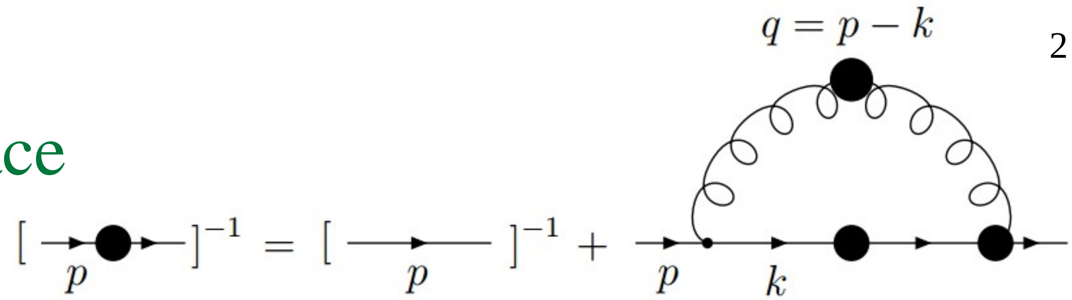
DSE and the pion in Minkowski space: new developments

Dyana C. Duarte

INTERNATIONAL SCHOOL OF NUCLEAR PHYSICS, 46th Course
QCD under extreme conditions - present and future

Erice, Sicily, September 16 – 22 (2025)

DSE in Minkowski space



Usually defined and solved in Euclidean space:

- Lattice gauge theory simulations and its numerical solutions;
- QCD perturbation theory are strictly valid only at spacelike-momenta, the only possibility for Euclidean formulation.

Why Minkowski? Difficulties to deal with singular behavior of physical quantities...

- Dynamical observables defined in the light-front;
- Electromagnetic form-factors (singularities!);
- 3d imaging that may clarify the hadron content (EIC facility in the future);
- ...
- QCD at finite density?
- Finite magnetic field?

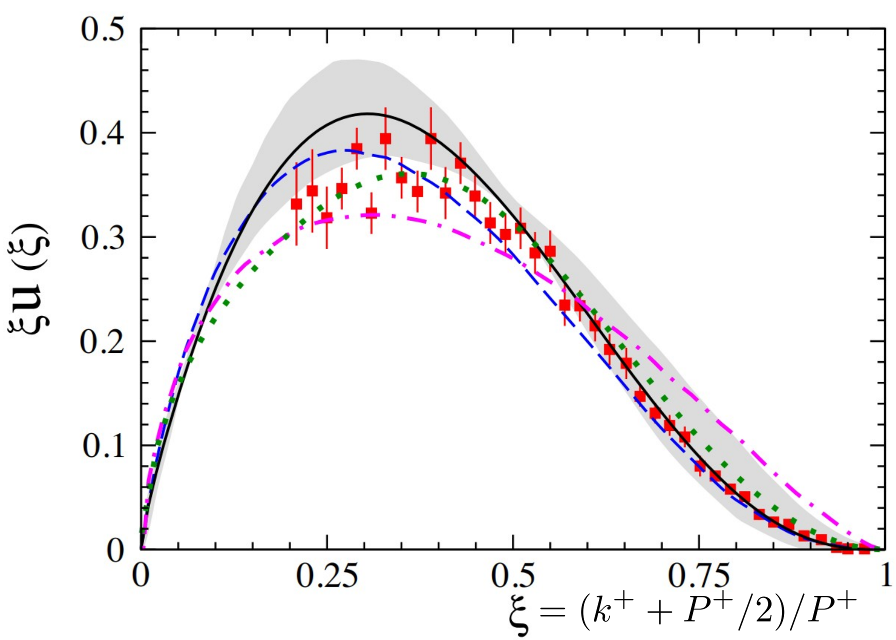
DSE in Minkowski space

Main Tool: Nakanishi
Integral Representation

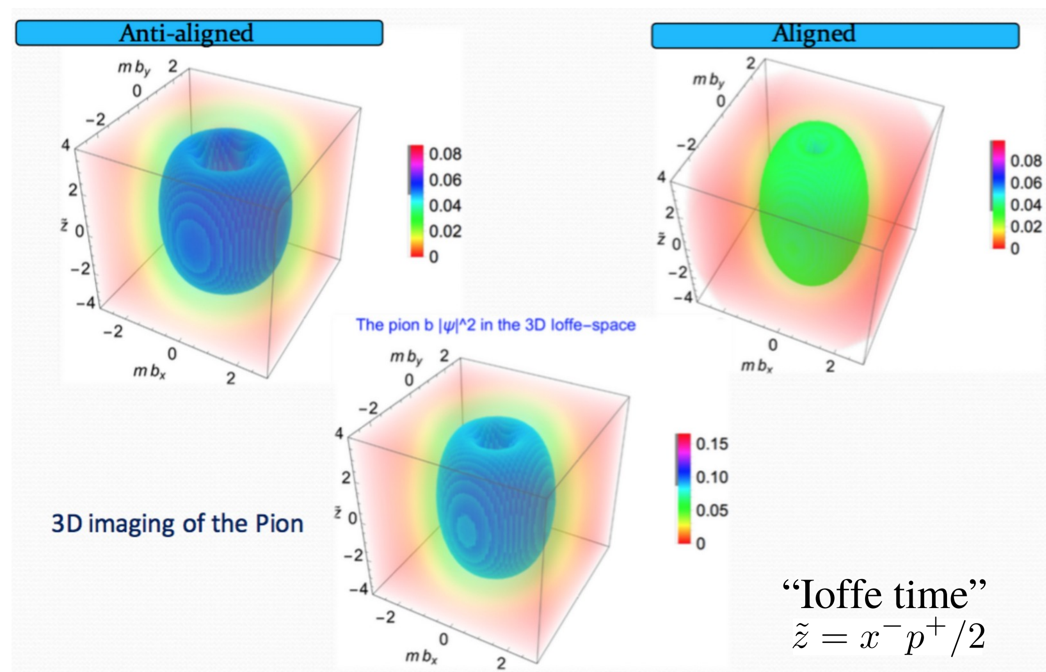
$$f_G(s) = \prod_h \int_0^1 dz_h \delta\left(1 - \sum_h z_h\right) \int_0^\infty d\chi \frac{\tilde{\phi}_G(z_h, \chi)}{\chi - \sum_h z_h s_h - i\varepsilon}$$

Information about the loops

Independent of the diagrams internal structure

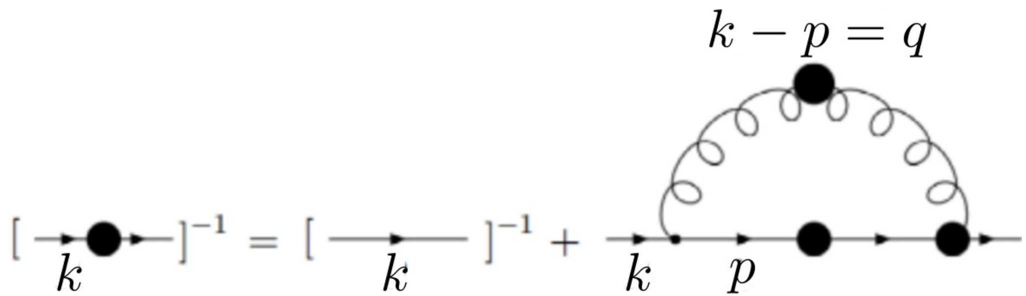


de Paula,Ydrefors,Nogueira,Frederico,Salmè, PRD 105, L071505 (2022).



W. de Paula@XXIX Int. Workshop on Deep-Inelastic Scattering and Related Subjects (2022)

Fermion Dyson Schwinger Equation (Rainbow-Ladder)



- DSE for the above schematic representation:

$$S_q^{-1}(k) = \not{k} - m_B + ig^2 \int \frac{d^4q}{(2\pi)^4} \Gamma_\mu(q,k) S_q(k-q) \gamma_\nu D^{\mu\nu}(q)$$

- Rainbow ladder approximation: $\Gamma_\mu(q,k) = \gamma_\mu$

Gauge fixing

- Gluon propagator: $D^{\mu\nu}(q) = \frac{1}{q^2 - m_\sigma^2 + \imath\epsilon}\left[g^{\mu\nu} - \frac{(1-\xi)q^\mu q^\nu}{q^2 - \xi m_\sigma^2 + \imath\epsilon}\right]$

- Quark propagator: $S_q(k) = \left[\not{k} A\left(k^2\right) - B\left(k^2\right) + i\epsilon\right]^{-1}$

$$S_q(k) = R\frac{\not{k} + \overline{m}_0}{k^2 - \overline{m}_0^2 + i\epsilon} + \not{k}\int_0^\infty ds \frac{\rho_v(s)}{k^2 - s + i\epsilon} + \int_0^\infty ds \frac{\rho_s(s)}{k^2 - s + i\epsilon}$$

Fermion Dyson Schwinger Equation

- Parameters: $\alpha = \dfrac{g^2}{4\pi}$, Λ , m_g , $\overline{m}_0$.

- Spectral densities are obtained from the IR of the self-energy:

$$\rho_A(\gamma) = -\frac{1}{\pi}\mathrm{Im}\left[A(\gamma)\right] \qquad A(k^2) = 1 + \int_0^\infty \frac{\rho_A(s)}{k^2 - s + i\epsilon}$$

$$\rho_B(\gamma) = -\frac{1}{\pi}\mathrm{Im}\left[B(\gamma)\right] \qquad B(k^2) = m_B + \int_0^\infty \frac{\rho_B(s)}{k^2 - s + i\epsilon}$$

- Solutions of DSE obtained writing the trivial relation $S_f^{-1} S_f = 1$:

$$\frac{R}{\gamma - \overline{m}_0^2 + i\epsilon} + \int_0^\infty ds \frac{\rho_v(s)}{\gamma - s + i\epsilon} = \frac{A(\gamma)}{\gamma A^2(\gamma) - B^2(\gamma) + i\epsilon}$$

$$\frac{R\,\overline{m}_0}{\gamma - \overline{m}_0^2 + i\epsilon} + \int_0^\infty ds \frac{\rho_s(s)}{\gamma - s + i\epsilon} = \frac{B(\gamma)}{\gamma A^2(\gamma) - B^2(\gamma) + i\epsilon}$$

Fermion Dyson Schwinger Equation

- Parameters: $\alpha = \dfrac{g^2}{4\pi}$, Λ , m_g , $\overline{m}_0$.

Bare mass m_B:
$$A^2(\overline{m}_0^2) - B^2(\overline{m}_0^2) = 0$$

- Spectral densities are obtained from the IR of the self-energy:

$$\rho_A(\gamma) = -\frac{1}{\pi}\mathrm{Im}\,[A(\gamma)] \qquad A(k^2) = 1 + \int_0^\infty \frac{\rho_A(s)}{k^2 - s + i\epsilon}$$
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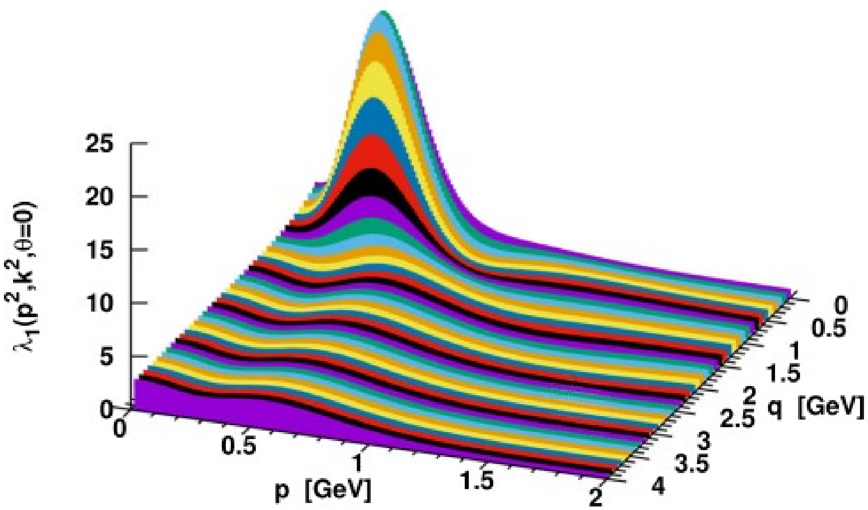
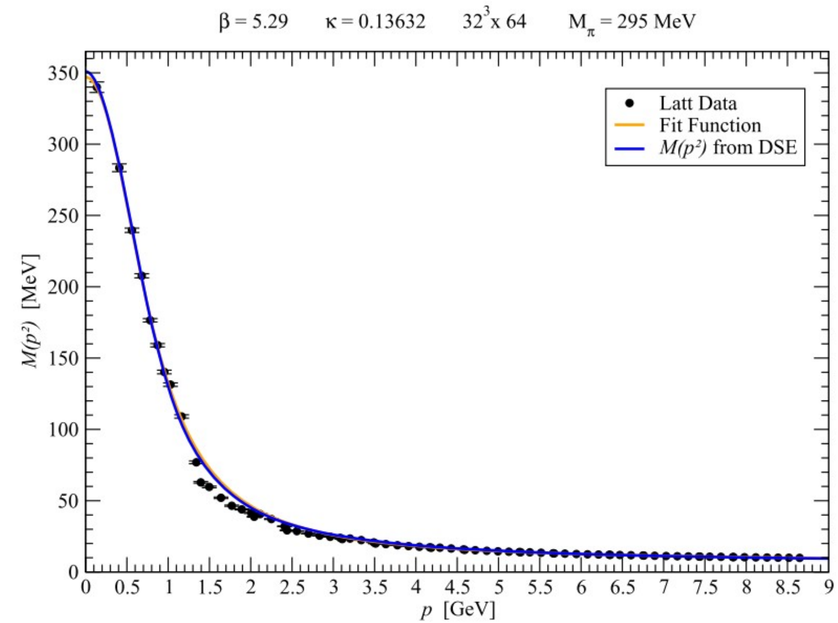
DSE+Lattice QCD propagators: Enhancement of the quark-gluon vertex at the infrared region![†]

[†]O. Oliveira et al., EPJC 80,484 (2020).



Pauli-Villars regulator can also be effectively associated with the form factor of the γ^μ component of the quark-gluon vertex:

$$\lambda_1(q^2) = \frac{m_g^2 - \Lambda^2}{q^2 - \Lambda + i\epsilon}$$



[†]Rojas et al., JHEP 10 (2013) 193;
O. Oliveira et al., EPJC 79, 116 (2019)

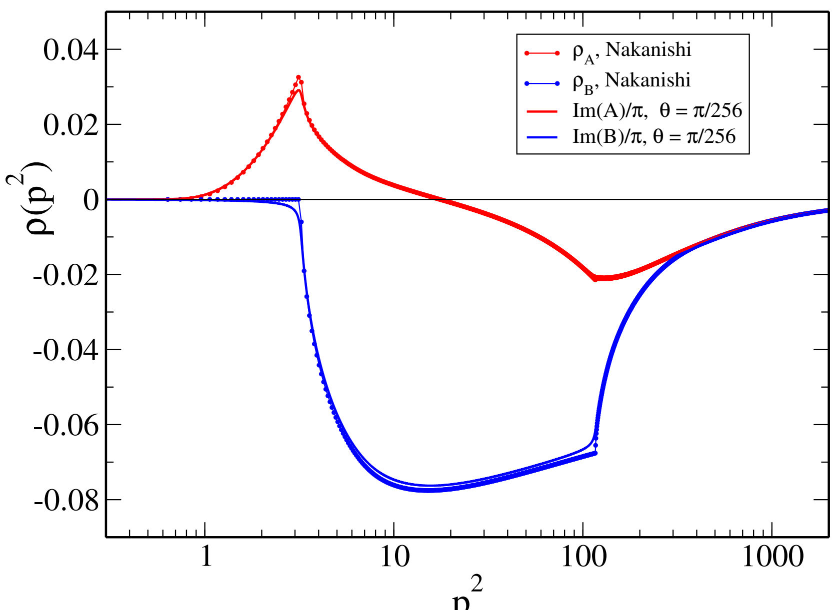
Comparison with Un-Wick rotated results

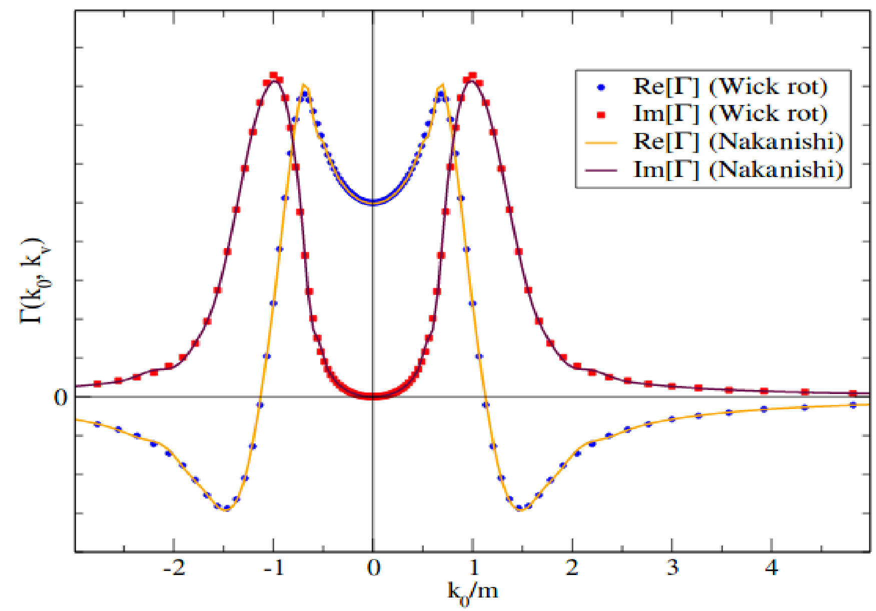
Wick rotation is the exact analytical continuation of the Minkowski space Nakanishi representation: Explorations in the complex plane.

$$p_0 \Longrightarrow \exp(-i\delta)p_0$$
$$k_0 \Longrightarrow \exp(-i\delta)k_0$$

$$\theta = 0, \begin{cases} p_0^2 = 0, & \vec{p}^2 > 0 \quad \text{spacelike region} \\ p_0^2 > 0, & \vec{p}^2 = 0 \quad \text{timelike region} \end{cases}$$

- Minkowski space: $\delta = \pi/2$, or $\Theta = \pi/2 - \delta$.





A. Castro et al., J.Phys.Conf.Ser.1291 012006 (2019)

In collaboration with T. Frederico, S. Jia, P. Maris, W. de Paula and E. Ydrefors

Large coupling regime: Phenomenological model

Calibration of the model: Possibility to explore the chiral symmetry breaking region!

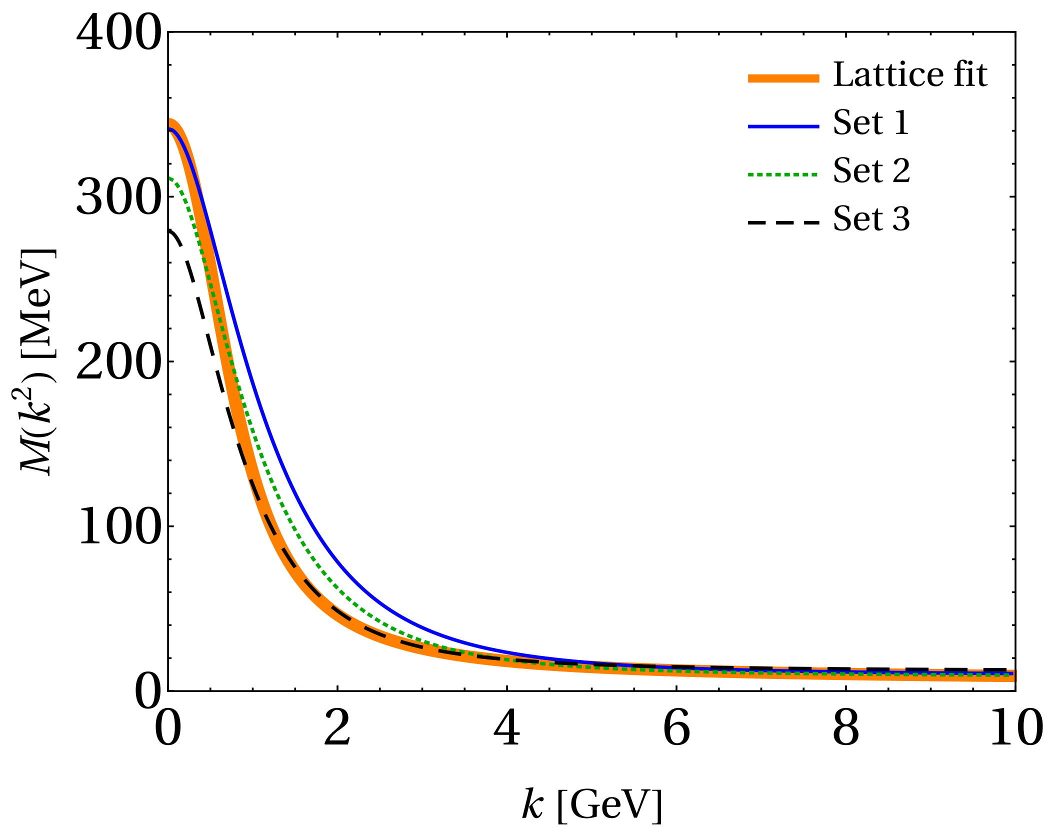
Set	$\bar{m}_0$ (GeV)	m_g (GeV)	Λ (GeV)	α
1	0.42	0.84	1.20	19.70
2	0.38	0.78	1.10	20.30
3	0.35	0.60	1.00	13.25

Set	(Outputs)	m_B (MeV)	R
1		9.06	2.22
2		8.53	2.09
3		12.25	2.64

Appropriate behavior in the infrared require a large enough Kernel

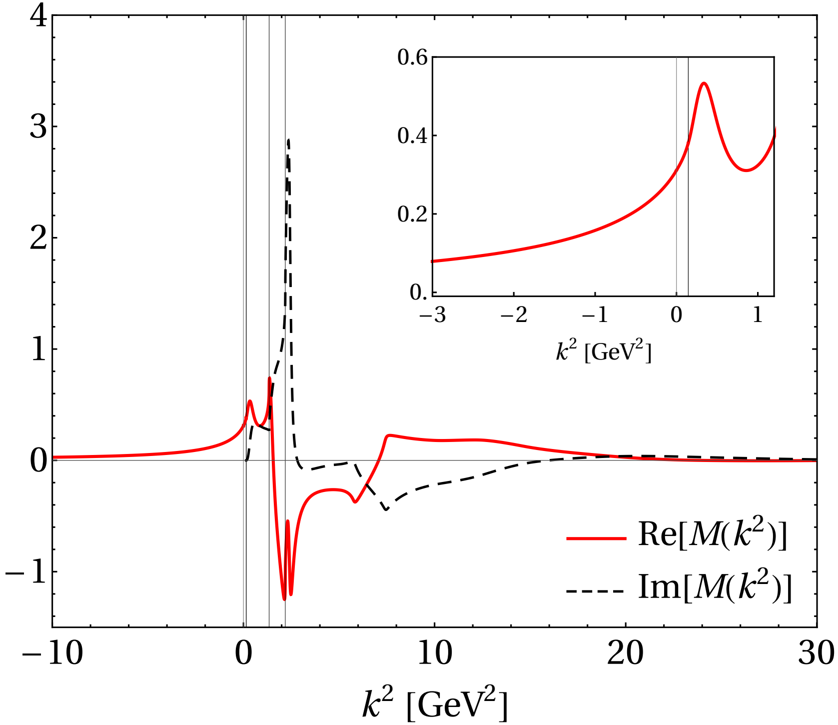
⇓

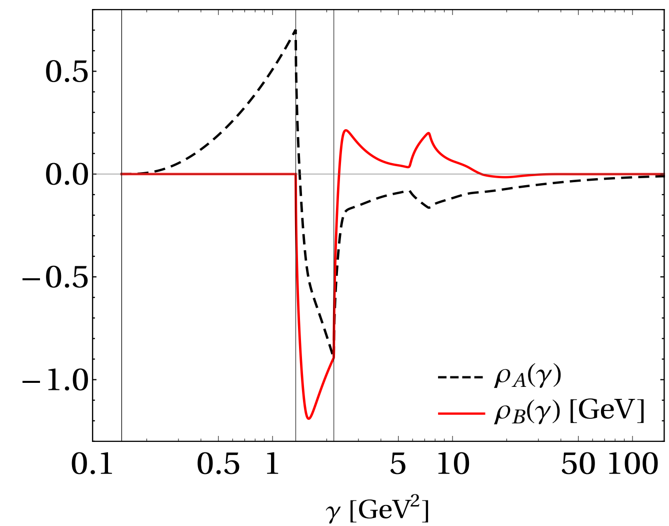
Λ cannot be large compared to m_g, and as a consequence, α must increase!

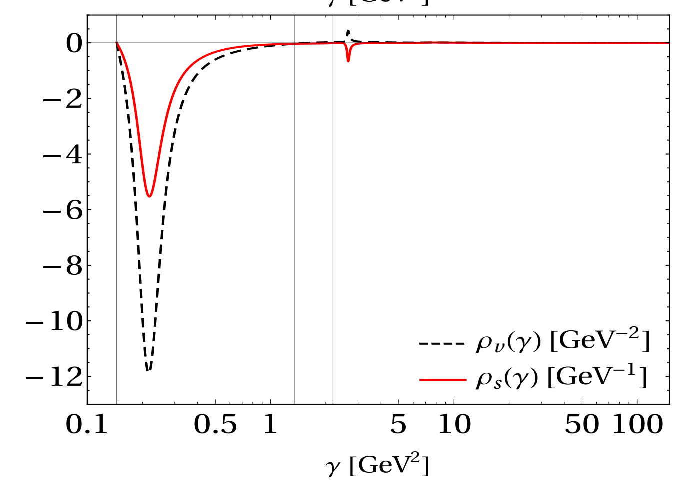


DD, Frederico, de Paula, Ydrefors PRD 105, 114055 (2022). Fit of lattice data from O. Oliveira, et al., PRD 99, 094506 (2019).

Large coupling regime: Phenomenological model







PRD 105, 114055 (2022)

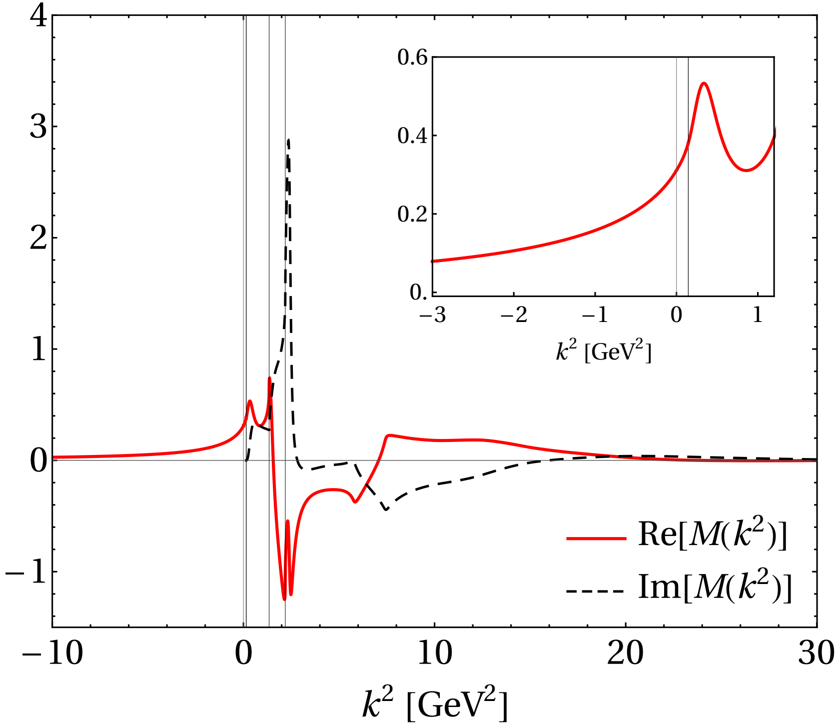
$\overline{m}_0^2$

$(\overline{m}_0 + m_g)^2$

$(\overline{m}_0 + \Lambda)^2$

DD, T. Frederico, W. de Paula, E. Ydrefors Phys. Rev. D 105, 114055 (2022).

Large coupling regime: Phenomenological model

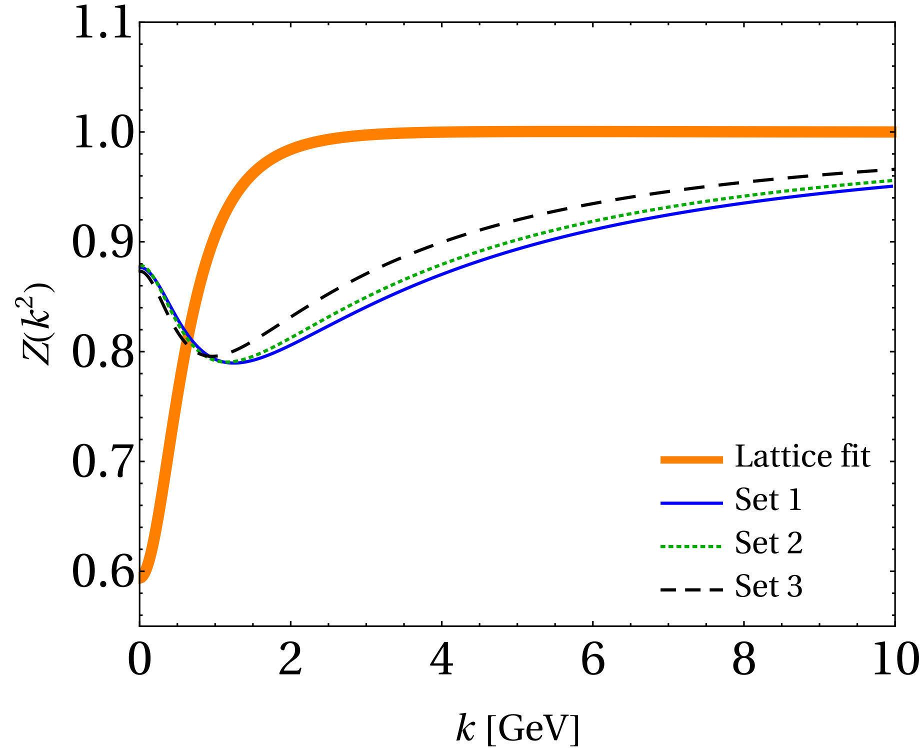


$\overline{m}_0^2$

$(\overline{m}_0 + m_g)^2$

$(\overline{m}_0 + \Lambda)^2$

Not enough to get a good fit for $Z(k^2)$

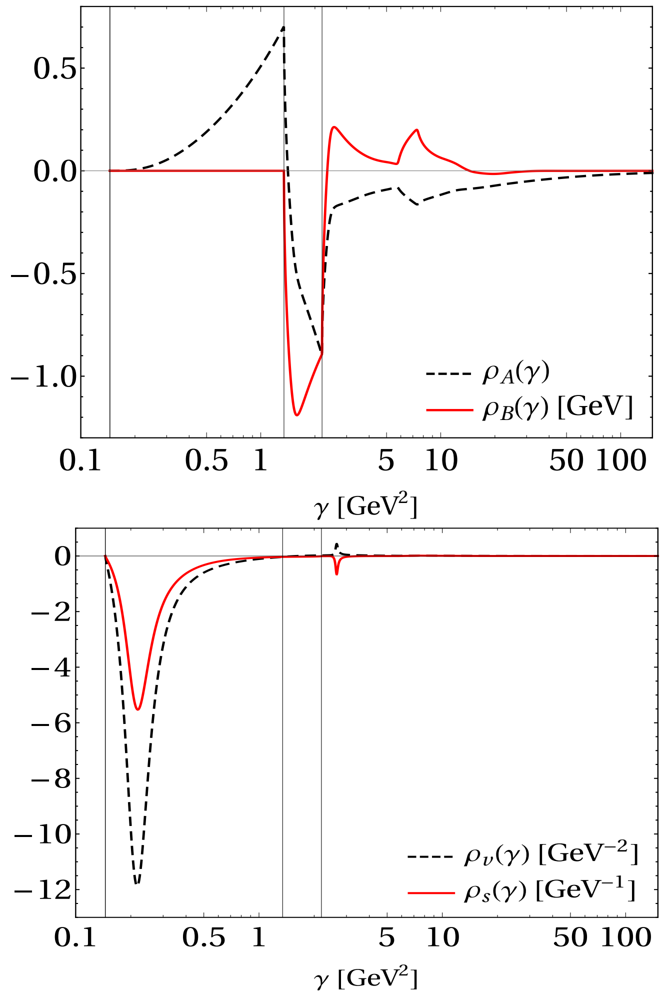


DD, T. Frederico, W. de Paula, E. Ydrefors Phys. Rev. D 105, 114055 (2022).

The pion

Spectral densities evaluated by solving the DSE the method described previously as inputs for the pion Bethe-Salpeter equation.

PRD 105, 114055 (2022)



$$\Psi_\pi(k;p) = S_F(k_q)\Gamma_\pi(k;p)S_F(k_{\bar{q}})$$

$$S_F(k) = \frac{1}{A(k^2)\not{k} - B(k^2)}$$
$$= S_v(k^2)\not{k} + S_s(k^2)$$

$$S_v(k^2) = \frac{R}{k^2 - \overline{m}_0^2 + i\epsilon} + \int_0^\infty ds \frac{\rho_v(s)}{k^2 - s + \imath\epsilon}$$

$$S_s(k^2) = \frac{R\overline{m}_0}{k^2 - \overline{m}_0^2 + i\epsilon} + \int_0^\infty ds \frac{\rho_s(s)}{k^2 - s + i\epsilon}$$

$$\Gamma_\pi(k,p) = \gamma_5[\imath E_\pi(k,p) + F_\pi(k,p)$$
$$+ k^\mu p_\mu\, G_\pi(k,p) + \sigma_{\mu\nu}k^\mu p^\nu H_\pi(k,p)]$$

The pion

- Bethe Salpeter equation: $\Psi_\pi(p,P) = S(q)\Gamma_\pi(p,P)S(\bar{q})$

$$\Gamma_\pi(p,P) = \gamma_5\left[iE_\pi(p,P) + \not{P}\,F_\pi(p,P) + p^\mu P_\mu \not{p}\,G_\pi(p,P) + \sigma_{\mu\nu}p^\mu P^\nu H_\pi(p,P)\right]$$

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- First approximation: Chiral limit! In this case, the pion quark-antiquark vertex is given by**

$$E_\pi(k) = -\frac{1}{f_\pi^0}B(k^2)\,, \qquad B(k^2) = \int_0^\infty ds\,\frac{\rho_B(s)}{k^2 - s + i\epsilon}$$

- Calculation of observables, more rigorous study of chiral symmetry breaking, ingredients from LQCD...

Pion decay constant: $ip^\mu f_\pi = N_c \int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^\mu\gamma^5\Psi_\pi(k,p)\right]$

**C. S. Mello, et al., Phys. Lett. B, 766 86–93 (2017), L. Chang et al., PRL 110, 132001 (2013).

The pion

Set	Pion decay constant (in MeV)
1	182.52
2	163.66
3	190.46

DD, T. Frederico, W. de Paula, E. Ydrefors Phys. Rev. D 105, 114055 (2022).

$$f_\pi^2 = \frac{4N_c}{\sqrt{2}(4\pi)^2}\int_0^\infty ds\int_0^\infty ds'\int_0^\infty ds''\rho_B(s'')\tilde{\rho}_v(s')\tilde{\rho}_s(s) \times\int_0^1 dv\int_0^{1-v} du\frac{1-u+v}{\frac{M_\pi^2}{4}(v-u)^2+s''(1-v-u)-\frac{M_\pi^2}{4}(v+u)+vs+us'+\imath\epsilon}$$

$$\tilde{\rho}_v(s) = \rho_v(s) + R\delta(s-\overline{m}_0^2)$$

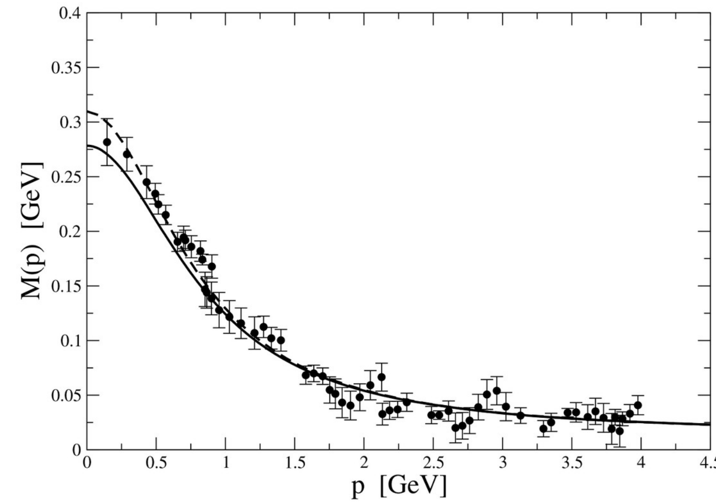
$$\tilde{\rho}_s(s) = \rho_s(s) + R\overline{m}_0\delta(s-\overline{m}_0^2)$$

Only the fit of running mass is not enough to generate good results for the observables!

In collaboration with T. Frederico, W. de Paula, S. Bortagaray and M. Pelaez:

- Inspired in a phenomenological model for the quark mass*...

$$S_F(k) = i\frac{(k^2-\lambda)^2(\not{k}+m_0)-(k^2-\lambda)m^3}{\Pi_{i=1,3}(k^2-m_i^2+i\epsilon)}$$



*Mello et al., PLB, **766** 86 (2017). Lattice data from Parappilly et al., PRD **73** 054504 (2006). **Pelaez et al., PRD 96, 114011 (2017).

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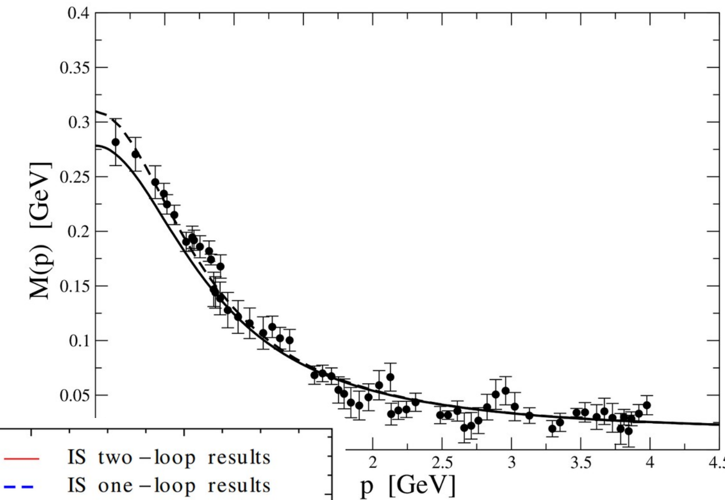
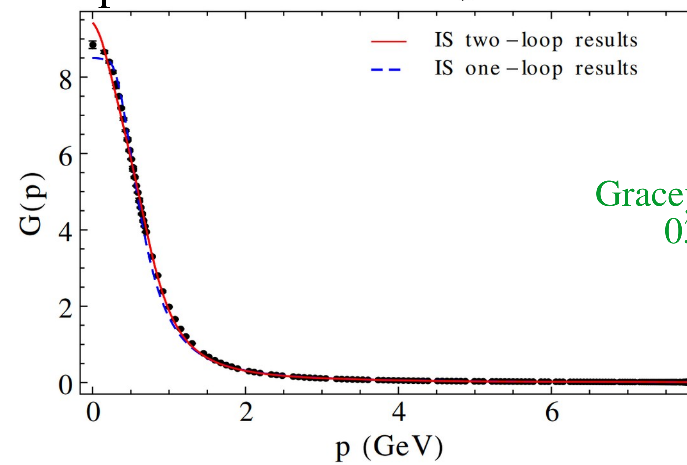


- ...we wrote the gluon propagator as a sum of three poles:

$$D^{\mu\nu}(q) = \sum_{i=1,3}\frac{R_i}{q^2-m_{g,i}^2+\imath\epsilon}\left[g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2+\imath\epsilon}\right]$$

$\left(\sum_{i=1,3}R_i = 1\right)$



Gracey et al., PRD **100**, 034023 (2019)

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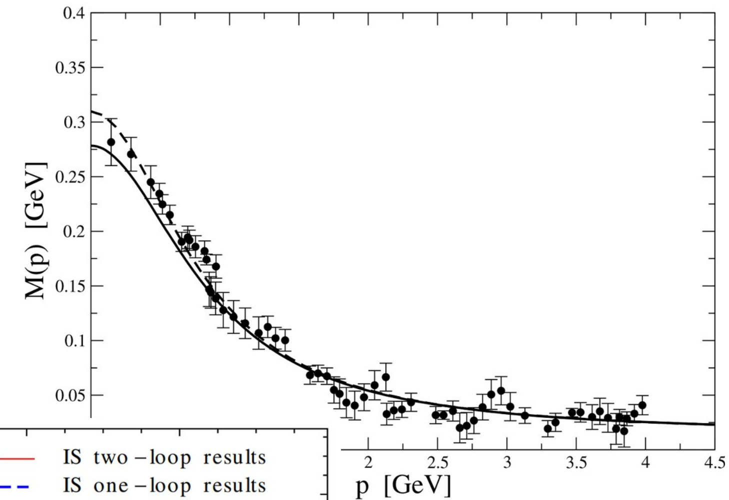

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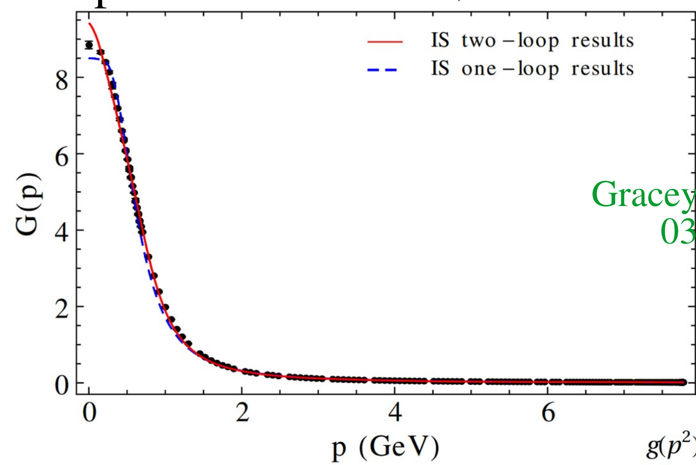
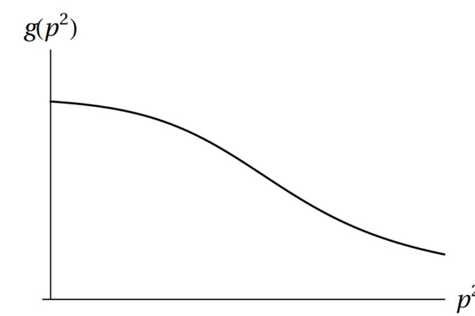


Gracey et al., PRD **100**, 034023 (2019)

- … and a running coupling**

$$g^2 \Longrightarrow g^2(\gamma) = \frac{g_0^2}{1+\beta_0 g_0^2\ln\left(\frac{\gamma+x^2\overline{m}_0^2}{x^2\overline{m}_0^2}\right)}$$



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Recent developments

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PRD 105, 114055

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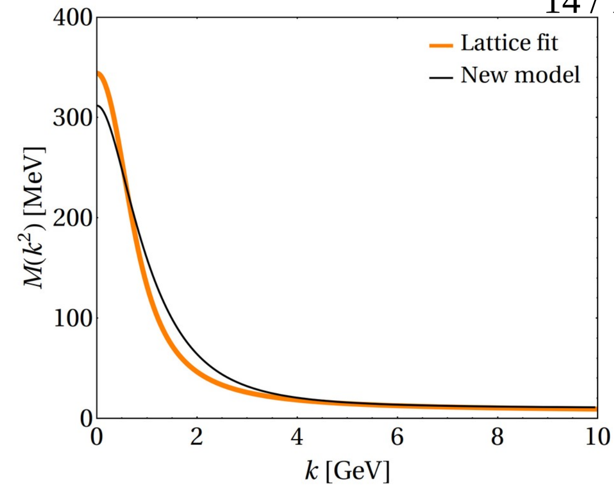

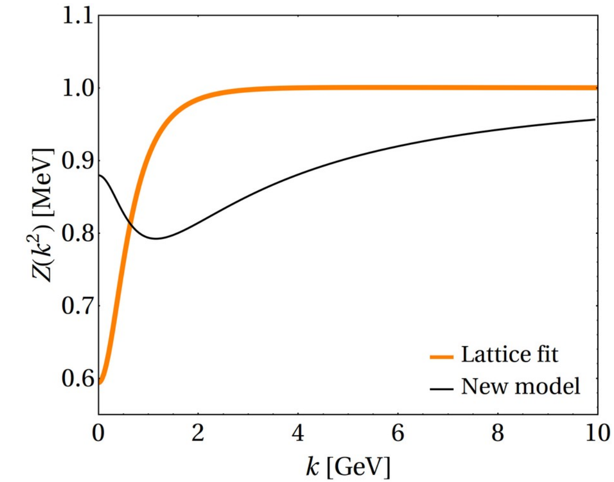

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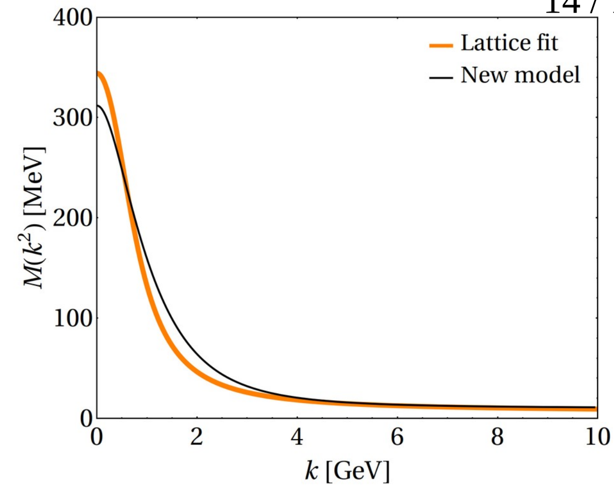
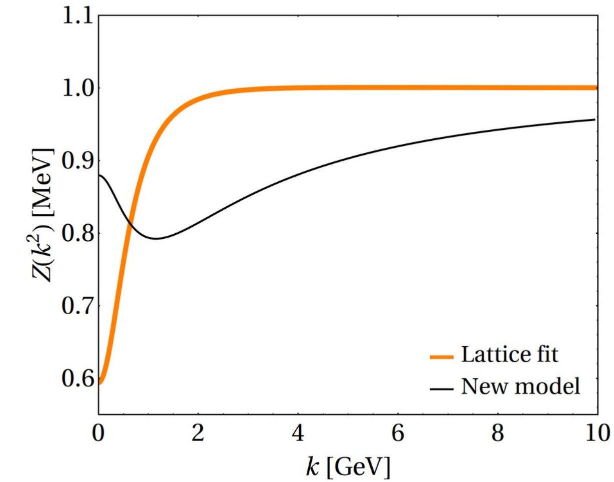




Next steps:

- Valence wave function, probability amplitudes, …….

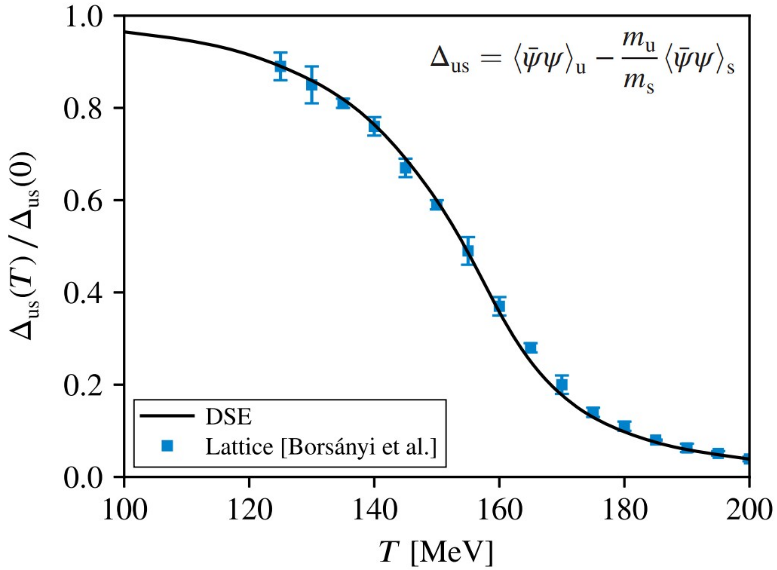
$$\psi_{\uparrow\downarrow} = -\frac{1}{4}\int \frac{dp^-}{2\pi} \mathrm{Tr}\left[\gamma^+\gamma_5\Psi(p,P)\right]$$

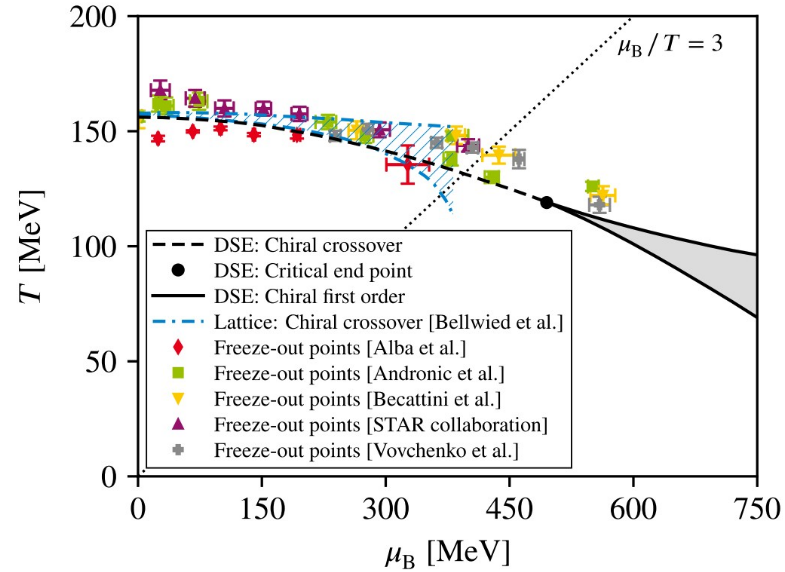
$$ip^i\psi_{\uparrow\uparrow}(p^+,\vec{p}_\perp) = \frac{\sqrt{\gamma}}{4}\int \frac{dp^-}{2\pi} \mathrm{Tr}\left[\sigma^{+i}\gamma_5\psi(p,P)\right]$$

- Calculation of more observables, more rigorous study of chiral symmetry breaking, ingredients from LQCD...

Medium effects: Challenges

⟶ Is there a way to write down the propagators in such a way that overall the analytic structure is carried out in the denominator? $\langle\bar{\psi}\psi\rangle = -\mathrm{Tr}\int \frac{d^4k}{(2\pi)^4} S_F(k)$





Isserstedt et al., PRD **100**, 074011 (2019)

- How to construct the EoS at finite density and/or temperature in Minkowski space? Mallik, Sarkar: EPJC 61:489-494(2009).
- Is it possible to retain the integral representation while including magnetic field effects?

Summary

- Formulation in Minkowski space: possibility of calculation of dynamical observables;
- Based on the NIR, which is as a very important tool to solve DSE and BSE;
- More sophisticated ingredients, as quark-gluon vertex, Lattice QCD (self energy, vertex, ...) $\Rightarrow$ more realistic theories!
- Wide range of applications: Form factors, parton distribution functions, analytic structure of pion, kaon, nucleon, Nakanishi weight functions ...
- Modifying the quark propagator to include T, μ, eB: Phase diagram and EoS and in Minkowski space?

Thanks for your attention!

FAPERGS

serrapilheira

Universidade Federal de Santa Maria 1960

Fermion Dyson Schwinger Equation (Rainbow-Ladder)

- In terms of vector and scalar components (Källén-Lehman SR):

$$
\begin{aligned}
S_q(k) &= \not{k} S_v(k^2) + S_s(k^2) \\
&= R\frac{\not{k} + \overline{m}_0}{k^2 - \overline{m}_0^2 + i\epsilon} + \not{k}\int_0^\infty ds \frac{\rho_v(s)}{k^2 - s + i\epsilon} + \int_0^\infty ds \frac{\rho_s(s)}{k^2 - s + i\epsilon}
\end{aligned}
$$

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- In terms of vector and scalar components (Källén-Lehman SR):

$$\begin{aligned} S_q(k) &= \not{k} S_v(k^2) + S_s(k^2) \\ &= R\frac{\not{k} + \overline{m}_0}{k^2 - \overline{m}_0^2 + i\epsilon} + \not{k}\int_0^\infty ds \frac{\rho_v(s)}{k^2 - s + i\epsilon} + \int_0^\infty ds \frac{\rho_s(s)}{k^2 - s + i\epsilon} \end{aligned}$$

$$k^2 A\left(k^2\right) = i\alpha \int \frac{d^4q}{4\pi^3} \left\{ \frac{R}{(k-q)^2 - \overline{m}_0^2 + i\epsilon} + \int_0^\infty ds \frac{\rho_v(s)}{(k-q)^2 - s + i\epsilon} \left[\frac{-2k^2 + 2(k\cdot q)}{q^2 - m_g^2 + i\epsilon} - \frac{(1-\xi)\left(2(k\cdot q)^2 - k^2 q^2 - q^2 (k\cdot q)\right)}{\left(q^2 - m_g^2 + i\epsilon\right)\left(q^2 - \xi m_g^2 + i\epsilon\right)} \right] \right\} - [m_g \to \Lambda]$$

$$B\left(k^2\right) = -i\alpha \int \frac{d^4q}{4\pi^3} \left\{ \frac{R\,\overline{m}_0}{(k-q)^2 - \overline{m}_0^2 + i\epsilon} + \int_0^\infty ds \frac{\rho_s(s)}{(k-q)^2 - s + i\epsilon} \times \frac{1}{q^2 - m_g^2 + i\epsilon} \left[4 - \frac{q^2(1-\xi)}{q^2 - \xi m_g^2 + i\epsilon} \right] \right\} - [m_g \to \Lambda]$$

Pauli-Villars regularization

Fermion DSE solution

$$\rho_A(\gamma) = R\mathcal{K}^{\xi}_{0A}(\gamma, \overline{m}_0^2, m_g^2) + \int_0^{\infty} ds\, \mathcal{K}^{\xi}_A(\gamma, s, m_g^2)\, \rho_v(s) - [m_g \to \Lambda]$$

$$\rho_B(\gamma) = R\,\overline{m}_0\, \mathcal{K}^{\xi}_{0B}(\gamma, \overline{m}_0^2, m_g^2) + \int_0^{\infty} ds\, \mathcal{K}^{\xi}_B(\gamma, s, m_g^2)\, \rho_s(s) - [m_g \to \Lambda]$$

- Driving term:

$$\mathcal{K}^{\xi}_{0A(0B)} = K_{A(B)} + m_g^{-2} \bar{K}^{\xi}_{A(B)}$$

- Kernel:

$$\mathcal{K}^{\xi}_A(\gamma, s, m_g^2) = K_A(\gamma, s, m_g^2)\, \Theta(s - (\overline{m}_0 + m_g)^2) + m_g^{-2} \bar{K}^{\xi}_A(\gamma, s, m_g^2)\, \Theta(s - (\overline{m}_0 + \sqrt{\xi} m_g)^2)$$

$$f_A(\gamma) = 1 + \int_0^{\infty} ds \frac{\rho_A(s)}{\gamma - s}$$

$$f_B(\gamma) = m_B + \int_0^{\infty} ds \frac{\rho_B(s)}{\gamma - s}$$

$$d(\gamma) = \left[\gamma f_A^2(\gamma) - \pi^2\, \gamma\, \rho_A^2(\gamma) - f_B^2(\gamma) + \pi^2\, \rho_B^2(\gamma)\right]^2 + 4\pi^2 \left[\gamma \rho_A(\gamma)\, f_A(\gamma) - \rho_B(\gamma)\, f_B(\gamma)\right]^2$$

Connection Formulas

$$\rho_v(\gamma) = -2\frac{f_A(\gamma)}{d(\gamma)} \left[\gamma \rho_A(\gamma) f_A(\gamma) - \rho_B(\gamma) f_B(\gamma)\right] + \frac{\rho_A(\gamma)}{d(\gamma)} \left[\gamma\, f_A^2(\gamma) - \pi^2 \gamma \rho_A^2(\gamma) - f_B^2(\gamma) + \pi^2 \rho_B^2(\gamma)\right]$$

$$\rho_s(\gamma) = -2\frac{f_B(\gamma)}{d(\gamma)} \left[\gamma \rho_A(\gamma) f_A(\gamma) - \rho_B(\gamma) f_B(\gamma)\right] + \frac{\rho_B(\gamma)}{d(\gamma)} \left[\gamma\, f_A^2(\gamma) - \pi^2 \gamma \rho_A^2(\gamma) - f_B^2(\gamma) + \pi^2 \rho_B^2(\gamma)\right]$$

Feynman gauge kernel ($\xi = 1$):

$$K_A\left(\gamma, s, m_g^2\right) = -\frac{\alpha}{4\pi}\frac{\gamma - m_g^2 + s}{\gamma^2}\sqrt{(\gamma - m_g^2 + s)^2 - 4\gamma\, s}\;\;\Theta\left[\gamma - (m_g + \sqrt{s})^2\right]$$
$$K_B\left(\gamma, s, m_g^2\right) = -\frac{\alpha}{4\pi}\frac{4}{\gamma}\sqrt{(\gamma - m_g^2 + s)^2 - 4\gamma s}\;\;\Theta\left[\gamma - (m_g + \sqrt{s})^2\right]$$

Remaining arbitrary ξ-gauge contribution:

$$\bar{K}_A^\xi(\gamma, s, m_g^2) = -\frac{\alpha}{4\pi}\frac{(\gamma - s)^2 - m_g^2(\gamma + s)}{2\gamma^2}\sqrt{(\gamma - m_g^2 + s)^2 - 4\gamma s}$$
$$\times\,\Theta\left[\gamma - (m_g + \sqrt{s})^2\right] - [m_g^2 \to \xi m_g^2]$$
$$\bar{K}_B^\xi(\gamma, s, m_g^2) = \frac{\alpha\, m_g^2}{4\pi}\frac{\sqrt{(\gamma - m_g^2 + s)^2 - 4\gamma s}}{\gamma}\Theta\left[\gamma - (m_g + \sqrt{s})^2\right] - [m_g^2 \to \xi m_g^2]$$

Bare mass:

$$m_B = \overline{m}_0 + \overline{m}_0 \int_0^\infty ds\,\frac{\rho_A(s)}{\overline{m}_0^2 - s} - \int_0^\infty ds\,\frac{\rho_B(s)}{\overline{m}_0^2 - s}$$

Residue:

$$R^{-1} = 1 + \int_0^\infty ds\,\frac{\rho_A(s)}{\overline{m}_0^2 - s} - 2\overline{m}_0^2 \int_0^\infty ds\,\frac{\rho_A(s)}{(\overline{m}_0^2 - s)^2}$$
$$+\, 2\overline{m}_0 \int_0^\infty ds\,\frac{\rho_B(s)}{(\overline{m}_0^2 - s)^2}$$