Exploring Dense QCD through Hamiltonian Lattice Simulation in (1+1) Dimensions

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Erice: INTERNATIONAL SCHOOL OF NUCLEAR PHYSICS, 46th COURCE
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Hayata, KN and Hidaka, JHEP 07 (2024) 106


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Sign problem

- Monte Carlo can not be directly applied to QCD at finite baryon density.
  - QCD at finite baryon density with Nc=2 and Nf=(even number)

    Kogut, Stephanov and Toublan (1999); Kogut, Sinclair, Hands and Morrison (2001); Muroya, Nakamura, Nonaka and Takaishi (2003); Hands, Kogut, Lombardo and Morrison (1999); Hands, Sitch and Skullerud (2008); Cotter, Giudice, Hands and Skullerud (2013); Braguta, Ilgenfritz, Kotov, Molochkov and Nikolaev (2016); Wilhelm, Holicki, Smith, Wellegehausen and von Smekal (2019); Iida, Itou and Lee (2020); Boz, Giudice, Hands and Skullerud (2020); Braguta (2023)………

  - Three-color QCD with isospin chemical potential

    Son and Stephanov (2001); Kogut and Sinclair (2002, 2004); Forcrand, Stephanov, Wenger (2007); Detmold, Orginos and Shi (2012); Endrődi (2014); Brandt, Endrődi, and Schmalzbauer (2018); Bornyakov et al. (2021); Brandt, Cuteri, Endrődi (2023)……

- Investigation of numerical calculation without sign problem is important!
  - Neutron stars and heavy ion collision experiments = Three-color QCD at μ_B

Hamiltonian formalism

- Hamiltonian formalism may be good for sign problem free calculation.
  - Lattice Hamiltonian is defined on a spatial lattice, and time is continuous.

- Minimization of the free energy without Monte Carlo→No sign problem
  - We use good variational ansatz and efficient algorithm for searching the ground state.

- There are many application of Hamiltonian formalism to gauge theories.
  - SU(2) or SU(3) lattice gauge theory in (1+1) dim: Kuhn et.al., (2015); Silvi et.al., (2017); Banuls et.al., (2017); Sala et.al., (2018); Cataldi et.al. (2024); Calajo et.al. (2024), Ciavarella and Bauer (2024)……
  - Schwinger model: Silvi et.al., (2017); Rigobello et.al., (2023); Buyens et al. (2016); Bañuls et al. (2017); Farrell et al. (2024); Belyansky et al. (2023); Chen, Yan and Shi (2024)……

- Numerical calculation of thermodynamic quantities in (1+1)-d QCD at μ_B.

Hayata, KN and Hidaka (2024); Pai, Akiyama and Todo (2025); Cataldi et.al. (2024)

Matrix Product State and Density Matrix Renormalization Group

- **Matrix Product State (MPS):** Perez-Garcia, et.al., (2007) ; McCulloch, et.al., (2007)

  - This is frequently used for quantum spin systems in 1+1 dim.

- **Density Matrix Renormalization Group (DMRG)** White, PRL (1992)

  - Efficient algorithm for searching the ground states represented by MPS.

- **iTensor = A wonderful package that performs MPS and DMRG!**

  https://itensor.github.io/ITensors.jl/stable/ITensorType.html

  Also see reference by Fishman, White and Stoudenmire (2022)

Hamiltonian formalism for lattice gauge theory in (1+1) dimension

- **Kogut-Susskind Hamiltonian:** $H = H_{\rm E} + H_{\rm hop} + H_{\rm m}$ Kogut and Susskind, PRD (1975)

$$H_{\rm E} = \frac{ag_0^2}{2}\sum_{n=1}^{N-1} E_i^2(n),$$

$$H_{\rm hop} = \frac{1}{2a}\sum_{n=1}^{N-1}\left(\chi^\dagger(n+1)U(n)\chi(n) + \chi^\dagger(n)U^\dagger(n)\chi(n+1)\right),$$

$$H_{\rm m} = m_0\sum_{n=1}^{N}(-1)^n\chi^\dagger(n)\chi(n)$$

- **Quark (baryon) number:** $N_{\rm q} = \sum_{n=1}^{N}\chi^\dagger(n)\chi(n)$

- **Full Hamiltonian:** $H_{\rm tot} \equiv H_{\rm E} + H_{\rm hop} + H_{\rm m} - \mu_{0\rm B}N_{\rm B}$

Dimensionless Hamiltonian

- **Full Hamiltonian w/ n_B normalized g_0:** $\tilde{H}_{\rm tot} \equiv H_{\rm tot}/g_0$

$$H_{\rm E}/g_0 = J \sum_{n=1}^{N-1} E_i^2(n) \equiv \tilde{H}_{\rm E},$$

$$H_{\rm hop}/g_0 = \epsilon \sum_{n=1}^{N-1} \left(\chi^\dagger(n+1)U(n)\chi(n) + \chi^\dagger(n)U^\dagger(n)\chi(n+1)\right) \equiv \tilde{H}_{\rm hop},$$

$$H_{\rm m}/g_0 = m \sum_{n=1}^{N} (-1)^n \chi^\dagger(n)\chi(n) \equiv \tilde{H}_{\rm m},$$

$$\mu_{0\rm B} N_{\rm B}/g_0 = \mu_{\rm B} \frac{1}{N_{\rm c}} \sum_{n=1}^{N} \chi^\dagger(n)\chi(n) \equiv \mu_{\rm B} N_{\rm B}.$$

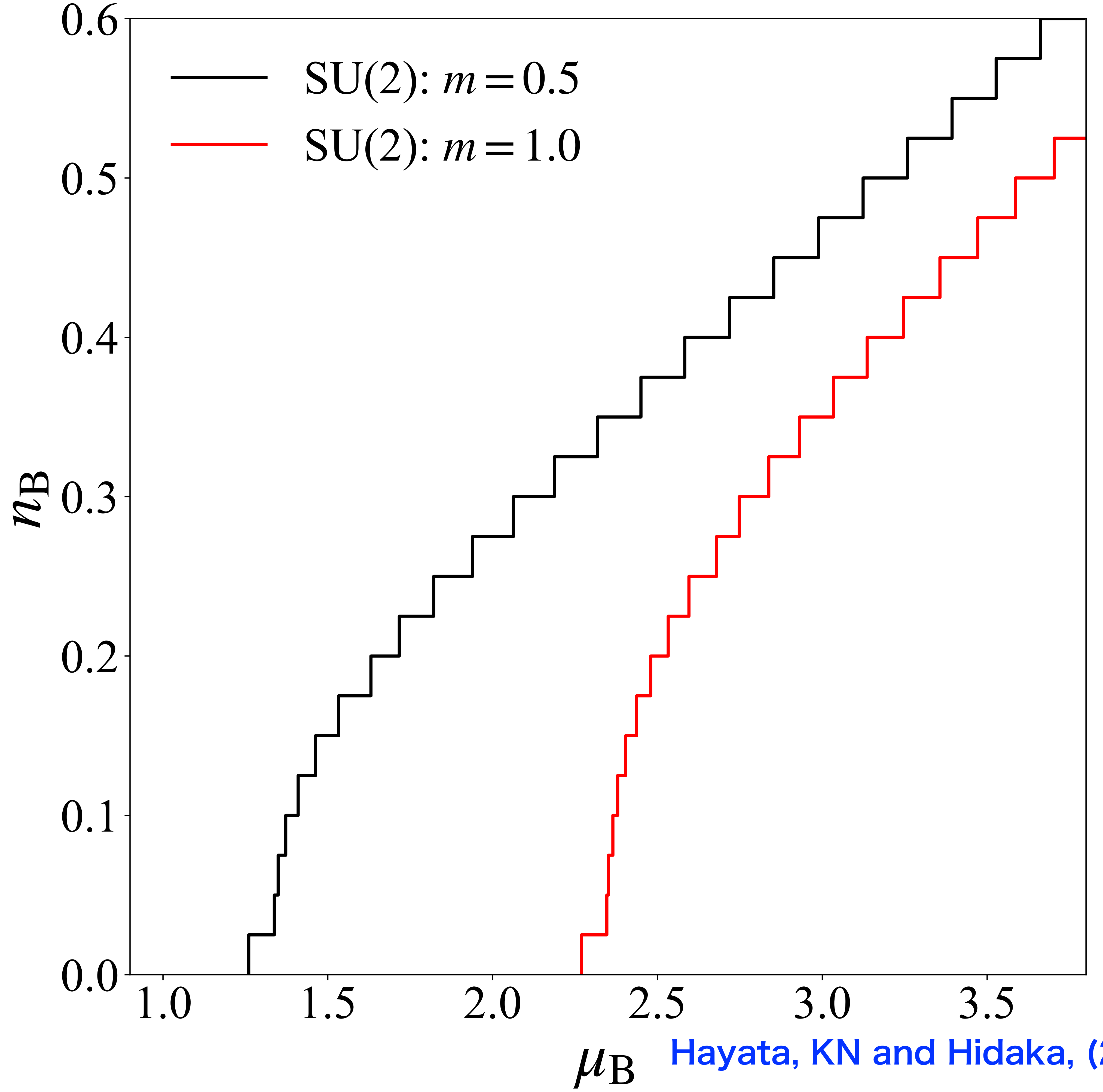
- **Parameters:** $J = ag_0/2\,,\ \epsilon = 1/(2ag_0)\,,\ m = m_0/g_0\,,\ \mu = \mu_0/g_0$

- **Since I can solve Gauss law constraint in open boundary condition, SU(2) can be eliminated and represented by Staggared fermions.**

- **After Wigner-Jordan transformation, our Hamiltonian is represented by Pauli matrices.**

Baryon number

- N=160, N_c=2, and w=2.0



Hayata, KN and Hidaka, (2024)

- Stepwise behavior
- Threshold value:

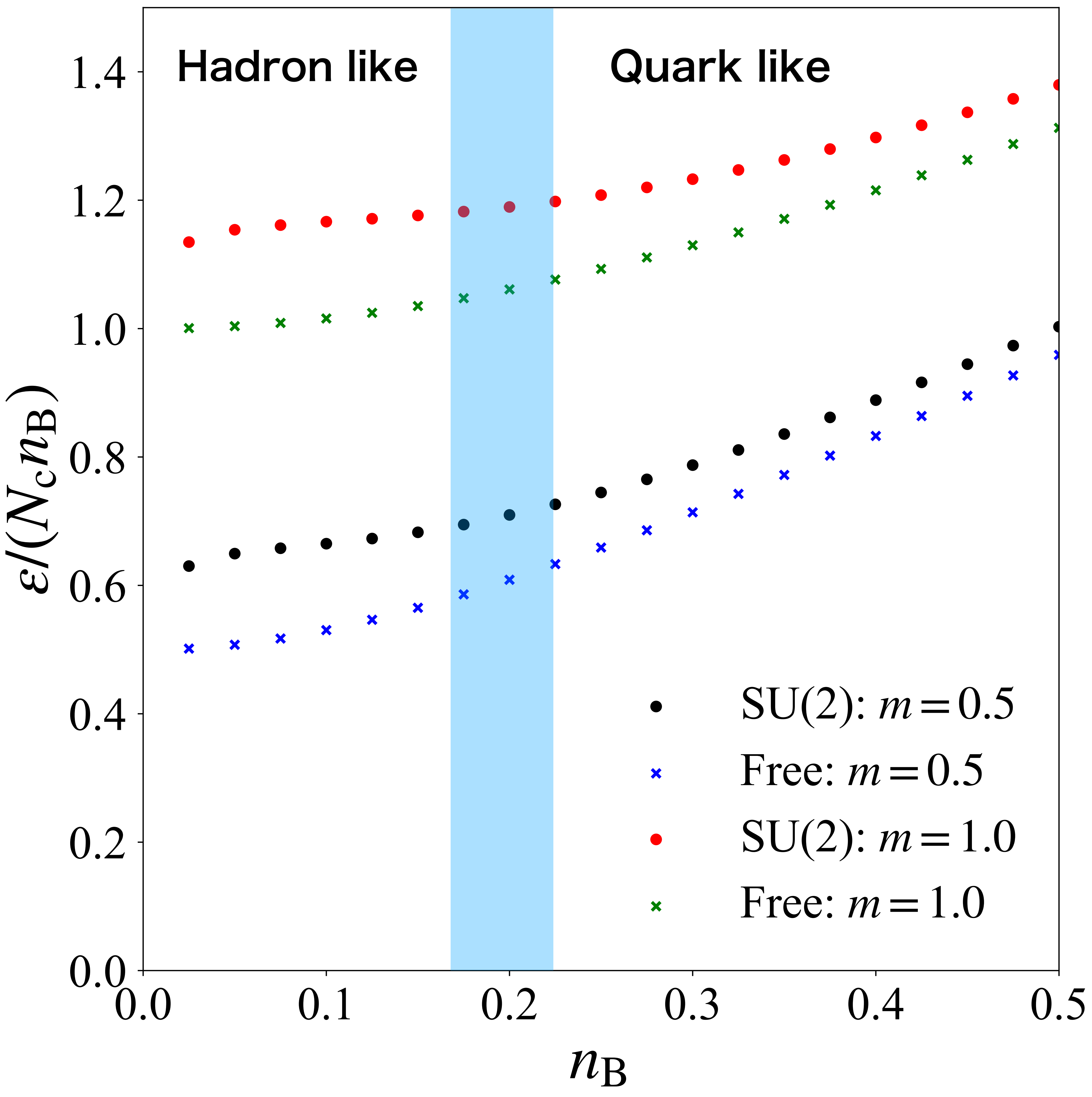
$$\mu_c = \begin{cases} 1.26 > 2 \times 0.5 & (m = 0.5) \\ 2.27 > 2 \times 1.0 & (m = 1.0) \end{cases}$$

- At high density g_0/μ<<1 and m/μ<<1, the contribution from g_0 and m can be negligible→Free-quark!

$$n_B = N_c \mu / \pi = \mu_B / \pi$$

- How does it behave at low density?

From hadron to quark



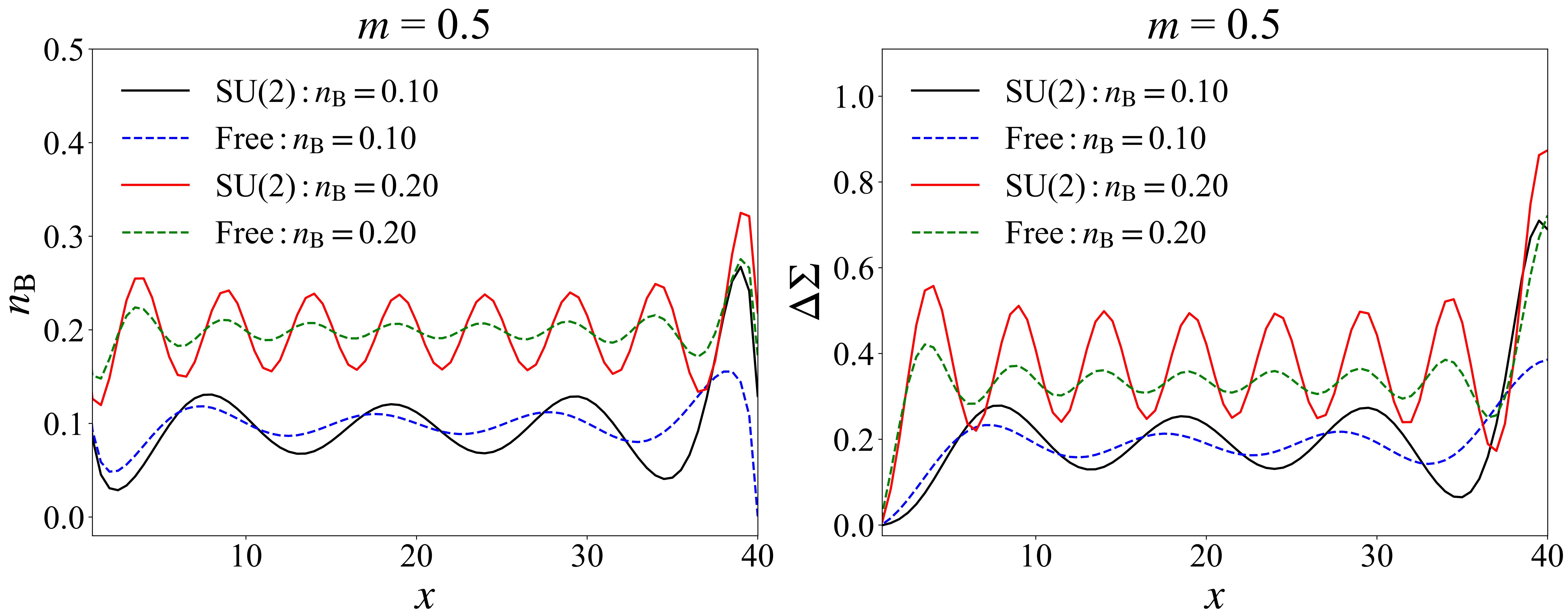
Hayata, KN and Hidaka, (2024)

- Energy per one quark
- At high density, ε/n_q behaves like the free theory.
- At low density, ε/n_q is larger due to the confining energy.
- The behavior changes to free theory around n_B~0.2.
- If interaction can be negligible, baryons will degenerate to the lowest energy state.

$$\epsilon/(N_c n_B) = \mathrm{const}$$

Inhomogeneous phase (m=0.5)

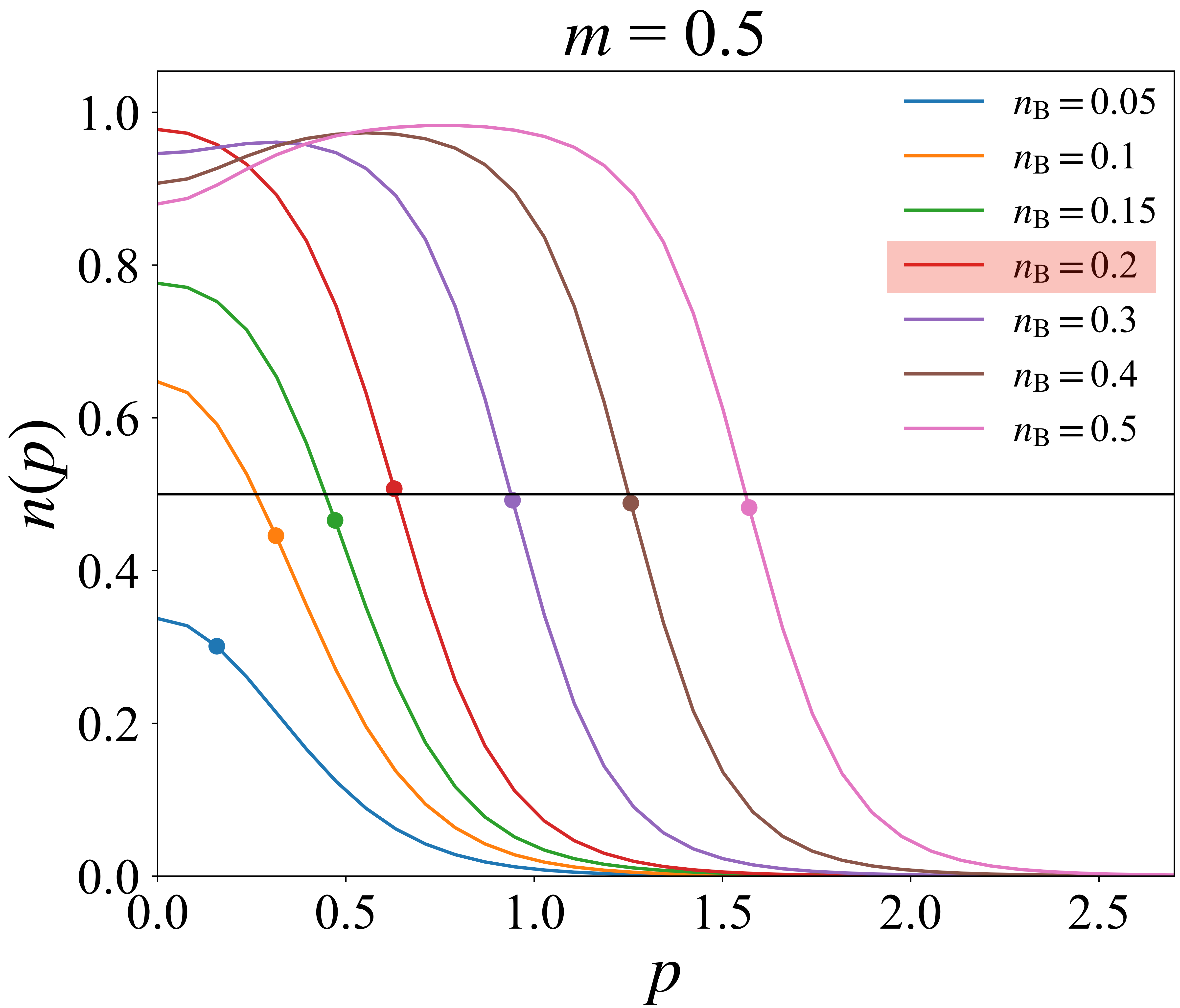
- **The explicit translational symmetry leads to spatial dependence of n_B and Σ.**
  - This explicit breaking is induced by the open boundary condition and finite lattice spacing.



- **SSB of continuous symmetry is prohibited.**
  - Taking large volume and continuum limit, the above inhomogeneous state will vanish,
  - Important direction: calculation with larger site, investigation of quasi-long range order
  - In free theory, the modulations vanish proportionally to 1/L.

Distribution function

- N=240, w=2.0



- At low density, n(p) has a peak at p=0.

- As n_B increases, maximum of n(p) increases and forms Fermi sea.

cf) Quarkyonic matter in (3+1) dim
BCS-BEC crossover of ultracold atom gases

Kojo, (2012); Astrakharchik et.al., (2005); Regal et.al., (2005)

- Fermi sea forms near n_B~0.2, which is consistent with behavior of ε/n_q

- Fermi surface is smooth, not sharp.
→Instability of the Fermi surface

Summary

- Variational numerical calculation based on Hamiltonian formalism.
  - The transition from baryonic matter to quark matter occurs near n_B~0.2.
  - Fermi surface forms about n_B~0.2.

Thank you for your attention!

Back up

Solving gauss law

- In (1+1) dim, the gauge field are non-dynamical and can be eliminated from the Hamiltonian in open boundary condition.

- We choose an appropriate unitary matrix to eliminate the link variables in hopping term.

$$\Theta \tilde{H}_{\mathrm{hop}} \Theta^{\dagger} = \tilde{H}'_{\mathrm{hop}} = \frac{1}{2a} \sum_{n=1}^{N-1} (\chi^{\dagger}(n+1)\chi(n) + \chi^{\dagger}(n)\chi(n+1))$$

- Solving the Gauss law, we can represent the hopping term in terms of KS fermions.

$$\Theta \tilde{H}_{\mathrm{E}} \Theta^{\dagger} = \tilde{H}'_{\mathrm{E}} = \frac{ag^2}{2} (\sum_{n=1}^{N-1} (N-n) Q_i^2(n) + 2 \sum_{n=1}^{N-2} \sum_{m=n+1}^{N-1} (N-m) Q_i(n) Q_i(m))$$

$$\chi^{\dagger}(n) T_i \chi(n)$$

- The mass term and baryon number do not change under the unitary transformation.

$$\Theta \tilde{H}_{\mathrm{m}} \Theta^{\dagger} = \tilde{H}'_{\mathrm{m}} = \tilde{H}_{\mathrm{m}} \,, \quad \Theta N_{\mathrm{B}} \Theta^{\dagger} = N'_{\mathrm{B}} = N_{\mathrm{B}}$$

Wigner Jordan transformation

- For numerical calculations, we map the Hamiltonian to the spin system.

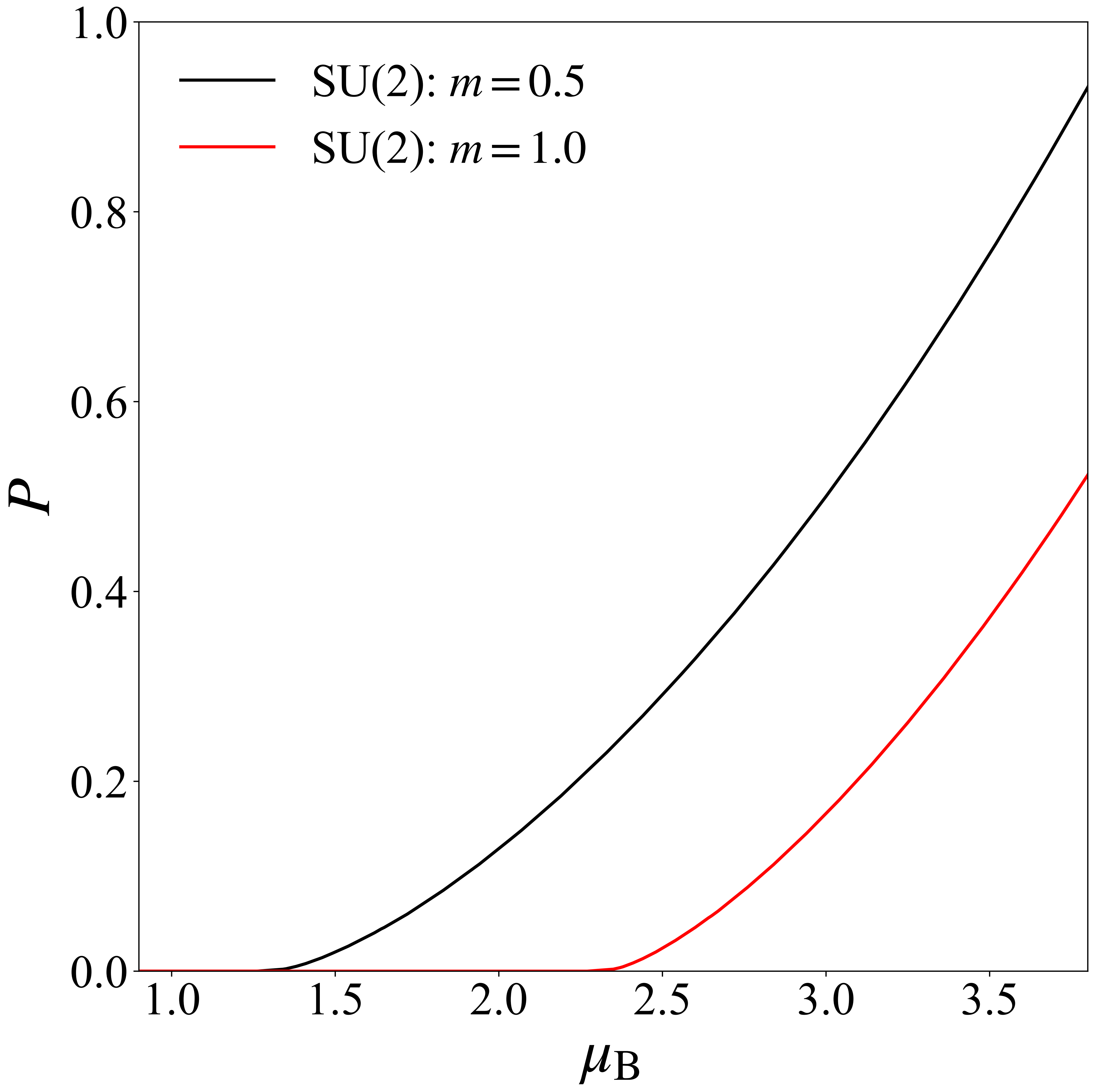
- Using Pauli matrices, construct something that satisfies the fermionic anti-commutation relations.

$$\chi_{f,c}(n) = \left[\prod_{k<\ell(n,f,c)} (-\sigma_k^z)\right] \sigma^-_{\ell(n,f,c)}\,,\ \chi^\dagger_{f,c}(n) = \left[\prod_{k<\ell(n,f,c)} (-\sigma_k^z)\right] \sigma^+_{\ell(n,f,c)}$$

- The Hamiltonian in terms of fermions can be expressed using Pauli matrices (below is an example of the hopping term).

$$\tilde{H}_{\rm hop}/g_0 = \epsilon \sum_{n=1}^{N-1} \sum_{f=1}^{N_{\rm f}} \sum_{c=1}^{N_{\rm c}} (\sigma^+_{\ell(n+1,f,c)} \Sigma_{f,c}(n) \sigma^-_{\ell(n,f,c)} + \sigma^+_{\ell(n,f,c)} \Sigma^\dagger_{f,c}(n) \sigma^-_{\ell(n+1,f,c)})$$

Pressure



Hayata, KN and Hidaka, (2024)

- **N=160, N_c=2, and w=2.0**

- **Grand potential and Pressure:**

$$J = \langle \mathrm{GS} | (H - \mu_\mathrm{B} N_\mathrm{B}) | \mathrm{GS} \rangle$$

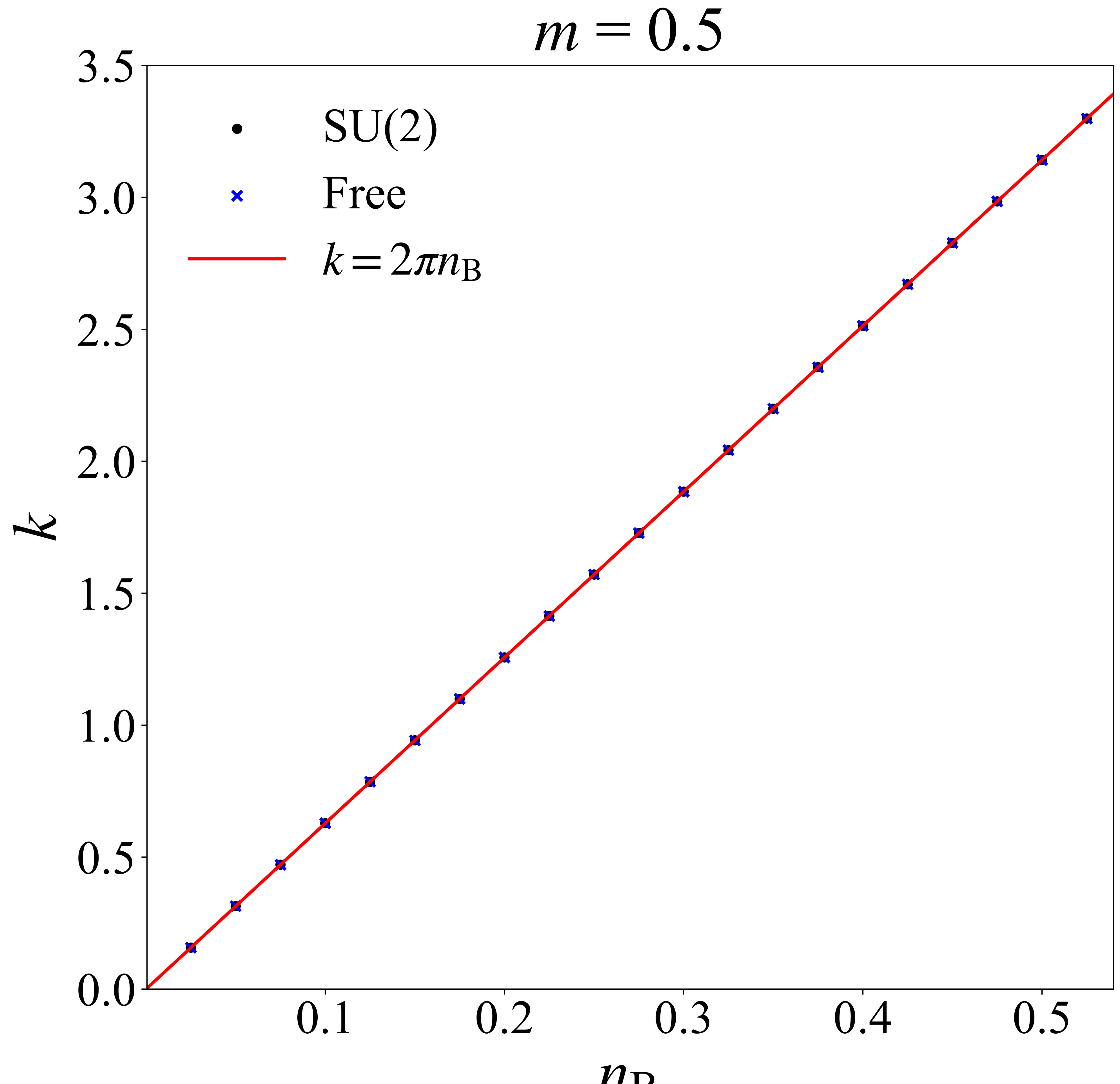
$$P = -J/V$$

- **First derivative = baryon number:**

$$n_\mathrm{B} = \frac{\mathrm{d}P(\mu_\mathrm{B})}{\mathrm{d}\mu_\mathrm{B}} = \frac{1}{V} \langle N_\mathrm{B} \rangle$$

Wave number (period)

- Fourier transform on $n_B(x)$ and wave number of the largest modulation.

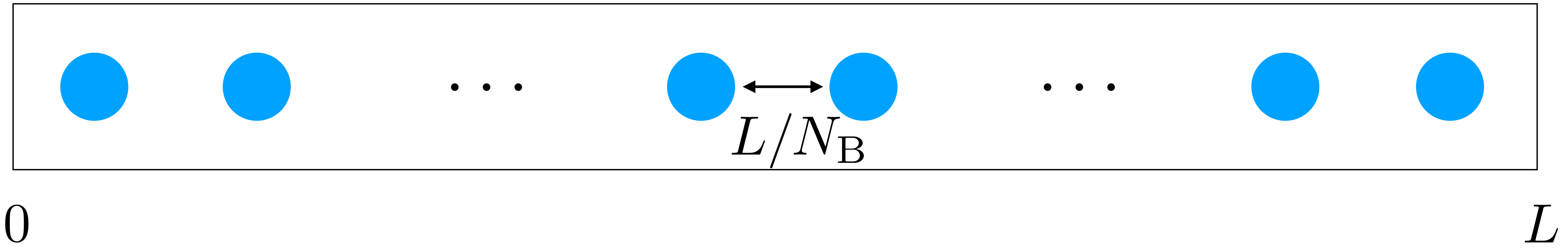


Hayata, KN and Hidaka, (2024)

Baryons crystal and chiral density wave

- If I assume repulsive force, the baryons are periodically aligned.

cf) Bosonization with large-N_c expansion Kojo, Nucl.A (2012)



- $k=2\pi/(\text{period})=2\pi/(L/N_B)=2\pi n_B$

- Due to the Peierls instability, the particle and hole form the condensate.

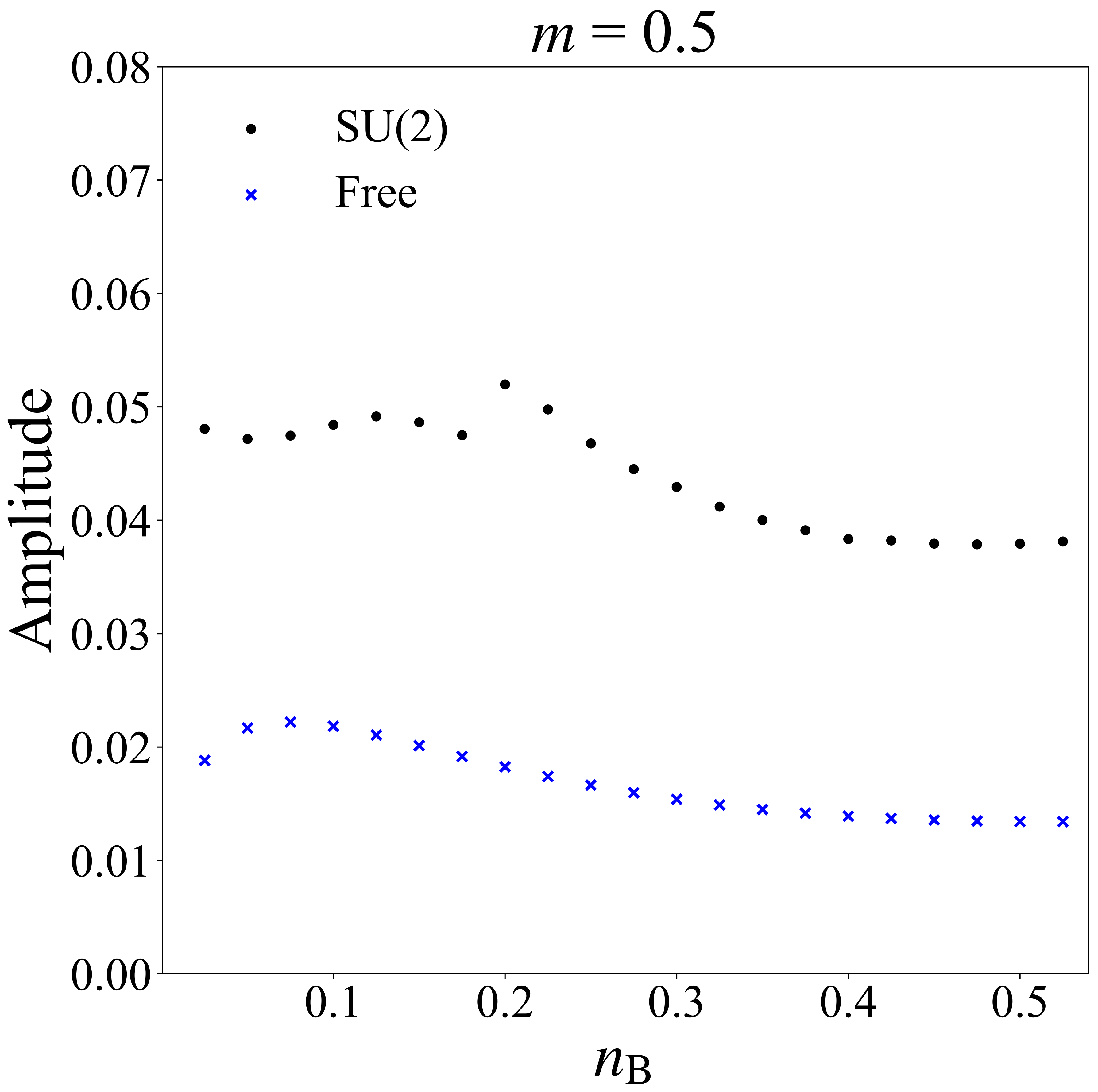
- Both of the particle and hole have p_F, then the total momentum is $2p_F$.

- $k=2p_F=2\pi n_B$ - In (1+1) dim, the fermi momentum is πn_B.

Back up for SU(2) results

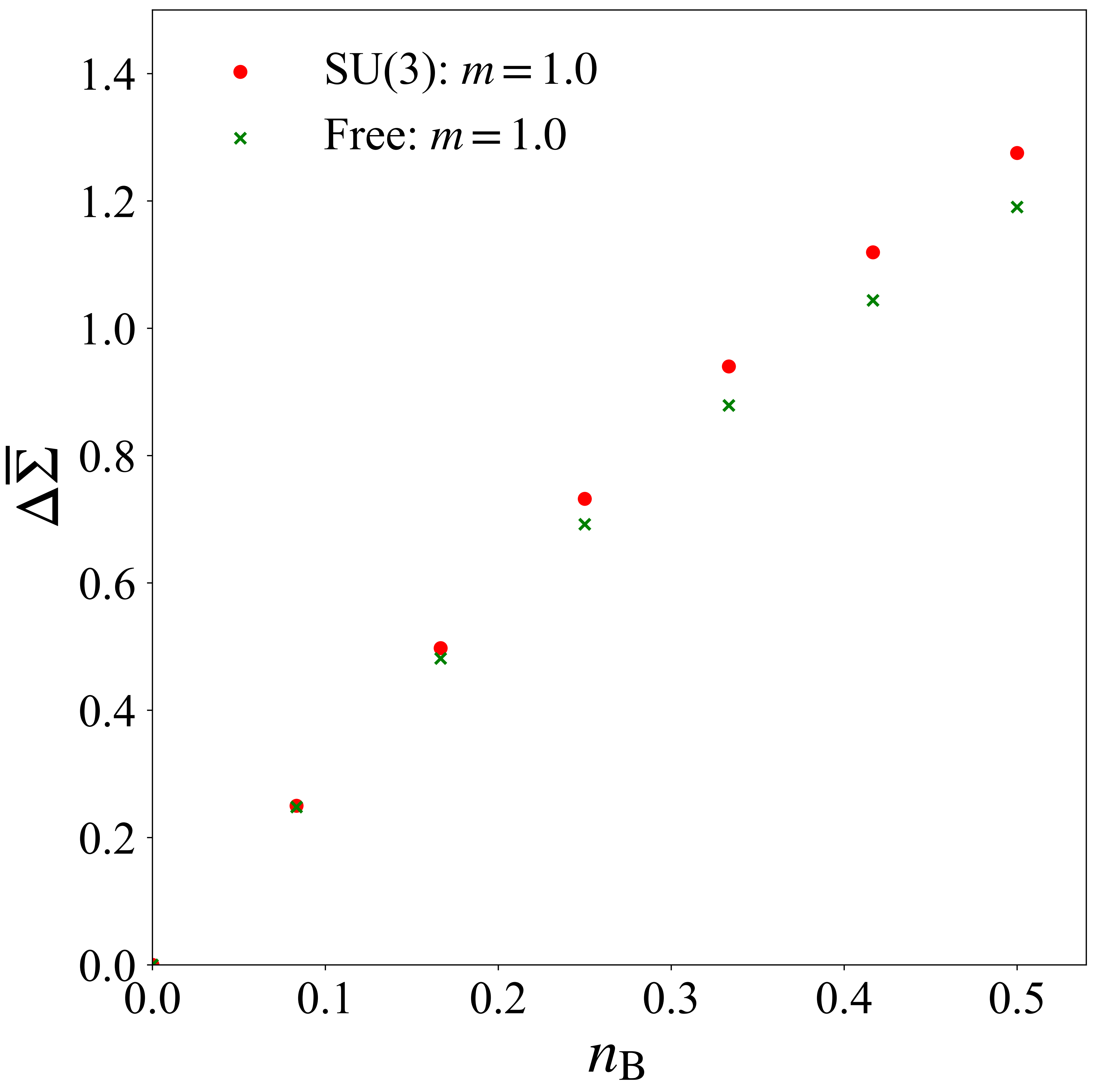
Amplitude

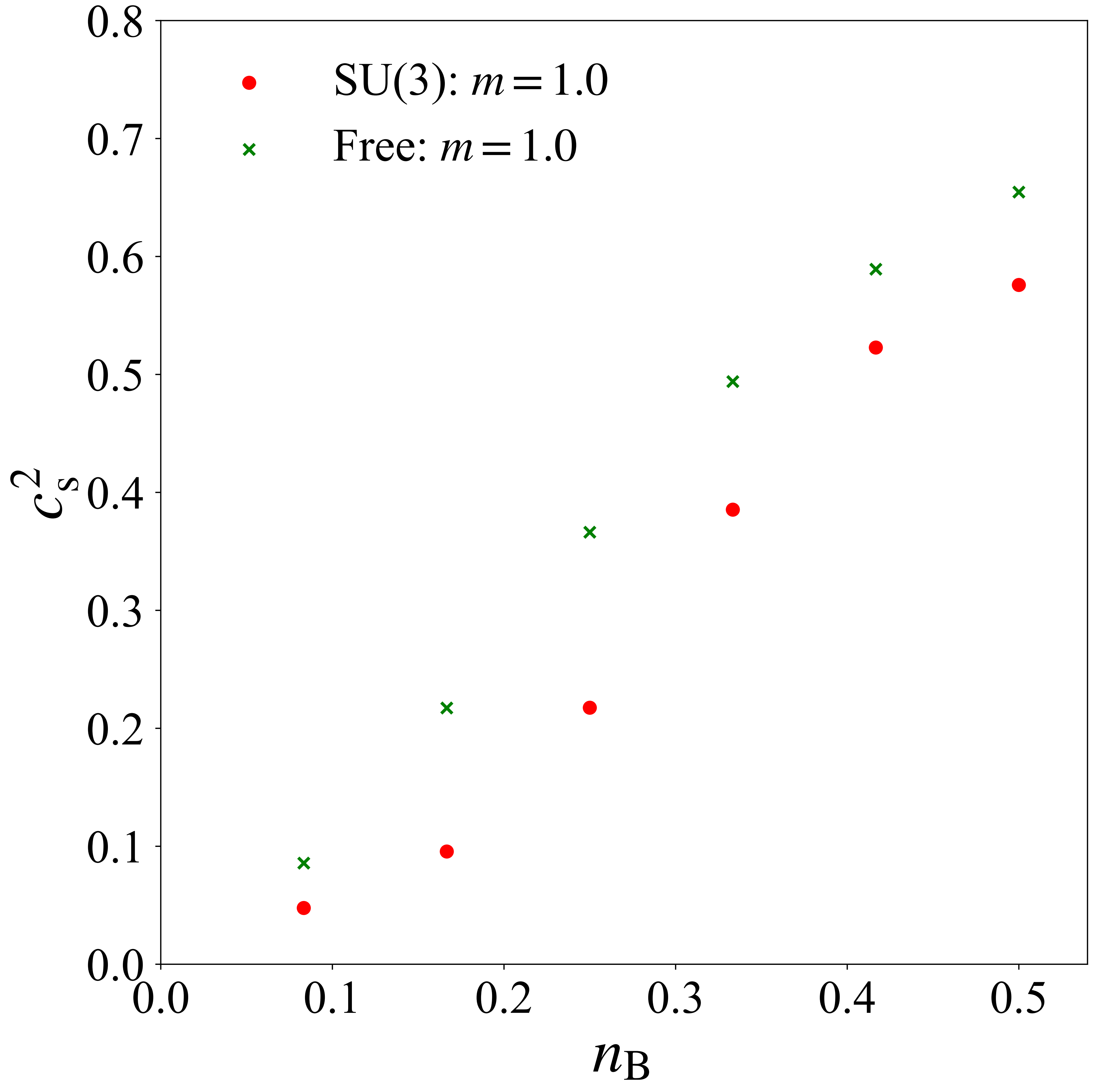
- The largest amplitude of the modulation with k in the previous slide.



- It is difficult to ascertain numerically whether the modulation persist in large V.

Chiral condensate and sound velocity

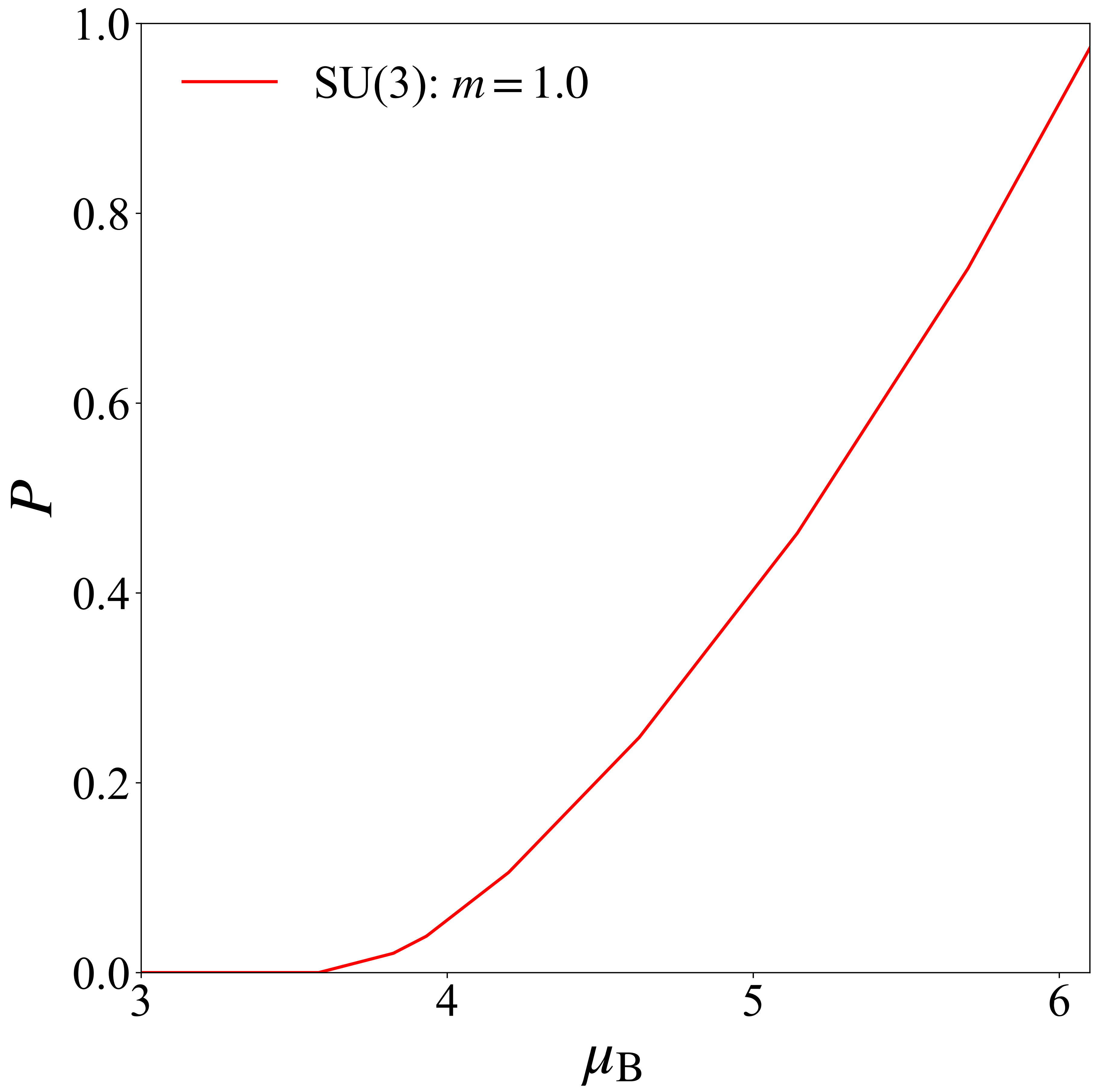


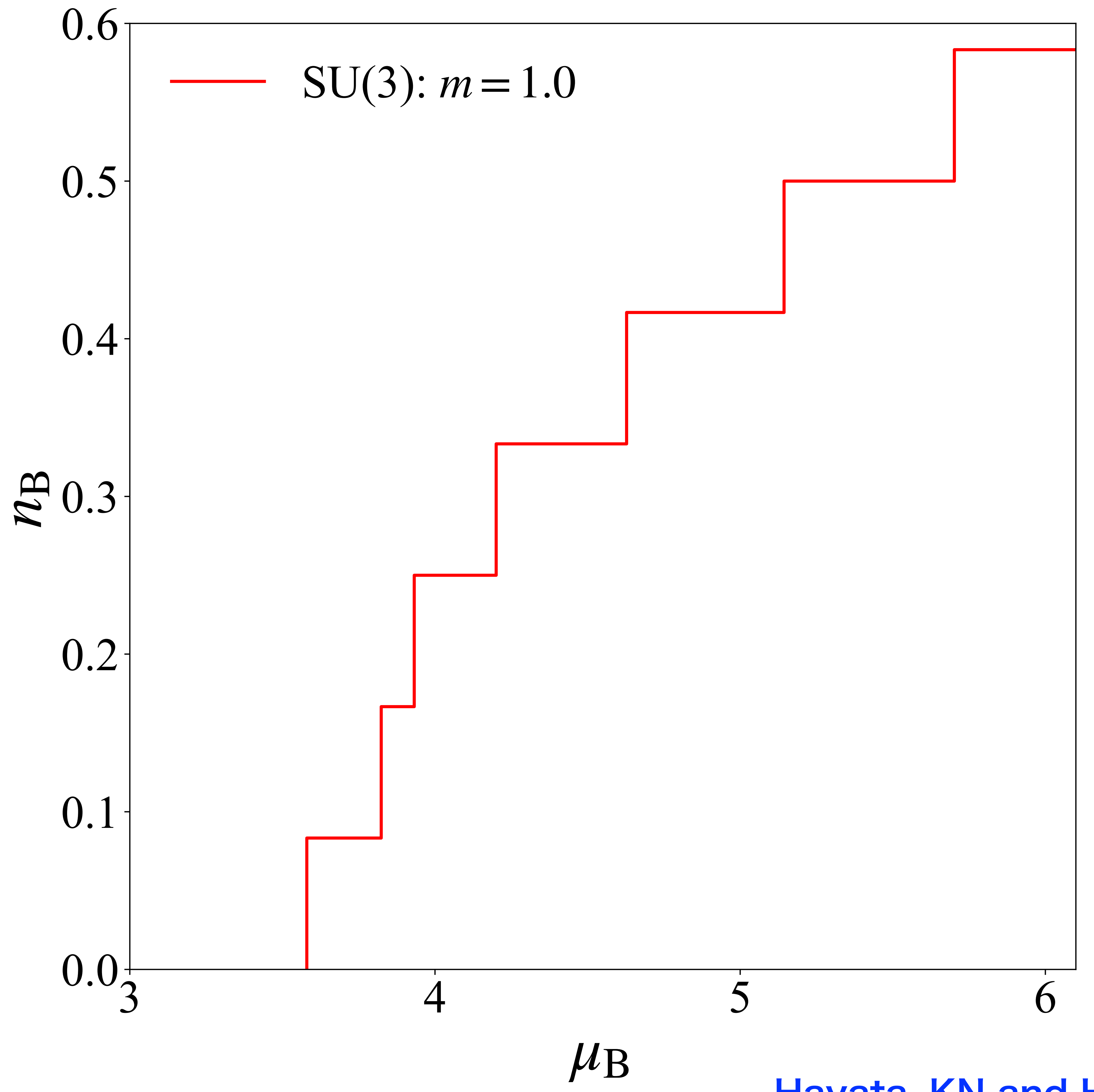


Back up for SU(3) results

Pressure and baryon number

- **N=48, w=2. Threshold value is 3.58.**





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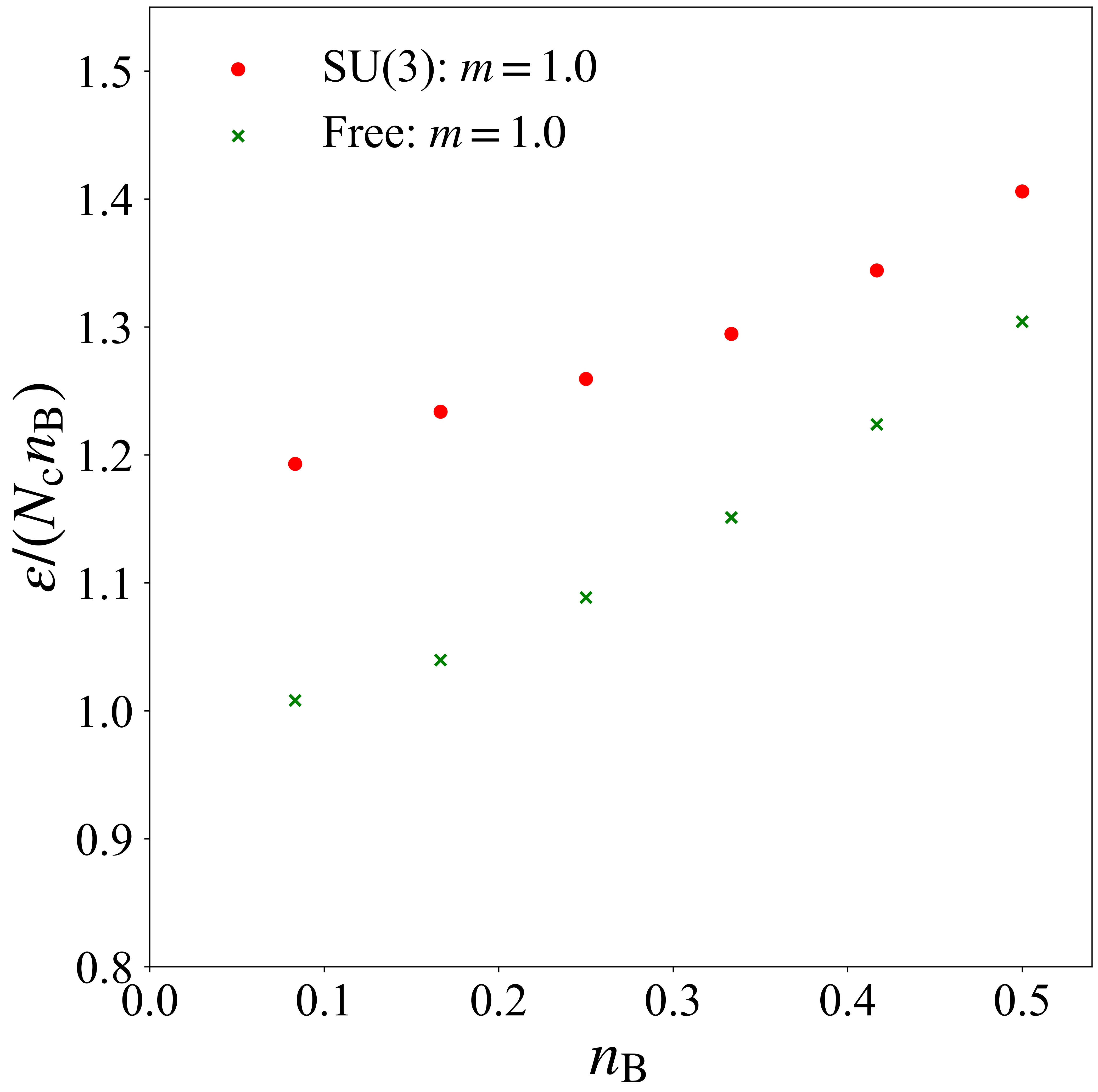
Energy per one quark



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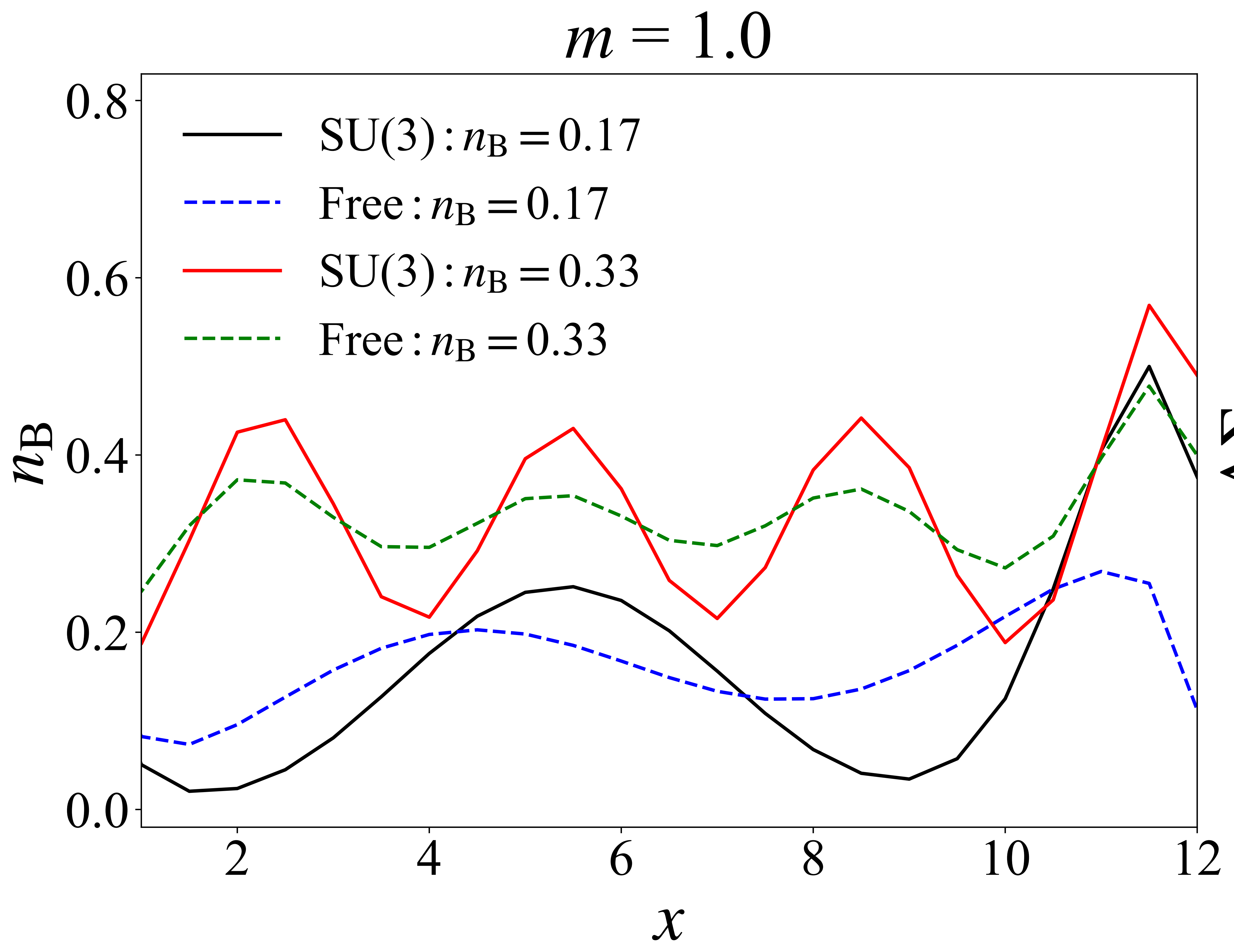
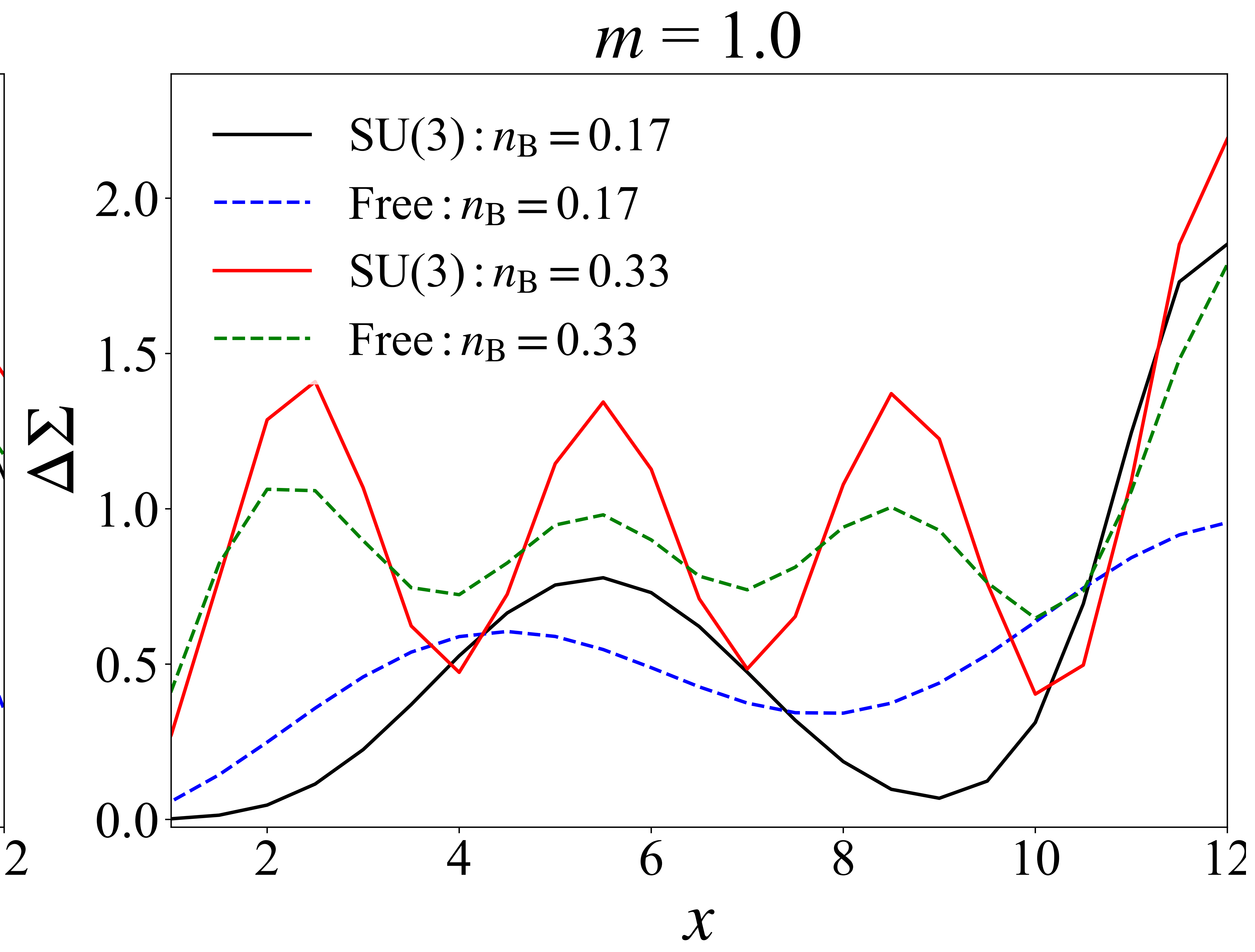
Spatial modulation





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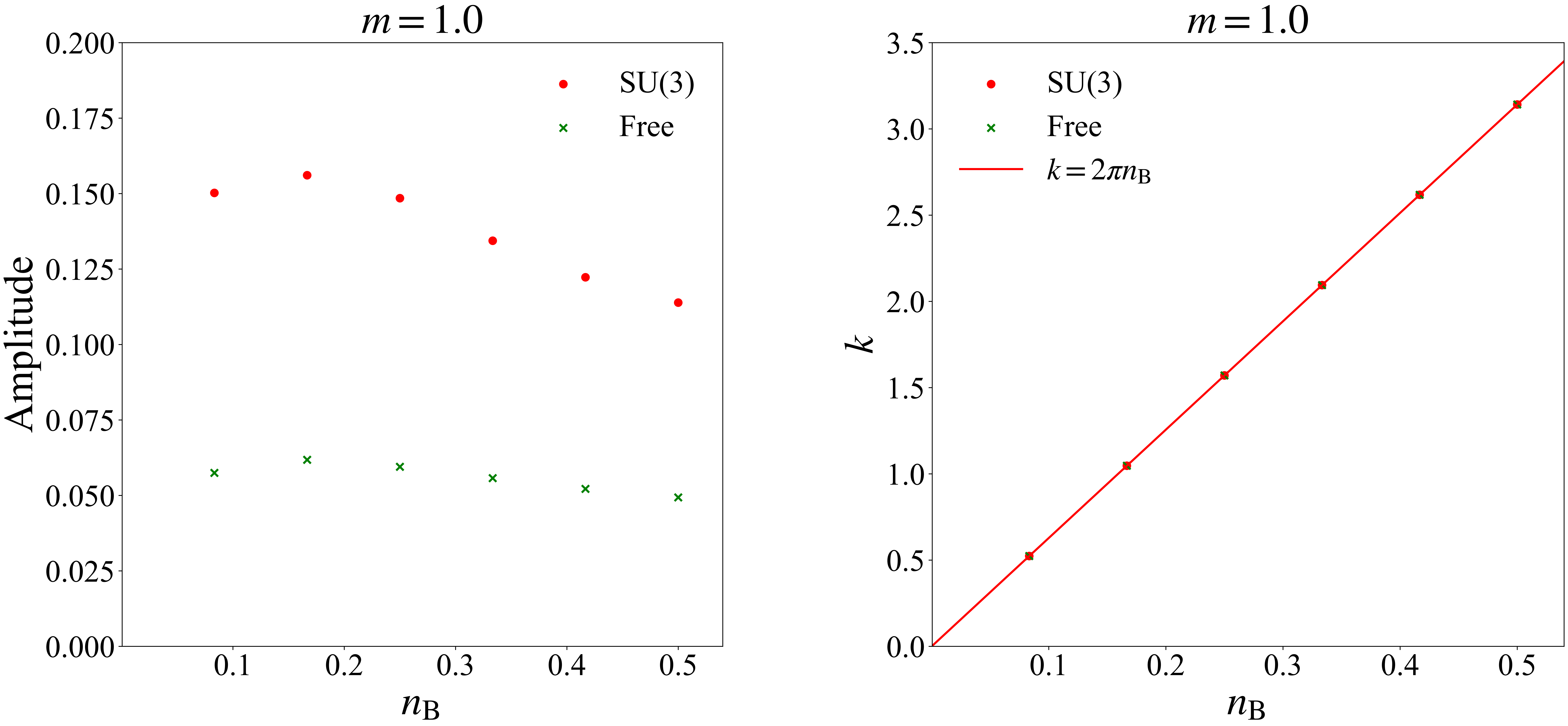
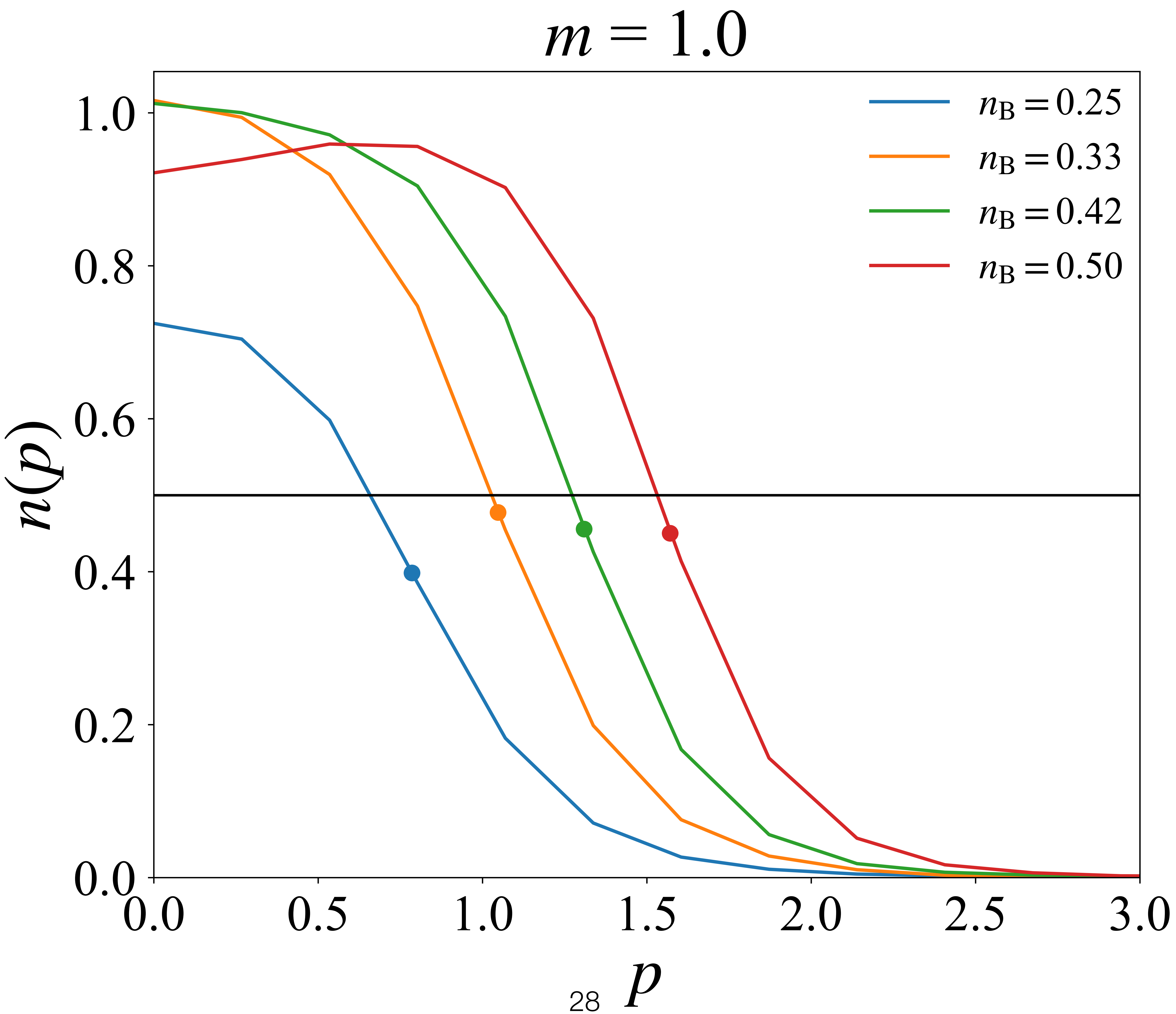
Wave number and amplitude



Quark distribution function



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