Helmholtz Forschungsakademie Hessen für FAIR

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

CRC-TR 211 Strong-interaction matter under extreme conditions

EMMI Workshop: Collective phenomena and Equation-of-State of dense baryonic matter @ GSI

Wigner Phase-Space Densities of Nuclear Clusters and Hypernuclei

Jiaxing Zhao

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In collaboration with: Joerg Aichelin and Elena Bratkovskaya

Based on: arXiv: 2508.05814, accepted by PRC.

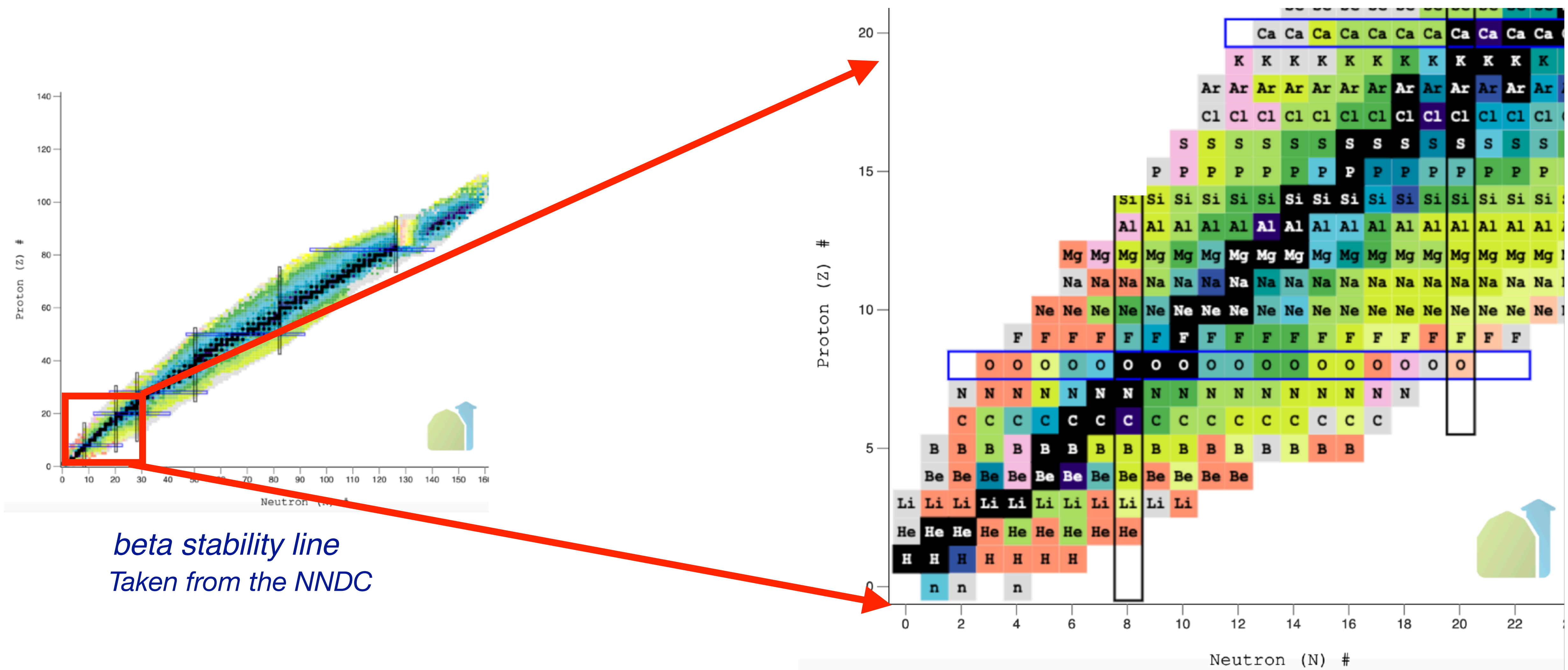
Outline

- ➡ *Brief introduction about nuclear cluster and hypernuclei*
- ➡ *N-body Schrödinger equation*
- ➡ *Wigner function and probability*
- ➡ *Summary and outlook*

Nuclear cluster and hypernuclei

Nuclear cluster : bound state of a few nucleons

$$d(pn),\ t(pnn),\ {}^3He(2p+n),\ {}^4He(2p+2n), \ldots$$



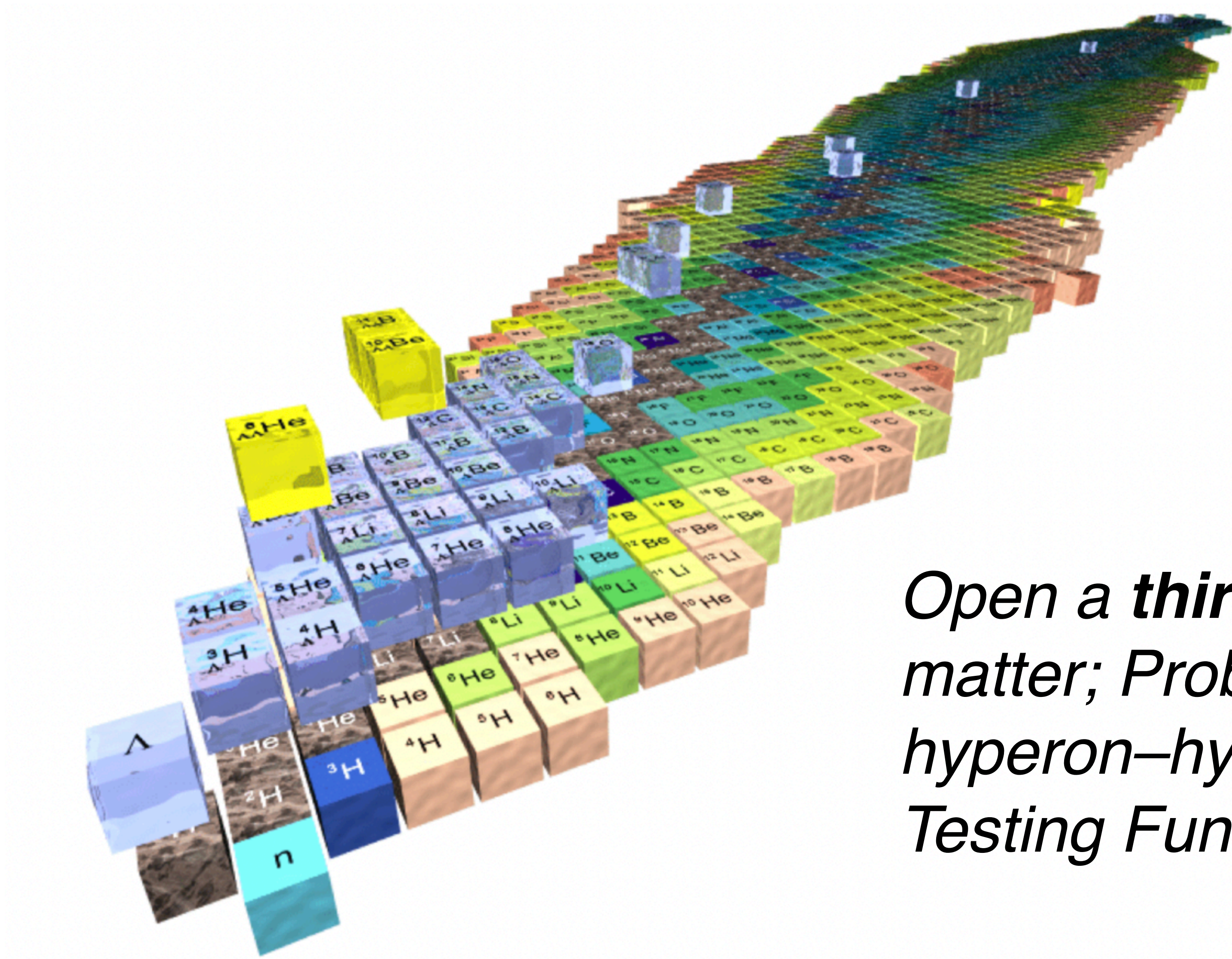
beta stability line
Taken from the NNDC

An ideal system for investigating nuclear forces, nuclear structure, and few-body physics, as well as for revealing nucleon correlations.

Nuclear cluster and hypernuclei

Hypernuclei: nuclei in which one or more nucleons are replaced or accompanied by hyperons ($\Lambda, \Sigma, \Xi, \Omega$)

$$^{3}_{\Lambda}H(p+n+\Lambda),\ ^{4}_{\Lambda}H(p+2n+\Lambda),\ ^{4}_{\Lambda}He(2p+n+\Lambda),\ldots$$

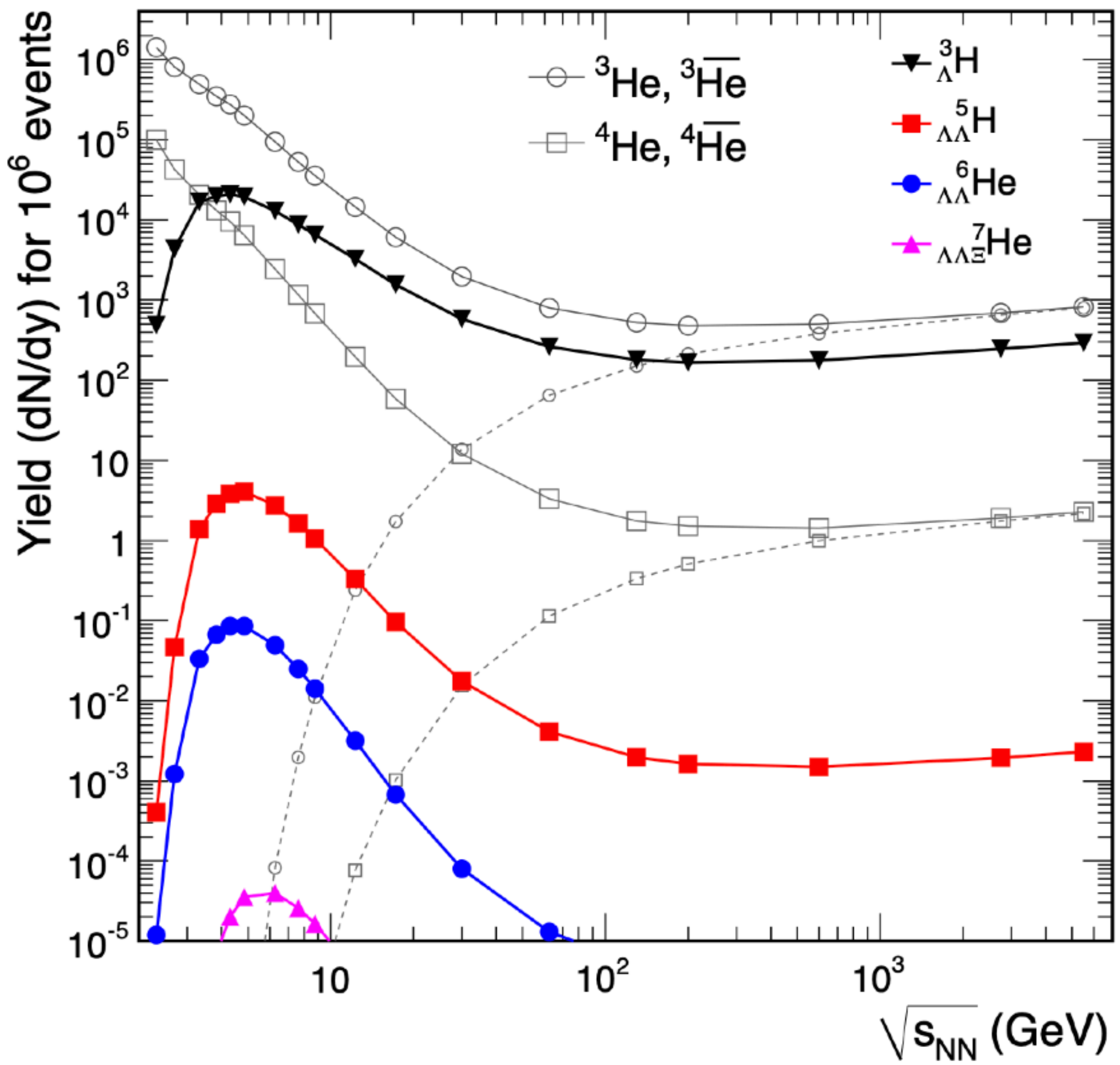


*Open a **third dimension** in the study of nuclear matter; Probe the hyperon–nucleon (Y–N) and hyperon–hyperon (Y–Y) interactions; Testing Fundamental Symmetries.*

Taken from: IX International Conference on Hypernuclear and Strange Particle Physics

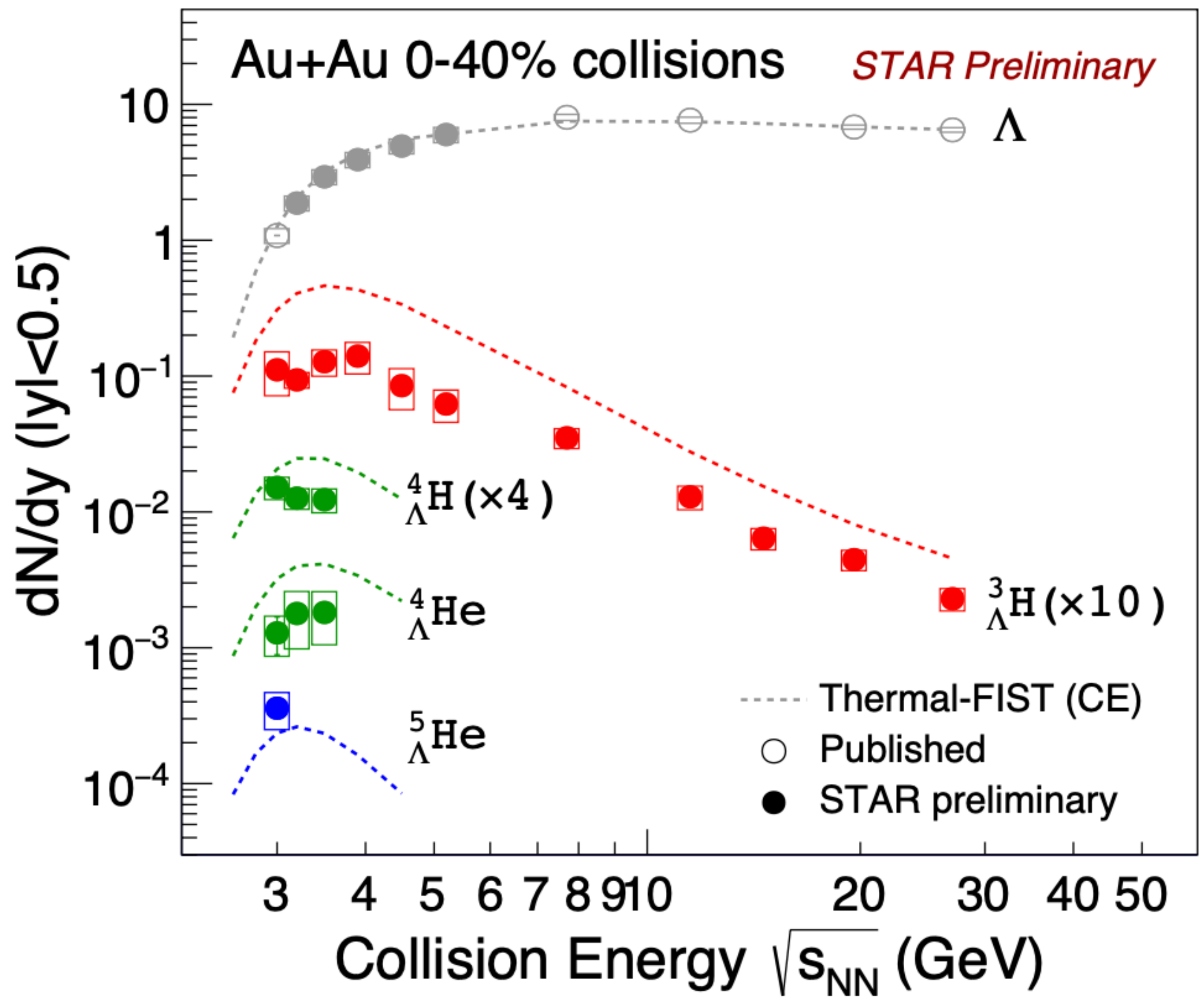
Nuclear cluster and hypernuclei in heavy ion collisions

The production of nuclear cluster and hypernuclei are largely enhanced in heavy ion collisions at RHIC BESII, CBM, HAIF, NICA etc. energies.



A. Andronic, et al. Phys.Lett.B 697 (2011) 203-207

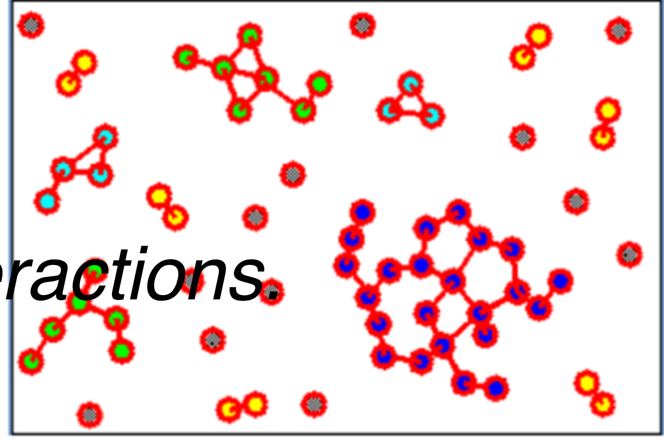
J. Steinheimer, et al. Phys.Lett.B 714 (2012) 85-91



Yingjie's slides, QM2025

Sensitive to the Equation-of-State (EoS); study the clusterization mechanism; hyperon–nucleon (Y–N) and hyperon–hyperon (Y–Y) interaction; to solve the "Hyperon Puzzle" in Neutron Stars, and so on.

Clusterization mechanism



Crucial to probe the EoS to reveal the correlations and interactions.

Elena's talk, Mon

☑ *Statistical model*

Clusters are in thermal equilibrium on the chemical freeze-out surface.

No dynamical infromations.

A. Andronic, et al. Phys.Lett.B 697 (2011) 203-207
J. Steinheimer, et al. Phys.Lett.B 714 (2012) 85-91

☑ *"Potential" model*

Clusters are identified during the dynamical evolution of the system by the Minimum Spanning Tree (MST and aMST).

$|\vec{r_i} - \vec{r_j}| \leq 4$ fm *& negative binding energy*

J. Aichelin, Phys. Rept. 202, 233 (1991).
J. Aichelin et al, Phys. Rev. C 101, 044905 (2020) (PHQMD)

☑ *Kinetic model*

Clusters are formed dynamically via reactions like: $NNN \rightarrow d + N; NN\pi \rightarrow d + \pi, \ldots$

Hard to include all channels especially for heavy clusters. Finite-size effect —> wave functions.

D. Oliinychenko, et al. Phys.Lett.B 714 (2012) 85-91 (SMASH
G. Coci et al. Phys. Rev. C 108, 014902 (2023) (PHQMD)
R. Wang et al. Phys.Rev.C 108 (2023) 3, L031601.

☑ *Coalescence model*

Clusters are formed from nucleons (hyperons) that are close together in phase space.

Coalescence model

PHYSICAL REVIEW VOLUME 129, NUMBER 2 15 JANUARY 1963

Deuterons from High-Energy Proton Bombardment of Matter

S. T. BUTLER AND C. A. PEARSON
The Daily Telegraph Theoretical Department, School of Physics, University of Sydney, Sydney, N.S.W., Australia
(Received 13 August 1962) 1962

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Coalescence and flow in ultra-relativistic heavy ion collisions

Rüdiger Scheibl and Ulrich Heinz 1999
Institut für Theoretische Physik, Universität Regensburg, D-93040 Regensburg, Germany
(March 16, 1999 — published in Physical Review C 59 (1999), 1585-1602)

Using a density matrix approach to describe the process of coalescence, we calculate the coalescence probabilities and invariant momentum spectra for deuterons and antideuterons. We evaluate our expressions with a hydrodynamically motivated parametrization for the source at freeze-out which implements rapid collective expansion of the collision zone formed in heavy ion collisions. We find that the coalescence process is governed by the same *lengths of homogeneity* which can be extracted from HBT interferometry. They appear in the absolute cluster yield via an *effective volume* factor as well as in a quantum mechanical correction factor which accounts for the internal structure of the deuteron cluster. Our analysis provides a new interpretation for the parameters in the popular

Coalescence is also a popular model used to describe the parton hadronization in QGP (~2002 to now)!

$$P^A(t) = \mathrm{Tr}[\rho^A \hat{\rho}_{\mathrm{tot}}]$$

density matrix of the cluster A *density matrix of the system*

$$N_A = g_A \int \prod_{i=1}^{A} d^3 r_i \, d^3 p_i \; f_i(\mathbf{r}_i, \mathbf{p}_i) \; W_A\big(\{\mathbf{r}_i, \mathbf{p}_i\}\big)$$

Coalescence model

The Wigner function can self-consistently be determined by the wavefunction.

$$W_N(\mathbf{r}_1,\ldots,\mathbf{r}_N;\mathbf{p}_1,\ldots,\mathbf{p}_N) = \int d^3\mathbf{s}_1\ldots d^3\mathbf{s}_N e^{-i\sum_{k=1}^{N}\mathbf{p}_k\cdot\mathbf{s}_k}\Psi^*\left(\mathbf{r}_1+\frac{\mathbf{s}_1}{2},\ldots,\mathbf{r}_N+\frac{\mathbf{s}_N}{2}\right)\Psi\left(\mathbf{r}_1-\frac{\mathbf{s}_1}{2},\ldots,\mathbf{r}_N-\frac{\mathbf{s}_N}{2}\right)$$

Coalescence model

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Three main ways to approximate the Wigner function for cluster formation:

- *within a certain relative spatial and relative momentum (UrQMD)*

$$|\mathbf{r}_1-\mathbf{r}_2|<\Delta r_{max}$$

$$|\mathbf{p}_1-\mathbf{p}_2|<\Delta p_{max}$$

	deuteron	^{3}H or ^{3}He	^{4}He	$^3_\Lambda$H	$^4_\Lambda$H
spin-isospin	3/8	1/12	1/96	1/12	1/96
$\Delta r_{\max}$ [fm]	4.0	3.5	3.5	9.5	9.5
$\Delta p_{\max}$ [GeV]	0.33	0.45	0.55	0.15	0.25

T. Reichert, et al. arXiv: 2504.17389

- *Gaussian-form of the Wigner density*

R. Scheibl and U. Heinz, Phys. Rev. C 59, 1585 (1999)
L. Zhu, C. Ko, and X. Yin, Phys. Rev. C 92, 064911 (2015)
K. Sun and C. Ko, Phys. Rev. C 103, 064909 (2021)
D. Liu, et al. Phys. Lett. B 855, 138855.
Q. Lin, et al. arXiv:2503.01128.
R. Wang, et al. Phys.Rev.C 112 (2025) 3, 034908 (2024)

$$W_2^{1S}(\mathbf{r},\mathbf{q})=8e^{-\frac{\mathbf{r}^2}{\sigma^2}-\mathbf{q}^2\sigma^2}.$$

width parameter is either related to the root-mean-square (rms) radius of the cluster or a free parameter tuned to experimental data

- *A realistic wavefunction or Wigner function for deuteron*

F. Bellini, et al. Phys. Rev. C 103, 014907 (2021)
M. Mahlein, et al. Eur. Phys. J. C 83, 804 (2023)

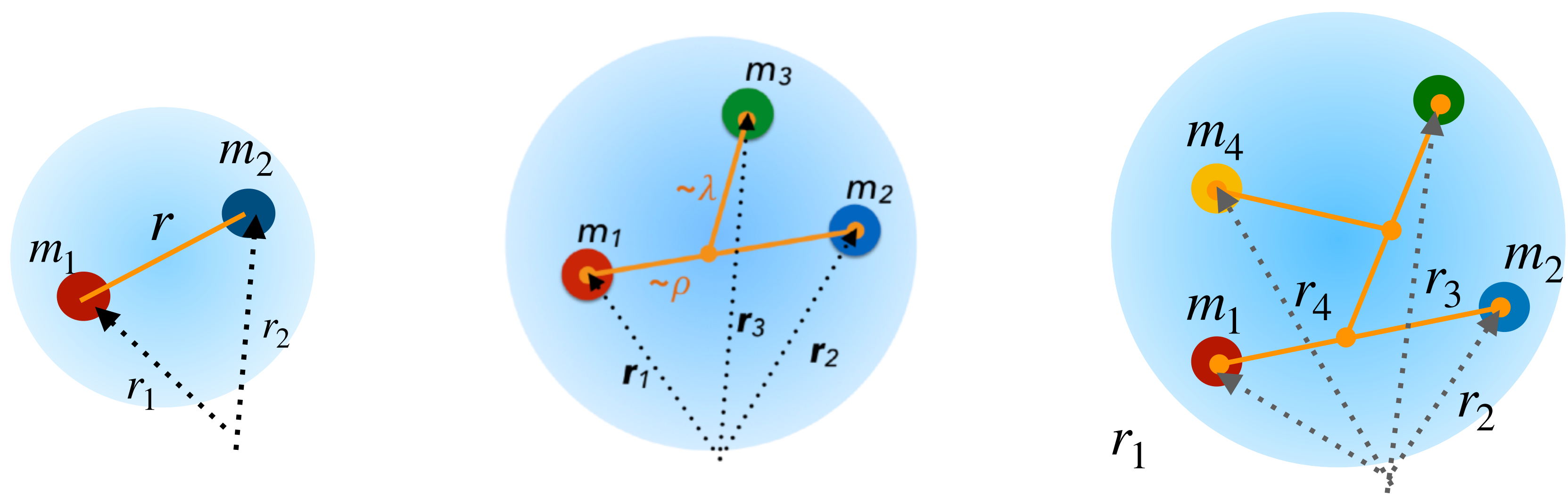
N-body Schrödinger equation

$$\left(\sum_{i=1}^{N} \frac{\hat{p}_i^2}{2m_i} + \sum_{i<j} V_{ij} \right) \Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) = E\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N)$$

Jacobi coordinates : Center of mass coordinate and N-1 Relative coordinates

$$\mathbf{R} = \frac{1}{M} \sum_{i=1}^{N} m_i \mathbf{r}_i, \qquad \mathbf{P} = \sum_{i=1}^{N} \mathbf{p}_i,$$

$$\boldsymbol{\chi}_{N-j} = \sqrt{\frac{M_j m_{j+1}}{M_{j+1}\mu}} \left(\mathbf{r}_{j+1} - \frac{1}{M_j} \sum_{i=1}^{j} m_i \mathbf{r}_i \right) \qquad \mathbf{q}_{N-j} = \sqrt{\frac{M_j m_{j+1}}{M_{j+1}\mu}} \left(\frac{\mu}{m_{j+1}} \mathbf{p}_{j+1} - \frac{\mu}{M_j} \sum_{i=1}^{j} \mathbf{p}_i \right)$$



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N-body Schrödinger equation

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Jacobi coordinates : *Center of mass coordinate and N-1 Relative coordinates*

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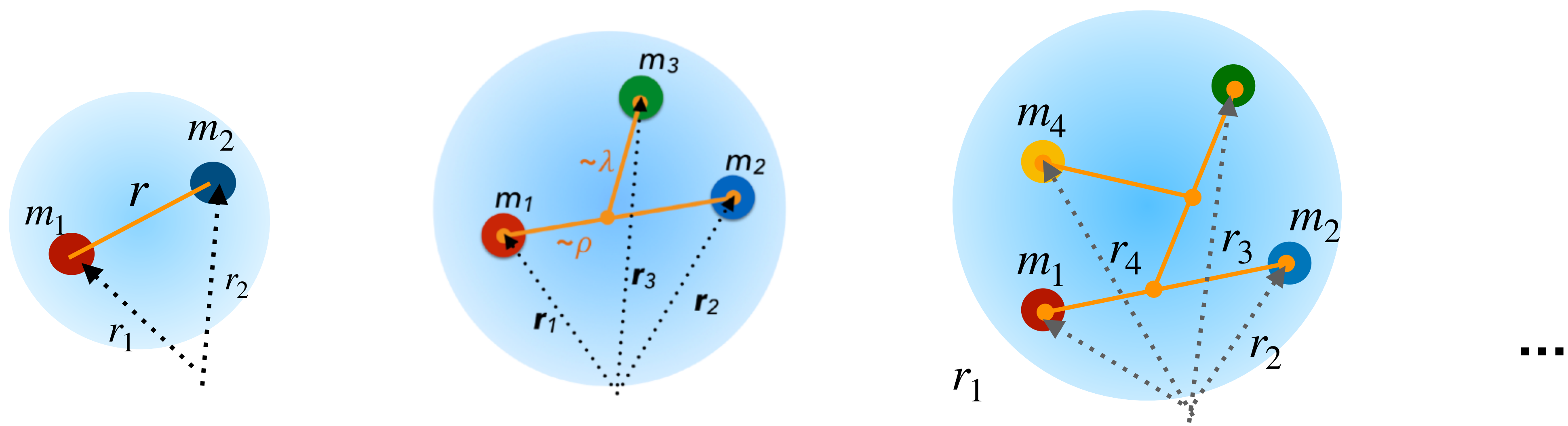

$$\sum_{i=1}^{N}\frac{\mathbf{p}_i^2}{2m_i} = \frac{\mathbf{P}^2}{2M}+\sum_{i=1}^{N-1}\frac{\mathbf{q}_i^2}{2\mu}$$

Then, factorize the N-body motion into a center-of-mass motion and a relative motion: $\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_N)=\phi(\mathbf{R})\Phi(\boldsymbol{\chi}_1,\ldots,\boldsymbol{\chi}_{N-1})$

N-body Schrödinger equation

Further, 3N-3 relative coordinates can be transformed to a single hyperradial coordinate ρ *and 3N-4 hyperangular coordinates* Ω.

$$(\chi_{\mathbf{1}}, \chi_{\mathbf{2}}, \ldots, \chi_{\mathbf{N-1}}) \rightarrow (\rho, \alpha_{N-1}, \ldots, \alpha_2, \theta_1, \phi_1, \ldots, \theta_{N-1}, \phi_{N-1})$$

$$\rho = \sqrt{\chi_{\mathbf{1}}{}^2 + \ldots + \chi_{\mathbf{N-1}}^{\mathbf{2}}} \quad \sin \alpha_i = |\chi_i| / \rho_i \quad \hat{\chi}_i = (\theta_i, \phi_i)$$

N. Barnea, et al. Phys. Rev. C 61.054001(2000)
FBS Colloquium. Few-Body System 25, 199-238(1998)

N-body Schrödinger equation

Further, 3N-3 relative coordinates can be transformed to a single hyperradial coordinate ρ and 3N-4 hyperangular coordinates Ω.

$$(\chi_1, \chi_2, \ldots, \chi_{N-1}) \to (\rho, \alpha_{N-1}, \ldots, \alpha_2, \theta_1, \phi_1, \ldots, \theta_{N-1}, \phi_{N-1})$$

$$\rho = \sqrt{\chi_1{}^2 + \ldots + \chi_{N-1}^2} \qquad \sin\alpha_i = |\chi_i|/\rho_i \qquad \hat{\chi}_i = (\theta_i, \phi_i)$$

The relative motion is controlled by :

N. Barnea, et al. Phys. Rev. C 61.054001(2000)
FBS Colloquium. Few-Body System 25, 199-238(1998)

$$\left[\frac{1}{2\mu}\left(-\frac{1}{\rho^{3N-4}}\frac{d}{d\rho}\rho^{3N-4}\frac{d}{d\rho} + \frac{\hat{K}_{N-1}^2}{\rho^2}\right) + V(\rho, \Omega)\right]\Phi(\rho, \Omega) = E_r\Phi(\rho, \Omega),$$

$$\hat{K}_{N-1}^2 = -\frac{\partial^2}{\partial\alpha_{N-1}^2} + \frac{(3N-9) - (3N-5)\cos(2\alpha_{N-1})}{\sin(2\alpha_{N-1})}\frac{\partial}{\partial\alpha_{N-1}} + \frac{1}{\cos^2\alpha_{N-1}}\hat{K}_{N-2}^2 + \frac{1}{\sin^2\alpha_{N-1}}\hat{l}_{N-1}^2,$$

$$\widehat{\mathbf{K}}_N^2\mathcal{Y}(\Omega) = K_N(K_N + 3N - 5)\mathcal{Y}(\Omega).$$ *hyper-angular momentum operator*

$$K_j = 2n_j + K_{j-1} + l_j, \qquad (j = 1, \ldots, N),$$

$$L = \sum_i^{N-1} l_i, \quad M = \sum_i^{N-1} m_i,$$

$$\begin{aligned}\mathcal{Y}_\kappa(\Omega) = & \Big[\sum_{m_1,\ldots,m_n}\langle l_1 m_1 l_2 m_2|L_2 M_2\rangle\langle L_2 M_2 l_3 m_3|L_3 M_3\rangle \\ & \times \ldots\langle L_{n-1}M_{n-1}l_n m_n|L_n M_n\rangle\prod_{j=1}^{n} Y_{l_j,m_j}(\theta_j, \phi_j)\Big] \\ & \times \Big[\prod_{j=2}^{n}\mathcal{N}_j(\sin\alpha_j)^{l_j}(\cos\alpha_j)^{K_{j-1}} \\ & \times P_{n_j}^{l_j+1/2, K_{j-1}+(3j-5)/2}(\cos 2\alpha_j)\Big],\end{aligned}$$

Orthogonal normalization basis!

N-body Schrödinger equation

$\Phi(\rho, \Omega) = \sum_\kappa R_\kappa(\rho)\mathcal{Y}_\kappa(\Omega)$ ***hyperspherical harmonic function*** *expansion*

$$\left[\frac{1}{2\mu}\left(\frac{1}{\rho^{3N-4}}\frac{d}{d\rho}\rho^{3N-4}\frac{d}{d\rho} - \frac{K(K+3N-5)}{\rho^2}\right) + E_r\right]R_\kappa = \sum_{\kappa'} V_{\kappa\kappa'}R_{\kappa'}$$

potential matrix element in angular momentum space

$$\begin{aligned} V_{\kappa\kappa'} &= \int \mathcal{Y}^*_\kappa(\Omega)V(\rho,\Omega)\mathcal{Y}_{\kappa'}(\Omega)d\Omega \\ &= \sum_{i<j}\int V^{(ij)}(|\mathbf{r}_i - \mathbf{r}_j|)\mathcal{Y}^*_\kappa(\Omega)\mathcal{Y}_{\kappa'}(\Omega)\mathrm{d}\Omega, \end{aligned}$$

$$d\Omega = \left[\prod_{j=1}^{N-1}\sin\theta_j d\theta_j d\phi_j\right]\prod_{j=2}^{N-1}(\sin\alpha_j)^2(\cos\alpha_j)^{3j-4}d\alpha_j$$

Now, we apply this tool to deal with nuclear cluster and hypernuclei!

- *For different states with total orbital angular momentum L, we can choose all possible hyperspherical harmonic functions in prinpicle. The truncation are needed which depends on the symmetry of the system.*
- *Numerically solve the coupled differential equations with inverse power method*

N-body Schrödinger equation

$\Phi(\rho, \Omega) = \sum_{\kappa} R_{\kappa}(\rho)\mathcal{Y}_{\kappa}(\Omega)$ ***hyperspherical harmonic function*** *expansion*

$$\left[\frac{1}{2\mu}\left(\frac{1}{\rho^{3N-4}}\frac{d}{d\rho}\rho^{3N-4}\frac{d}{d\rho} - \frac{K(K+3N-5)}{\rho^2}\right) + E_r\right]R_{\kappa} = \sum_{\kappa'} V_{\kappa\kappa'}R_{\kappa'}$$

potential matrix element in angular momentum space

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- *Num*

Computer Physics Communications

Volume 303, October 2024, 109284

COMPUTER PHYSICS COMMUNICATIONS

ELSEVIER

Computational Physics

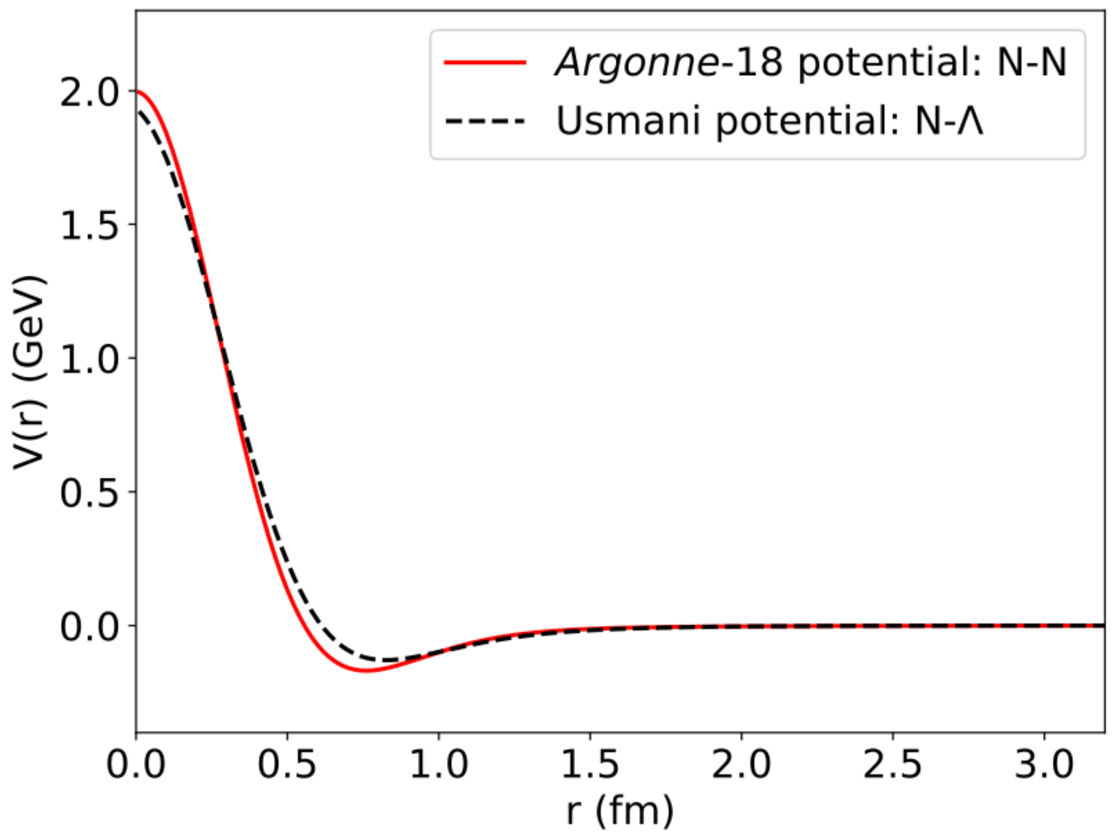
A numerical algorithm for solving the coupled Schrödinger equations using inverse power method ☆

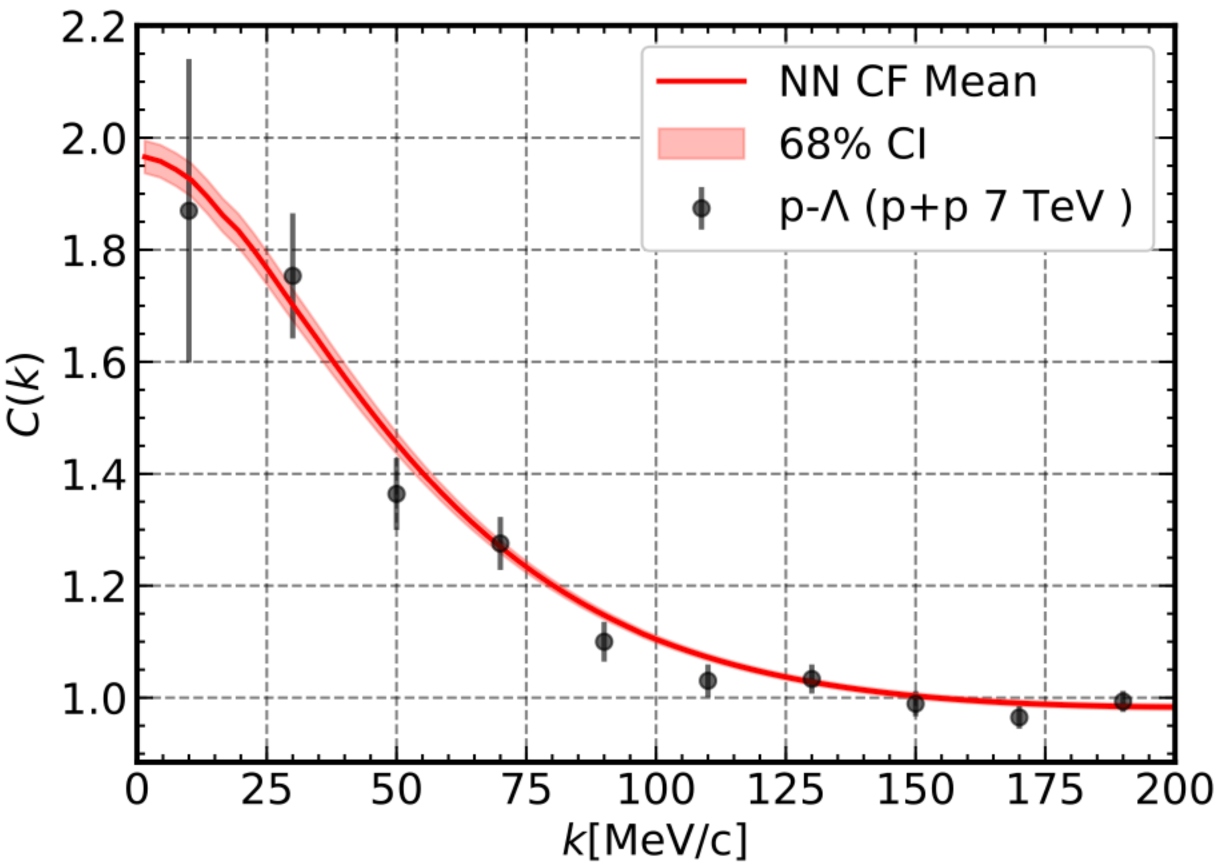
Jiaxing Zhao [a b], Shuzhe Shi [a]

Interaction potential

$$V = \sum_{i<j} V_{ij}$$

Assume that the interaction potential is the summation of the two-body interactions, and genuine three-body potentials are neglected (three-body Femtoscopy?).





N-N interaction: Paris potential, Argonne-18 potential, Reid Soft-Core potential, Nijmegen potential (ESC08),…by fitting the N-N elastic scattering data.

Y-N interaction: Usmani potential (ML+Femtoscopy), (NLO) χEFT, HALQCD,..

*L. Wang and **J. Zhao**, arXiv: 2411.16343.*

Results: Mass Spectra and rms

Cluster	d	t	^{3}He	$^3_\Lambda$H	^{4}He	$^4_\Lambda$He	$^4_\Lambda$H	$^5_\Lambda$He	$^5_{\Lambda\Lambda}$He
Constitutes	pn	pnn	ppn	pnΛ	ppnn	ppnΛ	pnnΛ	ppnnΛ	ppn$\Lambda\Lambda$
J^P	1^+	$\frac{1}{2}^+$	$\frac{1}{2}^+$	$\frac{1}{2}^+$	0^+	0^+	0^+	$\frac{1}{2}^+$	$\frac{1}{2}^+$
$M_{\rm exp.}$(GeV)	1.875	2.809	2.808	2.991	3.727	3.923	3.923	4.731	-
$M_{\rm theo.}$(GeV)	1.873	2.813	2.812	2.993	3.746	3.927	3.929	4.847	5.028
rms(fm)	2.790	1.561	1.567	4.332	1.590	1.810	1.809	1.509	1.595

The definition of the cluster mass and binding energy are:

$$\mathrm{M} \equiv \sum_i^N m_i + E_r$$

$$\mathrm{B.E.} \equiv -E_r.$$

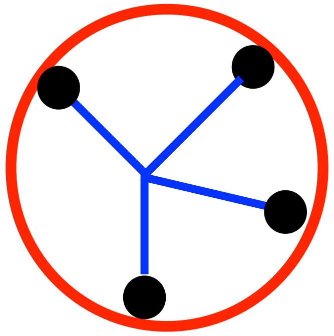
The root-mean-squared radius (rms) of the N-body cluster is defined as:

Mercedes-Benz

$$r^2_{\rm rms} \equiv \langle \rho^2 \rangle = \int \sum_\kappa |R_\kappa(\rho)|^2 \rho^{3N-2} d\rho.$$

$$\langle \rho^2 \rangle = \langle \sum_{j=1}^{N-1} \chi_j^2 \rangle = \frac{1}{\mu M} \sum_{i<j} m_i m_j \langle \mathbf{r}_{ij}^2 \rangle = \frac{1}{\mu} \sum_{i=1}^{N} m_i \langle (\mathbf{r}_i - \mathbf{R})^2 \rangle.$$

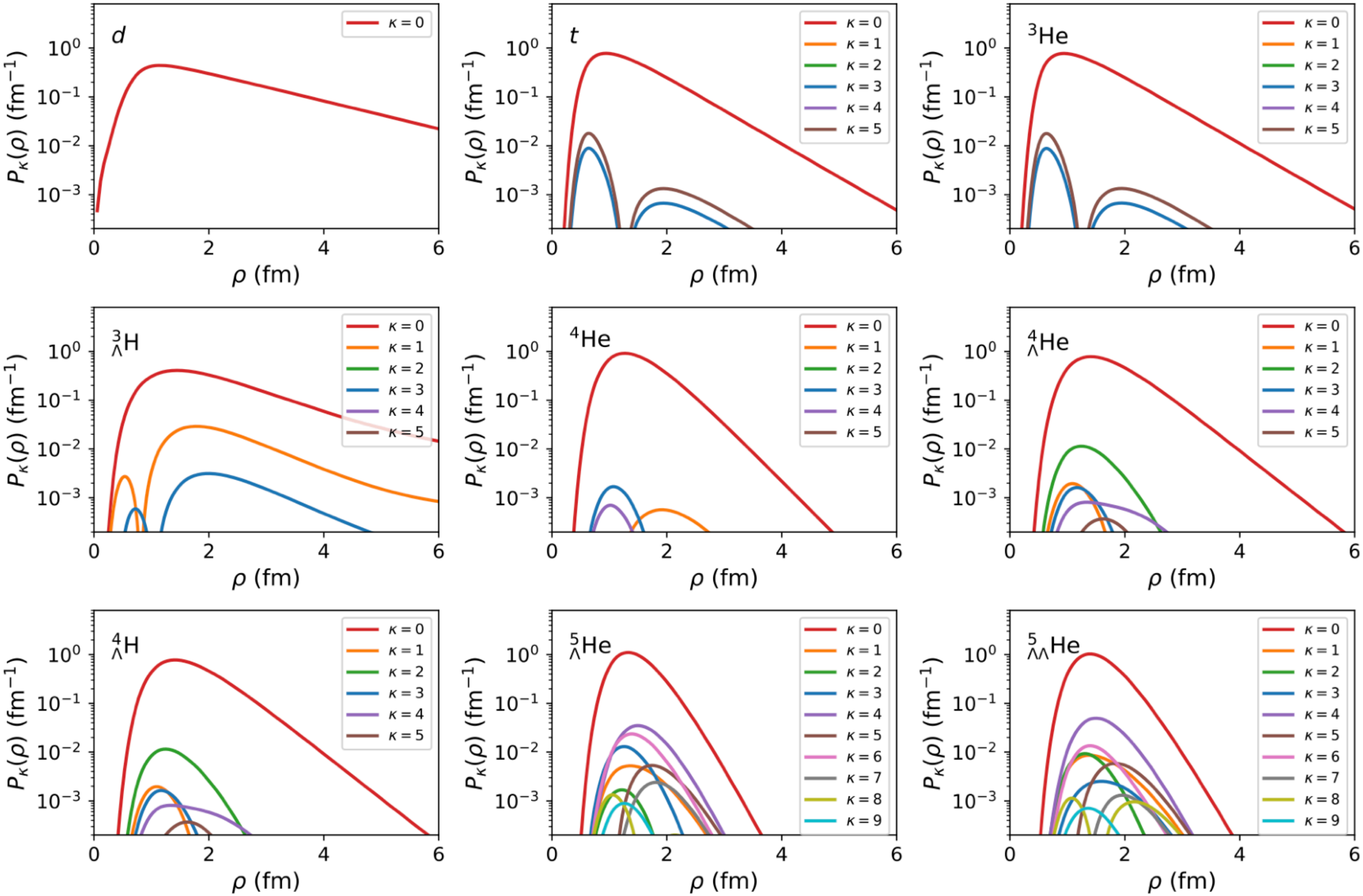
If the particles have the same mass, the rms represents the geometric radiu



The theoretical masses are close to the experimental data.

P - wave excited states are not founded.

Results: Wavefunctions



The radial probability of each component: $P_\kappa(\rho) \equiv |R_\kappa(\rho)|^2 \rho^{3N-4}.$

For the ground state, the truncation is good enough and we don't need too many basis due to the symmetry.

Wigner function

For two-body case, if assume a 3-D isotropic harmonic oscillator potential:

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_{l,m}(\theta,\phi) \qquad R_{nl}(r) = \left[\frac{2(n!)}{\sigma^3\Gamma(n+l+3/2)}\right]^{\frac{1}{2}} \left(\frac{r}{\sigma}\right)^l e^{-\frac{r^2}{2\sigma^2}} L_n^{l+1/2}\left(\frac{r^2}{\sigma^2}\right)$$

$$W_2^{1S}(\mathbf{r},\mathbf{q}) = 8e^{-\frac{\mathbf{r}^2}{\sigma^2}-\mathbf{q}^2\sigma^2}.$$

For N-body case, an analytical wavefunction if the interaction potential has the form

$$V(\mathbf{r}_{ij}) = \frac{1}{2\mu\sigma_{ij}^4}\frac{m_i m_j}{\mu\sum_{i=1}^{N} m_i}\mathbf{r}_{ij}^2.$$

separate the relative wavefunction as aproduct of N – 1 wavefunctions of a 3-D isotropic harmonic oscillator

$$\sum_{i=1}^{N-1}\left(\frac{\mathbf{q}_i^2}{2\mu}+\frac{1}{2\mu\sigma^4}\boldsymbol{\chi}_i^2\right)\Phi = E_r\Phi. \qquad \Phi = \prod_{i=1}^{N-1}\psi_{n_{\boldsymbol{\chi}_i},l_{\boldsymbol{\chi}_i},m_{\boldsymbol{\chi}_i}}(\boldsymbol{\chi}_i)$$

$$W_N^{1S}(\boldsymbol{\chi}_1,\ldots,\boldsymbol{\chi}_{N-1};\mathbf{q}_1,\ldots,\mathbf{q}_{N-1}) = \prod_{i=1}^{N-1} W_2^{1S}(\boldsymbol{\chi}_i,\mathbf{q}_i).$$

$$W_3^{1S}(\boldsymbol{\chi}_1,\boldsymbol{\chi}_2;\mathbf{q}_1,\mathbf{q}_2) = W_2^{1S}(\boldsymbol{\chi}_1,\mathbf{q}_1)W_2^{1S}(\boldsymbol{\chi}_2,\mathbf{q}_2).$$

A product of two-body Wigner functions—> interaction potential differs for pairs with different masses !!!

Wigner function

The Wigner function can self-consistently be determined by the wavefunction.

$$W_N(\chi_1,\ldots,\chi_{N-1};\mathbf{q}_1,\ldots,\mathbf{q}_{N-1}) = \int d^3\mathbf{s}_1\ldots d^3\mathbf{s}_{N-1}e^{-i\sum_{k=1}^{N-1}\mathbf{q}_k\cdot\mathbf{s}_k}\Phi^*\left(\chi_1+\frac{\mathbf{s}_1}{2},\ldots,\chi_{N-1}+\frac{\mathbf{s}_{N-1}}{2}\right)\Phi\left(\chi_1-\frac{\mathbf{s}_1}{2},\ldots,\chi_{N-1}-\frac{\mathbf{s}_{N-1}}{2}\right).$$

From the wavefunction, we can see the ground state is dominate.

$$W_N(\boldsymbol{\rho},\mathbf{q}) = \int d^{3(N-1)}\mathbf{s}e^{-i\mathbf{q}\cdot\mathbf{s}}R_0^*(|\boldsymbol{\rho}+\frac{\mathbf{s}}{2}|)(\mathcal{Y}_0^N)^*R_0(|\boldsymbol{\rho}-\frac{\mathbf{s}}{2}|)\mathcal{Y}_0^N. \qquad \mathcal{Y}_0^N = \sqrt{\frac{\Gamma[3(N-1)/2]}{2\pi^{3(N-1)/2}}}.$$

$$d^{3(N-1)}\mathbf{s} = s^{3N-4}ds(\sin\theta)^{3N-5}d\theta d\Omega_{3N-4}.$$

If integrated over all angles, we get an angle-independent Wigner function

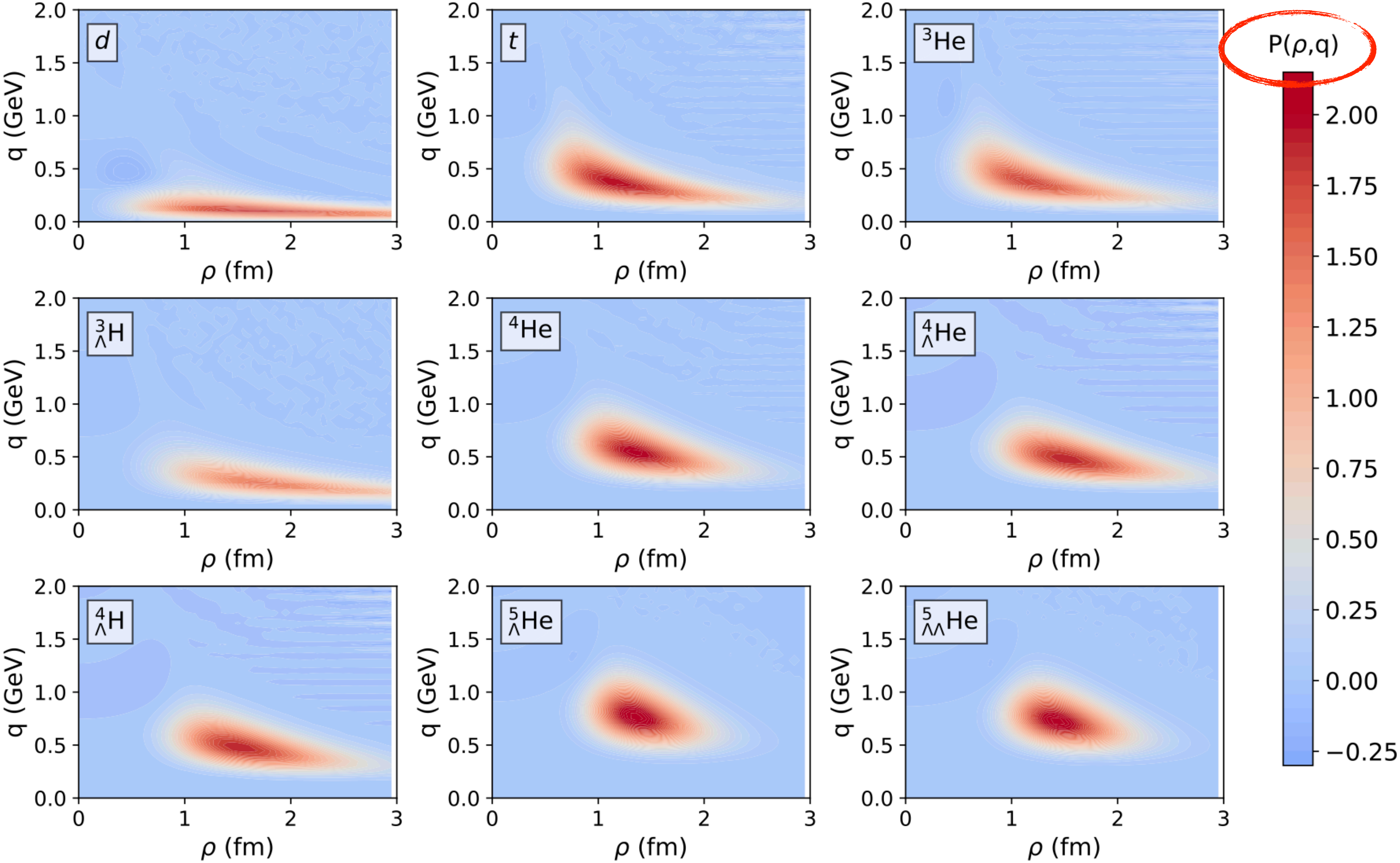
$$W_N(\rho,q) = \frac{\mathcal{S}_{3N-4}}{\mathcal{S}_{3N-3}}\int(\sin\theta_{\rho s})^{3N-5}d\theta_{\rho s}\int s^{3N-4}ds(\sin\theta_{qs})^{3N-5}d\theta_{qs}e^{-iqs\cos\theta_{qs}}R_0^*(\rho_+)R_0(\rho_-).$$

$$\rho_\pm = \sqrt{\rho^2+s^2/4\pm\rho s\cos\theta_{\rho s}} \qquad \mathcal{S}_d \equiv \int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma[d/2]}.$$

The probability to find the cluster in a hyperspherical shell in coordinate and momentum

$$P_N(\rho,q) \equiv \rho^{3N-4}q^{3N-4}\frac{\mathcal{S}_{3N-4}\mathcal{S}_{3N-3}}{(2\pi)^{3(N-1)}}W_N(\rho,q) \qquad \int P_N(\rho,q)d\rho dq = 1.$$

Results: Wigner functions



Clearly beyond the Gaussian form!

Summary

We solve the Schrödinger equation for few-body systems to obtain the wave function for light nuclear clusters and hypernuclei:

(d, t, ^3He, ${}^{3}_{\Lambda}\mathrm{H}$, $^4\mathrm{He}$, ${}^{4}_{\Lambda}\mathrm{H}$, ${}^{4}_{\Lambda}\mathrm{He}$, ${}^{5}_{\Lambda}\mathrm{He}$, ${}^{5}_{\Lambda\Lambda}\mathrm{He}$).

The Wigner densities obtained will allow to improve the present coalescence approaches to identify clusters, created in heavy-ion collisions.

Outlook

- *Consider the spin-spin interaction and search for the excited states.*
- *Employ these Wigner functions for transport model to investigate the production of nuclear clusters and hypernuclei (on-going).*

Thanks for your attention!