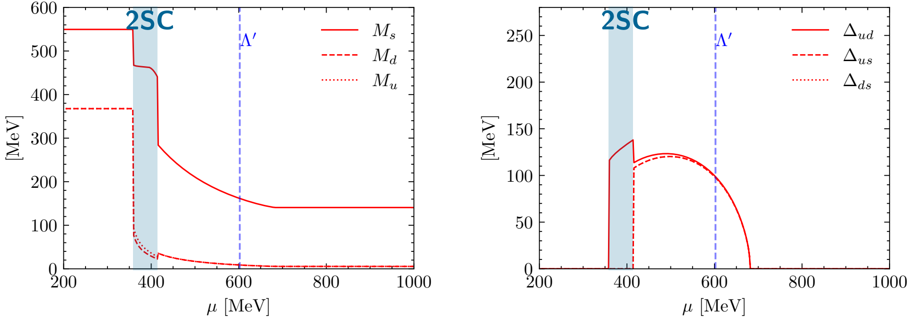

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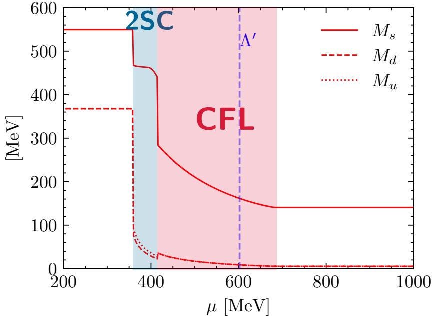

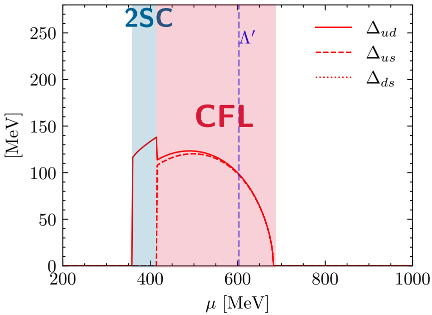

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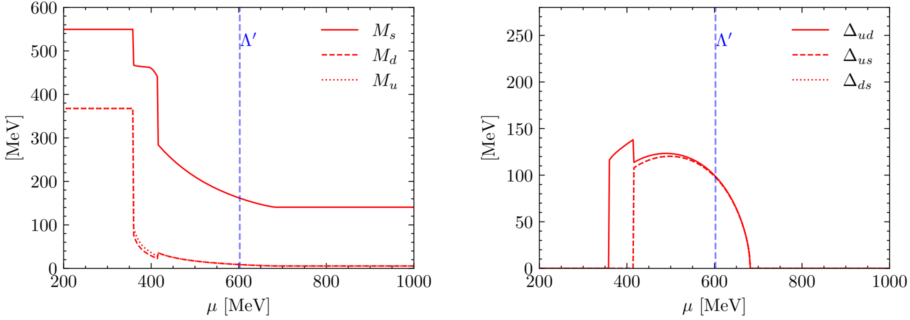

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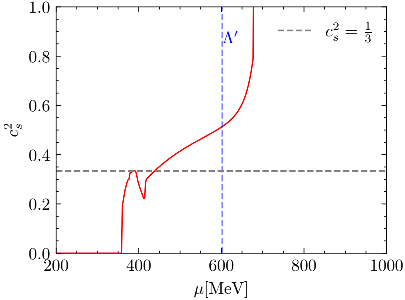
Cutoff Artefacts

- speech of sound: $c_s^2 = \frac{dp}{d\epsilon}\Big|_{T=0}$

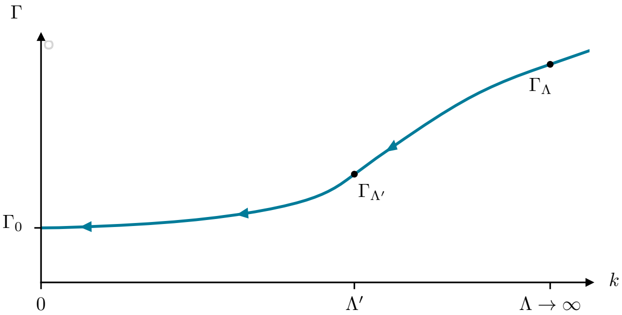


- Cutoff artefacts
  - Diquark gaps decrease in value at $\mu \sim \Lambda^{'}$
  - causality violation $c_s^2 > 1$
- One should not rely on this model hight densities $\rightarrow$ One can not rely on the astrophysical aspects of this model

Renormalization-Group consistent treatment

- Functional Renormalization Group (FRG)
  - Central object is Γ_k: scale-dependent one-particle irreducible effective action with the RG scale k: It contains physics at scales above k
  - Goal is to achieve full quantum effective action Γ_0, by solving the the Wetterich equation and flowing down to $k = 0$

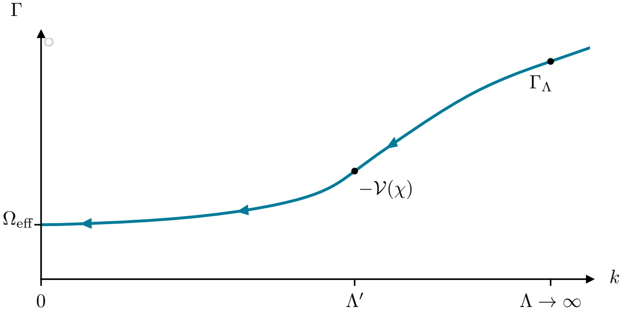
$$\partial_k \Gamma_k = -\frac{1}{2} \mathrm{Tr} \left[\left(\Gamma_k^{(2)} + R_k \right)^{-1} \partial_k R_k \right]$$



FRG and MFA

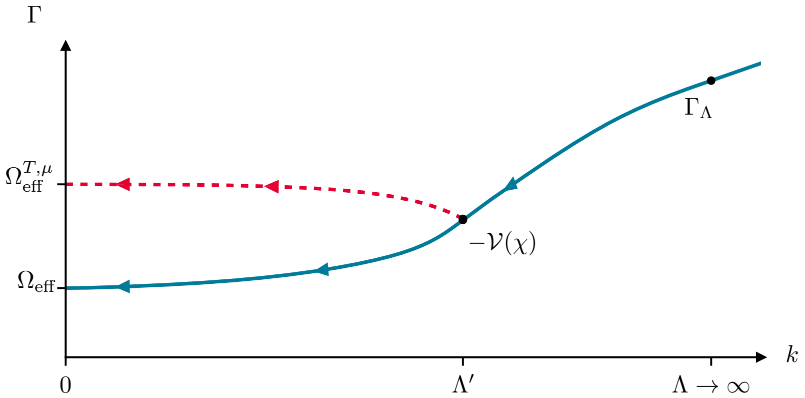
$$\Omega_{\text{eff}} = \frac{1}{2}\int_0^{\Lambda'} \text{tr}\left[\ln\left(S^{(2)}\right)\right] - \mathcal{V}(\chi) \rightleftharpoons \Gamma_0 = \frac{1}{2}\int_0^{\Lambda'} \text{tr}\left[\ln\left(S^{(2)}\right)\right] + \Gamma_{\Lambda'}$$

- A regularized MFA model is defining an RG scale Λ' at which all correlators are "zeroed out" except for the ones defined in $\mathcal{V}(\chi)$
- A realization: All points on this trajectory are equivalent $\rightarrow$ starting from any point, will lead to the same Ω_{eff}



Renormalization Group-Consistency

- RG consistency: $\frac{d}{d\{\mu,T,\chi\}}\frac{d}{d\log\Lambda'}\Omega_{\text{eff}}=0$
  A model's predictions should be independent of the regularization or renormalization scheme employed.
- **Problem**: Λ' isn't large enough with respect to all scales of the system when in medium: $\Lambda' \sim \mu, T, \Delta_i, M_j$.
- **Solution:** *Renormalization group-consistent* treatment : start from $\Gamma_{\Lambda\to\infty}$



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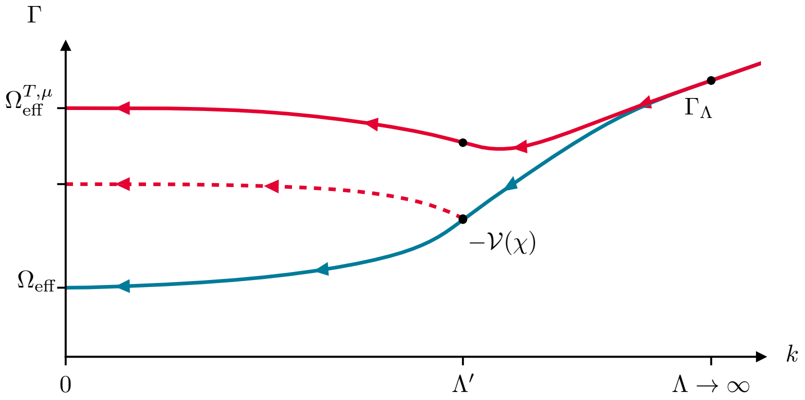
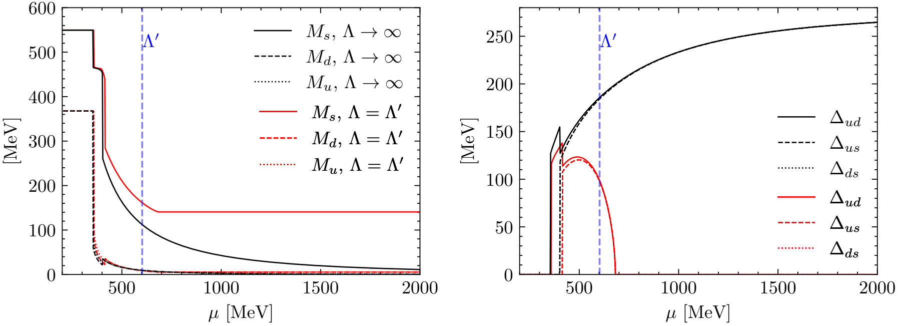


RG consistent treatment of the NJL model



- Cutoff artefacts
  - Diquark gaps decrease in value at $\mu \sim \Lambda^{'}$ ✓
  - causaulity violation $c_s^2 > 1$

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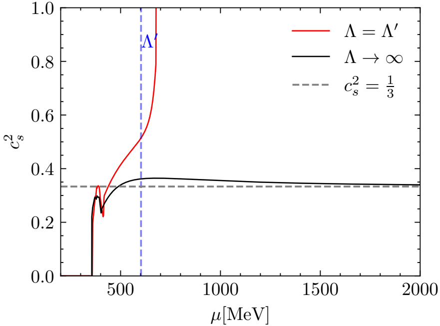
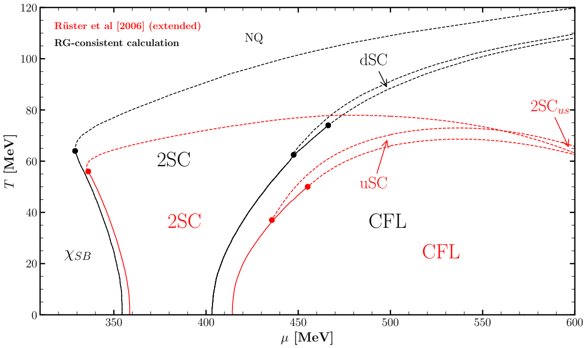


- Cutoff artefacts
  - Diquark gaps decrease in value at $\mu \sim \Lambda'$ ✓
  - causaulity violation $c_s^2 > 1$ ✓
- Spurious larges values of speed of sound vs. $c_s^2 \to \frac{1}{3}^+$ that is a genuine effect of color-superconductivity

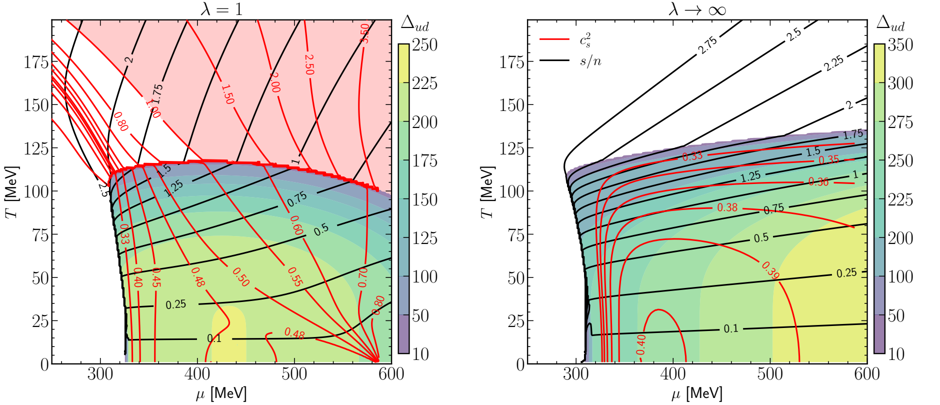
RG consistent treatment of the NJL model

- RG consistent treatment of color superconductivity in the NJL model
  [HG, M. Hofmann, M. Buballa, Phys. Rev. D 111, 014006 (2025), arXiv:2408.06704]
  [HG, M. Hofmann, M. Buballa, Quark Matter 2025 Proceedings, arXiv:2508.21735]



- Old puzzles [Iida et al. (2004)], [Fukushima (2005)], [Rüster et al. (2005)] resolved via RG consistent treatment.

Why it matters for neutron stars/mergers?



- Without this procedure, these subtle artefacts can distort all the conclusions drawn from the model.

Challenges and Road Map

Effective models come with parameters: η_D, η_V How do we fix them?

Challenges:

- Diquarks are not asymptotic states $\rightarrow$ cannot be directly observed in experiment
- Non-existent experimental input on diquark interactions

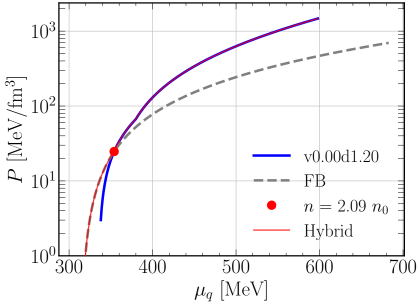
Strategy:

- First: Study what could be the possibilities from the model
- For that, we need to connect micro to macro physics
  EoS $\rightarrow$ TOV equations $\rightarrow$ Mass-Radius relations

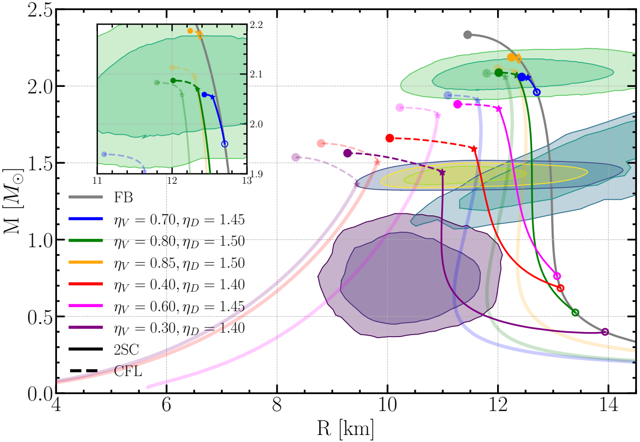
Astrophysical Constraints: Procedure

Maxwell construction:

- Pick a hadronic matter EoS compatible with nuclear physics and astrophysical constraints
- First-order phase transition: $P_{\mathsf{HM}}(\mu) = P_{\mathsf{QM}}(\mu)$ at transition point
- Mixed phase between pure hadronic and pure quark matter



Astrophysical Constraints: Results



- Hybrid stars with CSC cores can reach $2M_\odot$ and satisfy observational constraints

[HG, I. A. Rather, M. Hofmann, M. Buballa, J. Schaffner-Bielich, Phys. Rev. D 111, 103034 (2025)]

[J. E. Christian, I. A. Rather, HG, M. Hofmann, Astron. Astrophys. 701, A145 (2025)]

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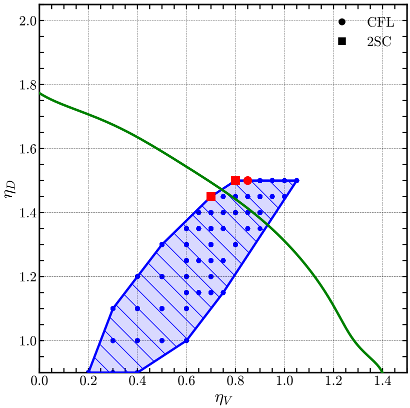

- Hybrid stars with CSC cores can reach $2M_\odot$ and satisfy observational constraints $\rightarrow$ more on Jan-Erik's talk

[HG, I. A. Rather, M. Hofmann, M. Buballa, J. Schaffner-Bielich, Phys. Rev. D 111, 103034 (2025)]

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Next Steps and Open Questions

With these parameter selections:

- We can now start searching for physics of CSC in these models
- Most interesting: **Merger simulations**
  - Next step: Making tables from this model suitable for merger simulations
  - Transport properties → Marco's talk

But the key question remains:

- Is there a way to justify/enhance these parameter values?

Outlook: Constraining from First Principles QCD

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Constraining Color Superconducting Low-energy Models from First Principle QCD

The idea:

- Diquarks are not asymptotic states, but one can calculate *off-shell* diquark correlators in vacuum:
  - Diquark curvature mass
  - Diquark-diquark scattering amplitude
  - Quark-diquark Yukawa coupling
  - ...
- RG-consistent low-energy models can be matched to these vacuum correlators

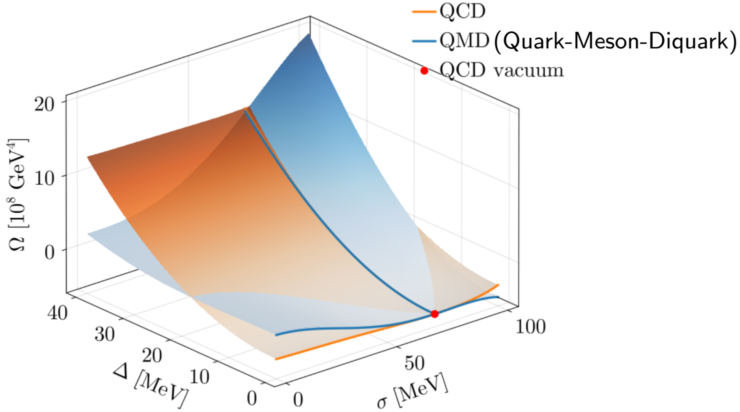
[HG, L. Kurth, U. Mire, M. Buballa, B.-J. Schaefer, arXiv:2505.22542]

Matching to QCD

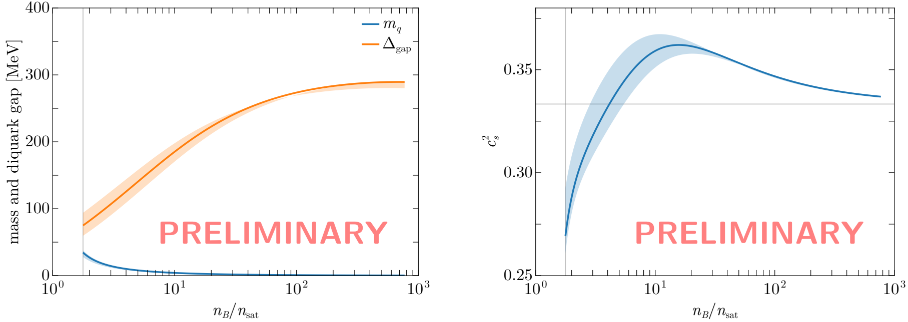
Vacuum matching procedure:

- Calculate off-shell diquark correlators from FRG QCD in vacuum
- Match the low-energy model parameters to these QCD results

Based on HG, U. Mire, F. Rennecke, B.-J. Schaefer, S. Yin (in preparation)

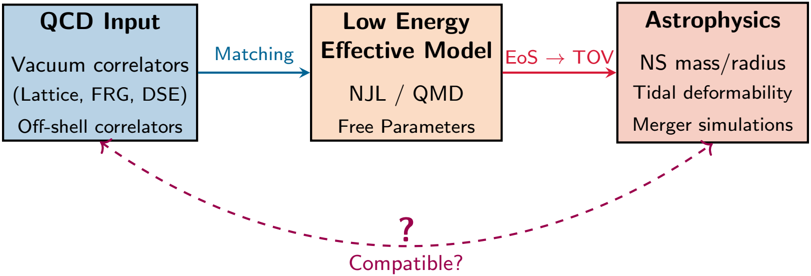


Preliminary Results



- Diquark gap and speed of sound from a LEM model matched to quantities computed from QCD in vacuum

Bridging QCD and Astrophysics: Diquarks



- Next step: Would QCD-anchored parameters be compatible with astrophysical constraints?
- This will help us systematically improve our models

The MUSES NJL Module

[with M. Hofmann, M. R. Pelicer and Y. Yang]

- The NJL module is open-source and publicly available.
- Implements the RG-consistent NJL model including diquark and vector degrees of freedom.
- Enables EoS calculations for neutron star properties and integrates with other MUSES modules to compute hybrid star observables.

Summary and Outlook

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Summary

- Color superconductivity is expected in dense QCD matter → relevant for neutron stars and neutron star mergers
- We can use Low-energy effective models to incorporate effects of color superconductivity
- RG-consistent treatment removes cutoff artifacts and resolves old puzzles
  [HG, M. Hofmann, M. Buballa, Phys. Rev. D 111, 014006 (2025)]
  [HG, L. Kurth, U. Mire, M. Buballa, B.-J. Schaefer, arXiv:2505.22542]
- Astrophysical constraints bound the parameter space
  [HG, I. A. Rather et al., Phys. Rev. D 111, 103034 (2025)]
  [J. E. Christian, I. A. Rather, HG, M. Hofmann, Astron. Astrophys. 701, A145 (2025)]

Outlook

- Matching to first-principles QCD vacuum correlators (in preparation)
- 3D tables for merger simulations
- Bridging QCD and astrophysics: Can we improve our models?

Thank you for your attention!