

Flow studies with multiparticle correlations in the SIS18/100 energy regime

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“EMMI Workshop: Collective Phenomena and EoS of dense baryonic matter”,

GSI, Darmstadt

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Outline

- QCD phase diagram and Quark-Gluon Plasma
- Heavy-ion collisions
 - Collective phenomena
 - Multi-particle correlations
- Bayesian studies
- Multi-harmonic observables in **ALICE**
 - Symmetric Cumulants (SC)
 - Asymmetric Cumulants (AC)
- Status of feasibility studies in **CBM**

C. Mordasini, AB, D. Karakoç, F. Taghavi, Phys. Rev. C **102**, 024907 (2020),
e-Print: [1901.06968](#)

ALICE Collaboration, Phys. Rev. Lett. 127 (2021) 9, 092302,
e-Print: [2101.02579](#)

AB, M. Lesch, C. Mordasini, F. Taghavi, Phys. Rev. C **105** (2022) 2, 024912,
e-Print: [2101.05619](#)

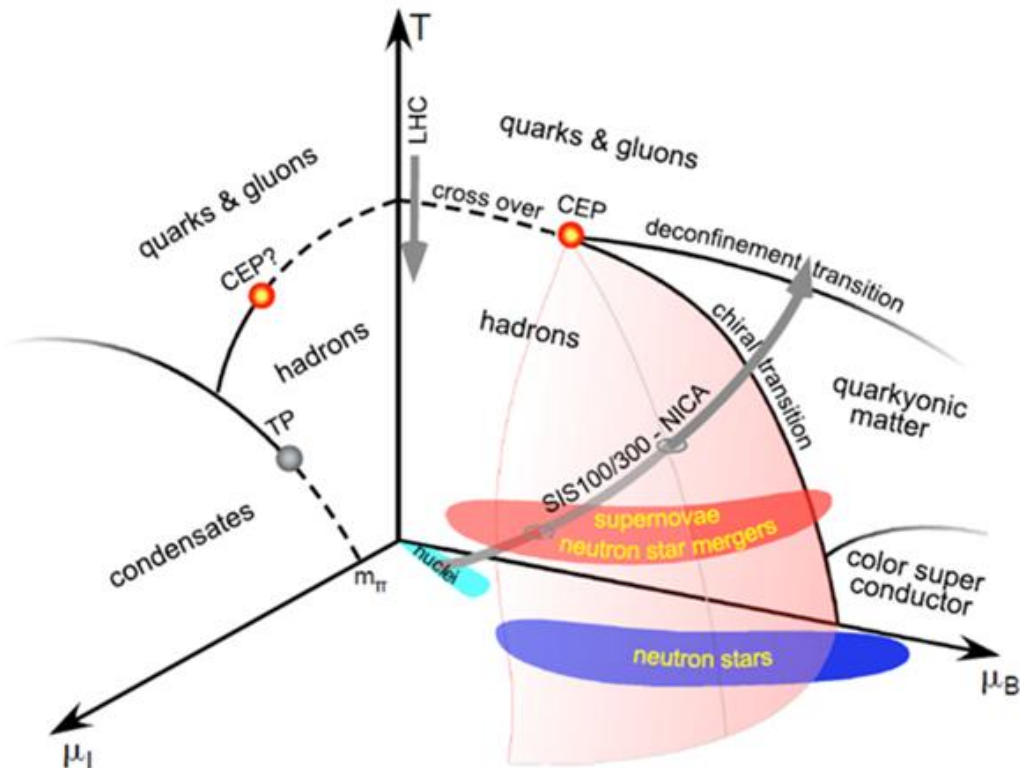
ALICE Collaboration, Phys. Rev. C **108** (2023) 5, 055203,
e-Print: [2303.13414](#)

ALICE Collaboration, Phys. Rev. C **112** (2025) 2, 024905,
e-Print: [2412.15873](#)

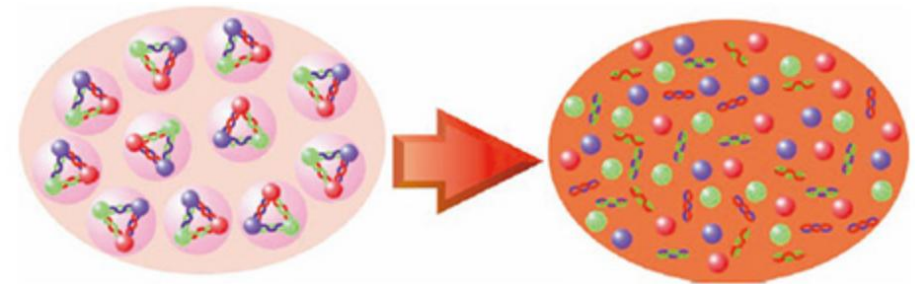


QCD phase diagram and Quark-Gluon Plasma

- Phase diagram of Quantum Chromodynamics (QCD)

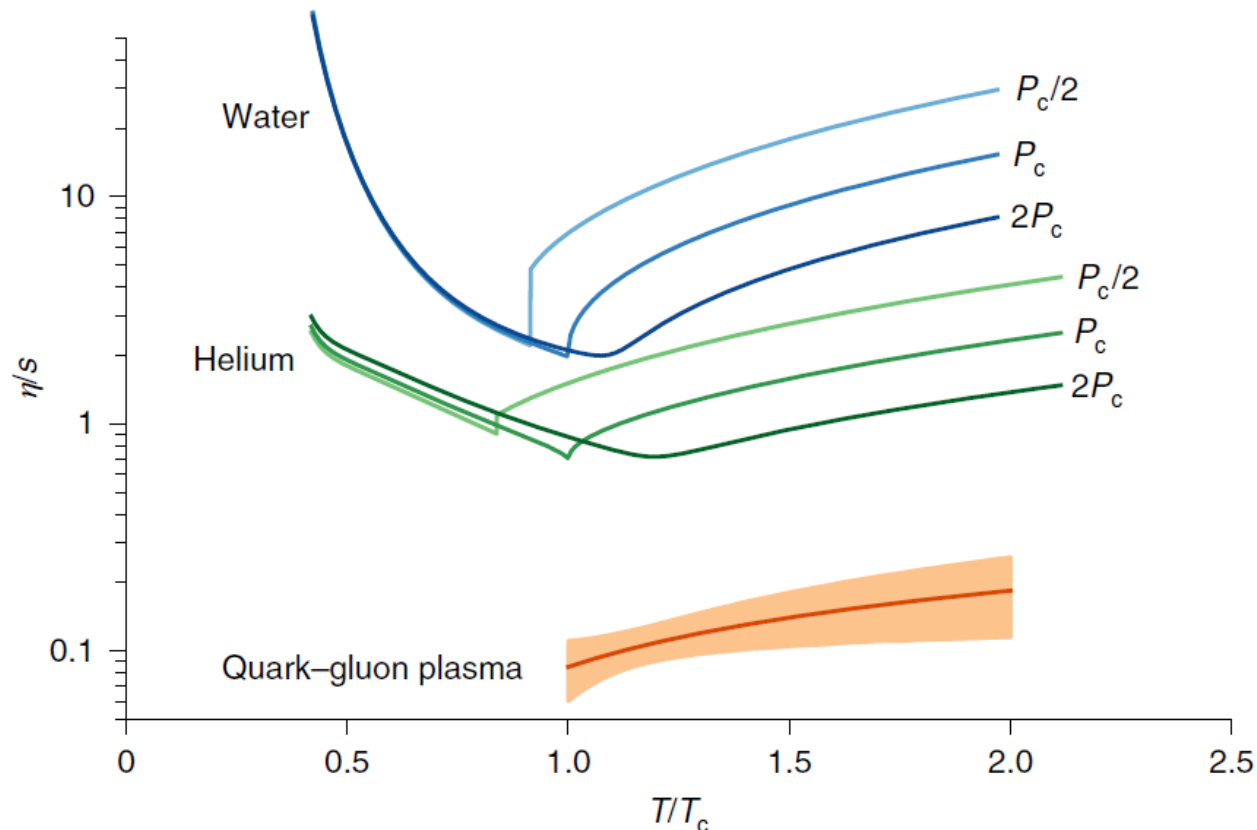


- Quark-Gluon Plasma (QGP): Extreme state of matter in which quarks and gluons can move freely over distances comparable to the size of hadrons



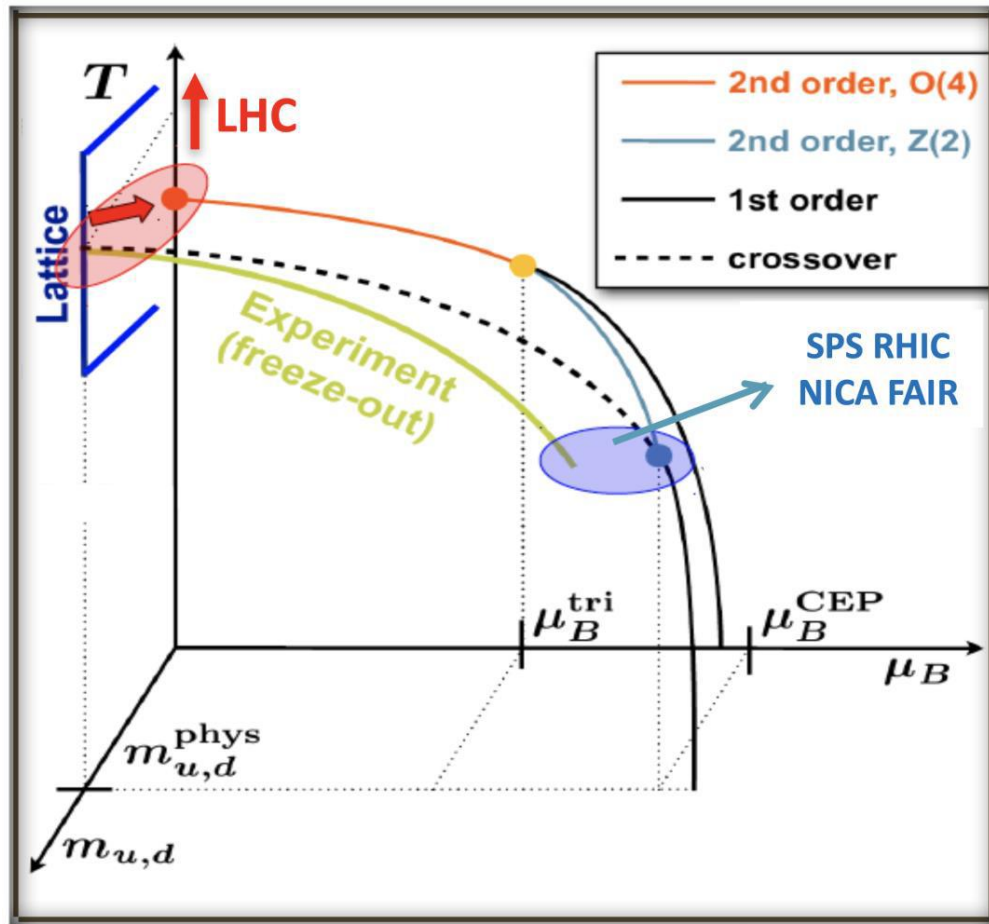
Example #1: Transport properties of QGP

- Temperature dependence of QGP's specific shear viscosity (η/s) is smallest of all known substances



Bernhard, J.E., Moreland, J.S. & Bass, S.A.
Nat. Phys. **15**, 1113–1117 (2019)

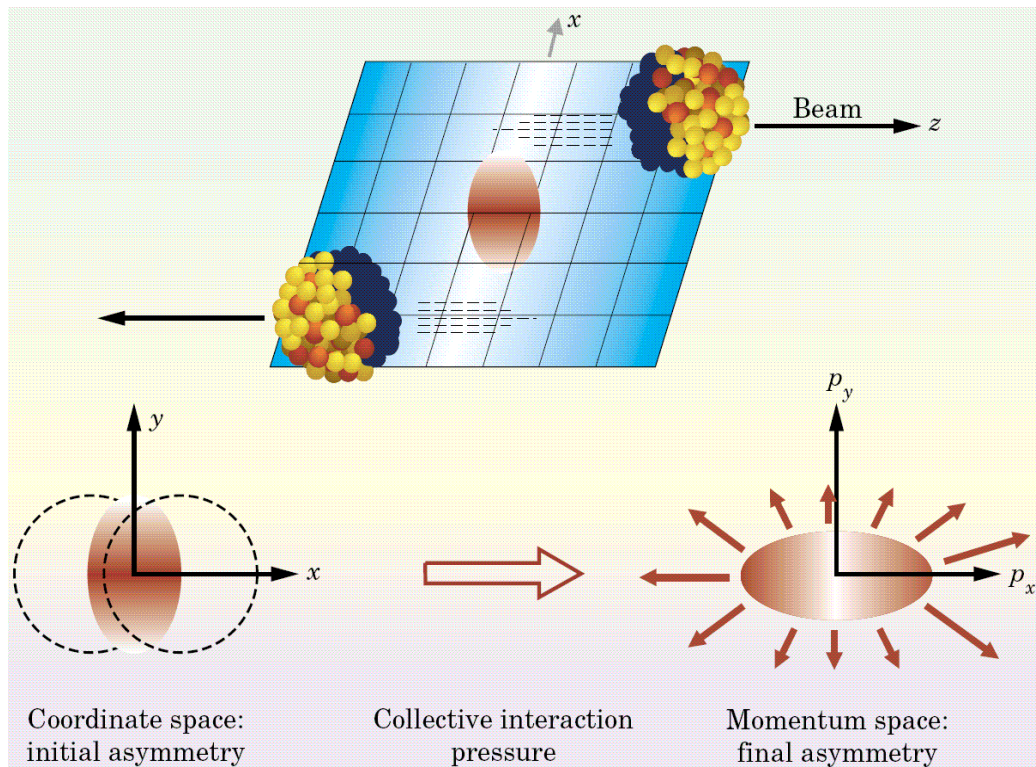
Example #2: Nature of QCD phase transitions



- QCD phase diagram of strongly interacting nuclear matter can be explored in ultrarelativistic heavy-ion collisions
- QGP phase is probed as a function of temperature and baryon chemical potential
- What is the nature of phase transitions in QCD phase diagram (smooth cross over, 1st or 2nd order phase transition, etc.)?
- Existence of critical point?

Collective anisotropic flow

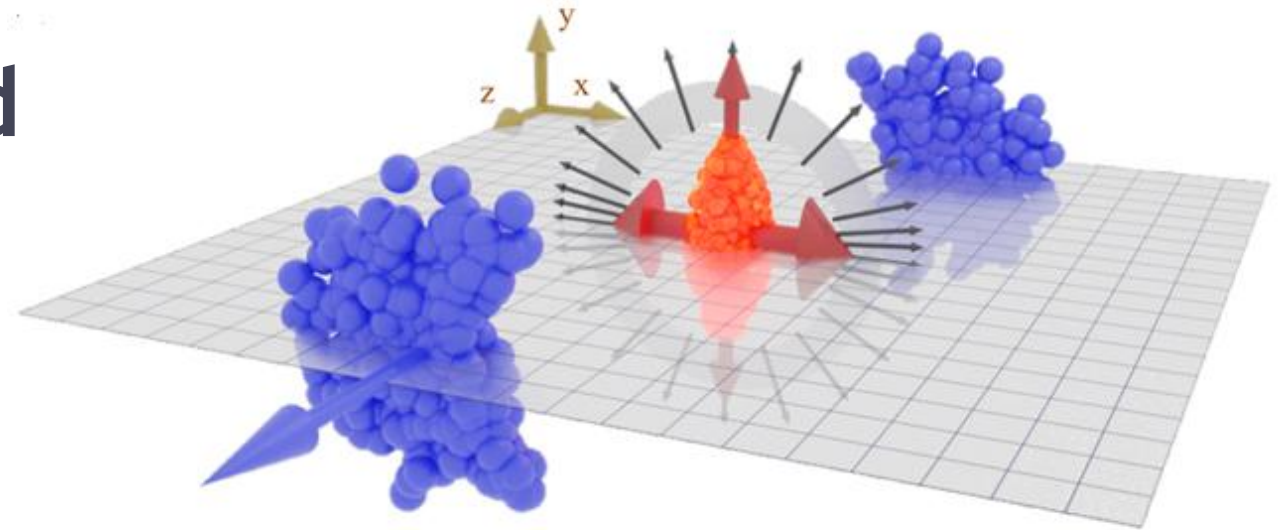
- Transfer of anisotropy from the initial coordinate space into the final momentum space via the thermalized medium:



- Anisotropic flow will develop in heavy-ion collisions only if both of the following two requirements are met:
 - Initial anisotropic volume in coordinate space
 - Thermalized medium

J.Y. Ollitrault, Phys. Rev. D **46** (1992) 229

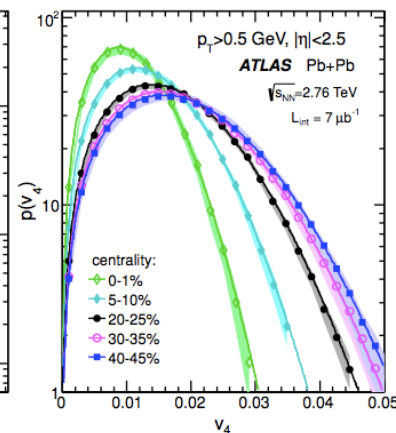
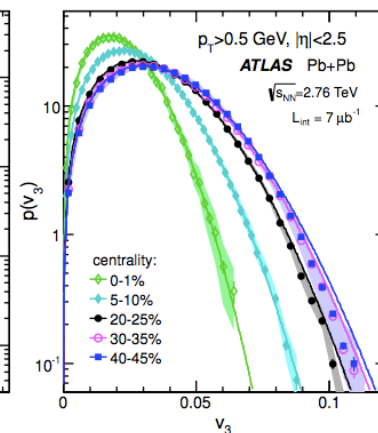
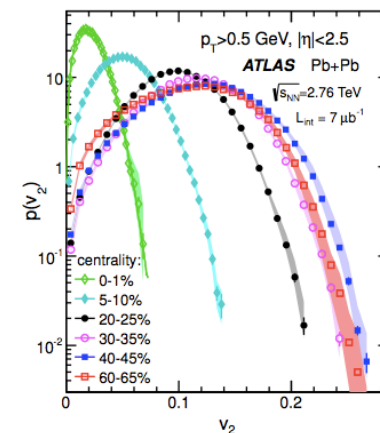
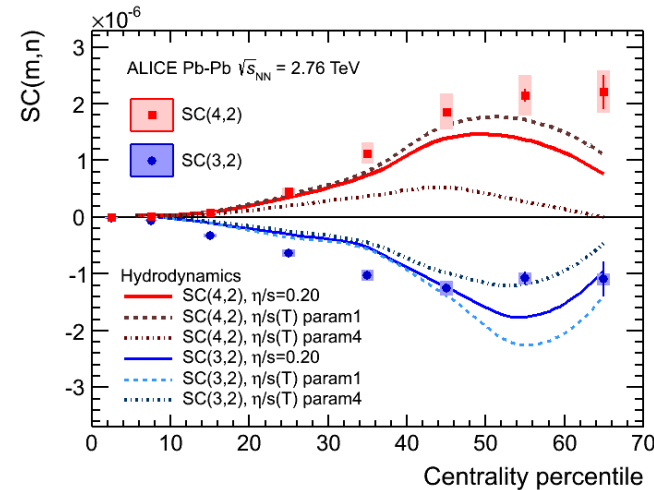
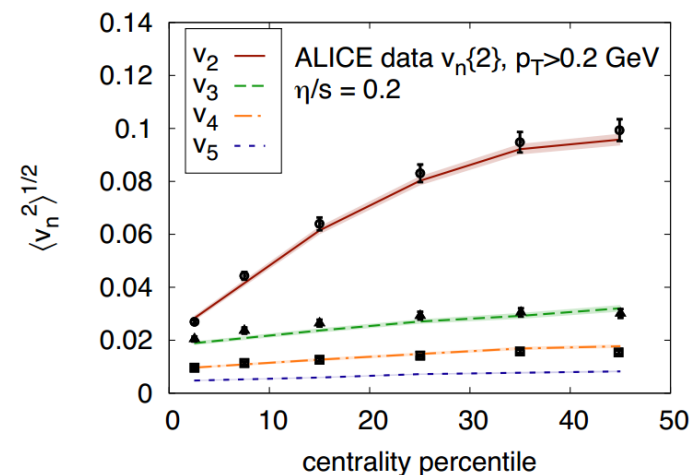
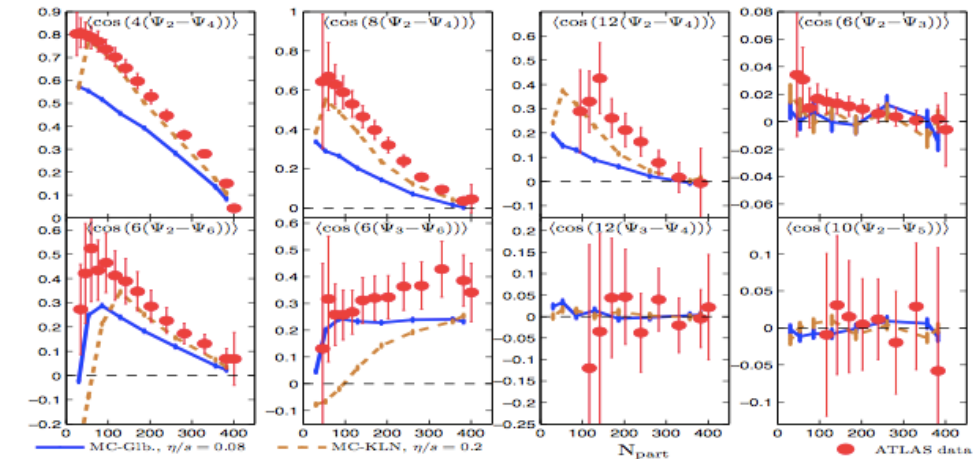
Heavy-ion collisions and collective phenomena



Credits: D.D. Chinellato, ICHEP 2020

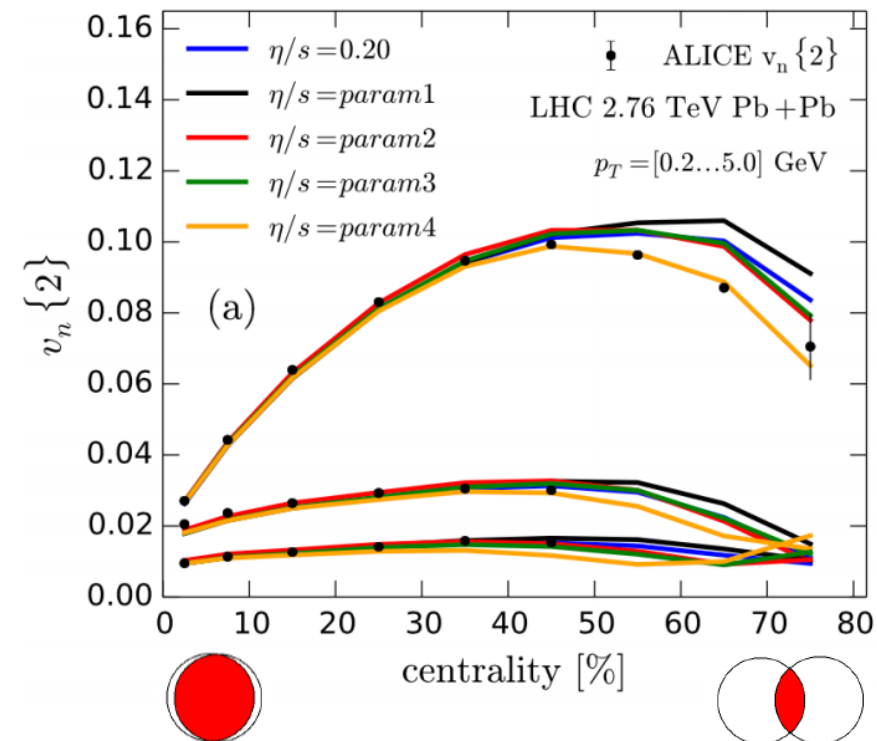
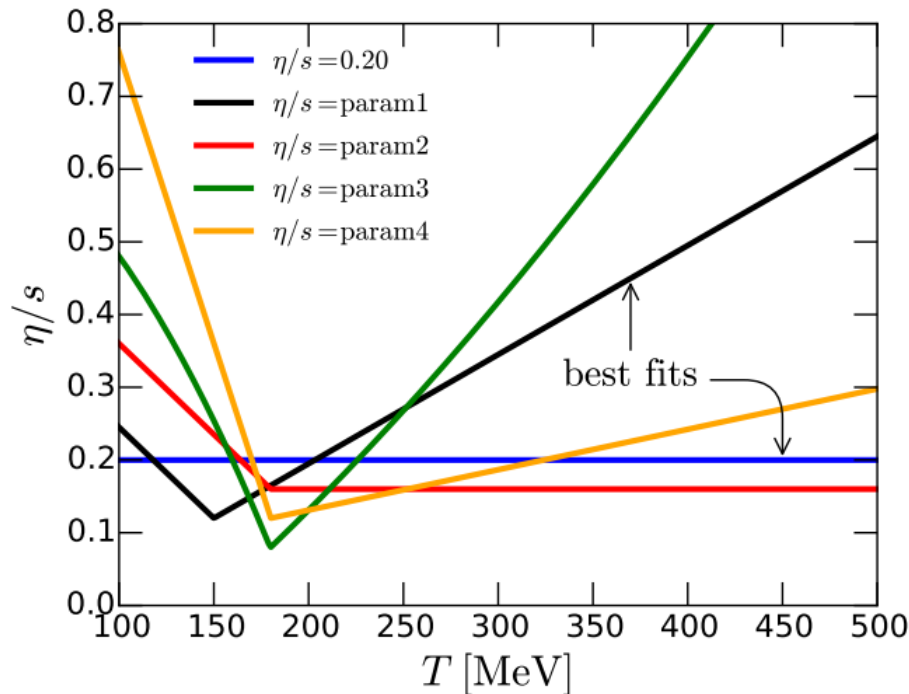
Flow observables

- Individual flow harmonics: $v_1, v_2, v_3, v_4, \dots$
- Correlations between harmonics: $\langle v_m^2 v_n^2 \rangle - \langle v_m^2 \rangle \langle v_n^2 \rangle$
- Symmetry plane correlations: $\langle \cos[mn(\Psi_m - \Psi_n)] \rangle$
- Probability density function: $P(v_n)$
- ...



‘Classical’ flow observables

- Individual v_n harmonics are sensitive only to average values of η/s
 - Insensitivity to the differential temperature dependence of η/s



Historical account

- New observables were introduced in 2012, independently by theorists and experimentalists:
 - Theorists: p_T dependence of covariance between v_m and v_n
 - Experimentalists: even moments of v_n^{2k} and their intercorrelations => Symmetric Cumulants (SC)
- Two sets of observables were proposed on exactly the same day 5th of December 2012!

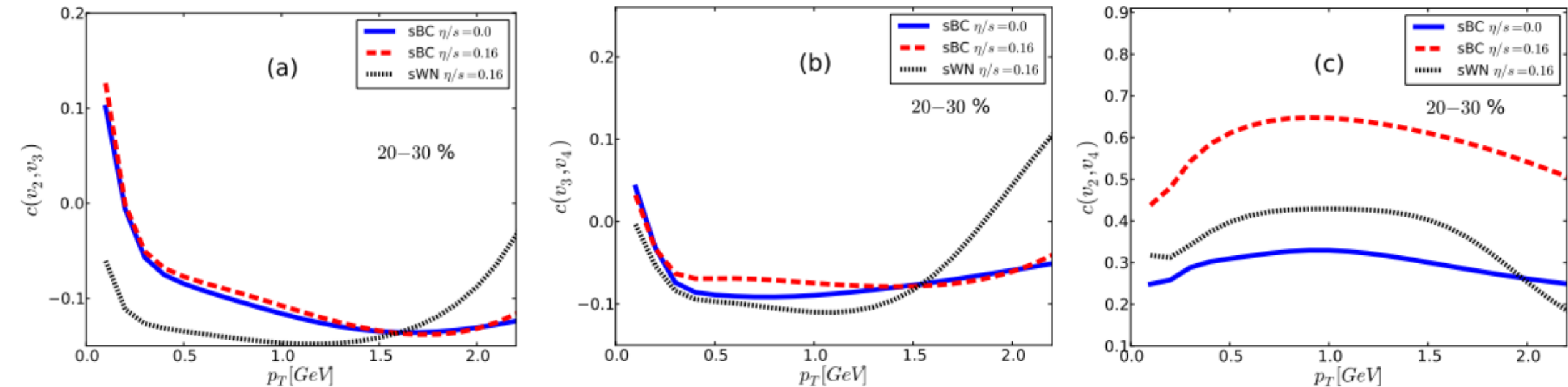
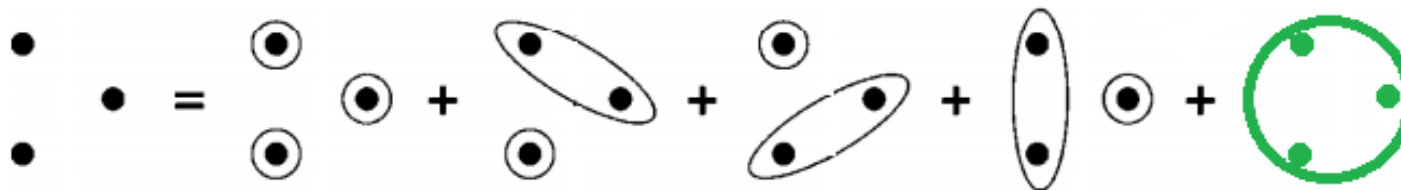


Figure 9 from H. Niemi, G. S. Denicol, H. Holopainen and P. Huovinen,
Phys. Rev. C **87** (2013) 5, 054901 313, e-Print: [1212.1008](https://arxiv.org/abs/1212.1008)

Multiparticle correlations and cumulants

R. S. Bhalerao, M. Luzum and J.-Y. Ollitrault, Phys. Rev. C **84** 034910 (2011), e-Print: [1104.4740](https://arxiv.org/abs/1104.4740)

Kubo, Journal of the Physical Society of Japan, Vol. 17, No. 7, (1962)



Independent information

- To constrain the functional form of $f(v_1, \dots, v_n, \Psi_1, \dots, \Psi_n)$, we have to measure all its **valid moments (or cumulants)**
- The most general mathematical result, which relates multiparticle azimuthal correlators and flow degrees of freedom:

$$\left\langle e^{i(n_1 \varphi_1 + \dots + n_k \varphi_k)} \right\rangle = v_{n_1} \cdots v_{n_k} e^{i(n_1 \Psi_{n_1} + \dots + n_k \Psi_{n_k})}$$

R. S. Bhalerao, M. Luzum and J.-Y. Ollitrault, Phys. Rev. C **84** 034910 (2011), e-Print: [1104.4740](#)

Open questions:

- What are the smallest collision systems and energies at which QGP can be formed?
- How to extract new and independent constraints from the available heavy-ion data?
- Is the observed universality of flow measurements in vastly different collision systems physical, or just a subtle artifact of using correlation techniques in the randomized data set?

Choice of fundamental observable

- Cumulants as used in flow analyses in the last ~25 years:
 1. Cumulant expansion is performed on azimuthal angles
 2. Azimuthal correlators which are not isotropic are dropped
 3. The final result is merely re-expressed in terms of flow degrees of freedom v_n and Ψ_n via the analytic relation

$$\left\langle e^{i(n_1\varphi_1 + \dots + n_k\varphi_k)} \right\rangle = v_{n_1} \cdots v_{n_k} e^{i(n_1\Psi_{n_1} + \dots + n_k\Psi_{n_k})}$$

R. S. Bhalerao, M. Luzum and J.-Y. Ollitrault, Phys. Rev. C **84** 034910 (2011), e-Print: [1104.4740](https://arxiv.org/abs/1104.4740)

- Few additional remarks:
 - Cumulants of v_n and v_n^2 are in general different
 - v_n and Ψ_n have different properties (e.g. with respect to Lorentz invariance)

The root of the problem

- General 2-particle cumulant

$$\langle X_1 X_2 \rangle_c = \langle X_1 X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle$$

- **Old paradigm:** fundamental observable is an angle

$$X_1 \rightarrow e^{in\varphi_1}, \quad X_2 \rightarrow e^{-in\varphi_2}$$

- **New paradigm:** fundamental observable is a flow amplitude

$$X_1 \rightarrow v_n^2, \quad X_2 \rightarrow v_m^2$$

- Both choices yielded **accidentally the same results** for $SC(m,n)$ observables
- But results for $SC(k,l,m)$, $SC(k,l,m,n)$, etc., are in general different
 - Which paradigm is correct in general?

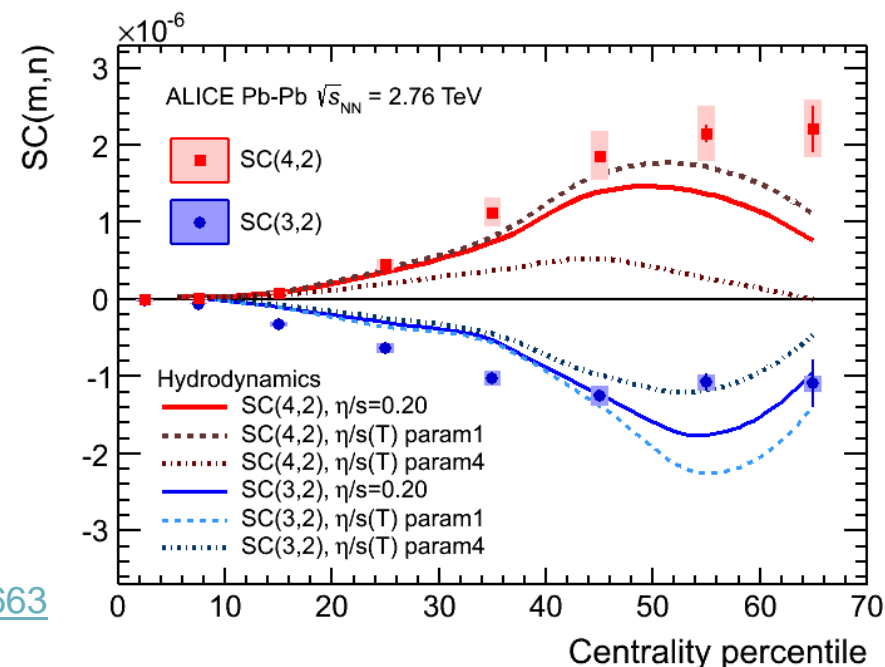
Symmetric Cumulants $SC(m,n)$

- How to quantify experimentally the correlation between two different v_n amplitudes?
 - Symmetric Cumulants** (Section IVC in Phys. Rev. C **89** (2014) no.6, 064904, e-Print: [1312.3572](#))

$$\begin{aligned}
 SC(m,n) &\equiv \langle\langle \cos(m\varphi_1 + n\varphi_2 - m\varphi_3 - n\varphi_4) \rangle\rangle_c \\
 &= \langle\langle \cos(m\varphi_1 + n\varphi_2 - m\varphi_3 - n\varphi_4) \rangle\rangle \\
 &\quad - \langle\langle \cos[m(\varphi_1 - \varphi_2)] \rangle\rangle \langle\langle \cos[n(\varphi_1 - \varphi_2)] \rangle\rangle \\
 &= \langle v_m^2 v_n^2 \rangle - \langle v_m^2 \rangle \langle v_n^2 \rangle
 \end{aligned}$$

← **Old paradigm definition**

- SC observables are sensitive to differential $\eta/s(T)$ parametrizations
- Individual flow amplitudes are dominated by averages $\langle \eta/s(T) \rangle$
- Independent constraints both on initial conditions and QGP properties



ALICE Collaboration, Phys. Rev. Lett. 117, 182301 (2016), e-Print: [1604.07663](#)

Symmetric Cumulants $SC(m,n)$

- How to quantify experimentally the correlation between two different v_n amplitudes?
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$$\begin{aligned}
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 \end{aligned}$$

 **Old paradigm definition**

$$\begin{aligned}
 SC(m,n) &\equiv \langle v_m^2 v_n^2 \rangle - \langle v_m^2 \rangle \langle v_n^2 \rangle \\
 &= \langle\langle e^{i(m\varphi_1 + n\varphi_2 - m\varphi_3 - n\varphi_4)} \rangle\rangle - \langle\langle e^{im(\varphi_1 - \varphi_2)} \rangle\rangle \langle\langle e^{in(\varphi_1 - \varphi_2)} \rangle\rangle
 \end{aligned}$$

 **New paradigm definition**

C. Mordasini, AB, D. Karakoç, F. Taghavi, Phys. Rev. C **102**, 024907 (2020), e-Print [1901.06968](#)

AB, M. Lesch, C. Mordasini, F. Taghavi, Phys. Rev. C **105** (2022) 2, 024912, e-Print: [2101.05619](#)

Generalization: $SC(k, l, m)$, $SC(k, l, m, n)$, ...

- New paradigm:
 - Cumulant expansion directly on flow amplitudes v^2 :

$$SC(k, l, m) \equiv \langle v_k^2 v_l^2 v_m^2 \rangle - \langle v_k^2 v_l^2 \rangle \langle v_m^2 \rangle - \langle v_k^2 v_m^2 \rangle \langle v_l^2 \rangle - \langle v_l^2 v_m^2 \rangle \langle v_k^2 \rangle + 2 \langle v_k^2 \rangle \langle v_l^2 \rangle \langle v_m^2 \rangle$$

- Azimuthal angles are used merely to build an estimator for the above expression, term-by-term:

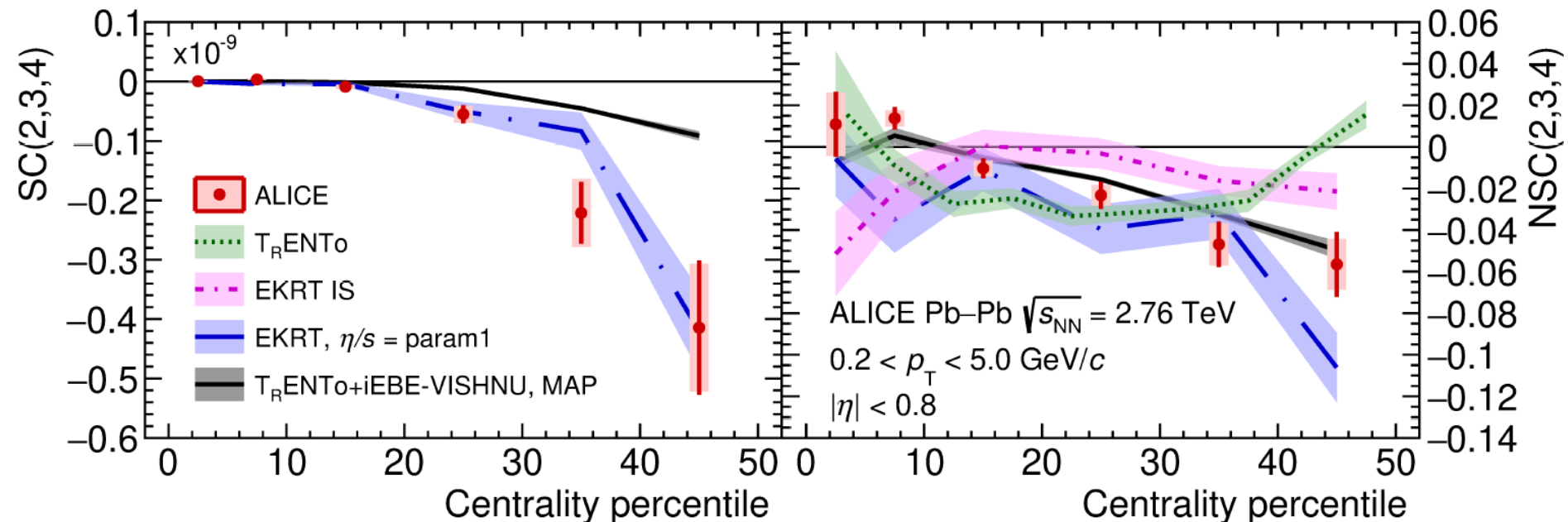
$$\begin{aligned} SC(k, l, m) = & \langle \langle \cos[k\varphi_1 + l\varphi_2 + m\varphi_3 - k\varphi_4 - l\varphi_5 - m\varphi_6] \rangle \rangle \\ & - \langle \langle \cos[k\varphi_1 + l\varphi_2 - k\varphi_3 - l\varphi_4] \rangle \rangle \langle \langle \cos[m(\varphi_5 - \varphi_6)] \rangle \rangle \\ & - \langle \langle \cos[k\varphi_1 + m\varphi_2 - k\varphi_5 - m\varphi_6] \rangle \rangle \langle \langle \cos[l(\varphi_3 - \varphi_4)] \rangle \rangle \\ & - \langle \langle \cos[l\varphi_3 + m\varphi_4 - l\varphi_5 - m\varphi_6] \rangle \rangle \langle \langle \cos[k(\varphi_1 - \varphi_2)] \rangle \rangle \\ & + 2 \langle \langle \cos[k(\varphi_1 - \varphi_2)] \rangle \rangle \langle \langle \cos[l(\varphi_3 - \varphi_4)] \rangle \rangle \langle \langle \cos[m(\varphi_5 - \varphi_6)] \rangle \rangle \end{aligned}$$

C. Mordasini, AB, D. Karakoç, F. Taghavi, Phys. Rev. C **102**, 024907 (2020), e-Print [1901.06968](https://arxiv.org/abs/1901.06968)

AB, M. Lesch, C. Mordasini, F. Taghavi, Phys. Rev. C **105** (2022) 2, 024912, e-Print: [2101.05619](https://arxiv.org/abs/2101.05619)

Higher-order Symmetric Cumulants $SC(k,l,m)$

- Generalization to more than two v_n harmonics is not trivial
- New definition for higher-order SC observables is always based on a **new paradigm** approach
AB, Marcel Lesch, Cindy Mordasini, Seyed Farid Taghavi, Phys. Rev. C **105** (2022) 2, 024912, e-Print: [2101.05619](https://arxiv.org/abs/2101.05619)
- All fundamental properties of cumulants are satisfied => **independent input for Bayesian studies**



ALI-DER-479271

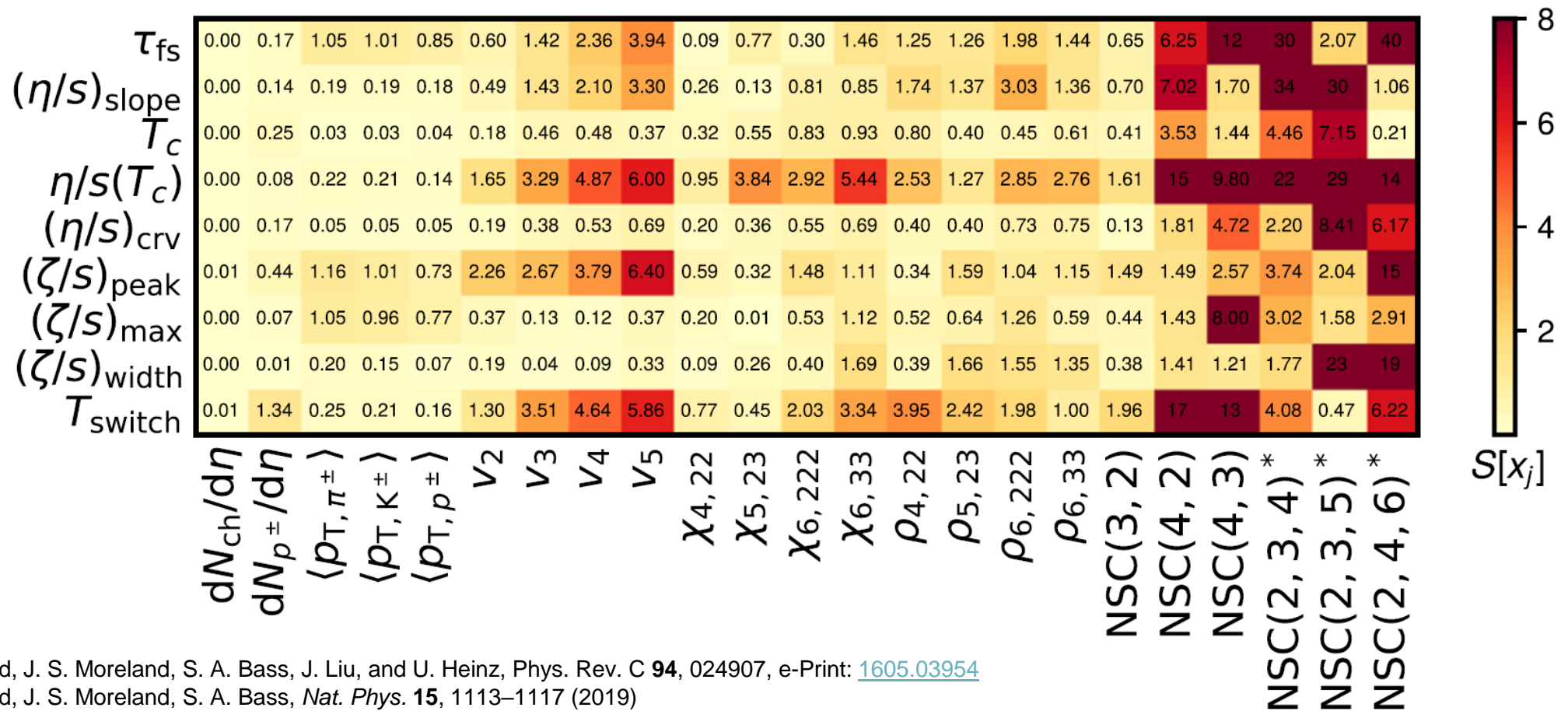
Constraining QGP properties with Bayesian estimation

J. E. Bernhard, J. S. Moreland, S. A. Bass, J. Liu, and U. Heinz, *'Applying Bayesian parameter estimation to relativistic heavy-ion collisions: Simultaneous characterization of the initial state and quark-gluon plasma medium,'* Phys. Rev. C **94**, 024907, e-Print: [1605.03954](https://arxiv.org/abs/1605.03954)

J. E. Bernhard, J. S. Moreland, S. A. Bass, *'Bayesian estimation of the specific shear and bulk viscosity of quark-gluon plasma,'* Nat. Phys. **15**, 1113–1117 (2019)

Bayesian studies: follow-up work

- The highest sensitivity on model parameters comes from higher-order SC observables!



J. E. Bernhard, J. S. Moreland, S. A. Bass, J. Liu, and U. Heinz, Phys. Rev. C **94**, 024907, e-Print: [1605.03954](https://arxiv.org/abs/1605.03954)

J. E. Bernhard, J. S. Moreland, S. A. Bass, Nat. Phys. **15**, 1113–1117 (2019)

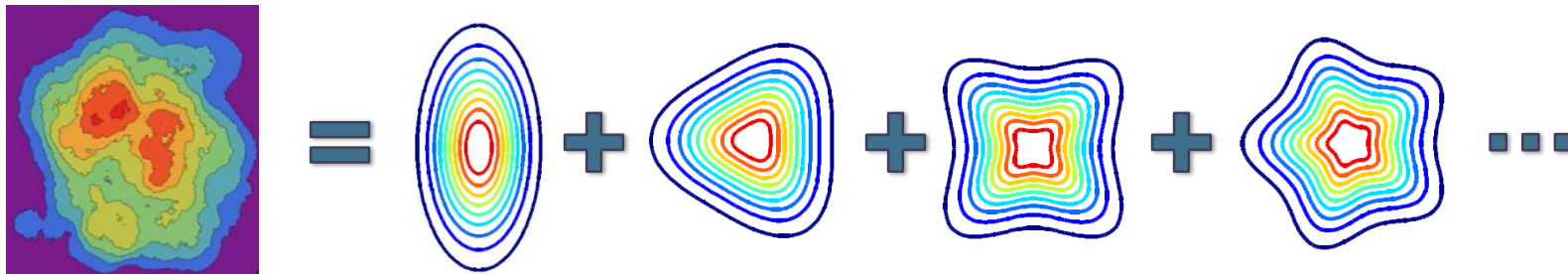
J.E.Parkkila, A.Onnerstad, S.F.Taghavi, C.Mordasini, AB, M.Virta, D.J.Kim, Phys. Lett. B **835** (2022), 137485, e-Print: [2111.08145](https://arxiv.org/abs/2111.08145)

Symmetric cumulants of hexagonal flow harmonics

ALICE Collaboration, Phys. Rev. C **112** (2025) 2, 024905, e-Print: [2412.15873](https://arxiv.org/abs/2412.15873)

Fourier series

- In **non-central** heavy-ion collisions, due to collision geometry the initial volume is almond shaped (ellipsoidal)
 - Dominant harmonic is v_2 (elliptic flow)
- In **most central (head-on)** collisions, due to fluctuations of participating nucleons, any shape can develop
 - All lower-order harmonics are equally probable



Linear and non-linear response

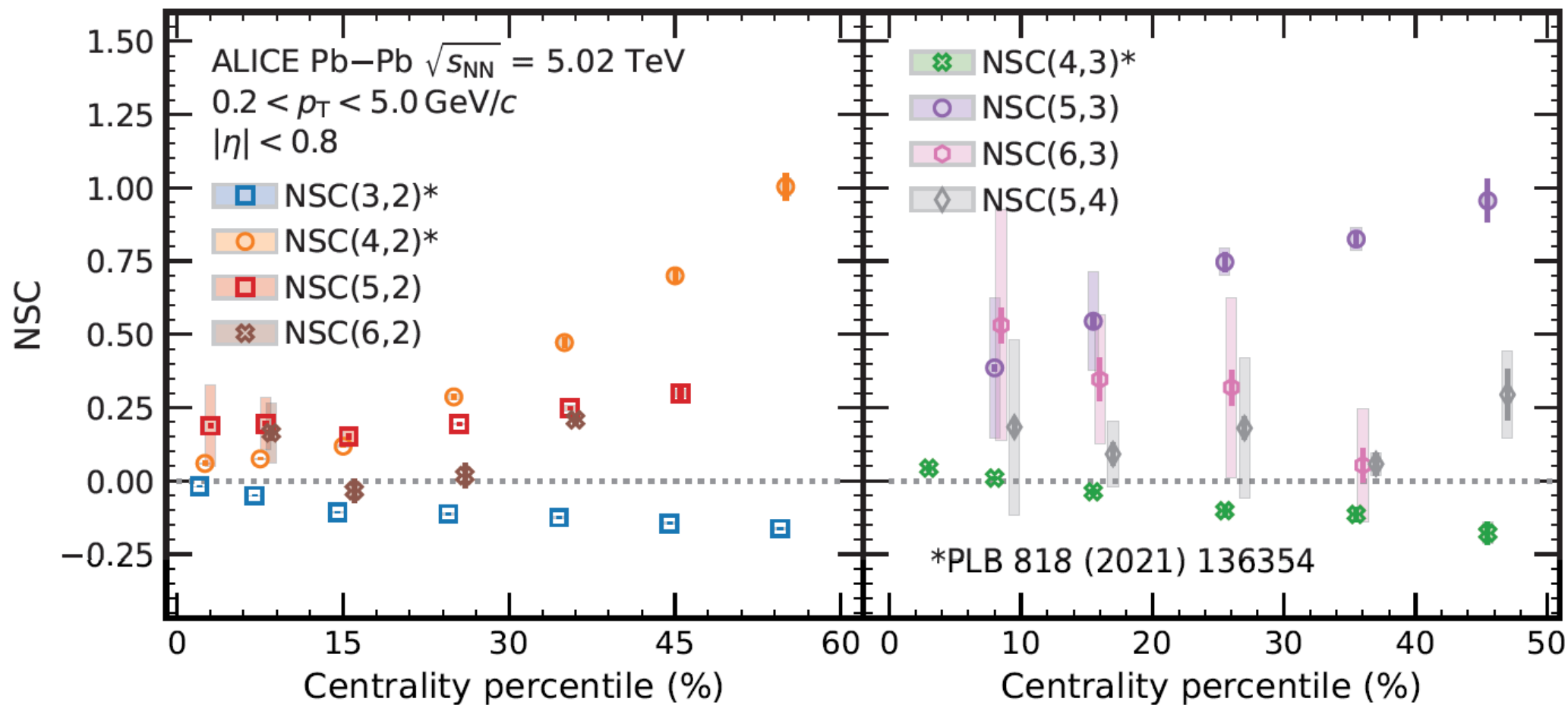
- It is possible to relate the observed \mathbf{v}_n coefficients to the initial-state spatial eccentricities defined as

$$\varepsilon_n e^{in\Phi_n} = -\frac{\{r^n e^{in\varphi}\}}{\{r^n\}}, \quad n \geq 2$$

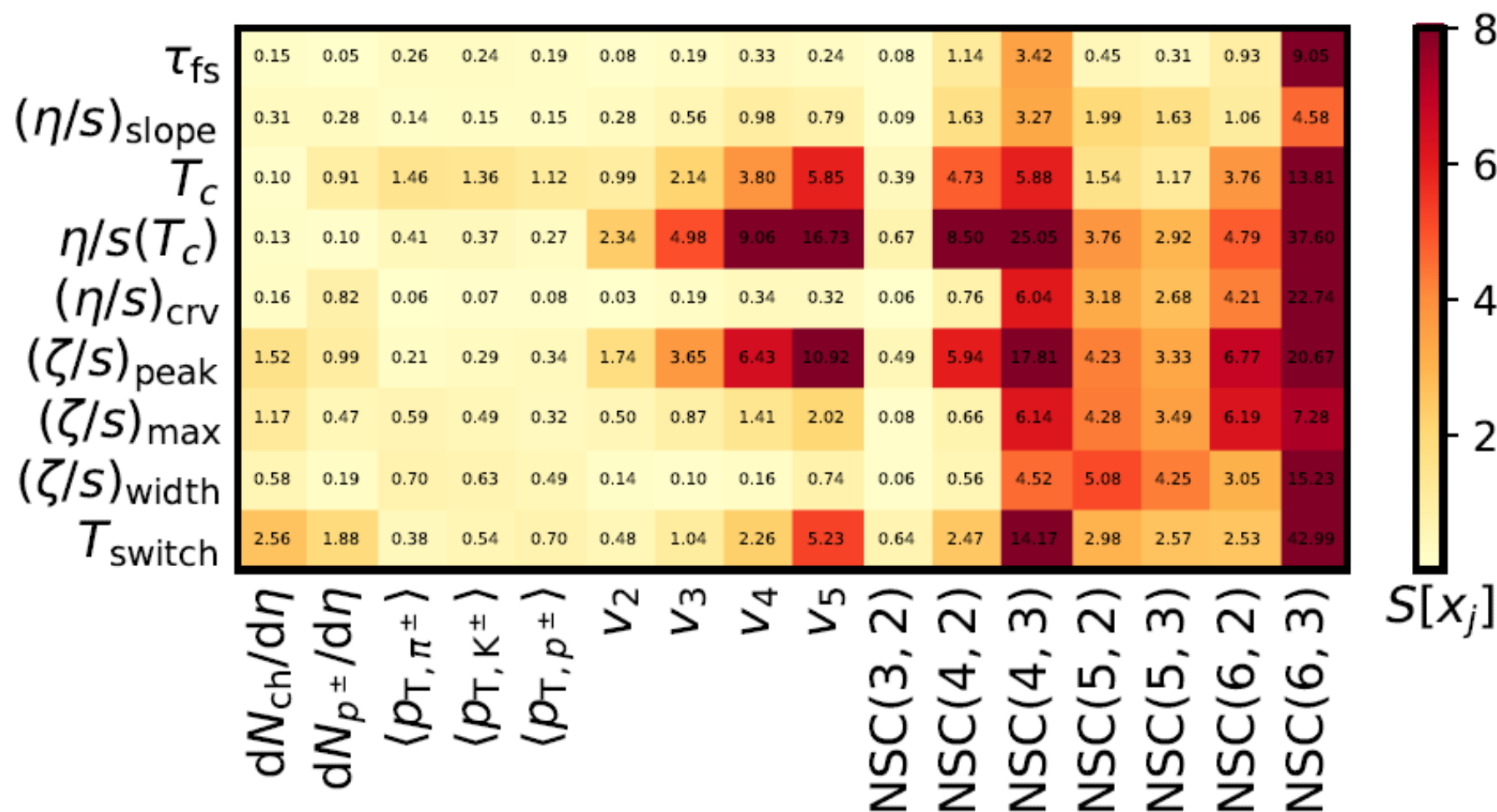
- Linear response (for small eccentricities): $\mathbf{v}_n \sim \varepsilon_n$
- Non-linear response via matrix equation : $\{\mathbf{v}_n\} = \{\varepsilon_n\}$
- Hexagonal harmonic \mathbf{v}_6 is particularly interesting due to different scaling of non-linear response to lower-order harmonics:

$$v_6 \sim v_2^3 \quad \text{and} \quad v_6 \sim v_3^2$$

Results

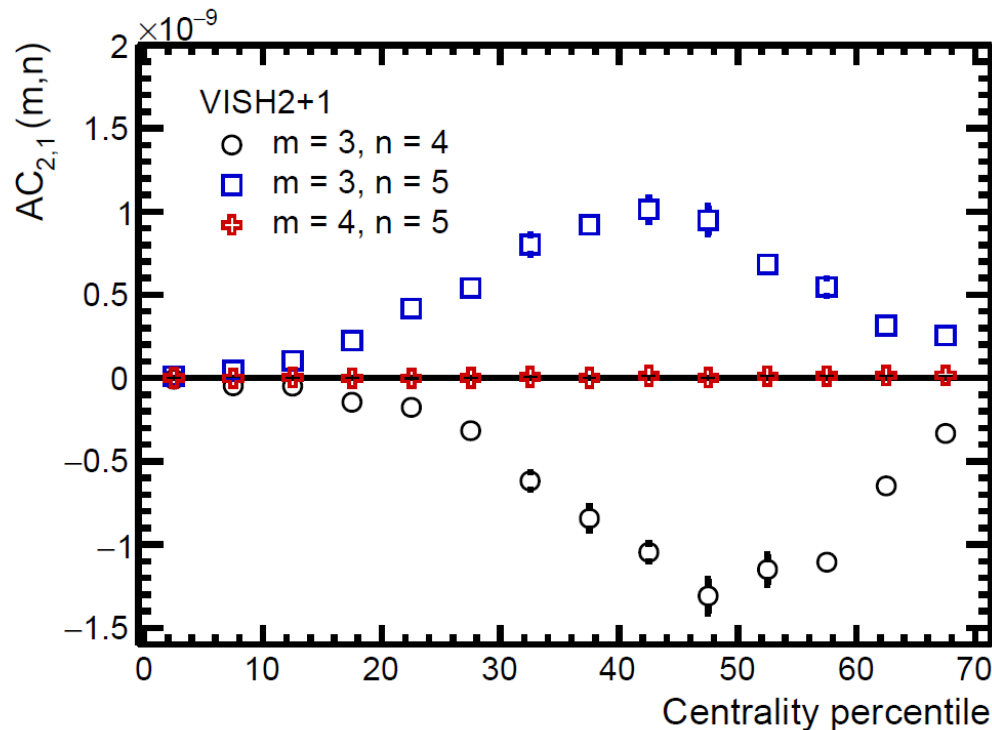


Input to Bayesian studies



Asymmetric Cumulants $AC_{k,l}(m,n)$

- Generalization of Symmetric Cumulants: $SC(m,n) = AC_{1,1}(m,n)$
- Fundamental stochastic observable is v^2
- Each of these observables is insensitive to lower-order correlations, because they satisfy all mathematical properties of cumulants



$$AC_{2,1}(m,n) = \langle v_m^4 v_n^2 \rangle - \langle v_m^4 \rangle \langle v_n^2 \rangle - 2 \langle v_m^2 v_n^2 \rangle \langle v_m^2 \rangle + 2 \langle v_m^2 \rangle^2 \langle v_n^2 \rangle,$$

$$AC_{3,1}(m,n) = \langle v_m^6 v_n^2 \rangle - \langle v_m^6 \rangle \langle v_n^2 \rangle - 3 \langle v_m^2 v_n^2 \rangle \langle v_m^4 \rangle - 3 \langle v_m^4 v_n^2 \rangle \langle v_m^2 \rangle + 6 \langle v_m^4 \rangle \langle v_m^2 \rangle \langle v_n^2 \rangle + 6 \langle v_m^2 v_n^2 \rangle \langle v_m^2 \rangle^2 - 6 \langle v_m^2 \rangle^3 \langle v_n^2 \rangle,$$

$$AC_{4,1}(m,n) = \langle v_m^8 v_n^2 \rangle - \langle v_m^8 \rangle \langle v_n^2 \rangle - 4 \langle v_m^2 v_n^2 \rangle \langle v_m^6 \rangle - 6 \langle v_m^4 v_n^2 \rangle \langle v_m^4 \rangle + 6 \langle v_m^4 \rangle^2 \langle v_n^2 \rangle - 4 \langle v_m^6 v_n^2 \rangle \langle v_m^2 \rangle + 8 \langle v_m^6 \rangle \langle v_m^2 \rangle \langle v_n^2 \rangle + 24 \langle v_m^2 v_n^2 \rangle \langle v_m^4 \rangle \langle v_m^2 \rangle + 12 \langle v_m^4 v_n^2 \rangle \langle v_m^2 \rangle^2 - 36 \langle v_m^4 \rangle \langle v_m^2 \rangle^2 \langle v_n^2 \rangle - 24 \langle v_m^2 v_n^2 \rangle \langle v_m^2 \rangle^3 + 24 \langle v_m^2 \rangle^4 \langle v_n^2 \rangle,$$

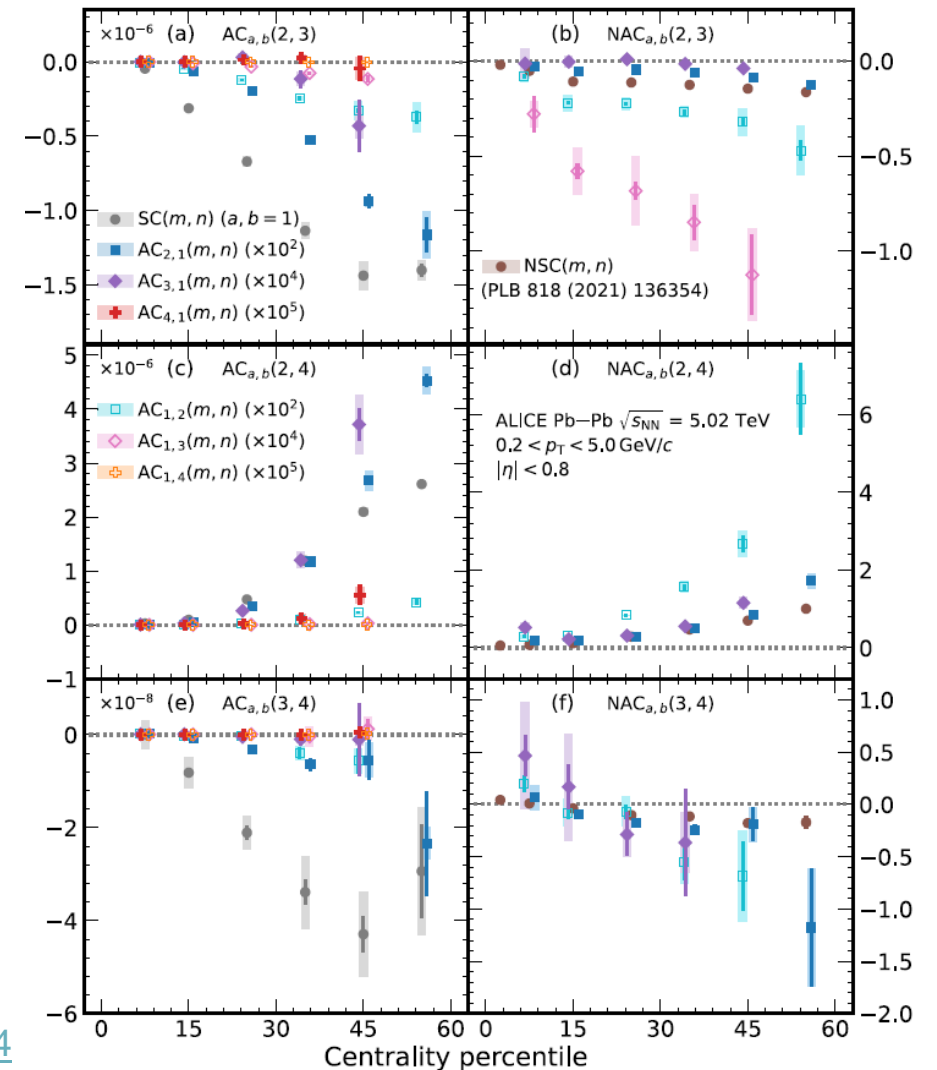
$$AC_{2,1,1}(k,l,m) = \langle v_k^4 v_l^2 v_m^2 \rangle - \langle v_k^4 v_l^2 \rangle \langle v_m^2 \rangle - \langle v_k^4 v_m^2 \rangle \langle v_l^2 \rangle - \langle v_k^4 \rangle \langle v_l^2 v_m^2 \rangle + 2 \langle v_k^4 \rangle \langle v_l^2 \rangle \langle v_m^2 \rangle - 2 \langle v_k^2 v_l^2 \rangle \langle v_k^2 v_m^2 \rangle - 2 \langle v_k^2 v_l^2 v_m^2 \rangle \langle v_k^2 \rangle + 4 \langle v_k^2 v_l^2 \rangle \langle v_k^2 \rangle \langle v_m^2 \rangle + 4 \langle v_k^2 v_m^2 \rangle \langle v_k^2 \rangle \langle v_l^2 \rangle + 2 \langle v_k^2 \rangle^2 \langle v_l^2 v_m^2 \rangle - 6 \langle v_k^2 \rangle^2 \langle v_l^2 \rangle \langle v_m^2 \rangle.$$

Asymmetric Cumulants $AC_{k,l}(m,n)$ in ALICE

- A new set of multiharmonic observables involving different moments of v_n amplitudes
 - Independent information about event-by-event v_n fluctuations, nonlinear response, etc. which is inaccessible to the SC observables

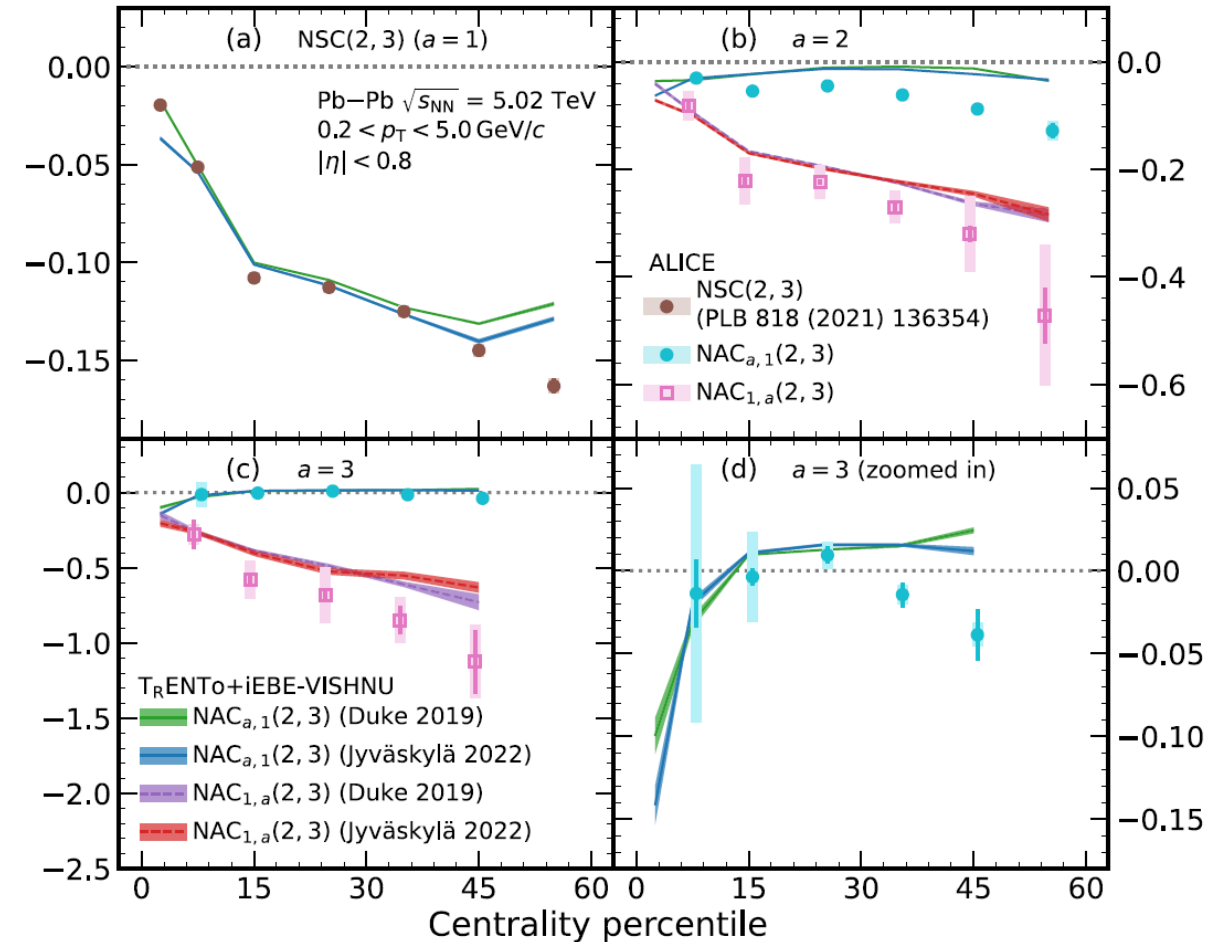
$$AC_{a,b}(m,n) = AC_{b,a}(n,m)$$

- The magnitudes of the measured AC observables show a dependence on the different moments as well as on the collision centrality, indicating the presence of nonlinear response in all even moments up to the eighth
- The higher-order asymmetric cumulants show different signatures than the symmetric and lower-order asymmetric cumulants



Asymmetric Cumulants $AC_{k,l}(m,n)$ in ALICE

- Splitting between ‘mirror’ AC observables $NAC_{a,1}(2,3)$ and $NAC_{1,a}(2,3)$ ($a = 2, 3$) is confirmed by the models
- The biggest discrepancy is for $NAC_{3,1}(2,3)$: close to zero in data while predicted to be positive by model parameterizations
- No sizable difference between predictions from the two model parameterizations
 - Correlations are dominated by initial state, and in both cases T_R ENTo is used to model it

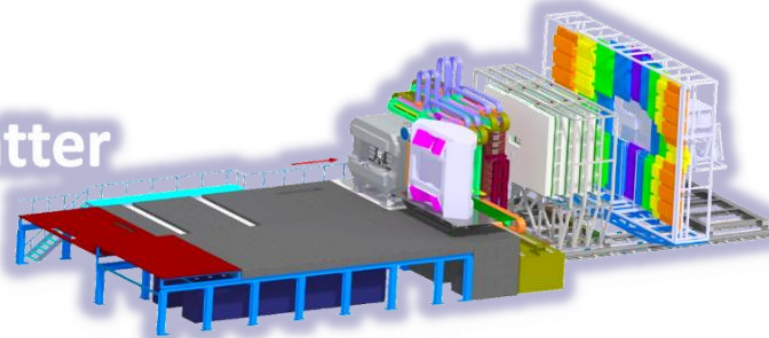


Status of feasibility studies for CBM

- QC, SC and AC observables

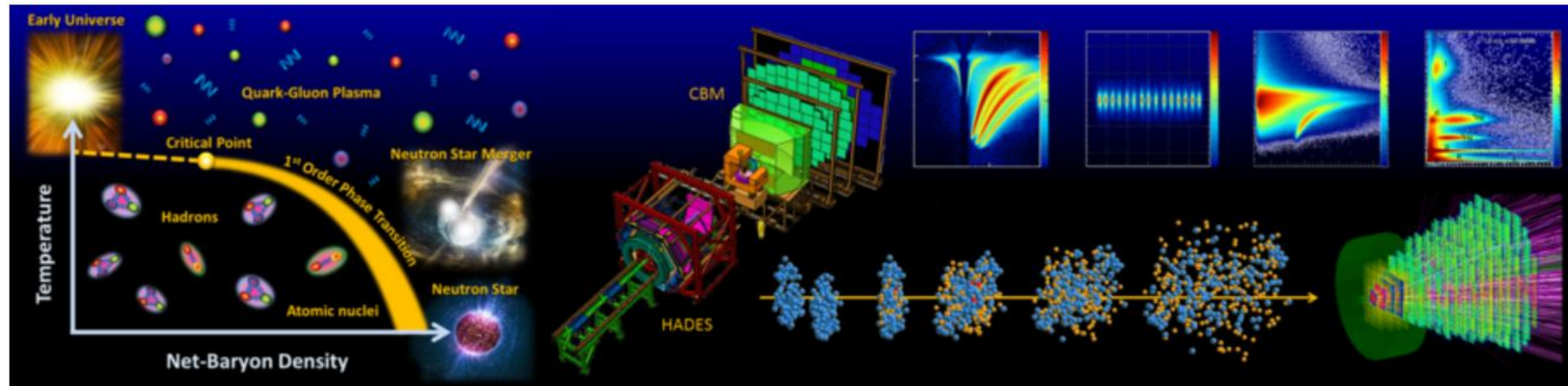


Compressed Baryonic Matter
experiment at FAIR



How can we produce Quark-Gluon Plasma?

- Alternatively, we can focus on the region of high baryon densities using fixed-target nucleus-nucleus collisions at GSI



- The **Compressed Baryonic Matter (CBM)** experiment will be one of the major scientific pillars of the future Facility for Antiproton and Ion Research (FAIR) at GSI in Darmstadt

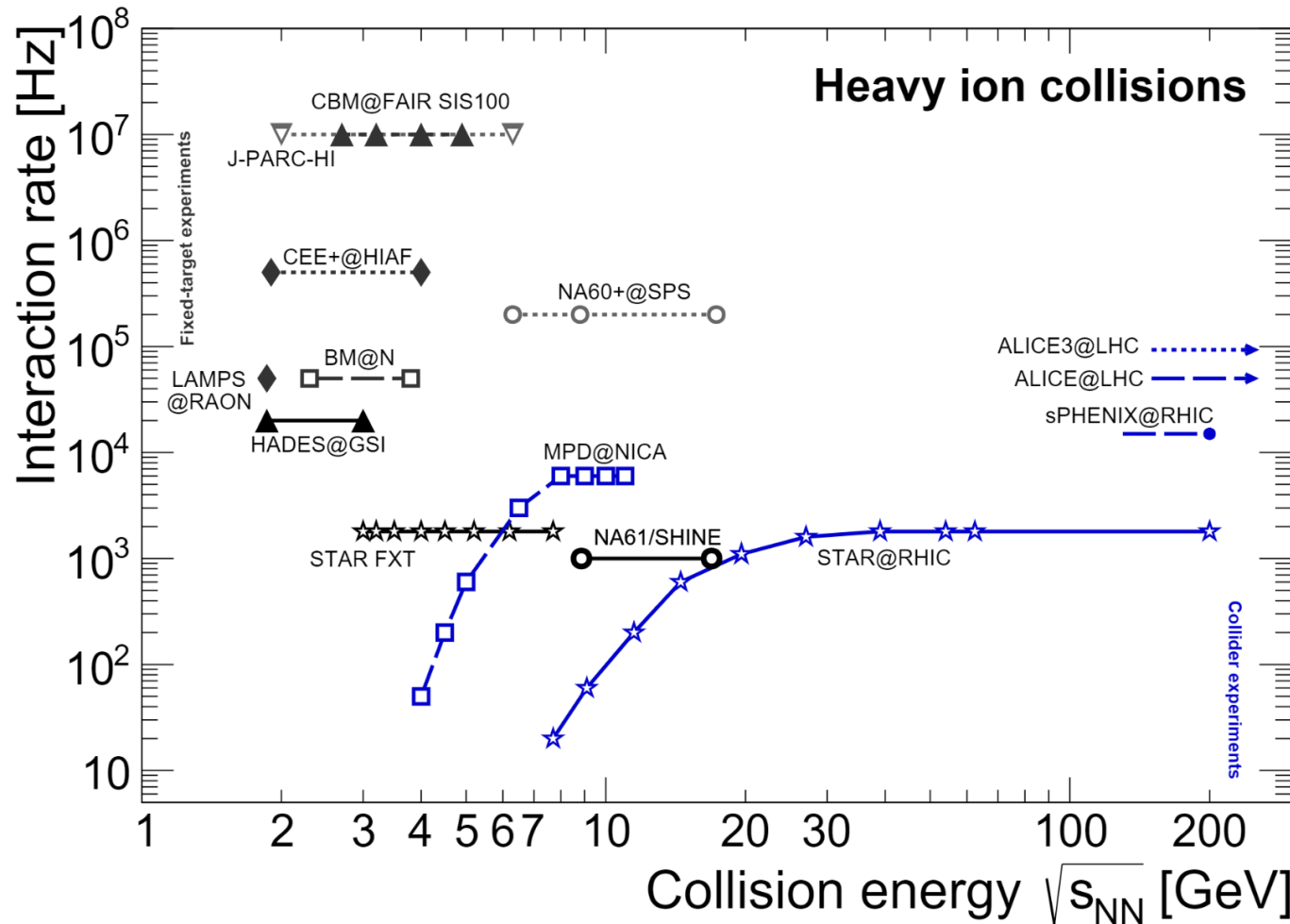
CBM: The physics case

- CBM is a unique high-intensity experiment with the capability of taking data at event rates up to 10 MHz
 - Unprecedentedly large statistics
 - Full mid-rapidity coverage
- Collision energies: $2.9 < s_{NN} < 4.9$ GeV
- Experimental investigation of region in QCD phase diagram with baryon chemical potential $500 < \mu_B < 850$ MeV
- Collision of a variety of nuclei, including isobaric species with varying isospin content

	$\sqrt{s_{NN}}$ [GeV]	μ_B [MeV]
SIS 18	2 – 2.5	830 – 760
SIS 100	2.3 – 5.3	785 – 520
SPS	5.1 – 17.3	530 – 220
STAR Collider	7.7 – 200	400 – 22
STAR FXT	3 – 13.7	700 – 265

$\mu_B(\sqrt{s_{NN}})$ from A. Andronic, P. Braun-Munzinger, K. Redlich and J. Stachel, Nature 561, no. 7723, 321 (2018)

CBM: The physics case



- CBM is ideal for studies of rare probes
- Systematic investigation of dependence on energy, size/centrality

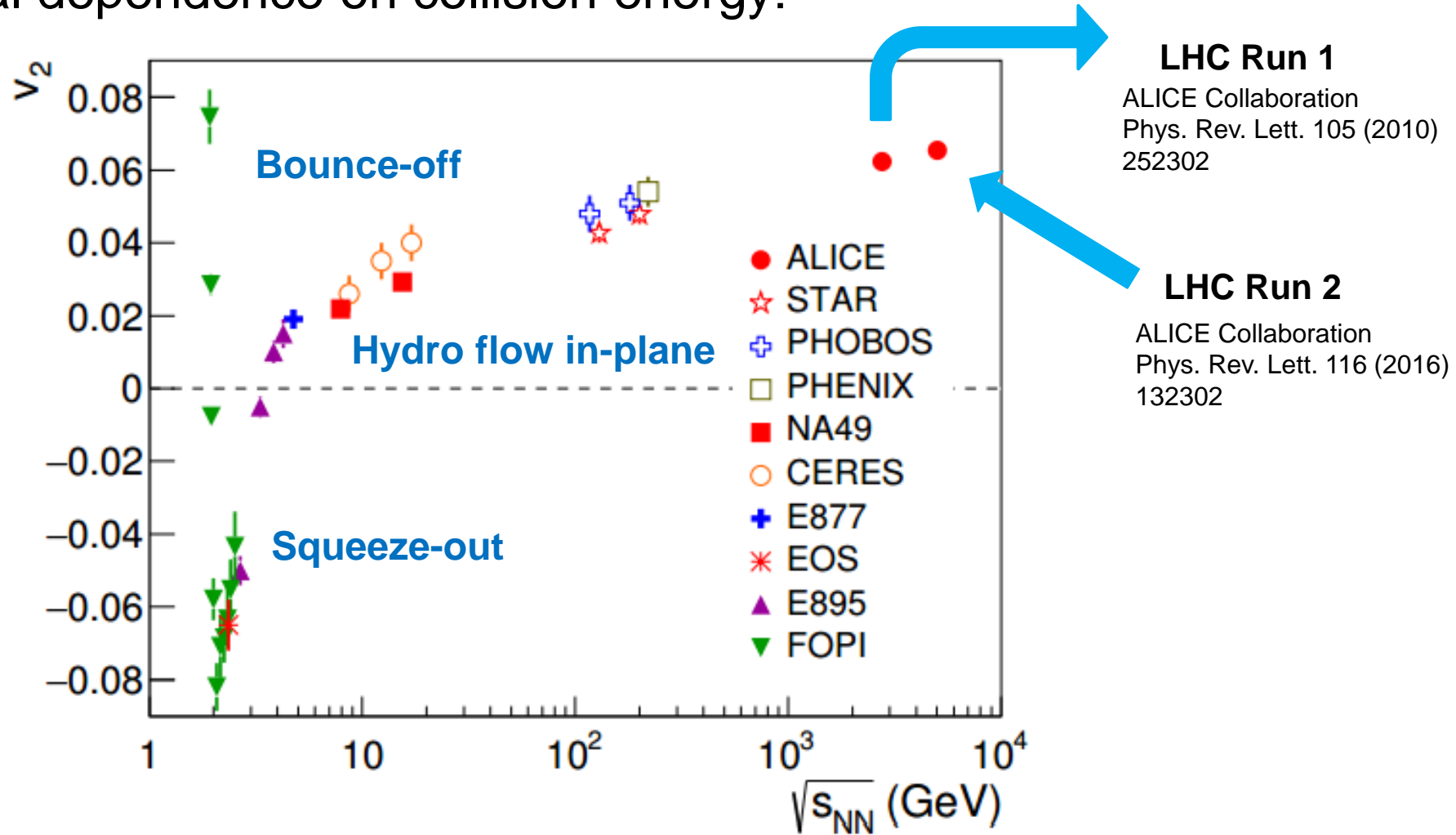
Claudia Höhne, Quark Matter 2023,
[‘The status of the CBM experiment at FAIR’](#)

CBM: The physics case

- Decisively confirm or disapprove the existence of the QCD critical point
- Interactions in a dense baryonic matter — understanding of the inner structure of compact stars and dynamics of neutron star mergers
 - Dynamics and the equation of state of dense nuclear matter
- High-statistics measurements of spectra, flow, and femtoscopic correlations from a systematic scan of beam energies and target-projectile combinations
 - Hyperon-nucleon and hyperon-hyperon interaction
- New phases in the QCD diagram?
 - Quarkyonic phase

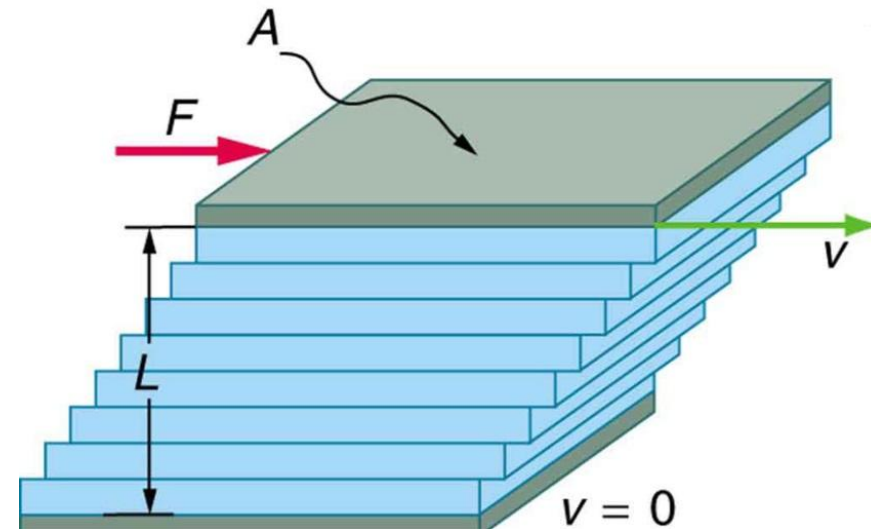
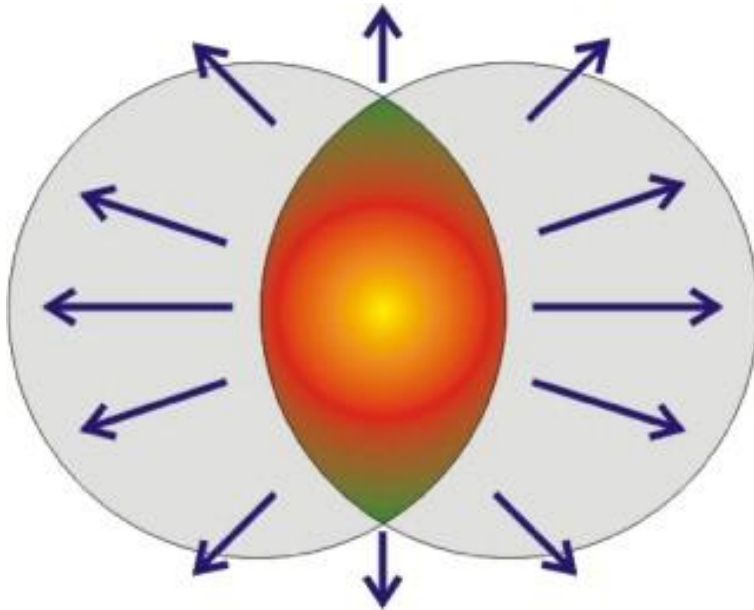
Historical snapshot

- Non-trivial dependence on collision energy:



Hydro flow in-plane

- Non-trivial effect which is sensitive to transport coefficients of QGP (e.g. its shear viscosity)

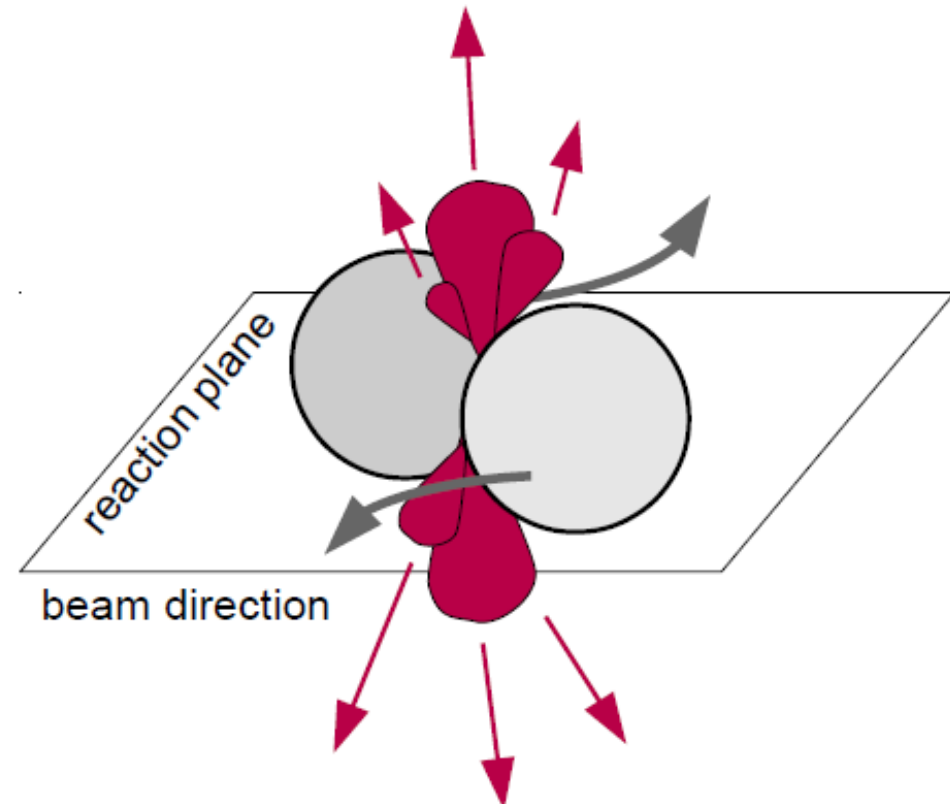
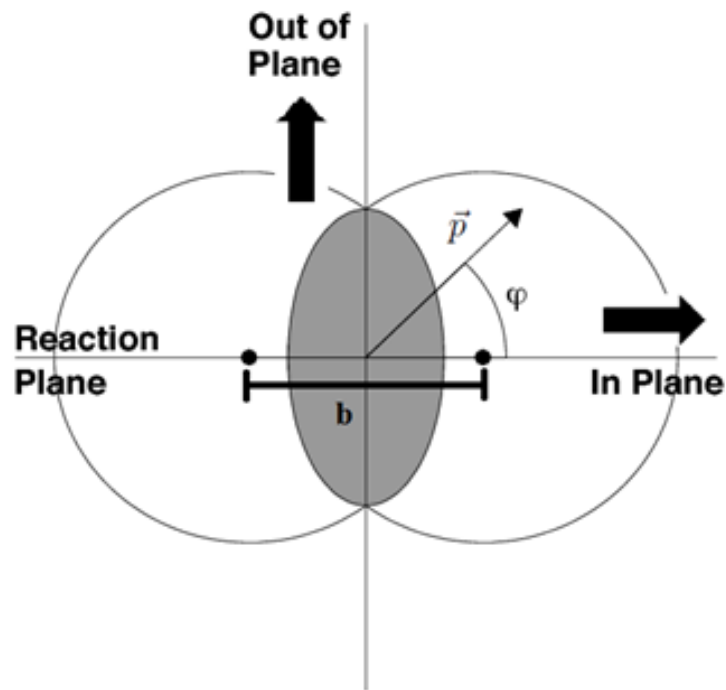


If anisotropic flow has developed, neighboring layers are moving at different relative velocities, parallel displacement is opposed by shear viscosity

large anisotropic flow \Leftrightarrow small shear viscosity

Squeeze-out flow

- ‘Squeeze-out’ a.k.a. elliptic flow ‘out-of-plane’
 - Can be both trivial (shadowing) and non-trivial (hydro)
 - Typically develops at intermediate energies: $\sqrt{s_{NN}} \sim 2-4 \text{ GeV}$
 - Negative flow values



Monte Carlo simulations

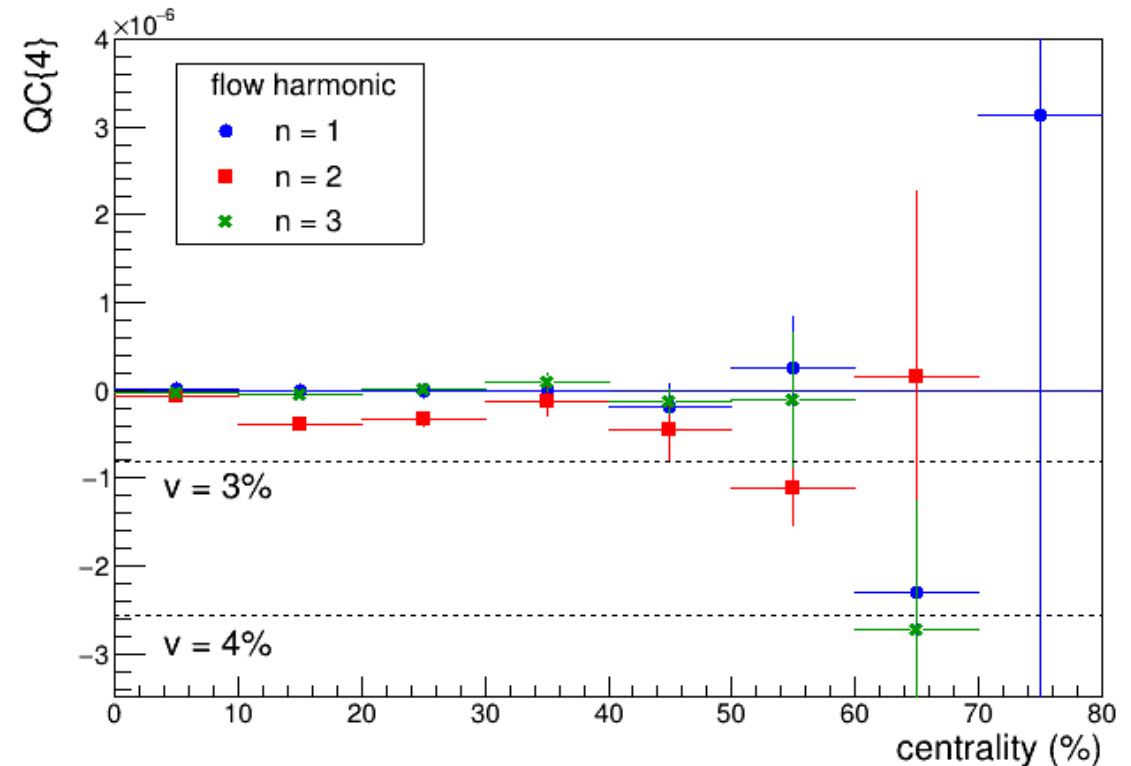
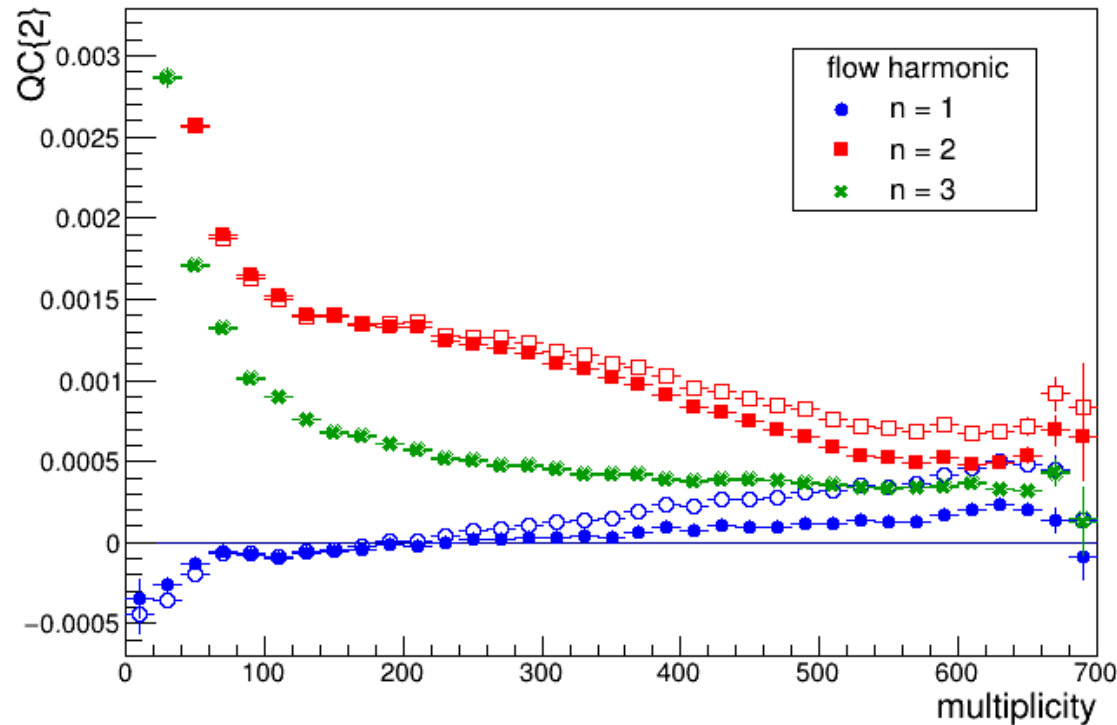
- Au+Au collisions at 12 AGeV
- **DCM-SMM**, based on Dubna Cascade Model + Quark-Gluon String Model (DCM-QGSM) and Statistical Multifragmentation Model (SMM)

Phys. Part. Nucl. Lett. 17 (2020) 3, 303-324, e-Print: [1912.09277](https://arxiv.org/abs/1912.09277)

- To make the DCM code applicable at higher energies (up to hundreds GeV/nucleon), it was merged with the QGSM
- New phenomena implemented: Extended coalescence, multifragmentation, hyperfragments production, vorticity of nuclear matter, Lambda polarization
- This study:
 - Basic cuts applied – total statistics ~ 4.2 M events

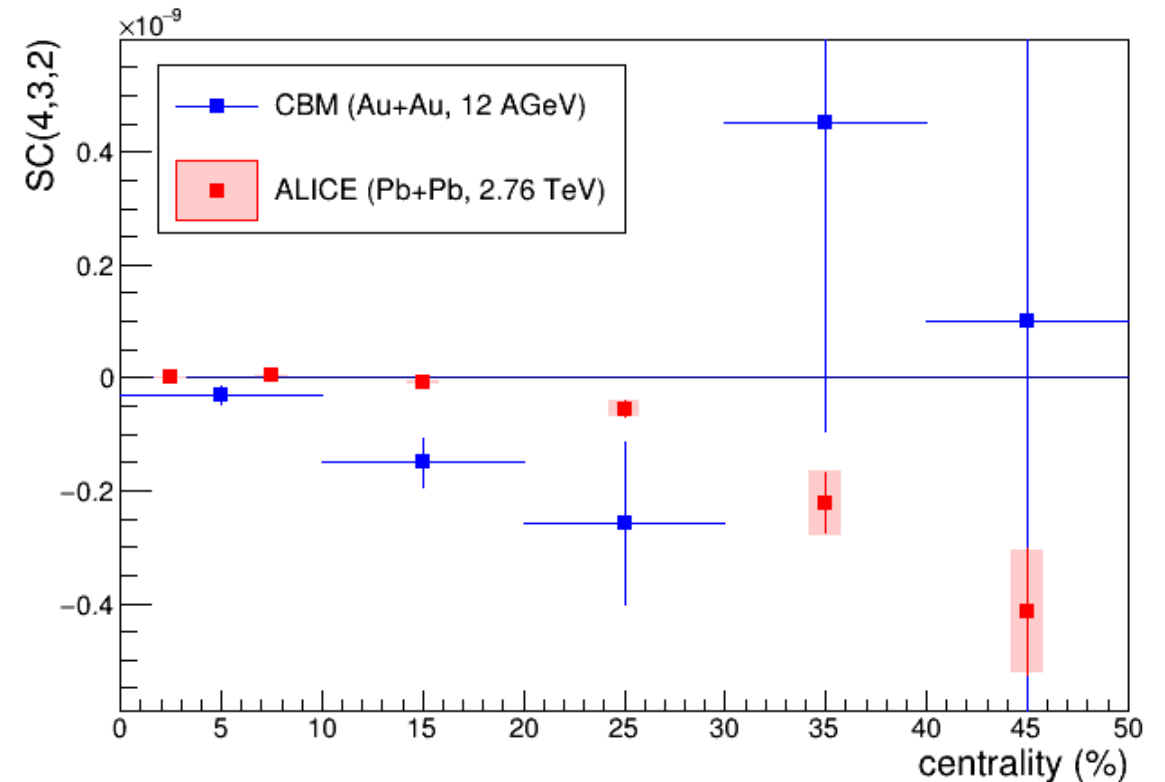
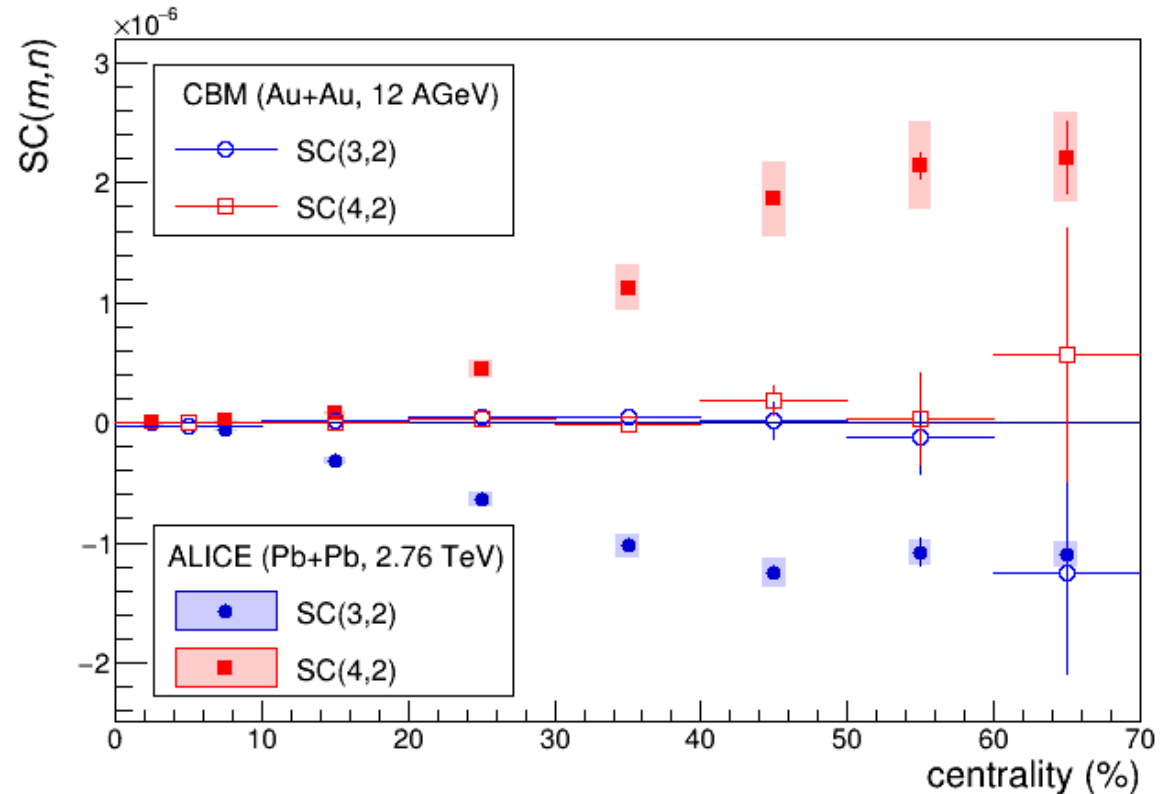
Multivariate flow observables in CBM

- Flow harmonics v_n estimated with Q-cumulants:



Multivariate flow observables in CBM

- Multiharmonic correlations estimated with Symmetric Cumulants (SC):



Multivariate flow observables in CBM

- Multiharmonic correlations estimated with Asymmetric Cumulants (AC):

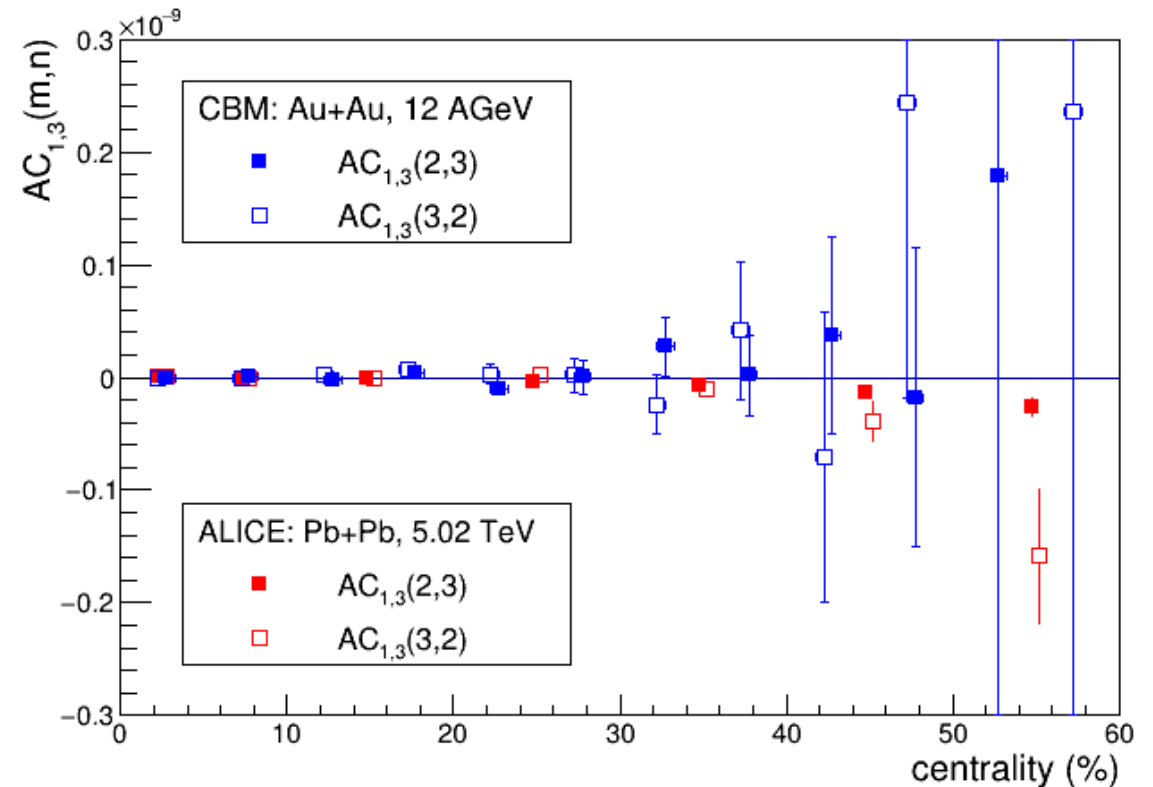
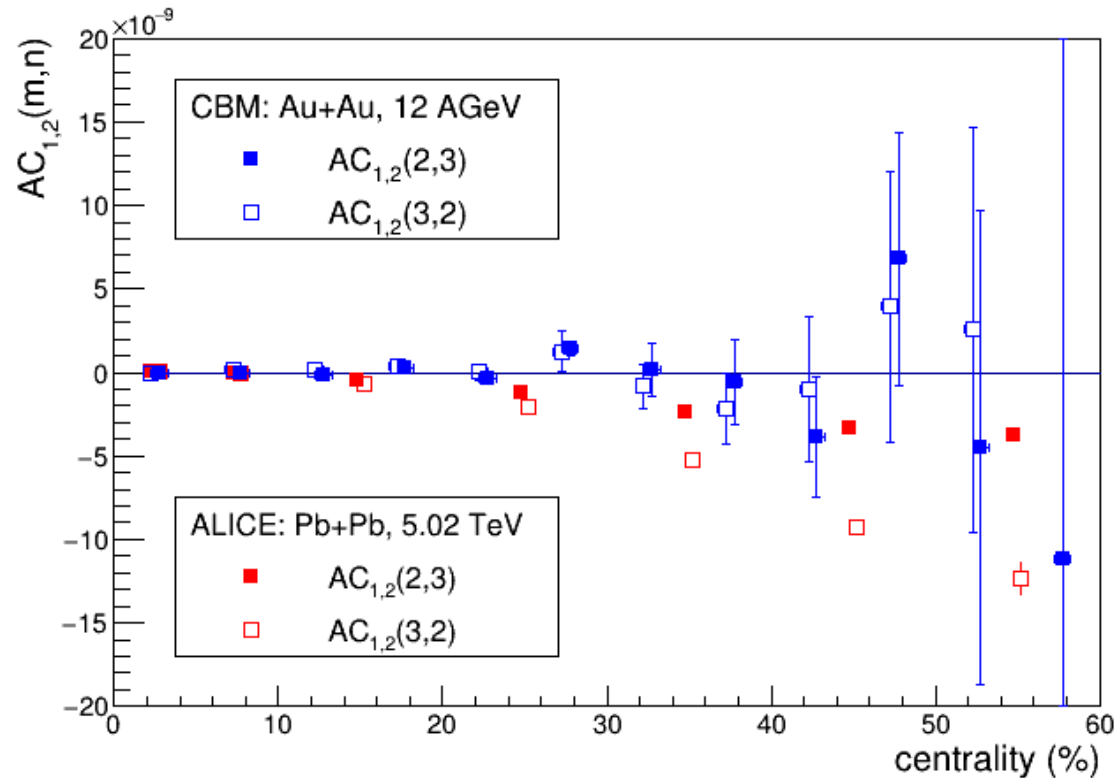
$$AC_{2,1}(m,n) = \langle v_m^4 v_n^2 \rangle - \langle v_m^4 \rangle \langle v_n^2 \rangle - 2 \langle v_m^2 v_n^2 \rangle \langle v_m^2 \rangle + 2 \langle v_m^2 \rangle^2 \langle v_n^2 \rangle,$$

$$AC_{3,1}(m,n) = \langle v_m^6 v_n^2 \rangle - \langle v_m^6 \rangle \langle v_n^2 \rangle - 3 \langle v_m^2 v_n^2 \rangle \langle v_m^4 \rangle - 3 \langle v_m^4 v_n^2 \rangle \langle v_m^2 \rangle \\ + 6 \langle v_m^4 \rangle \langle v_m^2 \rangle \langle v_n^2 \rangle + 6 \langle v_m^2 v_n^2 \rangle \langle v_m^2 \rangle^2 - 6 \langle v_m^2 \rangle^3 \langle v_n^2 \rangle,$$

- New higher-order flow observables, which quantify correlations of different higher-order moments of v_n , and satisfy all fundamental properties of multivariate cumulants
- By definition, AC extract new and independent information when compared to the previous lower-order flow observables
- In general, AC are not invariant under permutations of different v_n , therefore $AC_{1,2}(2,3)$ and $AC_{1,2}(3,2)$ are independent of each other

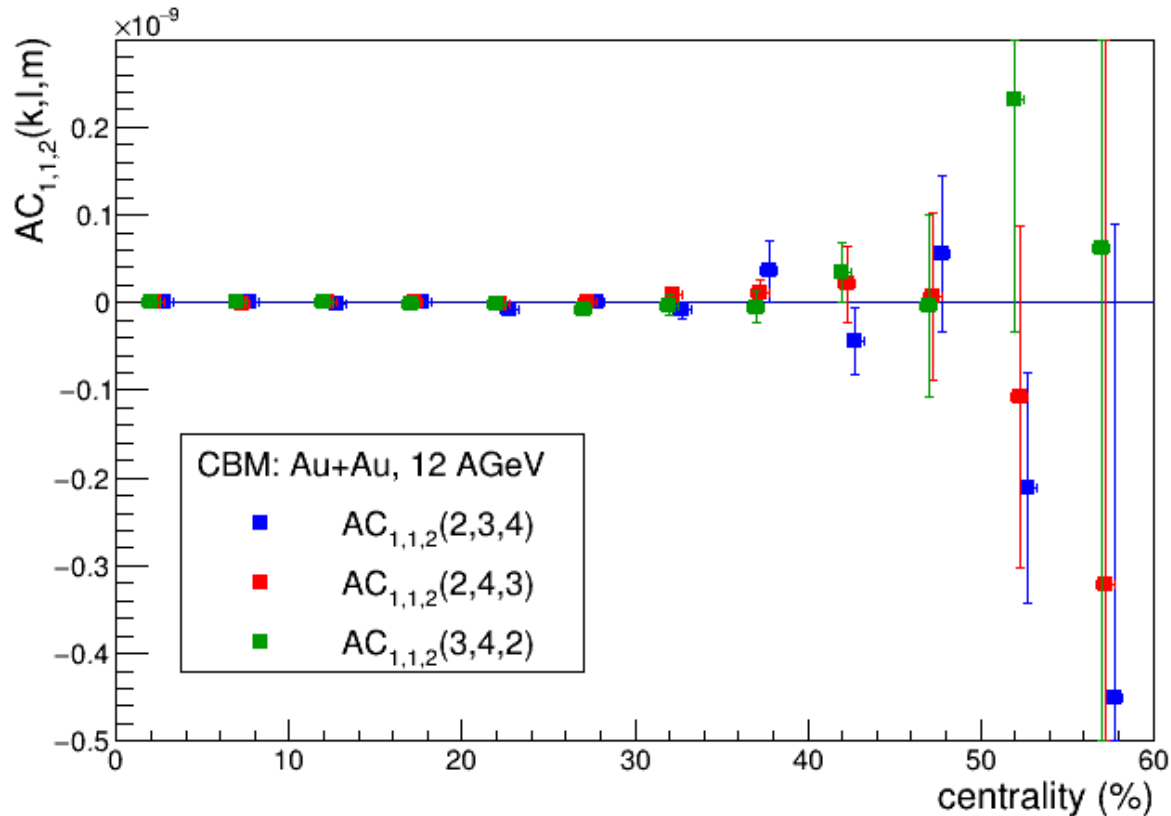
Multivariate flow observables in CBM

- Multiharmonic correlations estimated with Asymmetric Cumulants (AC):



Multivariate flow observables in CBM

- Multiharmonic correlations estimated with Asymmetric Cumulants (AC):



$$AC_{2,1,1}(k,l,m) = \langle v_k^4 v_l^2 v_m^2 \rangle - \langle v_k^4 v_l^2 \rangle \langle v_m^2 \rangle - \langle v_k^4 v_m^2 \rangle \langle v_l^2 \rangle - \langle v_k^4 \rangle \langle v_l^2 v_m^2 \rangle \\ + 2 \langle v_k^4 \rangle \langle v_l^2 \rangle \langle v_m^2 \rangle - 2 \langle v_k^2 v_l^2 \rangle \langle v_k^2 v_m^2 \rangle - 2 \langle v_k^2 v_l^2 v_m^2 \rangle \langle v_k^2 \rangle \\ + 4 \langle v_k^2 v_l^2 \rangle \langle v_k^2 \rangle \langle v_m^2 \rangle + 4 \langle v_k^2 v_m^2 \rangle \langle v_k^2 \rangle \langle v_l^2 \rangle \\ + 2 \langle v_k^2 \rangle^2 \langle v_l^2 v_m^2 \rangle - 6 \langle v_k^2 \rangle^2 \langle v_l^2 \rangle \langle v_m^2 \rangle.$$

Summary/Discussion

- Improving the built-in low sensitivity of correlation techniques for small multiplicities
- Bayesian studies: Establish a set of observables, which are mutually independent and well understood, both mathematically and physically
 - Cumulants: many observables in the field are named cumulants, even though they are not
- Signature problem
- Real gaps in the detector's acceptance

Scaling of stat. and sys. errors

- Scaling of the statistical uncertainty (N is a number of events, M is multiplicity, v is strength of Fourier harmonic, k is the order of correlator):

$$\sigma_v \sim \frac{1}{\sqrt{N}} \frac{1}{M^{k/2}} \frac{1}{v^{k-1}}$$

- Scaling of non-collective contribution in k -particle correlator:

$$\delta_k \sim \frac{1}{M^{k-1}}$$

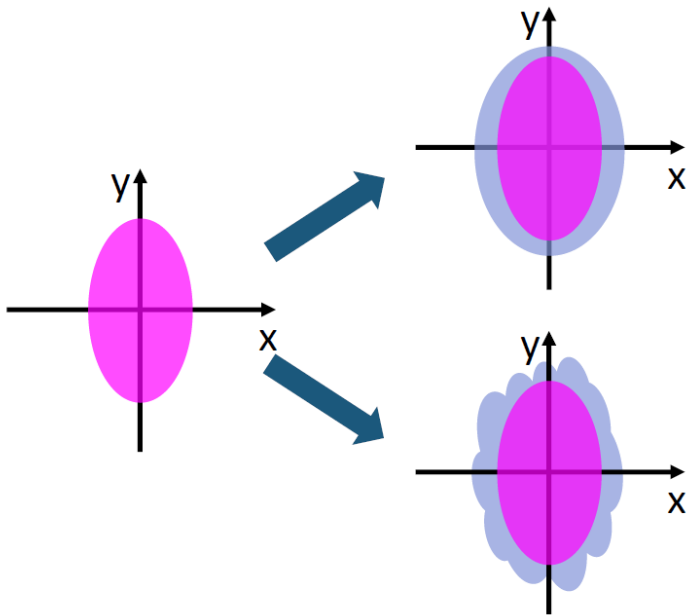
- For both reasons, multiparticle correlations are a precision technique only for:
 - a) large multiplicities
 - b) large values of Fourier harmonics v_n

Thanks!

Backup slides

Shear vs. bulk viscosities

- Can we separate the effects of shear (η) and bulk (ξ) viscosities?



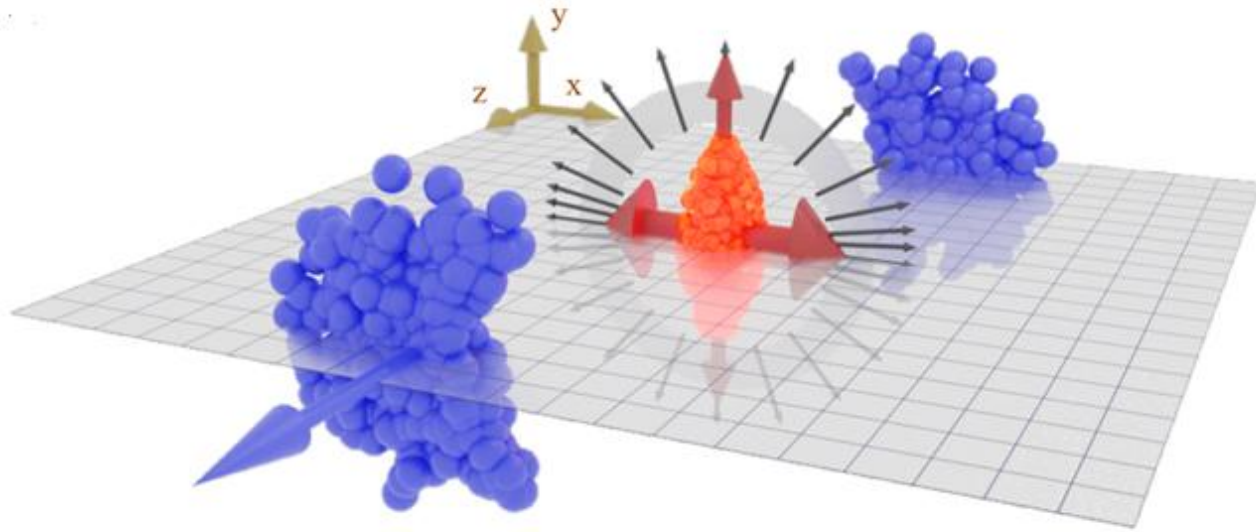
Credits: C. Mordasini

- Isotropic fluctuations
 - Neighbouring layers move at equal velocities
 - Generally preserves the ellipse shape
 - Main sensitivity to ξ/s
- Shape fluctuations
 - Neighbouring layers move at different velocities
 - Sensitivity to η/s

- We need new observables which can separate these different sources of fluctuations

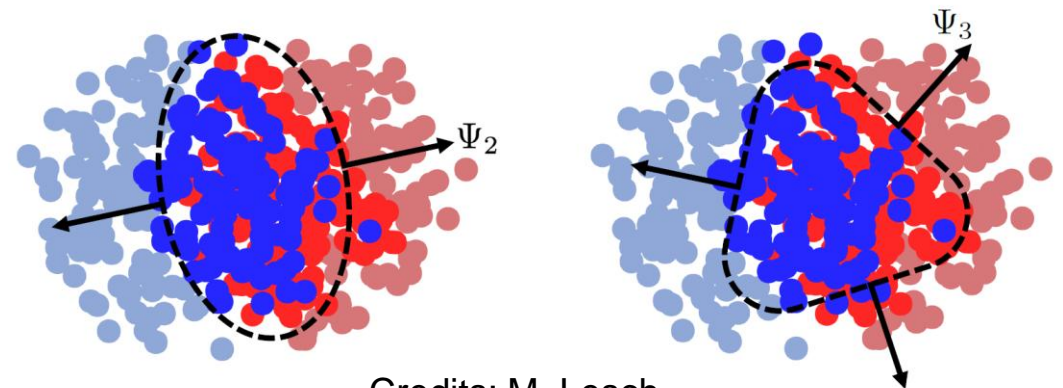
Initial anisotropic volume and thermalized medium

- **Non-central** heavy-ion collision is a prime example
 - Trivially (#1): Due to geometry of collision the resulting volume containing interacting matter is anisotropic in coordinate space
 - Trivially (#2): To leading order this anisotropic volume is ellipsoidal



Credits: D.D. Chinellato, ICHEP 2020

- More subtle cases of initial anisotropic volume can occur due to **fluctuations of participating nucleons**



Credits: M. Lesch

Fourier series

- In flow analysis, anisotropic emission of particles in the transverse plane after heavy-ion collision is described by:

$$f(\varphi) = \frac{1}{2\pi} \left[1 + 2 \sum_{n=1}^{\infty} v_n \cos[n(\varphi - \Psi_n)] \right] \longleftrightarrow \text{Contour Plot} = \text{Circular} + \text{Elliptical} + \text{Triangular} + \text{Quadrangular} + \dots$$

- v_n : flow amplitudes
- Ψ_n : symmetry planes
- Anisotropic flow is quantified with v_n and Ψ_n
 - v_1 is directed flow
 - v_2 is elliptic flow
 - v_3 is triangular flow
 - v_4 is quadrangular flow, etc.

S. Voloshin and Y. Zhang, Z.Phys. C70 (1996) 665-672

2-particle cumulants in general

- Cumulants are alternative to moments to describe stochastic properties of variable
- If 2 p.d.f.'s have same moments, they will also have same cumulants, and vice versa
 - True both for univariate and multivariate case
- X_i denotes the general i -th stochastic variable
- The most general decomposition of 2-particle correlation is:

$$\langle X_1 X_2 \rangle = \langle X_1 \rangle \langle X_2 \rangle + \langle X_1 X_2 \rangle_c$$

- By definition, the 2nd term on RHS is 2-particle cumulant
- Cumulants cannot be measured directly, however:

$$\langle X_1 X_2 \rangle_c = \langle X_1 X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle$$

3-particle cumulants in general

- The most general decomposition of 3-particle correlation is:



- Or written mathematically:

$$\begin{aligned}
 \langle X_1 X_2 X_3 \rangle &= \langle X_1 \rangle \langle X_2 \rangle \langle X_3 \rangle \\
 &+ \langle X_1 X_2 \rangle_c \langle X_3 \rangle + \langle X_1 X_3 \rangle_c \langle X_2 \rangle + \langle X_2 X_3 \rangle_c \langle X_1 \rangle \\
 &+ \langle X_1 X_2 X_3 \rangle_c
 \end{aligned}$$

- The key point: 2-particle cumulants were expressed independently in terms of measured correlations in previous step!

$$\langle X_1 X_2 \rangle_c = \langle X_1 X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle$$

3-particle cumulants in general

- Working recursively from higher to lower orders, we eventually have 3-particle cumulant expressed in terms of measured 3-, 2-, and 1-particle averages

$$\begin{aligned}\langle X_1 X_2 X_3 \rangle_c &= \langle X_1 X_2 X_3 \rangle \\ &- \langle X_1 X_2 \rangle \langle X_3 \rangle - \langle X_1 X_3 \rangle \langle X_2 \rangle - \langle X_2 X_3 \rangle \langle X_1 \rangle \\ &+ 2 \langle X_1 \rangle \langle X_2 \rangle \langle X_3 \rangle\end{aligned}$$

- In the same way, cumulants can be expressed in terms of measurable averages for any number of particles
 - The number of terms grows rapidly

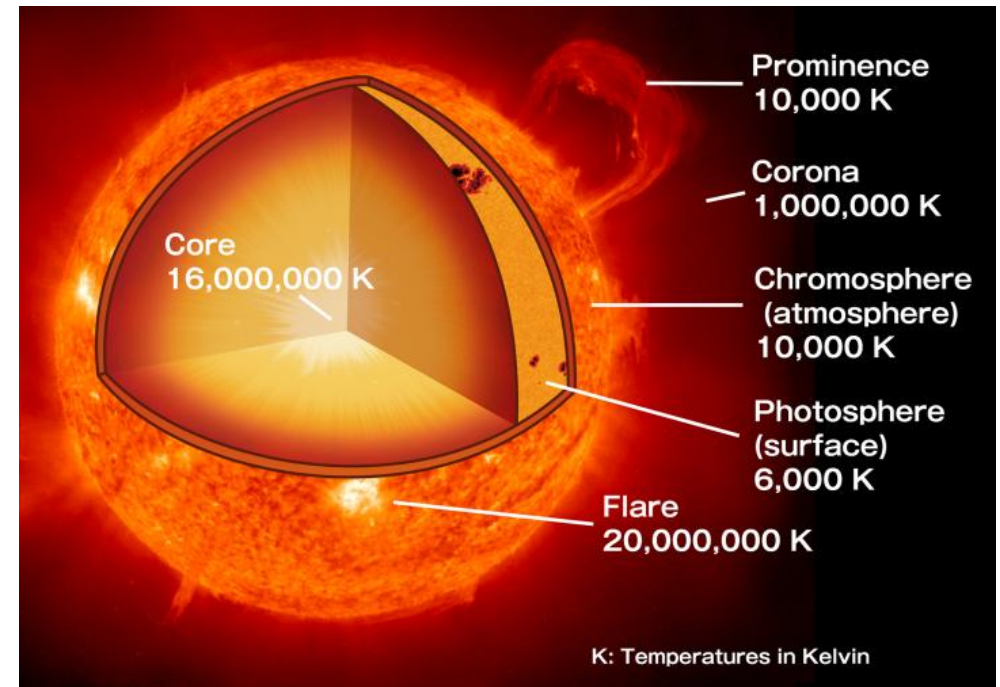
Setting up the stage

- Quantities for Bayesian estimation:
 - Vector of model parameters to be estimated: \mathbf{x}
 - Vector representing set of experimental data: \mathbf{y}
 - Likelihood ('how well the model describes the data'): $P(\mathbf{y}|\mathbf{x})$
 - Initial knowledge about parameters is given by a prior: $P(\mathbf{x})$
- The posterior distribution for the model parameters is obtained from the Bayesian theorem:

$$P(\mathbf{x}|\mathbf{y}) \propto P(\mathbf{y}|\mathbf{x})P(\mathbf{x})$$

Modeling heavy-ion collisions

- Theorists model heavy-ion collisions with (at least) 5 distinct stages:
 1. Initial conditions
 2. Pre-equilibrium
 3. Viscous relativistic hydrodynamics
 4. Particlization
 5. Boltzmann transport
- Smooth crossover at T_c between QGP state and ordinary hadronic state: **$T_c = 156.5 \pm 1.5 \text{ MeV}$**



This value is approximately $2 \times 10^{12} \text{ K}$
(i.e. 100,000 times hotter than in the
core of the Sun)

1. Initial conditions

- Monte Carlo Glauber model
 - $N_{\text{part}}(b)$ and $N_{\text{bin}}(b)$
 - When sampling nucleons, a minimum distance criterion is used for each pair of nucleons:

$$|\mathbf{x}_i - \mathbf{x}_j| > d_{\text{min}}$$

- Initial energy density is determined with the TRENTo model
 - The total density of participating nucleons is given by the superposition of individual nucleon densities, which are assumed to be Gaussian:

$$\begin{aligned}\tilde{\rho}_{A,B}(\mathbf{x}) &= \sum_{i=1}^{N_{\text{part}}} \gamma_i \rho_n(\mathbf{x} - \mathbf{x}_i) \\ \rho_n(\mathbf{x}) &= \frac{1}{(2\pi w^2)^{3/2}} \exp\left(-\frac{|\mathbf{x}|^2}{2w^2}\right)\end{aligned}$$

- Free parameters: d_{min} , $\gamma_i \rightarrow \sigma_{\text{fluct}}^2$ (controls multiplicity fluctuations), w (effective nucleon width)

1. Initial conditions

- Thickness function of participants – projection of total density of participating nucleons on the plane transverse to beam:

$$\tilde{T}_{A,B}(x,y) = \int dz \tilde{\rho}_{A,B}(x \pm b/2, y, z)$$

- Transition to Milne coordinates:

$$\mathbf{x}_{\perp} = (x, y) \quad (\text{position in transverse plane})$$

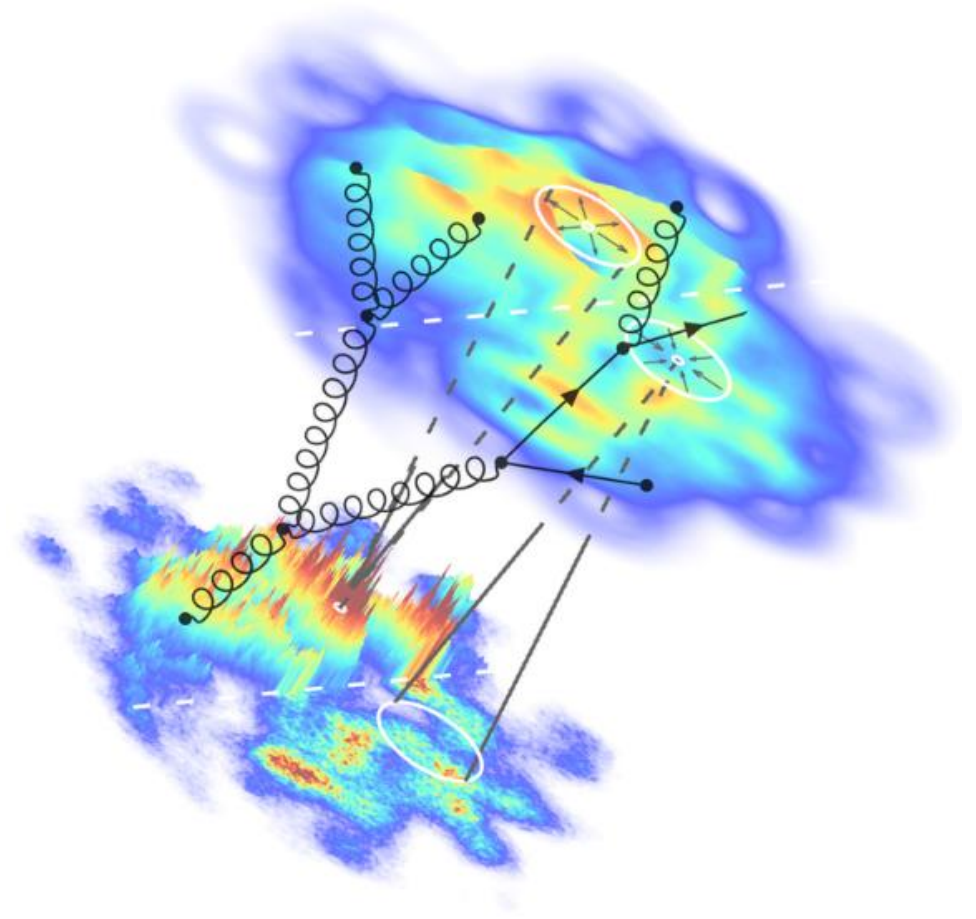
$$\eta_s = \frac{1}{2} \log[(t+z)/(t-z)] \quad (\text{spacetime rapidity})$$

$$\tau = \sqrt{t^2 - z^2} \quad (\text{proper time})$$

- Finally, the initial energy density and midrapidity ($\eta_s = 0$) and shortly after collision ($\tau = \tau_0$) is given by: $\left. \frac{dE}{d^2\mathbf{x}_{\perp} d\eta_s} \right|_{\tau=\tau_0} = \text{Norm} \times \left(\frac{\tilde{T}_A^p + \tilde{T}_B^p}{2} \right)^{1/p}$
- Free parameters: **Norm** and **p**

2. Pre-equilibrium

- After initial conditions are set, it is expected that within ~ 1 fm/c the produced matter will reach the equilibrium, at which point its evolution can be described with hydrodynamics
- This transient state, or pre-equilibrium, is not well understood
- Two regimes:
 - free streaming (zero coupling)
 - viscous hydrodynamics (strong coupling)
- Regimes matched at time $\tau = \tau_{fs} > \tau_0$
- Free parameter: τ_{fs}



A. Kurkela, A. Mazeliauskas, J-F. Paquet,
S. Schlichting, D. Teaney, Phys. Rev. Lett. 122 (2019)

3. Viscous relativistic hydrodynamics

- Dynamical evolution of QGP is simulated using **VISH2+1**:
 - boost invariance
 - zero net baryon density
 - viscous pressures are evolved using Israel-Stewart equations
 - accurate description at midrapidity

- Specific shear viscosity:

$$(\eta/s)(T) = (\eta/s)_{\min} + (\eta/s)_{\text{slope}}(T - T_c) \left(\frac{T}{T_c} \right)^{(\eta/s)_{\text{crv}}}$$

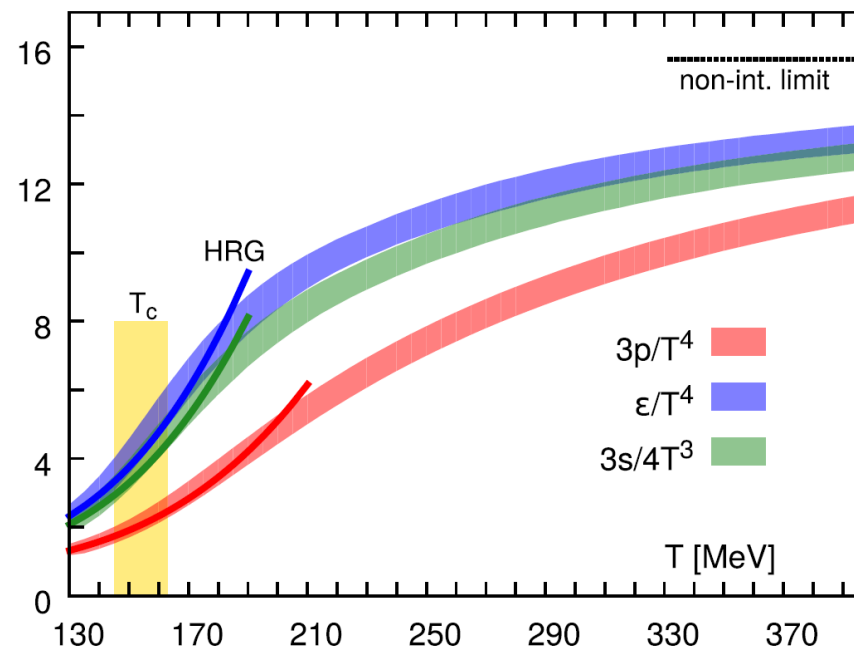
- Specific bulk viscosity: $(\zeta/s)(T) = \frac{(\zeta/s)_{\max}}{1 + \left(\frac{T - (\zeta/s)_{T_{\text{peak}}}}{(\zeta/s)_{\text{width}}} \right)^2}$

- Free parameters: $(\eta/s)_{\min}$, $(\eta/s)_{\text{slope}}$, $(\eta/s)_{\text{crv}}$, $(\zeta/s)_{\max}$, $(\zeta/s)_{T_{\text{peak}}}$ and $(\zeta/s)_{\text{width}}$

3. Viscous relativistic hydrodynamics

- Finally, the equation of state (EoS) needs to be specified, which relates energy density, pressure, entropy and temperature
- The choice for EoS: matching the solution from lattice QCD valid at high temperature with solution from hadron resonance gas calculation valid at low temperature

A. Bazavov et al (HotQCD Collaboration),
“Equation of state in (2 + 1)-flavor QCD”
Phys. Rev. D **90**, 094503 (2014)



4. Particlization

- As the system expands and cools down, number of particle interactions decreases
 - Switch from macroscopic description using hydrodynamic to microscopic description using Boltzmann transport
- How to convert the system from a continuous fluid into an ensemble of discrete particles at temperature T_{switch} ?
 - Cooper-Frye formula:

$$E \frac{dN_i}{d^3p} = \frac{g_i}{(2\pi)^3} \int_{\sigma} f_i(p) p^\mu d^3\sigma_\mu$$

F. Cooper and G. Frye, Phys. Rev. D **10**, 186 (1974)

- Probability density for particle species and momenta
- Free parameter: T_{switch}

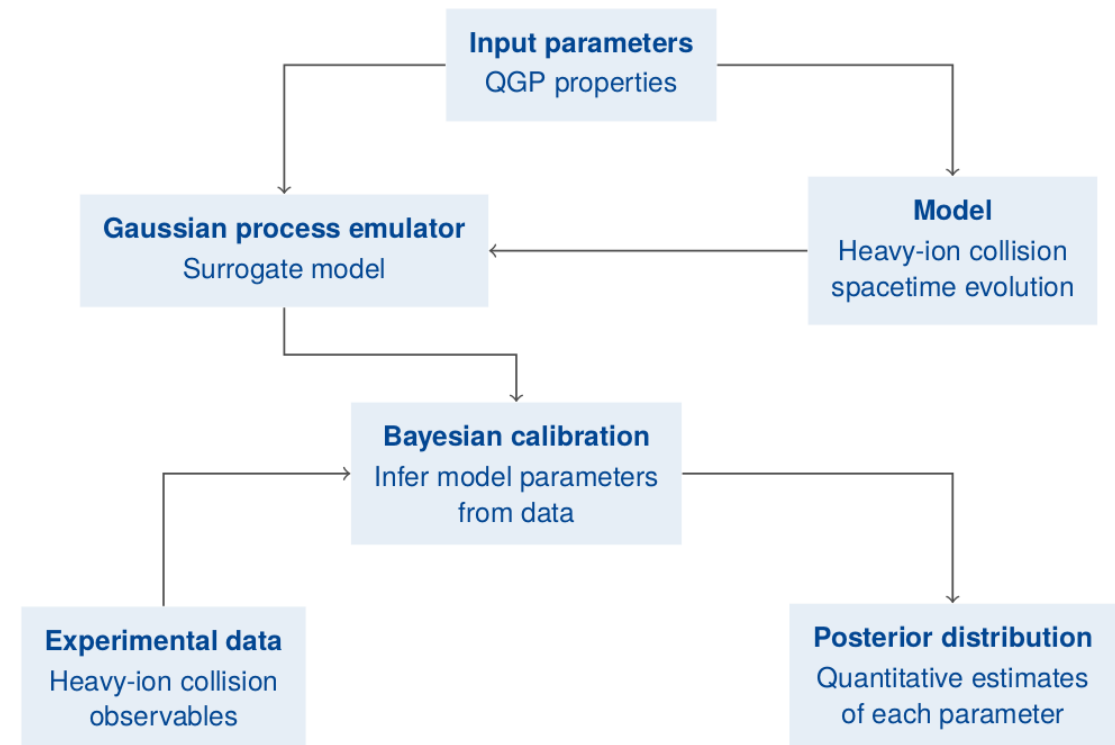
5. Boltzmann transport

- Once the fluid is converted into particles, a Boltzmann transport model simulates the subsequent interactions
 - Boltzmann equation:
$$\frac{df_i(x, p)}{dt} = C_i(x, p)$$
 - f_i and C_i are distribution function and collision kernel for specie i
- Popular model of choice: “Ultra-relativistic Quantum Molecular Dynamics (URQMD)”
 - Effectively solves the Boltzmann equation by propagating particles along classical (straight-line) trajectories, sampling stochastic binary collisions, and calculating formation and decay of intermediate resonances

URQMD website is at this [link](#)

Setting up the stage

- QGP: transient lifetime $\sim 10^{-23}\text{s}$ and microscopic size $\sim 10^{-14}\text{m}$
- Main idea: matching the output of computational collision models to experimental observations
- Main goals:
 - Temperature-dependent specific shear and bulk viscosity of QGP
 - Details of initial conditions in heavy-ion collisions



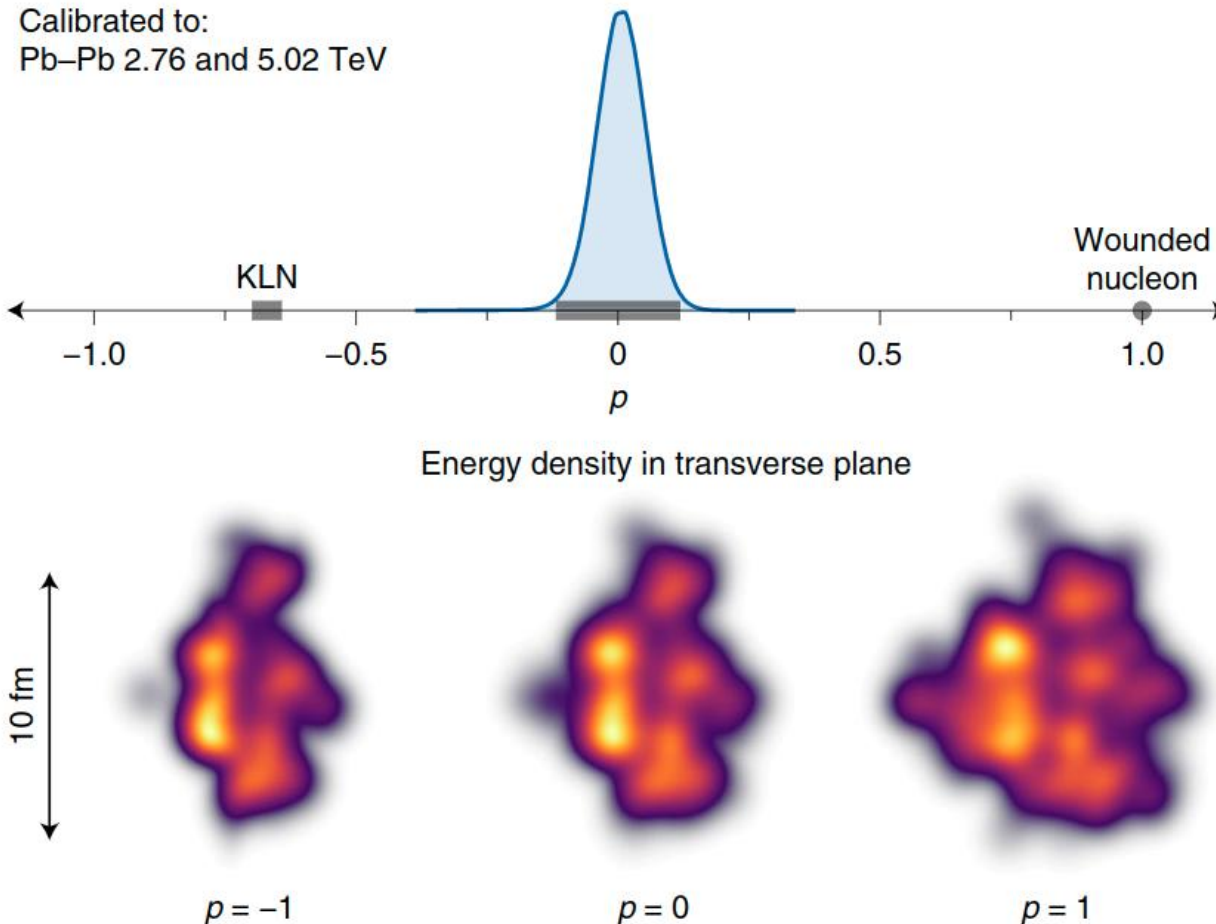
Our responsibility

Results for 14 free parameters

Parameter	Description	Prior range	Posterior estimate	MAP value
Norm [GeV]	Normalization factor, $\sqrt{s_{NN}} = 2.76$ TeV	8–20	$13.9^{+1.2}_{-1.1}$	13.94
Norm [GeV]	Normalization factor, $\sqrt{s_{NN}} = 5.02$ TeV	10–25	$18.5^{+1.8}_{-1.7}$	18.38
p	Energy deposition parameter	$-1/2$ to $+1/2$	$0.006^{+0.078}_{-0.078}$	0.007
σ_{fluct}	Std. dev. of nucleon multiplicity fluctuations	0–2	$0.90^{+0.24}_{-0.27}$	0.918
w [fm]	Gaussian nucleon width	0.4–1.0	$0.96^{+0.04}_{-0.05}$	0.956
d_{min} [fm]	Minimum inter-nucleon distance	0–1.7	$1.28^{+0.42}_{-0.53}$	1.27
τ_{fs} [fm/c]	Free streaming time	0–1.5	$1.16^{+0.29}_{-0.25}$	1.16
η/s min	Minimum value of η/s (at T_c)	0–0.2	$0.085^{+0.026}_{-0.025}$	0.081
η/s slope [GeV $^{-1}$]	Slope of η/s above T_c	0–8	$0.83^{+0.83}_{-0.83}$	1.11
η/s crv	Curvature of η/s above T_c	-1 to $+1$	$-0.37^{+0.79}_{-0.63}$	–0.48
ζ/s max	Maximum value of ζ/s	0–0.1	$0.037^{+0.040}_{-0.022}$	0.052
ζ/s width [GeV]	Width of ζ/s peak	0–0.1	$0.029^{+0.045}_{-0.026}$	0.022
ζ/s T_{peak} [GeV]	Temperature of ζ/s maximum	0.150–0.200	$0.177^{+0.023}_{-0.021}$	0.183
T_{switch} [GeV]	Switching / particlization temperature	0.135–0.165	$0.152^{+0.003}_{-0.003}$	0.151

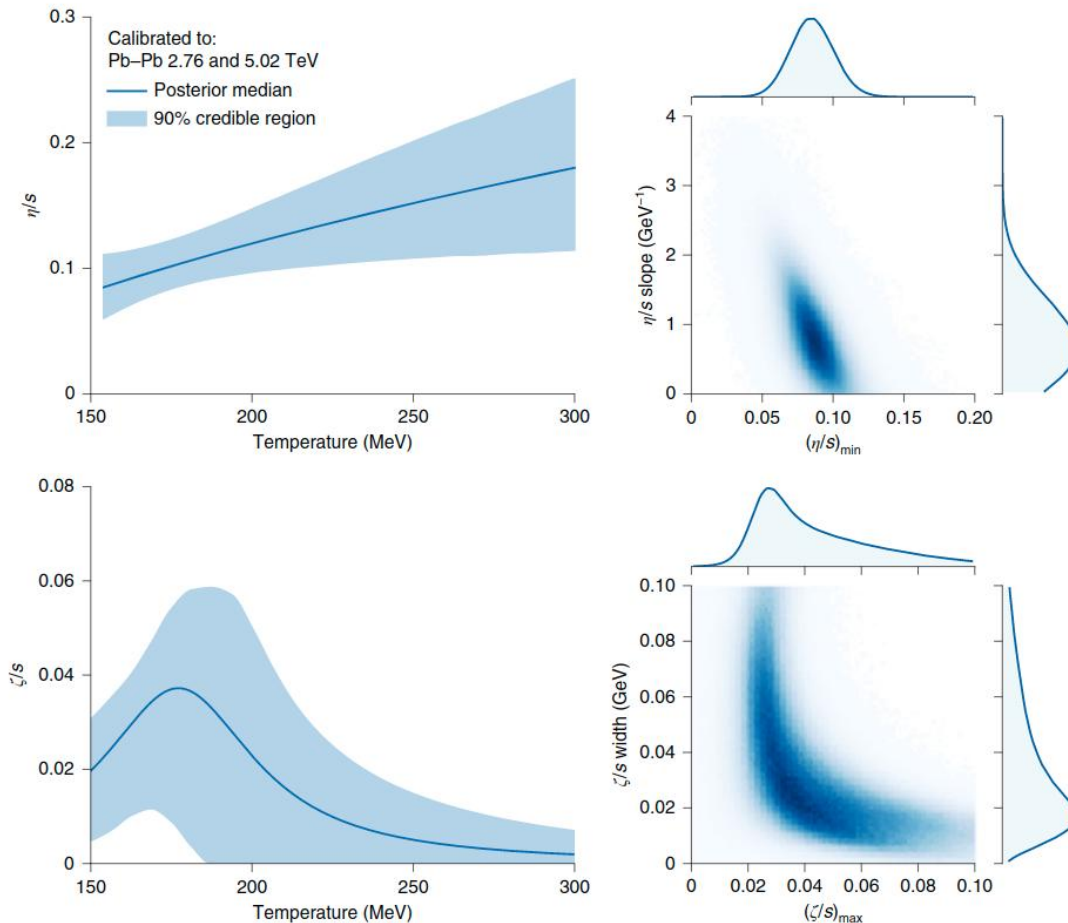
Posterior estimates: The central values are the medians; uncertainties are the 90% highest posterior density (HPD) credible intervals. The maximum a posteriori (MAP) values were determined by numerically maximizing the posterior probability.

Results for initial conditions



Value of $p = 0$ is clearly favoured \Rightarrow initial energy density $\sim \sqrt{\tilde{T}_A \tilde{T}_B}$

Results for shear and bulk viscosities



Shear viscosity:

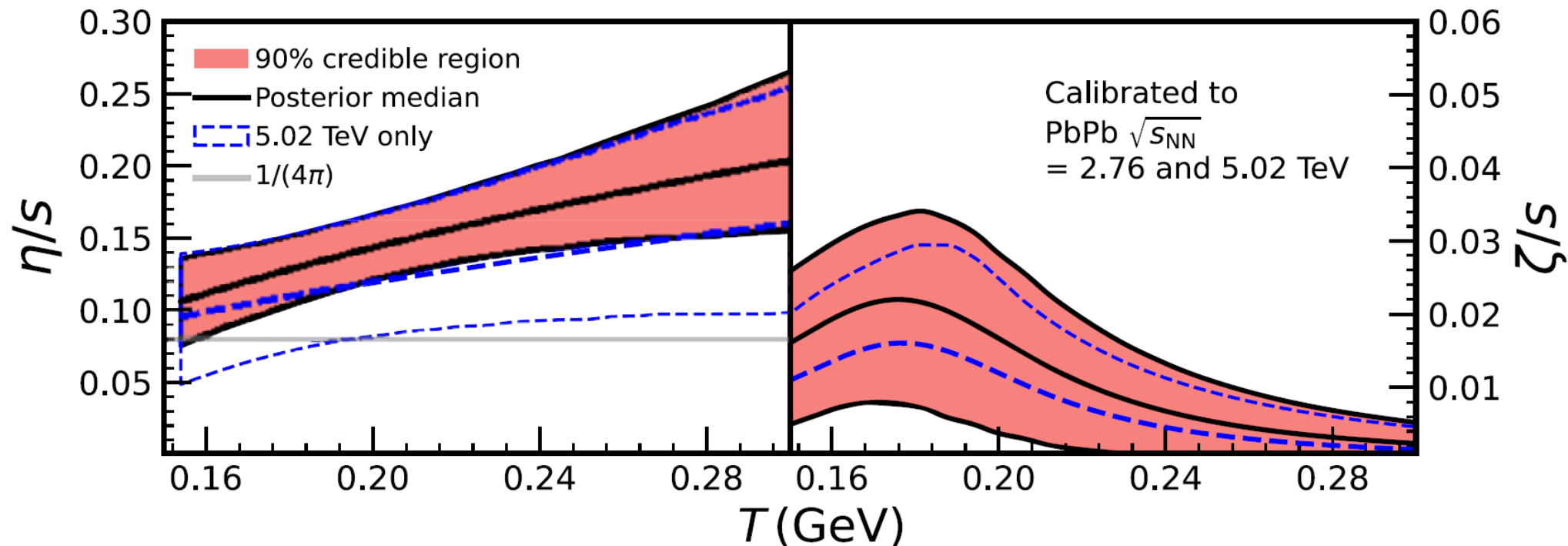
- Anticorrelation between minimum and the slope
- Constant $\eta/s(T)$ (i.e. zero slope) is not excluded

Bulk viscosity:

- The peak can be tall or wide, but not both
- Only integral of $\zeta/s(T)$ is constrained

Bayesian studies: follow-up work

- Constraints on the temperature dependence of specific shear and bulk viscosity change significantly, after including higher-order v_n observables in Bayesian studies



J. E. Bernhard, J. S. Moreland, S. A. Bass, J. Liu, and U. Heinz, Phys. Rev. C **94**, 024907, e-Print: [1605.03954](https://arxiv.org/abs/1605.03954)

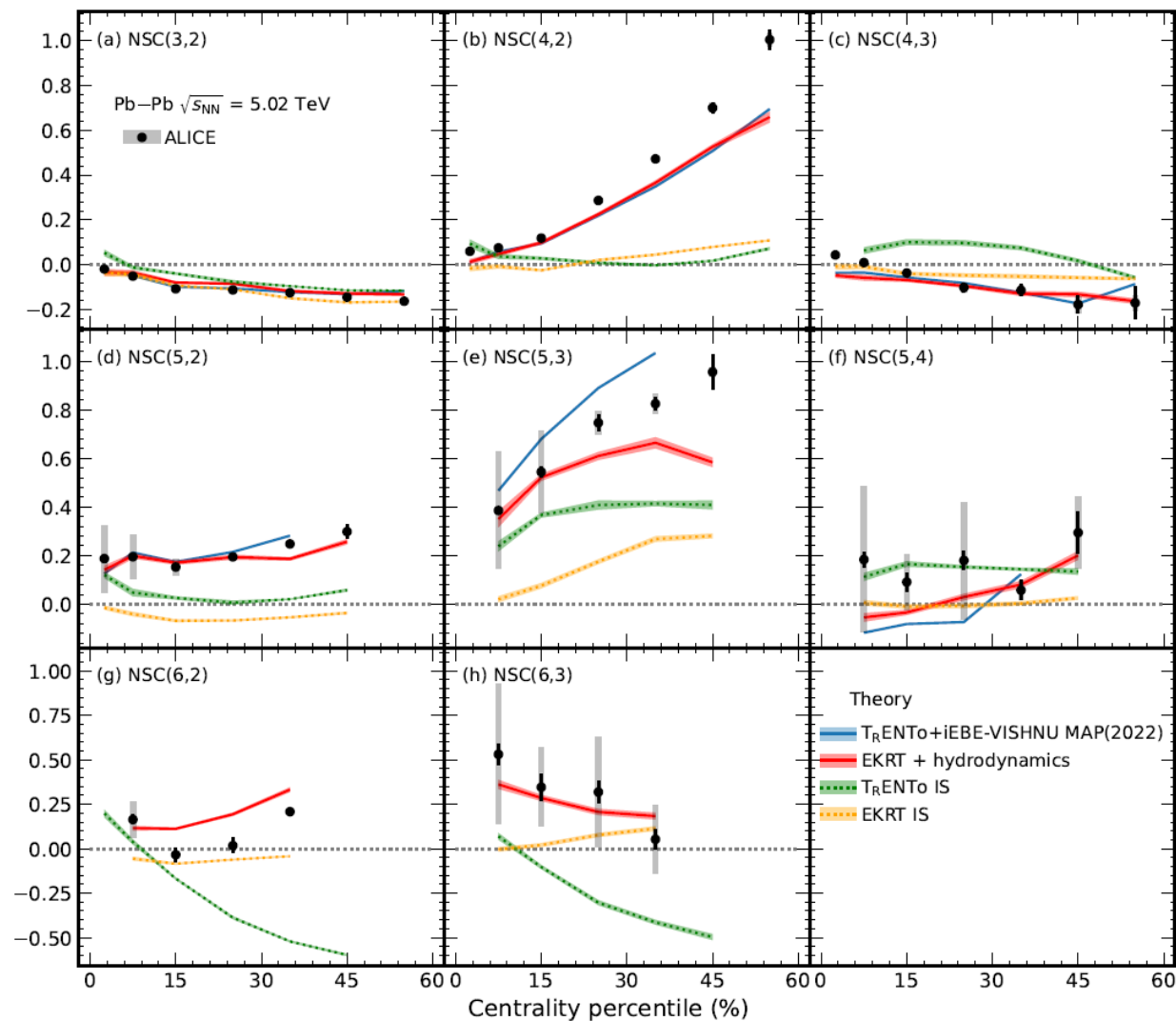
J. E. Bernhard, J. S. Moreland, S. A. Bass, *Nat. Phys.* **15**, 1113–1117 (2019)

J.E.Parkkila, A.Onnerstad, S.F.Taghavi, C.Mordasini, AB, M.Virta, D.J.Kim, Phys. Lett. B **835** (2022), 137485, e-Print: [2111.08145](https://arxiv.org/abs/2111.08145)

Latest SC/AC results and outlook

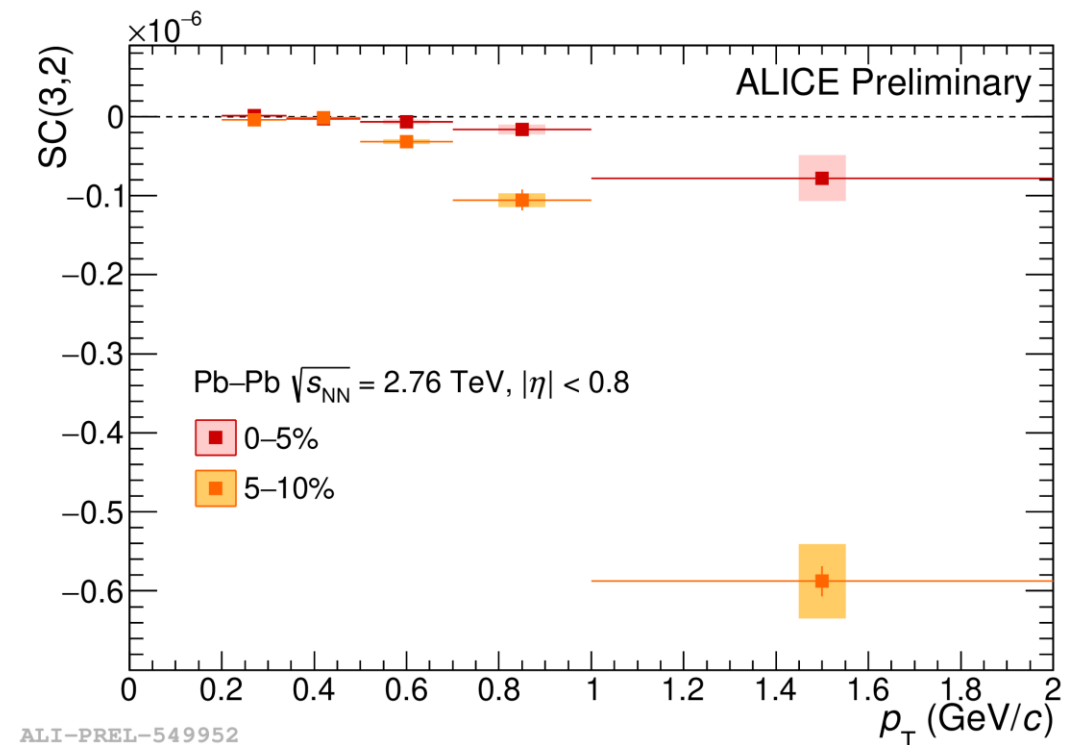
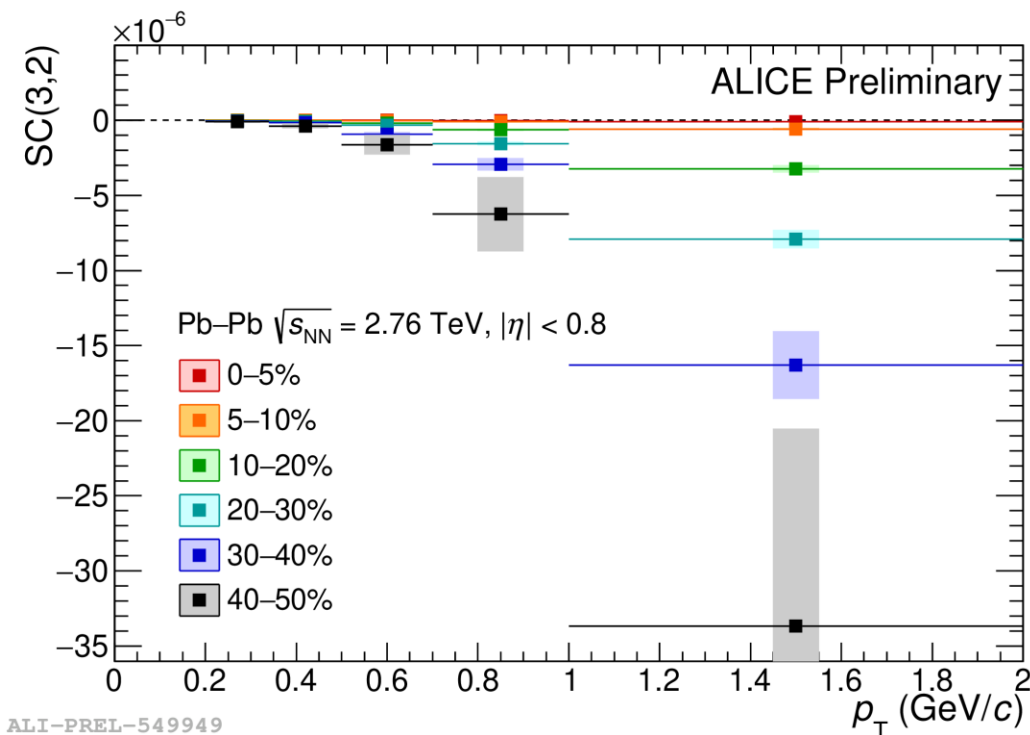
- Differential studies of SC and AC observables

Comparison to models



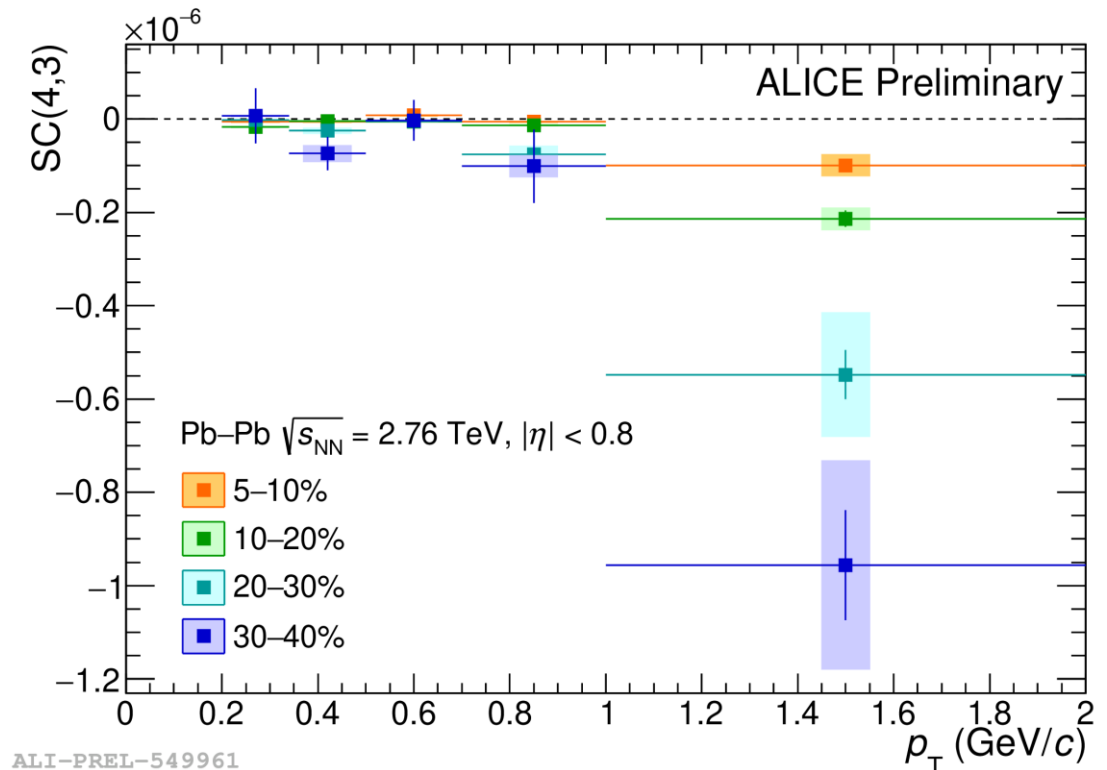
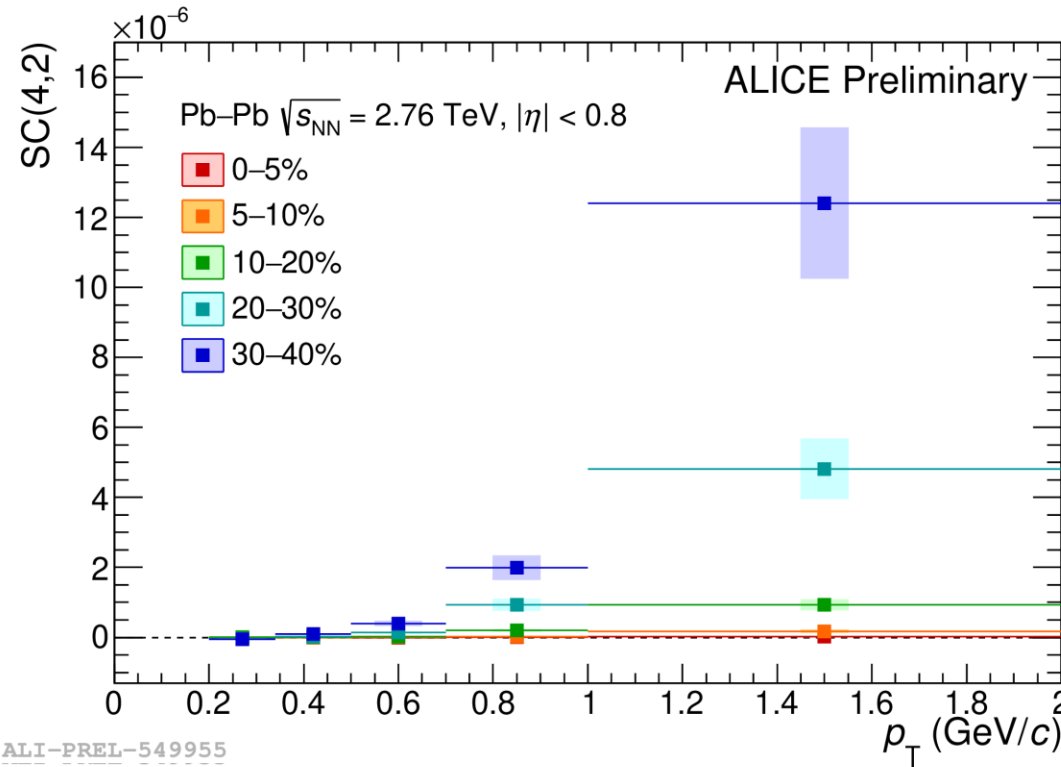
Differential measurements of SC observables

- First differential measurements of multiharmonic correlations as a function of p_T , using $SC(k,l)$ and $SC(k,l,m)$ observables
- Further independent input to Bayesian studies



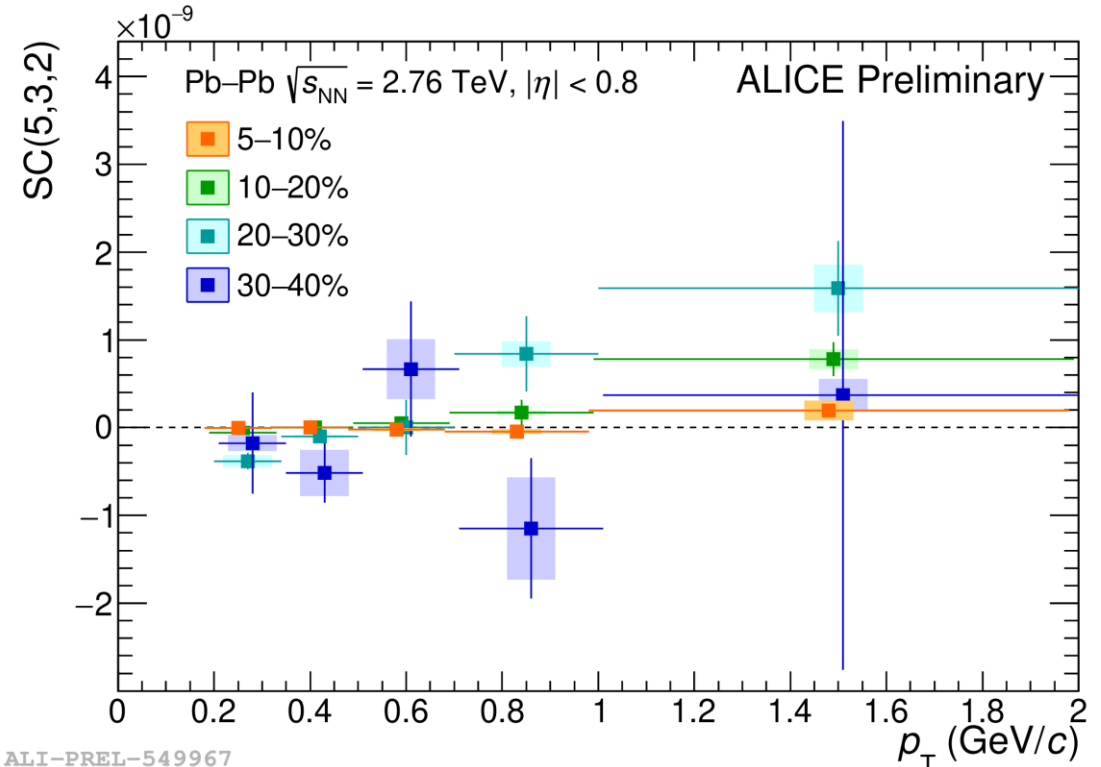
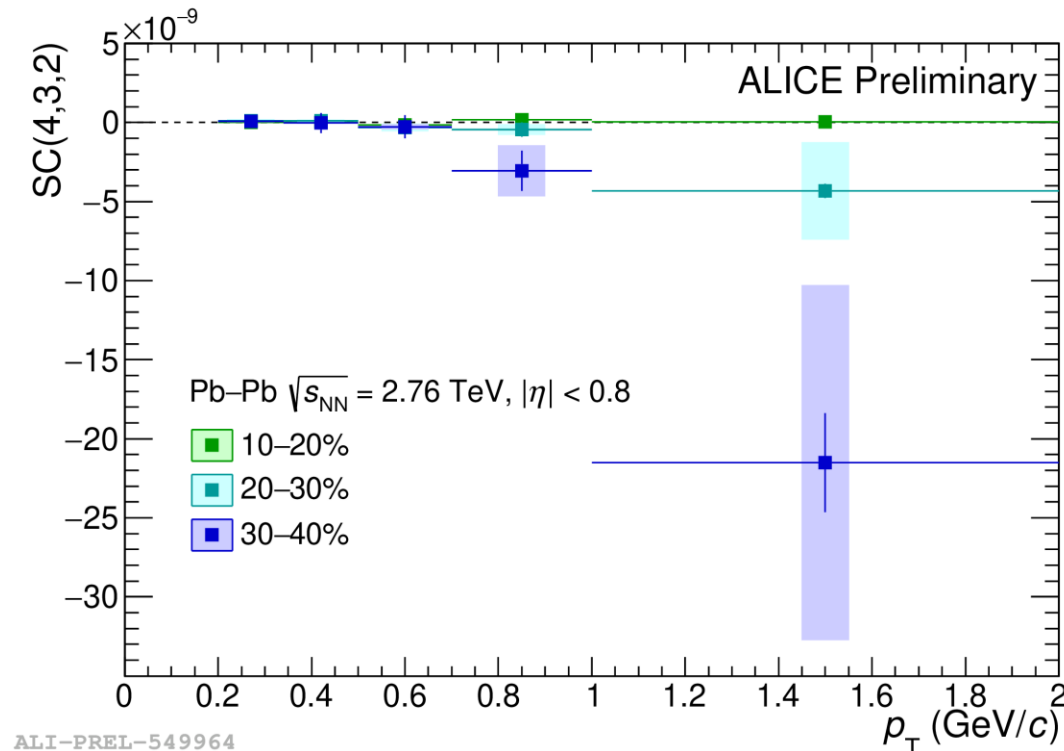
Differential measurements of SC observables

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- Further independent input to Bayesian studies



Differential measurements of SC observables

- First differential measurements of multiharmonic correlations as a function of p_T , using $SC(k,l)$ and $SC(k,l,m)$ observables
- Further independent input to Bayesian studies



Fundamental properties of cumulants

- We reviewed everything from scratch and supported proofs for:
 - Statistical independence
 - Reduction
 - Semi-invariance
 - Homogeneity
 - Multilinearity
 - Additivity
 - ...
- The main strategy in this technical paper is divided into two steps:
 - Confront all existing observables in the field named cumulants with these fundamental properties
 - For the ones which fail to satisfy them, provide the alternative definitions which do satisfy all fundamental properties of cumulants

For all technical details, see Section II and Appendix A
in e-Print: [2101.05619](https://arxiv.org/abs/2101.05619)

Main conclusions

- **The main conclusion #1:** One cannot perform cumulant expansion in one set of stochastic observables, then in the resulting expression perform the transformation to some new set of observables, and then claim that the cumulant properties are preserved in the new set of observables
 - After such transformation, the fundamental properties of cumulants are lost in general
- **The main conclusion #2:** The formal properties of cumulants are valid only if there are no underlying symmetries due to which some terms in the cumulant expansion would vanish identically
 - Due to symmetries, $\langle \langle e^{in\varphi_i} \rangle \rangle = 0$, $\langle \langle e^{in(\varphi_i + \varphi_j)} \rangle \rangle = 0$, etc., all vanish
 - There are no obvious symmetries for $\langle v_k^2 \rangle$, $\langle v_k^2 v_l^2 \rangle$, etc., to vanish

Necessary conditions for cumulants

- From the fundamental properties of cumulants (statistical independence, reduction, semi-invariance, homogeneity, multilinearity, additivity, etc.), we have established the following two simple necessary conditions:

1. We take temporarily that in the definition of $\lambda(X_1, \dots, X_N)$ all observables X_1, \dots, X_N are statistically independent and factorize all multivariate averages into the product of single averages \Rightarrow the resulting expression must reduce identically to 0;
2. We set temporarily in the definition of $\lambda(X_1, \dots, X_N)$ all observables X_1, \dots, X_N to be the same and equal to $X \Rightarrow$ for the resulting expression it must hold that

$$\lambda(aX + b) = a^N \lambda(X), \quad (23)$$

where a and b are arbitrary constants, and N is the number of observables in the starting definition of $\lambda(X_1, \dots, X_N)$.

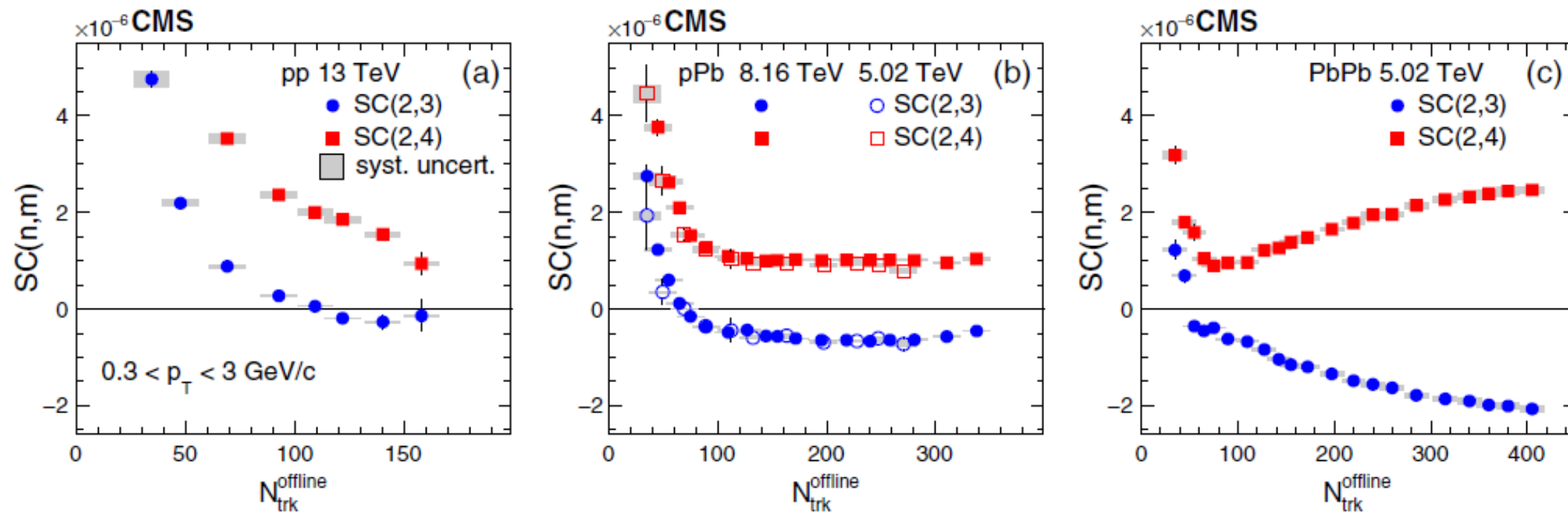
- A multivariate random variable is a multivariate cumulant only if it satisfies both requirements above

Reconciliation

- New flow observables ('The Next Generation') which do satisfy all formal mathematical properties of cumulants:
 - **'Symmetric and Asymmetric Cumulants'** (genuine multiharmonic correlations of flow amplitudes)
 - See arXiv:1901.06968 and Sec. V in arXiv:2101.05619
 - **'Cumulants of symmetry plane correlations'**
 - See Sec. VI in arXiv:2101.05619
 - **'Event-by-event cumulants of azimuthal angles'**
 - See Sec. IV in arXiv:2101.05619 and arXiv: 2106.05760
- **Open question:** Is it possible to build cumulants of complex flow vectors $v_n e^{in\Psi_n}$?
 - The underlying problem: $\langle v_n e^{in\Psi_n} \rangle_{\text{events}} = 0$
 - Properties of cumulants are lost at any order

Intermezzo: SC in small collision systems

- There is a big open question in the field: What is the smallest collision system in which QGP can be produced?

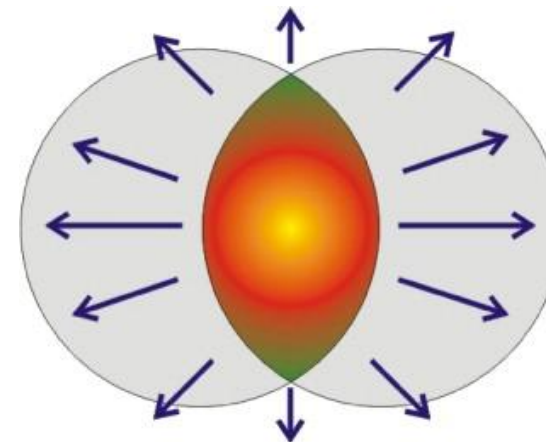
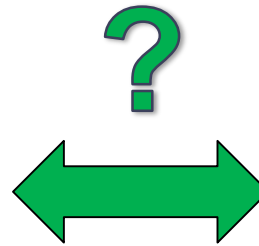
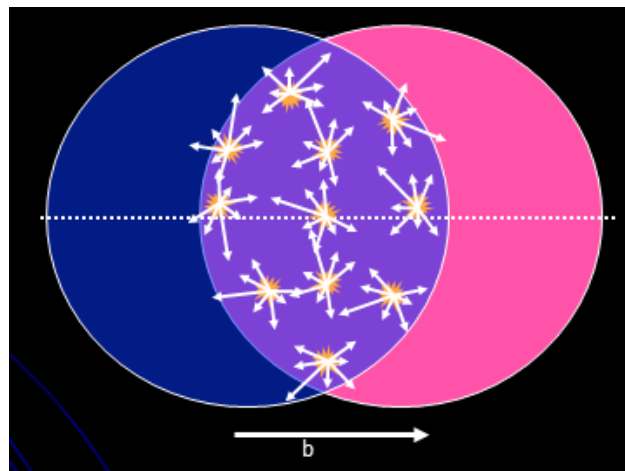


CMS Collaboration, Phys. Rev. Lett. 120, 092301 (2018), e-Print: [1709.09189](https://arxiv.org/abs/1709.09189)

SC observables couldn't be more different in pp, pPb and PbPb!

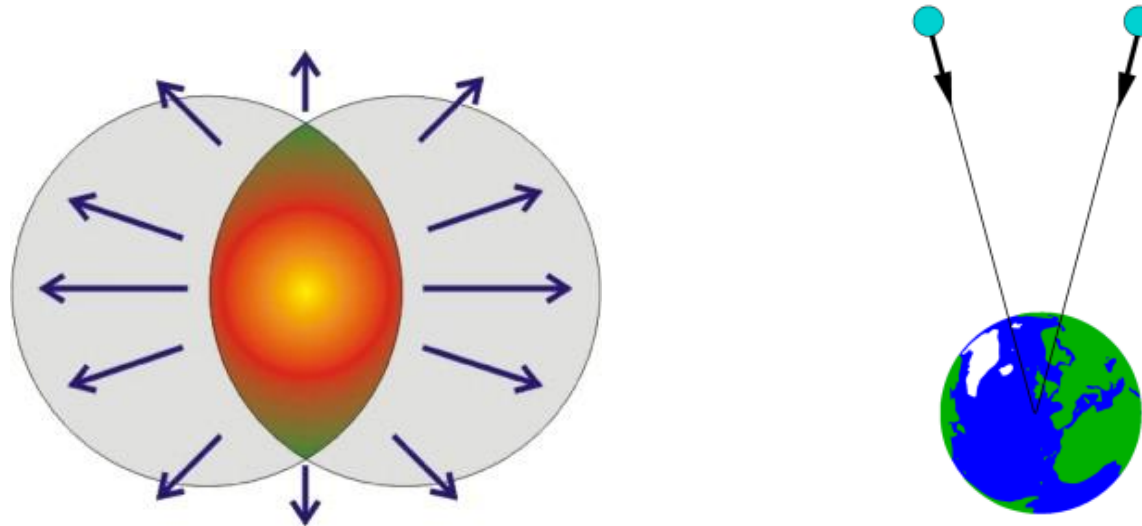
Transfer of anisotropy

- Two conceptually different notions of anisotropy:
 - **Coordinate space anisotropy:** Is the volume containing the interacting particles produced in a heavy-ion collision anisotropic or not?
 - **Momentum space anisotropy:** Is the final-state azimuthal distribution of resulting particles recorded in the detector anisotropic or not?
- A priori these anisotropies unrelated, in practice **correlated**



Flow measurements and observables

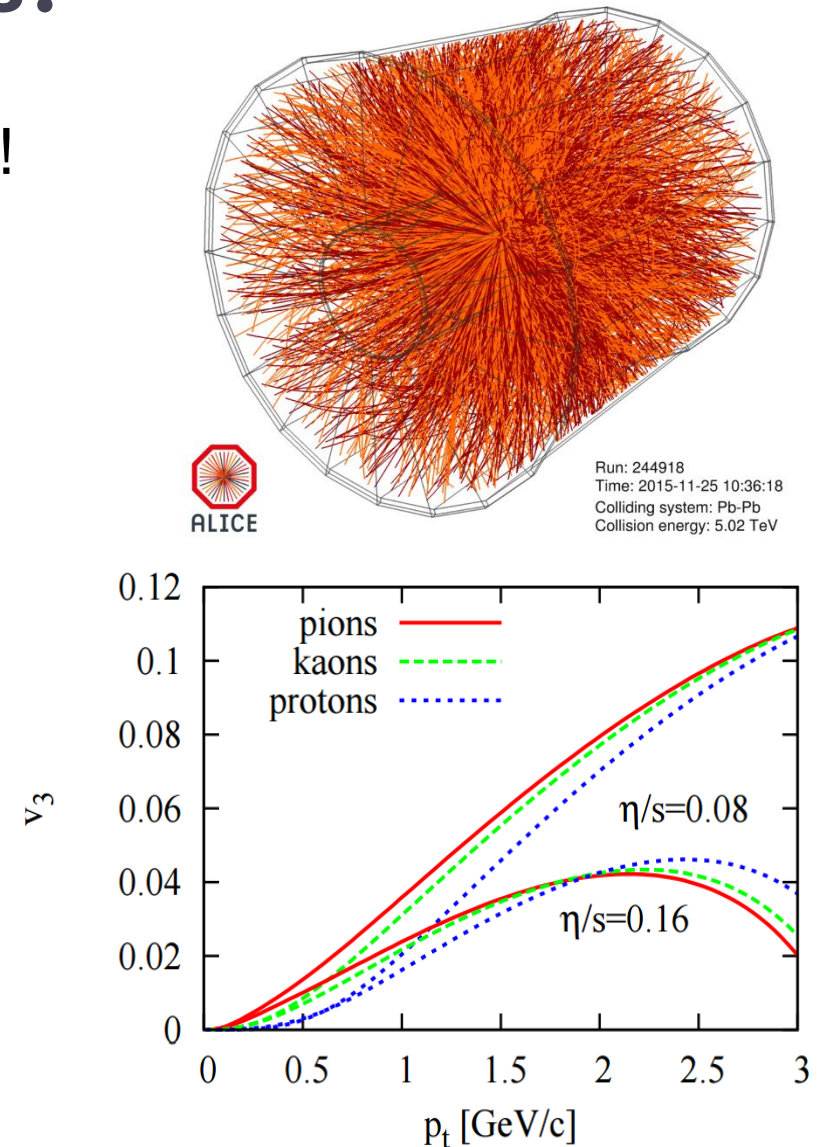
- The ‘flow principle’: Correlations among all produced particles are induced solely by correlation of each single particle to the collision geometry



- Analogy with the falling bodies in gravitational field (rhs)
- Whether or not particles are emitted simultaneously, or one by one, trajectories are the same

Why so many flow observables?

- All **independent** flow observables are welcome!
 - Heavy-ion collision is a rather complex system and we cannot describe everything only with few parameters
- Different observables exhibit different sensitivities to QGP properties
 - Example from theory: Transverse momentum dependence of triangular flow is different for different values of QGP's shear viscosity



Fluctuations, p.d.f., moments, cumulants

- Properties of random (stochastic) observable v of interest are specified by functional form of probability density function (p.d.f.) $f(v)$
- Different moments carry by definition independent information about the underlying p.d.f. $f(v_n)$

$$\langle v_n^k \rangle \equiv \int v_n^k f(v_n) dv_n$$

- Two completely different p.d.f.'s $f(v_n)$ can have first moment $\langle v_n \rangle$ to be the same, and all higher-order moments different
- Is it mathematically equivalent to specify functional form $f(v_n)$ and all its moments $\langle v_n^k \rangle$?
- A priori it is not guaranteed that a p.d.f. $f(v_n)$ is uniquely determined by its moments $\langle v_n^k \rangle$
 - Necessary and sufficient conditions have been worked out only recently

$$K[f] \equiv \int_0^\infty \frac{-\ln f(x^2)}{1+x^2} dx \quad \Rightarrow \quad K[f] = \infty$$

Krein-Lin conditions (1997)

$$L(x) \equiv -\frac{xf'(x)}{f(x)} \quad \Rightarrow \quad \lim_{x \rightarrow \infty} L(x) = \infty$$

J. Stoyanov, Section 3 in '*Determinacy of distributions by their moments*', Proceedings 2006