Finite-size effects
via volume-dependent free energy

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What are the typical sizes?

- Typical size of the fireball in **heavy ion collisions** is a few fm.
- Neutron stars and **compact stars** built up from strongly interacting matter (with extra structure) with a size $\sim$ 10 km.
- Several **models with finite** (different) size.
- In field theoretical calculations (LSM, NJL, DS, etc): **infinite size**.

Why does it matter?

- The properties of the system can change significantly.
- Criticality in a finite system?
- The CEP and the first-order region might "disappear".

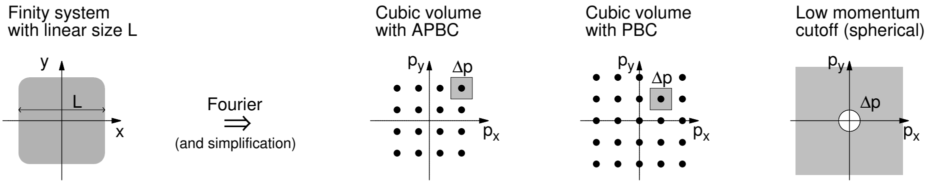
Might be studied in field theoretical models by implementing the finite size effects.

The vicinity of the CEP is accessible with models that are in the thermodynamic limit.

Finite size effects without losing the advantages of these models?

Modify only the momentum space.

- Discretization: $\int dp \to \sum_n$
- Low momentum cutoff: $\int_0^{\infty} dp \to \int_{\lambda}^{\infty} dp$



Volume dependence

Beyond momentum space: statistical factor

$$\mathcal{Z} = \int \mathcal{D}\phi e^{iS(\phi)}, \qquad S(\phi) = \int d^4x \, \mathcal{L}(\phi(x), x) \tag{1}$$

Keeping only one mode to integrate: degrees of freedom from ∞ to 1

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-S_E(\phi)} \xrightarrow{\text{single mode}} \int_{-\infty}^{\infty} d\phi \, e^{-S_E(\phi)} \tag{2}$$

Constant, homogenous field + local Lagrangian: $\int d^4x = \mathcal{V}_4 \xrightarrow{\text{fin } T} \beta V$

V dependence separates from the potential: $S_E = \beta V \cdot U(\phi)$

The expectation value:

$$\langle \phi \rangle = \frac{1}{\mathcal{Z}} \int_{-\infty}^{\infty} d\phi \, \phi \, e^{-\beta V U(\phi)} \tag{3}$$

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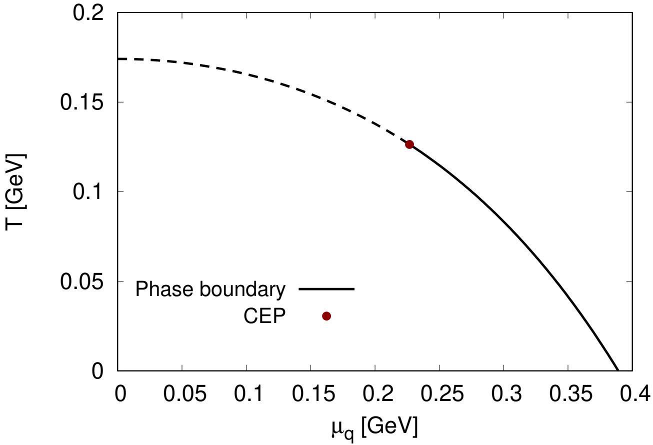
Simple quark-meson type model (classical potential + fermionic thermal fluct.):

$$U(\phi, T, \mu) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 - h\phi + 2TN_c \int_L \frac{dp^3}{(2\pi)^3} \left[\log(1 + e^{-\beta(E-\mu)}) + \log(1 + e^{-\beta(E+\mu)})\right] \tag{4}$$

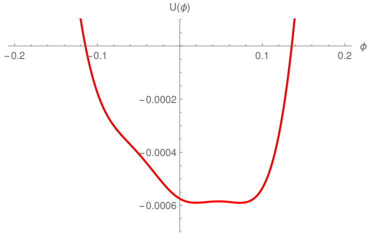
$V \to \infty$, $\frac{\partial U(\bar{\phi})}{\partial \bar{\phi}} = 0$ $\Rightarrow$ Crossover at $\mu = 0$, 1st order at $T = 0$, CEP at finite T and μ.

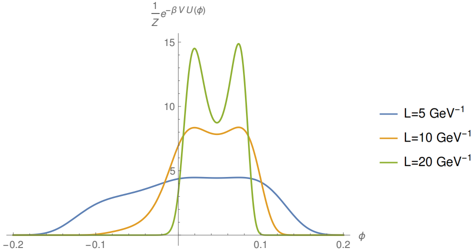


At finite V:

$$\langle\phi\rangle = \int_{-\infty}^{\infty} d\phi\, \phi\, P(\phi, V), \qquad P(\phi, V) = e^{-\beta V\, U(\phi)}/\mathcal{Z} \tag{5}$$

For a fixed T and μ with multiple solutions in MF (1 fm $\approx$ 5 GeV^{-1}):



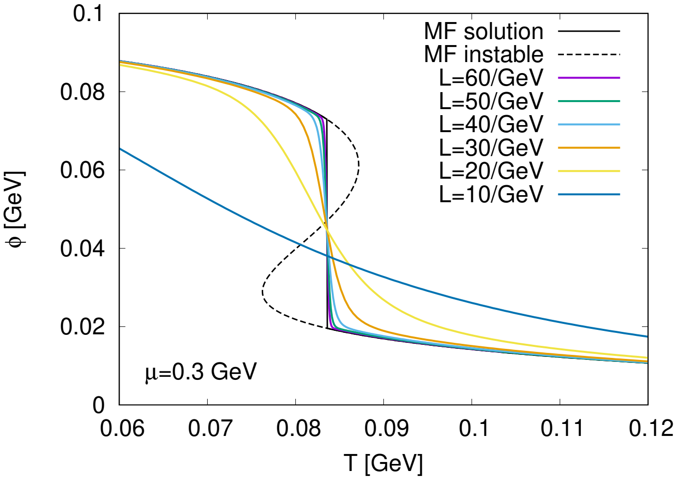


Mean-field:

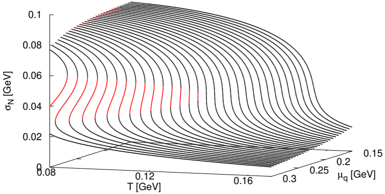
$$\langle\phi\rangle = \bar\phi \Leftarrow \frac{\partial U(\bar\phi)}{\partial\bar\phi} = 0$$

Full V dependence

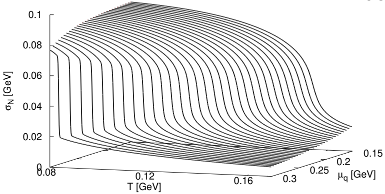
$$\begin{aligned}\langle\phi\rangle &= \frac{1}{\beta V}\frac{\partial \ln \mathcal{Z}}{\partial h}\\ &= \frac{1}{\mathcal{Z}}\int_{-\infty}^{\infty} d\phi\left(-\frac{\partial U}{\partial h}\right)e^{-\beta V U(\phi)}\\ &= \int_{-\infty}^{\infty} d\phi\, \phi\, P(\phi, V)\end{aligned}$$



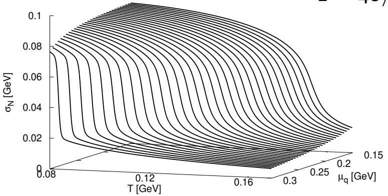
MF



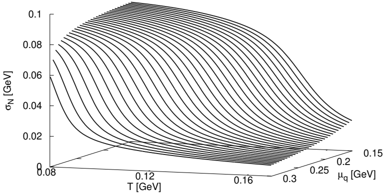
$L = 60/GeV$



$L = 40/GeV$

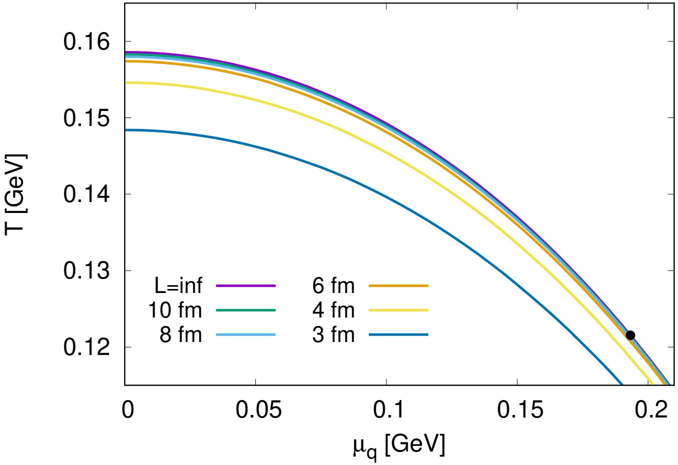


$L = 20/GeV$



Everything is smooth at finite L

Shift below $\approx 50/\text{GeV} \approx 10$ fm



V dependence of the free energy: $\Phi = -T \ln \mathcal{Z}$

Ecpectation value of ϕ and its fluctuations

$$\langle \phi \rangle = \frac{1}{\beta V} \frac{\partial \ln \mathcal{Z}}{\partial h} = -\frac{1}{V} \frac{\partial \Phi}{\partial h},$$
$$\chi_2 = \frac{\partial \langle \phi \rangle}{\partial h} = \frac{1}{\beta V} \left(\frac{1}{\mathcal{Z}} \frac{\partial^2 \mathcal{Z}}{\partial h^2} - \left(\frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial h} \right)^2 \right) = \beta V \left(\langle \phi^2 \rangle - \langle \phi \rangle^2 \right) \quad (6)$$
$$\chi_3 = \frac{\partial^2 \langle \phi \rangle}{\partial h^2}, \qquad \chi_4 = \frac{\partial^3 \langle \phi \rangle}{\partial h^3}$$

The pressure: $P = -\frac{\partial \Phi}{\partial V}$, The particle number: $\langle N \rangle = -\frac{\partial \Phi}{\partial \mu} = \frac{V}{\mathcal{Z}} \int_{-\infty}^{\infty} d\phi \frac{\partial U}{\partial \mu} e^{-\beta V U}$

Generally: $\langle \mathcal{A} \rangle = \mathcal{Z}^{-1} \int d\phi \, \mathcal{A}(\phi) \, e^{-\beta V \, U(\phi)}$, for a general observable $\mathcal{A}$

Scaling

Ising temperature (τ), field (η), and irrelevant/marginal direction (u).

The singular part of the free energy scales as

$$f_s(\tau, \eta, u) = L^{-d}\mathcal{F}(\tau L^{1/\nu}, \eta L^{\beta\delta/\nu}, uL^{-\omega})\,. \tag{7}$$

If u marginal $\omega = 0$, otherwise at leading order

$$f_s(\tau, \eta) = L^{-d}\mathcal{F}(\tau L^{1/\nu}, \eta L^{\beta\delta/\nu})\,. \tag{8}$$

The "magnetization" density and susceptibilities

$$m = \frac{\partial f_s}{\partial \eta} = L^{\beta\delta/\nu - d}\mathcal{F}(\tau L^{1/\nu}, \eta L^{\beta\delta/\nu}), \qquad \chi_n = \frac{\partial^n f_s}{\partial \eta^n} = L^{n\beta\delta/\nu - d}\mathcal{F}(\tau L^{1/\nu}, \eta L^{\beta\delta/\nu}) \tag{9}$$

Our phase diagram: CEP at finite T,μ_q, and h

Critical line in the T, μ_q, h space
The T, μ_q, and h direction show field-like scaling $\sim$ Nonzero overlap with η

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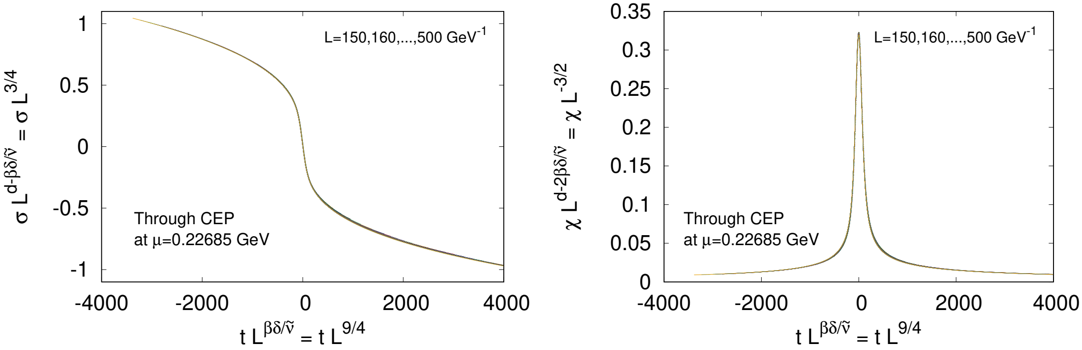
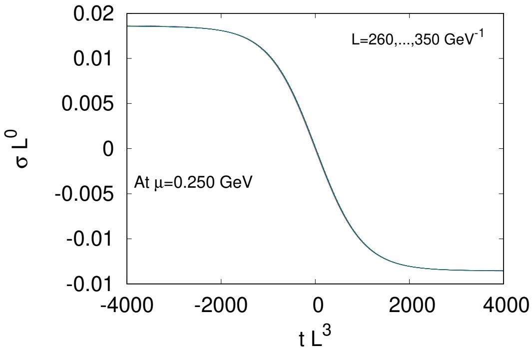
Mean-field approximation in 3D: hyperscaling violation
Use $\tilde{\nu} = (\gamma + 2\beta)/d = 2/3$ (c.f. $\nu = 1/2$ for MF)

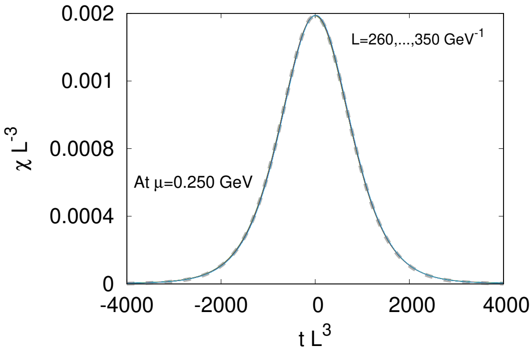
Expectation: $\sigma(tL^{\beta\delta/\tilde{\nu}}) \propto L^{\beta\delta/\tilde{\nu}-d}$, $\chi(tL^{\beta\delta/\tilde{\nu}}) \propto L^{2\beta\delta/\tilde{\nu}-d}$ $\quad \tilde{\nu} = (\gamma + 2\beta)/d = 2/3$



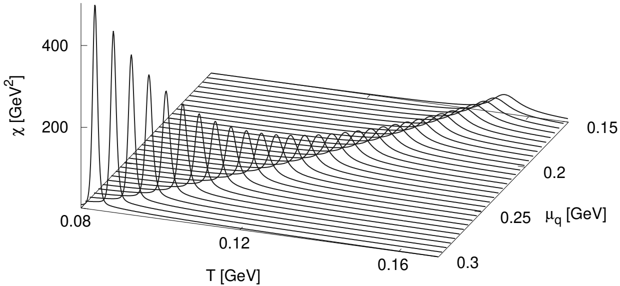
Expectation: $\sigma(tL^3) \propto L^0$, $\chi(tL^3) \propto L^3$

coexistance: $\chi = V\chi_\delta + (\chi_1 + \chi_2)/2$





In the 1st order region χ keeps increasing. E.g. at $L = 40/\text{GeV}$:



No maximum as apparent CEP

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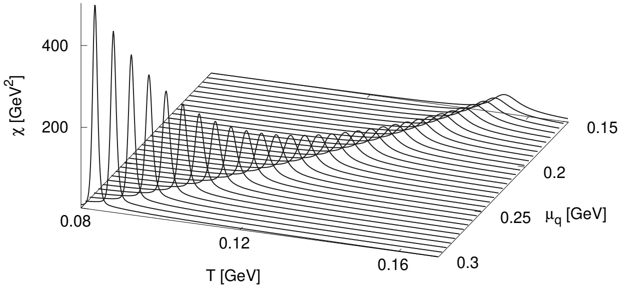


Thy system does not stay there long in reality

Using the double Gaussian approximation

$$P(\phi) = \frac{A}{(2\pi\tau)^{1/2}} \left[e^{-((\phi-\sigma_1)^2)-2\chi_1\phi\eta)L^d/(2\tau\chi_1)} + e^{-((\phi-\sigma_2)^2)-2\chi_2\phi\eta)L^d/(2\tau\chi_2)} \right] \tag{10}$$

with A is for the proper normalization $\int d\phi P(\phi) = 1$

Using $\langle\phi\rangle = \int d\phi\, \phi P(\phi)$, $\langle\phi^2\rangle = \int d\phi\, \phi^2 P(\phi)$, and $\chi = V/\tau(\langle\phi^2\rangle - \langle\phi\rangle^2)$ gives

$$\langle\phi\rangle = \lambda_1 U_1 + \lambda_2 U_2 \tag{11}$$

$$\chi = \chi_1 U_1 + \chi_2 U_2 + \frac{L^d}{\tau}(\lambda_1 - \lambda_2)^2 U_1 U_2 \tag{12}$$

width introducing $U_i = W_i/(W_1 + W_2)$, $W_i = \chi_i^{1/2} e^{\eta L^d(\chi_i\eta+2\sigma_i)/(2\tau)}$, $\lambda_i = \sigma_i + \eta\chi_i$
With $\sigma_{1/2} = \bar{\sigma} \pm \delta\sigma$, $\chi_{1/2} = \bar{\chi} \pm \delta\chi$, and assuming $\delta\chi$ is small:

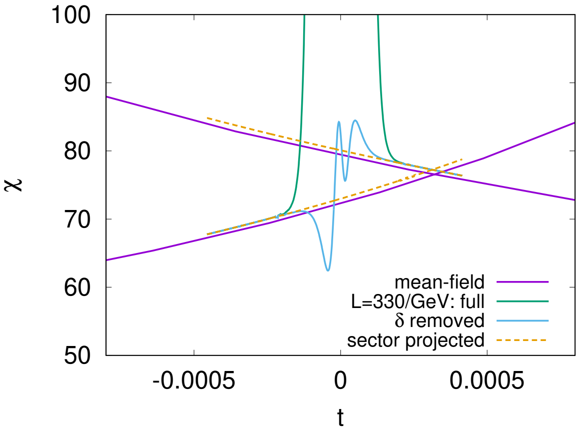
$$\chi\big|_{\delta\chi\to 0} = \bar{\chi} + \frac{L^d}{\tau}(\lambda_1 - \lambda_2)^2 \frac{1}{4}\cosh^{-2}\left(\eta L^d(\sigma_1 - \sigma_2)/(2\tau)\right) = \bar{\chi} + \frac{L^d}{\tau}\delta\sigma^2 \cosh^{-2}\left(\eta L^d \delta\sigma/\tau\right) \tag{13}$$

Binder: Lect. Notes Phys. 409, 59 (1992)

Peak from coexistence is clearly visible

The sector projected curves tend to MF

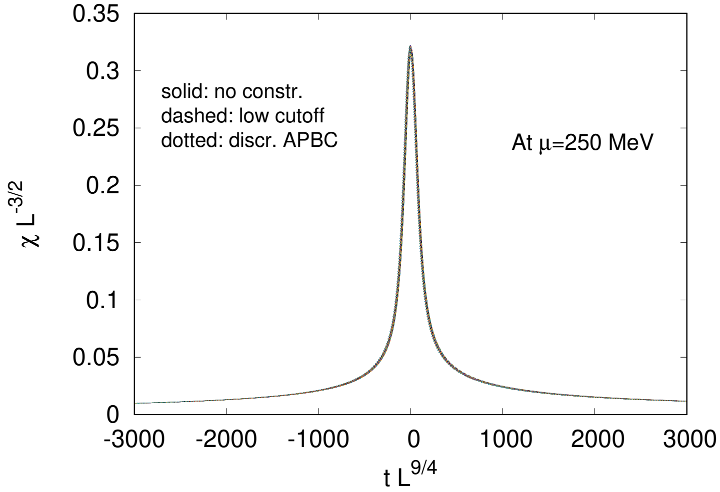
One may fit the χ_δ part and remove it from full (rough approximation)



Contraints:

- low momentum cutoff
- discretization with PBC/APBC

No change in the scaling for large *V*, as expected.



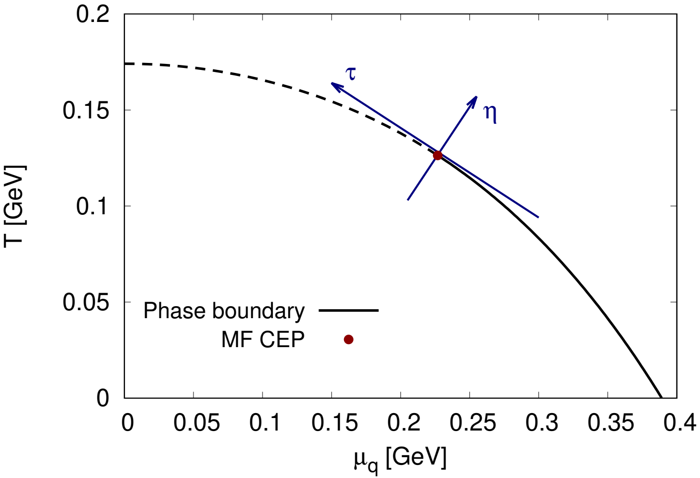
Scaling along special lines

Along the boundary τ direction with $\tau = sgn(T - T_C)\sqrt{(T - T_C)^2 + (\mu - \mu_C)^2}$

The phase boundary
almost straight
in such a small section

Binder cumulant:
$\kappa_B = \frac{\chi_4}{L^d \chi_2^2}$

$\kappa_B \propto L^0, \tau \propto L^{-1/\tilde{\nu}}$



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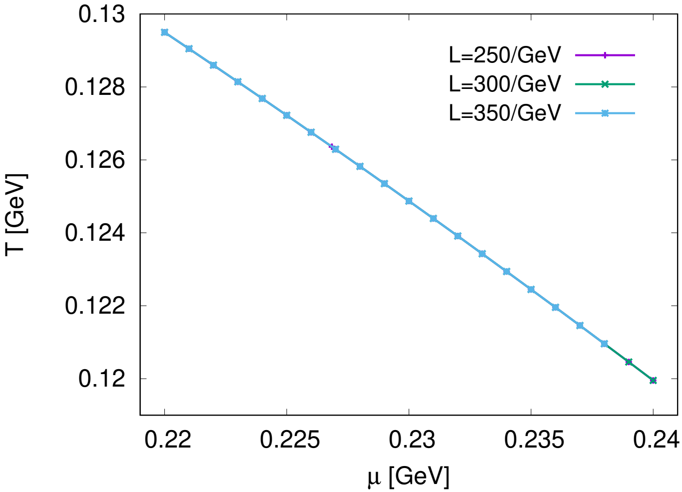

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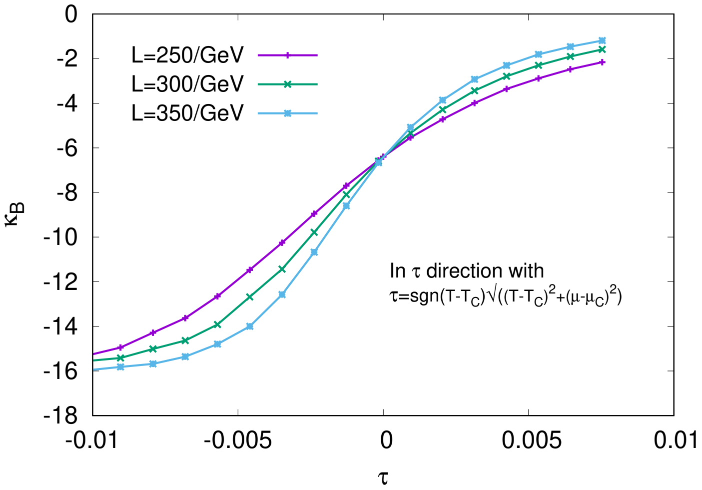

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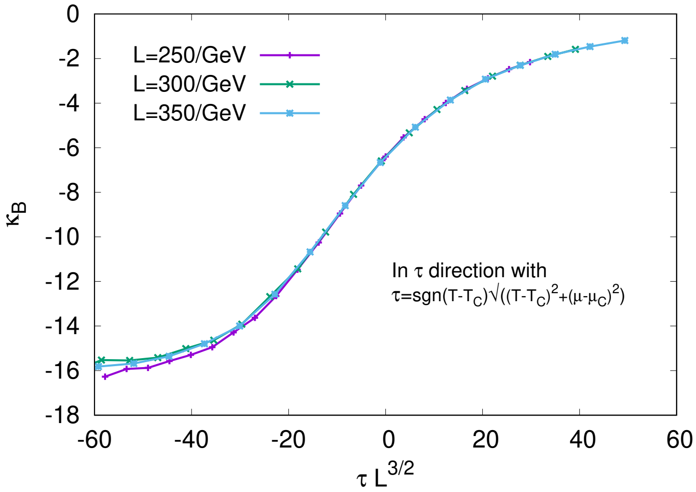
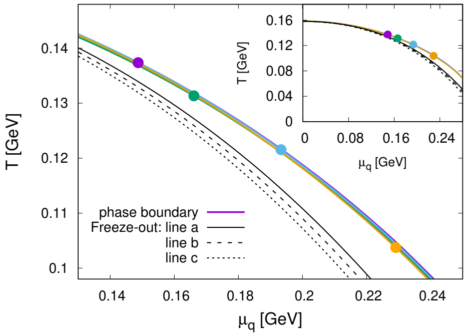
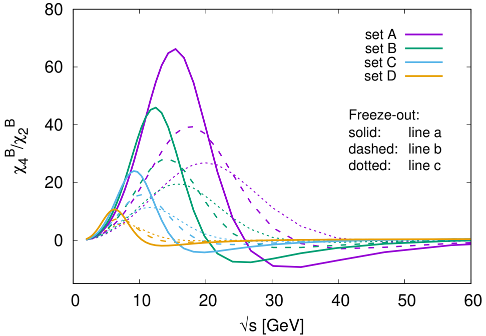
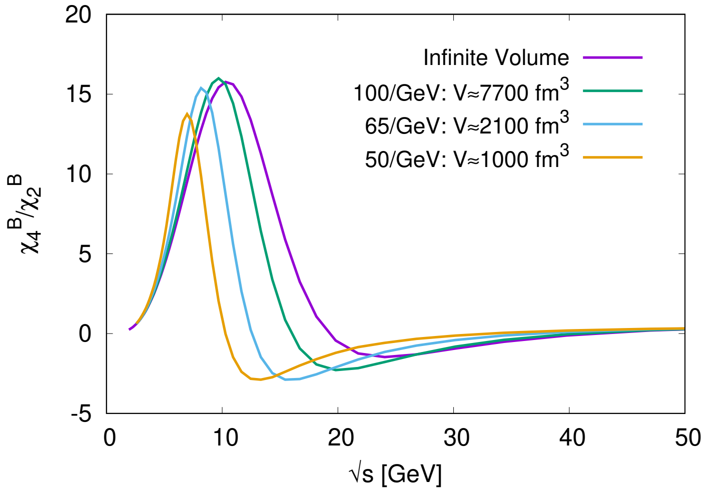






Very narrow window for $\chi_4^B/\chi_2^B < 0$ (model dependent, might be much wider)



Infinite Volume
100/GeV: V≈7700 fm^3
65/GeV: V≈2100 fm^3
50/GeV: V≈1000 fm^3
χ_4^B/χ_2^B
√s [GeV]
20
15
10
5
0
-5
0
10
20
30
40
50

- Simple approach for finite size effects in MF.
- Scaling properties can be reproduced
- FSS above $L \approx 20$ fm linear size
- No apparent CEP can be deduced as a maximum in the fluctuations.
- Along the transition through CEP: $\chi_4/(V\chi_2^2)$;   Along the freeze-out line: χ_4/χ_2
- Many other details....

Thank you!