Wigner function of spin-1/2 particles in equilibrium

by Sudip Kumar Kar[1]

Collaborators: Samapan Bhadury[1], Zbigniew Drogosz[1], Wojciech Florkowski[1] and Valeriya Mykhaylova[1]

[1]Institute of Theoretical Physics, Jagiellonian University,

At

Erice: INTERNATIONAL SCHOOL OF NUCLEAR PHYSICS, 46th COURSE

On September 20, 2025

Based on ArXiv:2505.02657

NARODOWE CENTRUM NAUKI

Outline

Outline

- Introduction

Outline

- Introduction
- Equilibrium distribution functions for spin-1/2 particles

Outline

- Introduction
- Equilibrium distribution functions for spin-1/2 particles
- ‘Spin’ thermodynamics

Outline

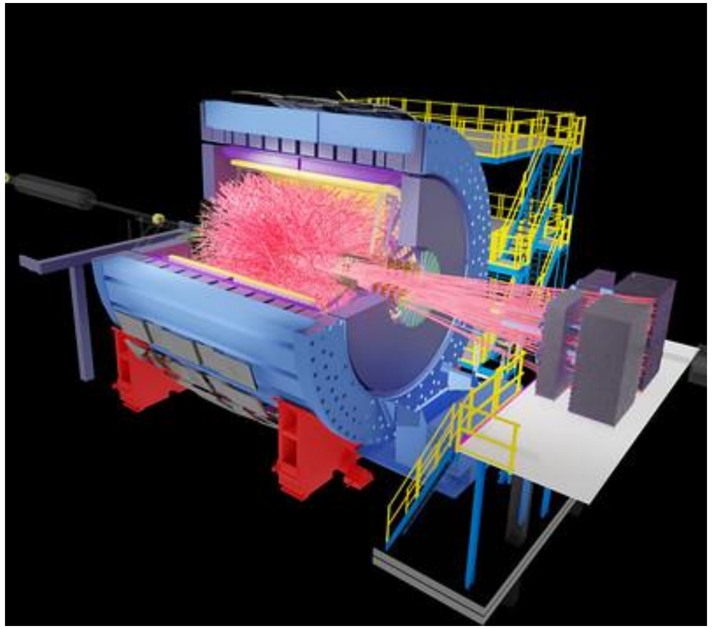
- Introduction
- Equilibrium distribution functions for spin-1/2 particles
- ‘Spin’ thermodynamics
- Compatibility with divergence type theories

Outline

- Introduction
- Equilibrium distribution functions for spin-1/2 particles
- ‘Spin’ thermodynamics
- Compatibility with divergence type theories
- Conclusions

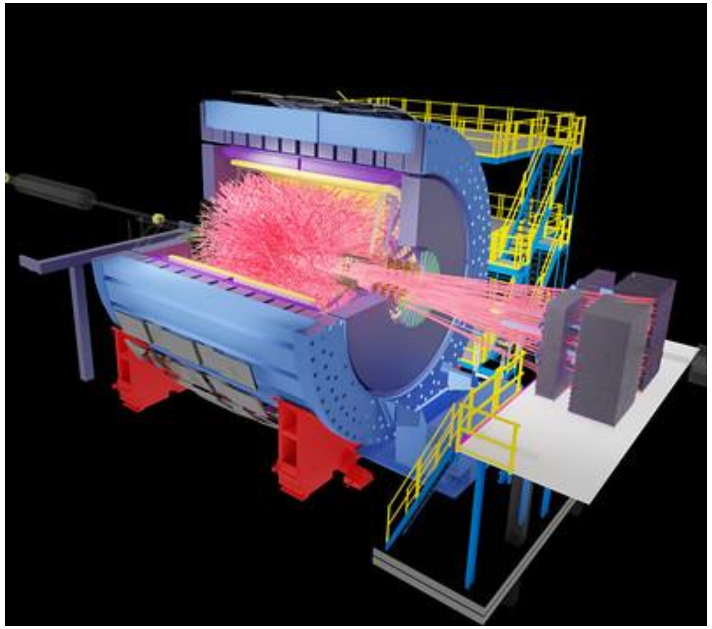
Spin hydrodynamics – a crucial tool to understand spin polarization in heavy ion collisions

Spin hydrodynamics – a crucial tool to understand spin polarization in heavy ion collisions



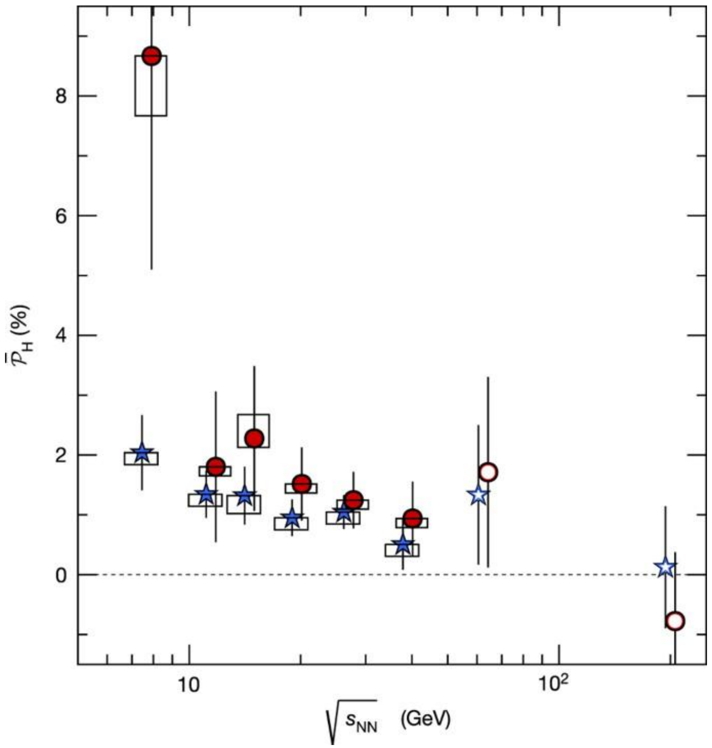
[Image from: Brookhaven National Laboratory]

Spin hydrodynamics – a crucial tool to understand spin polarization in heavy ion collisions



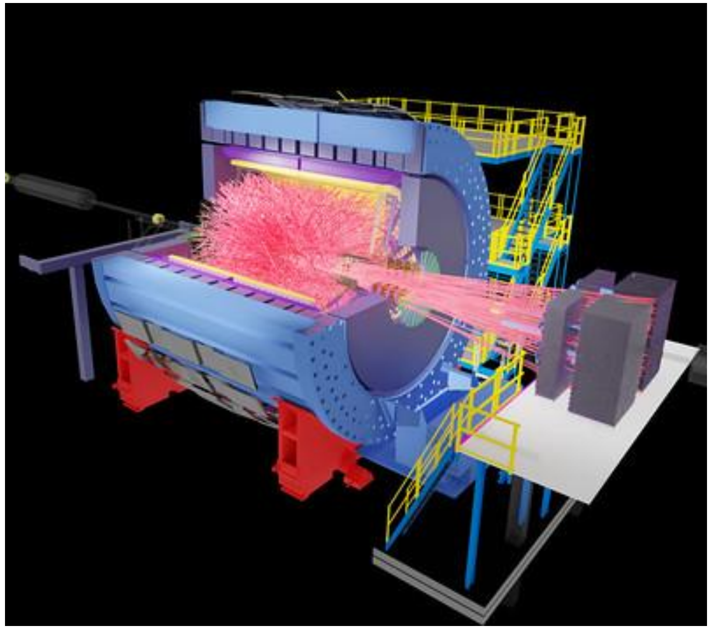
[Image from: Brookhaven National Laboratory]

Global spin polarization in Λ hyperons



[L. Adamczyk, et al., Nature 548 (2017) 62–65]

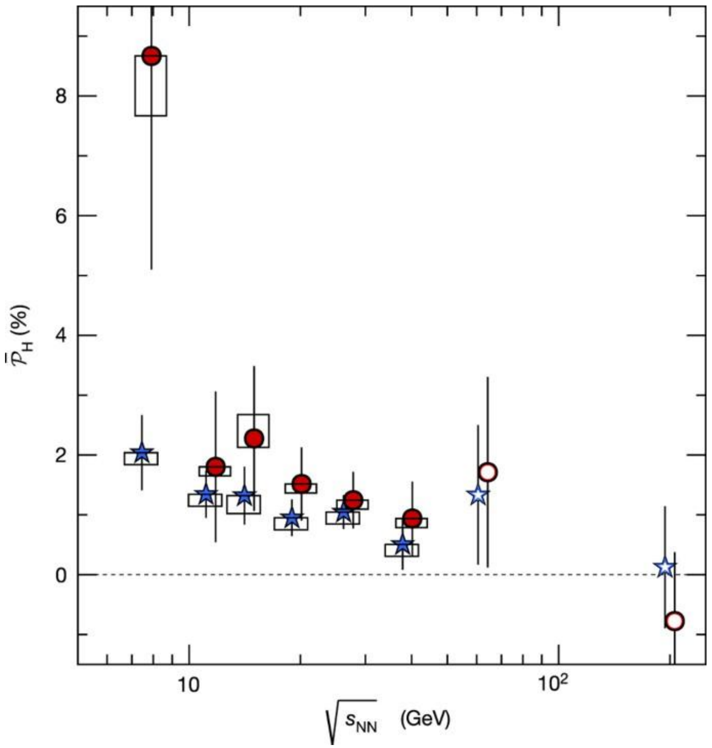
Spin hydrodynamics – a crucial tool to understand spin polarization in heavy ion collisions



[Image from: Brookhaven National Laboratory]

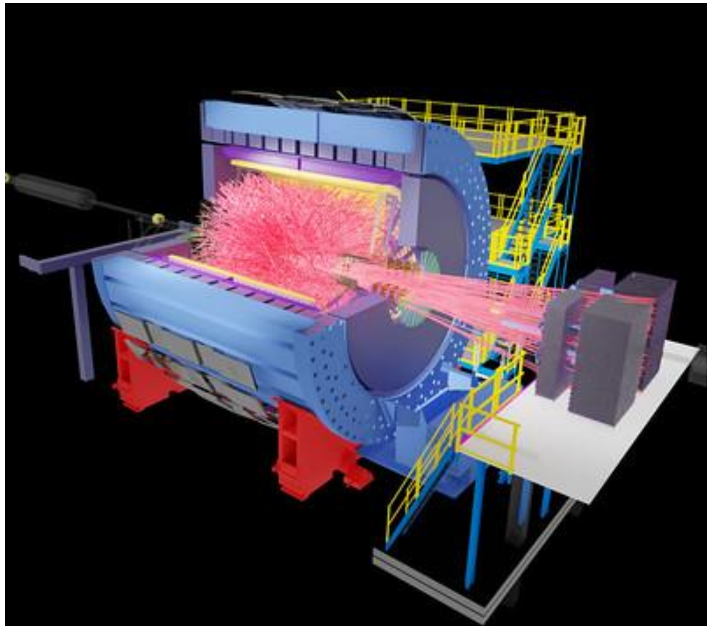
- Global spin polarization ☑

Global spin polarization in Λ hyperons



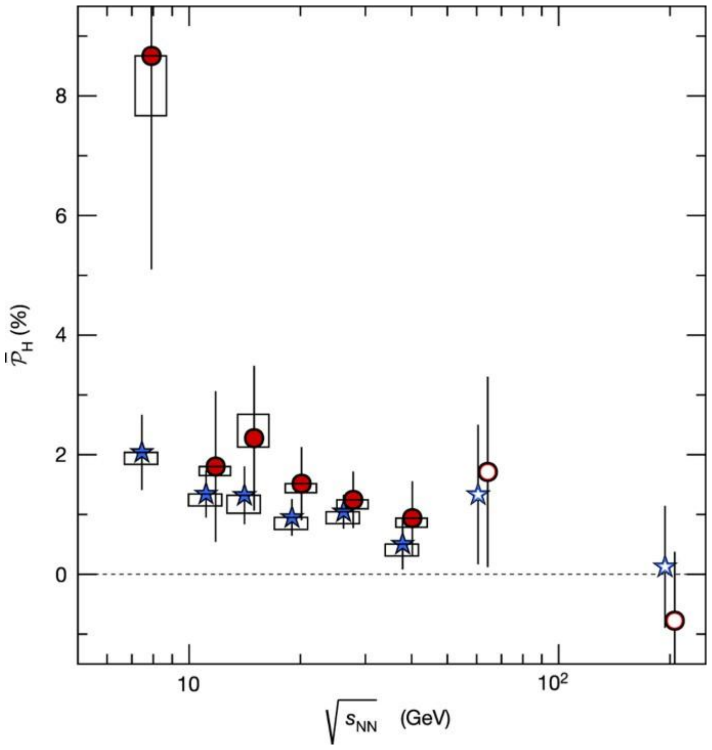
[L. Adamczyk, et al., Nature 548 (2017) 62–65]

Spin hydrodynamics – a crucial tool to understand spin polarization in heavy ion collisions



[Image from: Brookhaven National Laboratory]

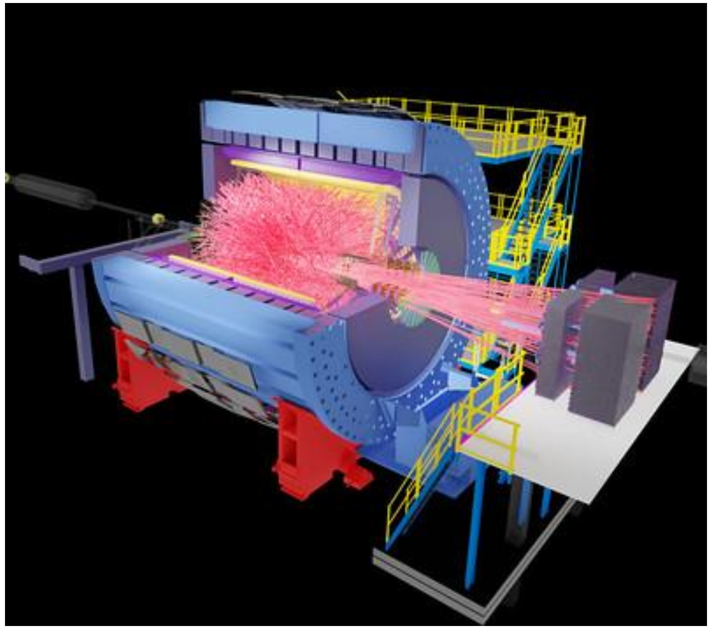
Global spin polarization in Λ hyperons



[L. Adamczyk, et al., Nature 548 (2017) 62–65]

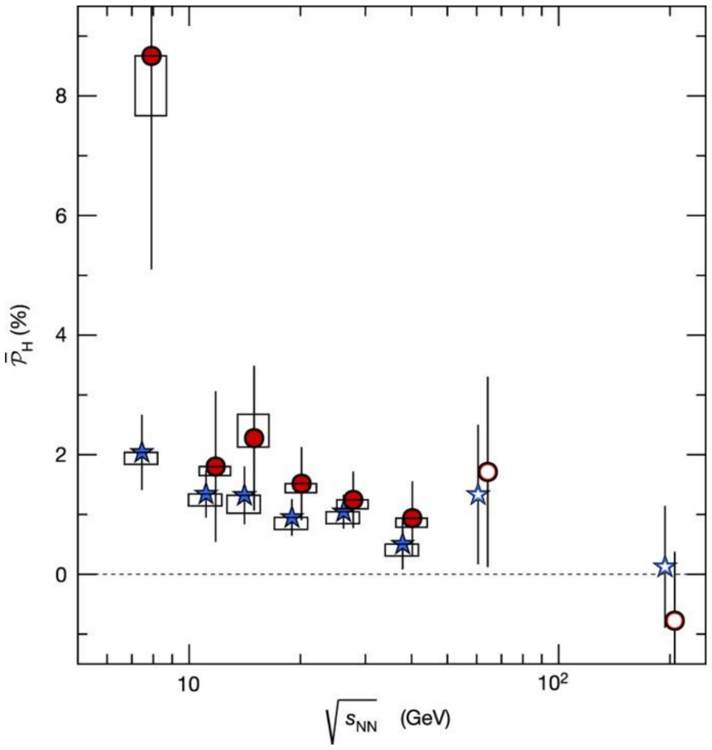
- Global spin polarization ☑
- Azimuthal dependence of the longitudinal spin polarization ?

Spin hydrodynamics – a crucial tool to understand spin polarization in heavy ion collisions



[Image from: Brookhaven National Laboratory]

Global spin polarization in Λ hyperons



[L. Adamczyk, et al., Nature 548 (2017) 62–65]

- Global spin polarization ☑
- Azimuthal dependence of the longitudinal spin polarization ?
- Need to build spin hydrodynamics: hydrodynamics with spin degrees of freedom

Wigner functions – a quantum approach to spin hydrodynamics

Wigner functions – a quantum approach to spin hydrodynamics

Spin-1/2 Fluid

Wigner functions – a quantum approach to spin hydrodynamics

Spin-1/2 Fluid

Classical spin

Phase space distribution function

(classical)

Wigner functions – a quantum approach to spin hydrodynamics

Spin-1/2 Fluid

Classical spin

Quantum spin

Phase space distribution function
(classical)

⟷

Quasiprobability distribution functions
(quantum)

Wigner functions – a quantum approach to spin hydrodynamics

Spin-1/2 Fluid

Classical spin | **Quantum spin**

Phase space distribution function *(classical)* ⟷ Quasiprobability distribution functions *(quantum)*

To formulate the quantum approach, we employ one such quasiprobability distribution function called the **Wigner function**.

[De Groot, Relativistic Kinetic Theory. Principles and Applications (1980)]

Wigner function for spin-1/2 particles

Wigner function for spin-1/2 particles

- The definition of the Wigner function for a spin-1/2 particle is

[De Groot, Relativistic Kinetic Theory. Principles and Applications (1980)]

$$\mathcal{W}^{+}_{\mathrm{eq}}(x,k) = \frac{1}{2}\sum_{r,s=1}^{2} \int dP\, \delta^{(4)}(k-p)\, u_r(p)\bar{u}_s(p) f^{+}_{rs}(x,p),$$

$$\mathcal{W}^{-}_{\mathrm{eq}}(x,k) = -\frac{1}{2}\sum_{r,s=1}^{2} \int dP\, \delta^{(4)}(k+p)\, v_s(p)\bar{v}_r(p) f^{-}_{rs}(x,p).$$

Wigner function for spin-1/2 particles

- The definition of the Wigner function for a spin-1/2 particle is

[De Groot, Relativistic Kinetic Theory. Principles and Applications (1980)]

$$\mathcal{W}^{+}_{\text{eq}}(x,k) = \frac{1}{2}\sum_{r,s=1}^{2}\int dP\,\delta^{(4)}(k-p)u_r(p)\bar{u}_s(p)f^{+}_{rs}(x,p),$$

$$\mathcal{W}^{-}_{\text{eq}}(x,k) = -\frac{1}{2}\sum_{r,s=1}^{2}\int dP\,\delta^{(4)}(k+p)v_s(p)\bar{v}_r(p)f^{-}_{rs}(x,p).$$

where,

$$[f^{+}(x,p)]_{rs} \equiv f^{+}_{rs}(x,p) = \frac{1}{2m}\,\bar{u}_r(p)X^{+}(x,p)u_s(p),$$

$$[f^{-}(x,p)]_{rs} \equiv f^{-}_{rs}(x,p) = -\frac{1}{2m}\,\bar{v}_s(p)X^{-}(x,p)v_r(p),$$

Wigner function for spin-1/2 particles

- The definition of the Wigner function for a spin-1/2 particle is

[De Groot, Relativistic Kinetic Theory. Principles and Applications (1980)]

$$\mathcal{W}^{+}_{\text{eq}}(x,k) = \frac{1}{2}\sum_{r,s=1}^{2}\int dP\,\delta^{(4)}(k-p)u_r(p)\bar{u}_s(p)f^{+}_{rs}(x,p),$$

$$\mathcal{W}^{-}_{\text{eq}}(x,k) = -\frac{1}{2}\sum_{r,s=1}^{2}\int dP\,\delta^{(4)}(k+p)v_s(p)\bar{v}_r(p)f^{-}_{rs}(x,p).$$

where,

Spin distribution functions (2 × 2 Hermitian matrices)

$$[f^{+}(x,p)]_{rs} \equiv f^{+}_{rs}(x,p) = \frac{1}{2m}\,\bar{u}_r(p)X^{+}(x,p)u_s(p),$$

$$[f^{-}(x,p)]_{rs} \equiv f^{-}_{rs}(x,p) = -\frac{1}{2m}\,\bar{v}_s(p)X^{-}(x,p)v_r(p),$$

Wigner function for spin-1/2 particles

- The definition of the Wigner function for a spin-1/2 particle is

[De Groot, Relativistic Kinetic Theory. Principles and Applications (1980)]

$$\mathcal{W}^{+}_{\text{eq}}(x,k) = \frac{1}{2}\sum_{r,s=1}^{2}\int dP\,\delta^{(4)}(k-p)u_r(p)\bar{u}_s(p)f^{+}_{rs}(x,p),$$

$$\mathcal{W}^{-}_{\text{eq}}(x,k) = -\frac{1}{2}\sum_{r,s=1}^{2}\int dP\,\delta^{(4)}(k+p)v_s(p)\bar{v}_r(p)f^{-}_{rs}(x,p).$$

where,

Spin distribution functions (2 × 2 Hermitian matrices)

$$\begin{cases} [f^{+}(x,p)]_{rs} \equiv f^{+}_{rs}(x,p) = \dfrac{1}{2m}\,\bar{u}_r(p)X^{+}(x,p)u_s(p), \\ [f^{-}(x,p)]_{rs} \equiv f^{-}_{rs}(x,p) = -\dfrac{1}{2m}\,\bar{v}_s(p)X^{-}(x,p)v_r(p), \end{cases}$$

- $X^{\pm}$ are 4×4 matrices called spinor distribution functions

Conserved currents from Wigner function

Conserved currents from Wigner function

$X^{\pm}$

Conserved currents from Wigner function

$X^{\pm} \longrightarrow f^{\pm}_{rs}(x,p)$

Conserved currents from Wigner function

$X^{\pm} \longrightarrow f^{\pm}_{rs}(x,p) \longrightarrow \mathcal{W}^{+}_{\text{eq}}(x,k)$

Conserved currents from Wigner function

$X^{\pm} \longrightarrow f^{\pm}_{rs}(x,p) \longrightarrow \mathcal{W}^{+}_{\text{eq}}(x,k) \longrightarrow$ Conserved currents

Conserved currents from Wigner function

$X^{\pm} \longrightarrow f^{\pm}_{rs}(x,p) \longrightarrow \mathcal{W}^{+}_{\text{eq}}(x,k) \longrightarrow$ Conserved currents

The conserved currents in terms of the spin distribution function are given as
[De Groot, Relativistic Kinetic Theory. Principles and Applications (1980)]

Conserved current	Lagrange multipliers
$N^{\mu}(x) = \sum_{r=1}^{2} \int dP\, p^{\mu}[f^{+}_{rr}(x,p) - f^{-}_{rr}(x,p)]$	$\xi = \frac{\mu}{T}$
$T^{\mu\nu}(x) = \sum_{r=1}^{2} \int dP\, p^{\mu}p^{\nu}[f^{+}_{rr}(x,p) + f^{-}_{rr}(x,p)]$	$\beta^{\mu} = \frac{u^{\mu}}{T}$
$S^{\lambda,\mu\nu}(x) = \frac{1}{2}\sum_{r,s=1}^{2} \int dP\, p^{\lambda}[\sigma^{+\mu\nu}_{sr} f^{+}_{rs}(x,p) + \sigma^{-\mu\nu}_{sr} f^{-}_{rs}(x,p)]$	$\omega^{\mu\nu}$

Where,

$$\sigma^{+\mu\nu}_{rs} = \frac{\bar{u}_r(p)\sigma^{\mu\nu}u_s(p)}{2M}$$

$$\sigma^{-\mu\nu}_{rs} = \frac{\bar{v}_s(p)\sigma^{\mu\nu}v_r(p)}{2M}$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu},\gamma^{\nu}]$$

Conserved currents from Wigner function

$X^{\pm} \longrightarrow f^{\pm}_{rs}(x,p) \longrightarrow \mathcal{W}^{+}_{\text{eq}}(x,k) \longrightarrow$ Conserved currents

The conserved currents in terms of the spin distribution function are given as
[De Groot, Relativistic Kinetic Theory. Principles and Applications (1980)]

Conserved current	Lagrange multipliers
$N^{\mu}(x) = \sum_{r=1}^{2} \int dP\, p^{\mu}[f^{+}_{rr}(x,p) - f^{-}_{rr}(x,p)]$	$\xi = \frac{\mu}{T}$
$T^{\mu\nu}(x) = \sum_{r=1}^{2} \int dP\, p^{\mu}p^{\nu}[f^{+}_{rr}(x,p) + f^{-}_{rr}(x,p)]$	$\beta^{\mu} = \frac{u^{\mu}}{T}$
$S^{\lambda,\mu\nu}(x) = \frac{1}{2}\sum_{r,s=1}^{2} \int dP\, p^{\lambda}[\sigma^{+\mu\nu}_{sr} f^{+}_{rs}(x,p) + \sigma^{-\mu\nu}_{sr} f^{-}_{rs}(x,p)]$	$\omega^{\mu\nu}$

Where,

$$\sigma^{+\mu\nu}_{rs} = \frac{\bar{u}_r(p)\sigma^{\mu\nu}u_s(p)}{2M}$$

$$\sigma^{-\mu\nu}_{rs} = \frac{\bar{v}_s(p)\sigma^{\mu\nu}v_r(p)}{2M}$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu},\gamma^{\nu}]$$

The particle current in terms of the spin distribution function is given as

$$\mathcal{N}^{\mu}(x) = \sum_{r=1}^{2} \int dP\, p^{\mu}[f^{+}_{rr}(x,p) + f^{-}_{rr}(x,p)],$$

Anti-symmetric nature of the spin Lagrange multiplier

Anti-symmetric nature of the spin Lagrange multiplier

$$S^{\lambda,\mu\nu}(x)$$

Anti symmetric in μ and ν

Anti-symmetric nature of the spin Lagrange multiplier

$S^{\lambda,\mu\nu}(x)$ ⟷ $\omega^{\mu\nu}$

Anti symmetric in μ and ν

Anti symmetric in μ and ν

Anti-symmetric nature of the spin Lagrange multiplier

$S^{\lambda,\mu\nu}(x)$ ⟷ $\omega^{\mu\nu}$

Anti symmetric in μ and ν

Anti symmetric in μ and ν

We construct the matrix of this 'spin' chemical potential in a manner similar to electrodynamics

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & e^1 & e^2 & e^3 \\ -e^1 & 0 & -b^3 & b^2 \\ -e^2 & b^3 & 0 & -b^1 \\ -e^3 & -b^2 & b^1 & 0 \end{bmatrix},$$

Anti-symmetric nature of the spin Lagrange multiplier

$S^{\lambda,\mu\nu}(x)$ ⟷ $\omega^{\mu\nu}$

Anti symmetric in μ and ν

Anti symmetric in μ and ν

We construct the matrix of this ‘spin’ chemical potential in a manner similar to electrodynamics

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & e^1 & e^2 & e^3 \\ -e^1 & 0 & -b^3 & b^2 \\ -e^2 & b^3 & 0 & -b^1 \\ -e^3 & -b^2 & b^1 & 0 \end{bmatrix},$$

With electric-like and magnetic-like vectors given as follows

$$\mathbf{e} = (e^1, e^2, e^3) \qquad \mathbf{b} = (b^1, b^2, b^3)$$

The traditional distribution function: a review of its use and limitations

The traditional distribution function: a review of its use and limitations

- A model of the equilibrium spin distribution function that has been widely was given by *[Becattini et al., Annals Phys. 338 (2013)]*

The traditional distribution function: a review of its use and limitations

- A model of the equilibrium spin distribution function that has been widely was given by *[Becattini et al., Annals Phys. 338 (2013)]*

$$X^{\pm} = \exp\left[\pm\xi(x) - \beta_{\mu}(x)p^{\mu} \pm \frac{1}{2}\omega_{\mu\nu}(x)\Sigma^{\mu\nu}\right],$$

The traditional distribution function: a review of its use and limitations

- A model of the equilibrium spin distribution function that has been widely was given by *[Becattini et al., Annals Phys. 338 (2013)]*

$$X^{\pm} = \exp\left[\pm\xi(x) - \beta_{\mu}(x)p^{\mu} \pm \frac{1}{2}\omega_{\mu\nu}(x)\Sigma^{\mu\nu}\right],$$

- The polarization vector in particle rest frame, given by

The traditional distribution function: a review of its use and limitations

- A model of the equilibrium spin distribution function that has been widely was given by *[Becattini et al., Annals Phys. 338 (2013)]*

$$X^{\pm} = \exp\left[\pm\xi(x) - \beta_\mu(x)p^\mu \pm \frac{1}{2}\omega_{\mu\nu}(x)\Sigma^{\mu\nu}\right],$$

- The polarization vector in particle rest frame, given by

$$\mathbf{P} = \frac{1}{2}\frac{\mathrm{tr}_2(f^{\pm}\boldsymbol{\sigma})}{\mathrm{tr}_2(f^{\pm})}$$

The traditional distribution function: a review of its use and limitations

- A model of the equilibrium spin distribution function that has been widely was given by *[Becattini et al., Annals Phys. 338 (2013)]*

$$X^{\pm} = \exp\left[\pm\xi(x) - \beta_{\mu}(x)p^{\mu} \pm \frac{1}{2}\omega_{\mu\nu}(x)\Sigma^{\mu\nu}\right],$$

- The polarization vector in particle rest frame, given by

$$\mathbf{P} = \frac{1}{2}\frac{\mathrm{tr}_2(f^{\pm}\boldsymbol{\sigma})}{\mathrm{tr}_2(f^{\pm})}$$

- Shows that, [$*$ in the subscript implies the quantity is in a frame where $p^{\mu} = (m, 0,0,0)$] *[Florkowski et al. PRD 97, 116017 (2018)]*

The traditional distribution function: a review of its use and limitations

- A model of the equilibrium spin distribution function that has been widely was given by [Becattini et al., Annals Phys. 338 (2013)]

$$X^{\pm} = \exp\left[\pm\xi(x) - \beta_{\mu}(x)p^{\mu} \pm \frac{1}{2}\omega_{\mu\nu}(x)\Sigma^{\mu\nu}\right],$$

- The polarization vector in particle rest frame, given by

$$\mathbf{P} = \frac{1}{2}\frac{\mathrm{tr}_2(f^{\pm}\boldsymbol{\sigma})}{\mathrm{tr}_2(f^{\pm})}$$

- Shows that, [$*$ in the subscript implies the quantity is in a frame where $p^{\mu} = (m, 0,0,0)$] [Florkowski et al. PRD 97, 116017 (2018)]

$$\mathbf{P} = \frac{-1}{2}\tanh\left[\frac{\sqrt{\mathbf{b}_* \cdot \mathbf{b}_* - \mathbf{e}_* \cdot \mathbf{e}_*}}{2}\right]\frac{\mathbf{b}_*}{\sqrt{\mathbf{b}_* \cdot \mathbf{b}_* - \mathbf{e}_* \cdot \mathbf{e}_*}},$$

The traditional distribution function: a review of its use and limitations

- A model of the equilibrium spin distribution function that has been widely was given by [Becattini et al., Annals Phys. 338 (2013)]

$$X^{\pm} = \exp\left[\pm\xi(x) - \beta_\mu(x)p^\mu \pm \frac{1}{2}\omega_{\mu\nu}(x)\Sigma^{\mu\nu}\right],$$

Where,
$\Sigma^{\mu\nu} = \sigma^{\mu\nu}/2 = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

- The polarization vector in particle rest frame, given by

$$\mathbf{P} = \frac{1}{2}\frac{\mathrm{tr}_2(f^{\pm}\boldsymbol{\sigma})}{\mathrm{tr}_2(f^{\pm})}$$

- Shows that, [$*$ in the subscript implies the quantity is in a frame where $p^\mu = (m, 0,0,0)$] [Florkowski et al. PRD 97, 116017 (2018)]

$$\mathbf{P} = \frac{-1}{2}\tanh\left[\frac{\sqrt{\mathbf{b}_*\cdot\mathbf{b}_* - \mathbf{e}_*\cdot\mathbf{e}_*}}{2}\right]\frac{\mathbf{b}_*}{\sqrt{\mathbf{b}_*\cdot\mathbf{b}_* - \mathbf{e}_*\cdot\mathbf{e}_*}},$$

may exceed the desired range of $0 \le |\mathbf{P}| \le 1/2$, restricting its use to only cases where $\mathbf{b}$ and $\mathbf{e}$ are small.

A new spin distribution function derived from general principles

A new spin distribution function derived from general principles

Any 2×2 Hermitian matrix may be written as

$$f^{\pm}_{rs}(x, p) = f^{\pm}_{0}(x, p)[\delta_{rs} + \boldsymbol{\zeta}^{\pm}_{*} \cdot \boldsymbol{\sigma}_{rs}],$$

A new spin distribution function derived from general principles

Any 2×2 Hermitian matrix may be written as

$$f_{rs}^{\pm}(x, p) = f_0^{\pm}(x, p)[\delta_{rs} + \boldsymbol{\zeta}_*^{\pm} \cdot \boldsymbol{\sigma}_{rs}],$$

Where, $\boldsymbol{\sigma}_{rs}$: Pauli matrices,

A new spin distribution function derived from general principles

Any 2×2 Hermitian matrix may be written as

$$f_{rs}^{\pm}(x, p) = f_0^{\pm}(x, p)[\delta_{rs} + \boldsymbol{\zeta}_*^{\pm} \cdot \boldsymbol{\sigma}_{rs}],$$

Where, $\boldsymbol{\sigma}_{rs}$: Pauli matrices,

$f_0^{\pm}(x, p)$: spin averaged phase space density

A new spin distribution function derived from general principles

Any 2×2 Hermitian matrix may be written as

$$f_{rs}^{\pm}(x, p) = f_0^{\pm}(x, p)[\delta_{rs} + \boldsymbol{\zeta}_*^{\pm} \cdot \boldsymbol{\sigma}_{rs}],$$

Where, $\boldsymbol{\sigma}_{rs}$: Pauli matrices,

$f_0^{\pm}(x, p)$: spin averaged phase space density

$\boldsymbol{\zeta}_*^{\pm}(x, \mathbf{p})$: is a vector such that $0 \leq |\boldsymbol{\zeta}_*^{\pm}| \leq 1$

A new spin distribution function derived from general principles

Any 2×2 Hermitian matrix may be written as

$$f_{rs}^{\pm}(x, p) = f_0^{\pm}(x, p)[\delta_{rs} + \boldsymbol{\zeta}_*^{\pm} \cdot \boldsymbol{\sigma}_{rs}],$$

Where, $\boldsymbol{\sigma}_{rs}$: Pauli matrices,

$f_0^{\pm}(x, p)$: spin averaged phase space density

$\boldsymbol{\zeta}_*^{\pm}(x, \mathbf{p})$: is a vector such that $0 \leq |\boldsymbol{\zeta}_*^{\pm}| \leq 1$

Recall that,

$$\mathbf{P} = \frac{1}{2}\frac{\mathrm{tr}_2(f^{\pm}\boldsymbol{\sigma})}{\mathrm{tr}_2(f^{\pm})}$$

So $\mathbf{P} = \boldsymbol{\zeta}_*^{\pm}/\mathbf{2}$

A new spin distribution function derived from general principles

Any 2 × 2 Hermitian matrix may be written as

$$f_{rs}^{\pm}(x, p) = f_0^{\pm}(x, p)[\delta_{rs} + \boldsymbol{\zeta}_*^{\pm} \cdot \boldsymbol{\sigma}_{rs}],$$

Where, $\boldsymbol{\sigma}_{rs}$: Pauli matrices,

$f_0^{\pm}(x, p)$: spin averaged phase space density

$\boldsymbol{\zeta}_*^{\pm}(x, \mathbf{p})$: is a vector such that $0 \leq |\boldsymbol{\zeta}_*^{\pm}| \leq 1$

Define $\zeta_*^{\pm\mu} = (0, \boldsymbol{\zeta}_*^{\pm})$

Recall that,

$$\mathbf{P} = \frac{1}{2}\frac{\mathrm{tr}_2(f^{\pm}\boldsymbol{\sigma})}{\mathrm{tr}_2(f^{\pm})}$$

So $\mathbf{P} = \boldsymbol{\zeta}_*^{\pm}/\mathbf{2}$

A new spin distribution function derived from general principles

Any 2 × 2 Hermitian matrix may be written as

$$f^{\pm}_{rs}(x, p) = f^{\pm}_0(x, p)[\delta_{rs} + \boldsymbol{\zeta}^{\pm}_* \cdot \boldsymbol{\sigma}_{rs}],$$

Recall that,

$$\mathbf{P} = \frac{1}{2}\frac{\mathrm{tr}_2(f^{\pm}\boldsymbol{\sigma})}{\mathrm{tr}_2(f^{\pm})}$$

So $\mathbf{P} = \boldsymbol{\zeta}^{\pm}_*/\mathbf{2}$

Where, $\boldsymbol{\sigma}_{rs}$: Pauli matrices,

$f^{\pm}_0(x, p)$: spin averaged phase space density

$\boldsymbol{\zeta}^{\pm}_*(x, \mathbf{p})$: is a vector such that $0 \leq |\boldsymbol{\zeta}^{\pm}_*| \leq 1$

Define $\zeta^{\pm\mu}_* = (0, \boldsymbol{\zeta}^{\pm}_*)$ —Boost→ $\zeta^{\mu}_{\pm} = \Lambda^{\mu}{}_{\nu}(\mathbf{v}_p)\zeta^{\nu}_{\pm *} = \left(\frac{\mathbf{p}\cdot\boldsymbol{\zeta}^{\pm}_*}{m}, \boldsymbol{\zeta}^{\pm}_* + \frac{\mathbf{p}\cdot\boldsymbol{\zeta}^{\pm}_*}{m(E_p + m)}\mathbf{p}\right),$

A new spin distribution function derived from general principles

Any 2 × 2 Hermitian matrix may be written as

$$f_{rs}^{\pm}(x, p) = f_0^{\pm}(x, p)[\delta_{rs} + \boldsymbol{\zeta}_*^{\pm} \cdot \boldsymbol{\sigma}_{rs}],$$

Recall that,

$$\mathbf{P} = \frac{1}{2}\frac{\mathrm{tr}_2(f^{\pm}\boldsymbol{\sigma})}{\mathrm{tr}_2(f^{\pm})}$$

So $\mathbf{P} = \boldsymbol{\zeta}_*^{\pm}/\mathbf{2}$

Where, $\boldsymbol{\sigma}_{rs}$: Pauli matrices,

$f_0^{\pm}(x, p)$: spin averaged phase space density

$\boldsymbol{\zeta}_*^{\pm}(x, \mathbf{p})$: is a vector such that $0 \leq |\boldsymbol{\zeta}_*^{\pm}| \leq 1$

Define $\zeta_*^{\pm\mu} = (0, \boldsymbol{\zeta}_*^{\pm})$ —Boost→ $\zeta_{\pm}^{\mu} = \Lambda^{\mu}{}_{\nu}(\mathbf{v}_p)\zeta_{\pm *}^{\nu} = \left(\frac{\mathbf{p}\cdot\boldsymbol{\zeta}_*^{\pm}}{m}, \boldsymbol{\zeta}_*^{\pm} + \frac{\mathbf{p}\cdot\boldsymbol{\zeta}_*^{\pm}}{m(E_p + m)}\mathbf{p}\right),$

Furthermore,

$$\bar{u}_r(p)\gamma_5\zeta_{\mu}^{+}\gamma^{\mu}u_s(p) = 2m\,\boldsymbol{\zeta}_*^{+} \cdot \boldsymbol{\sigma}_{rs},$$
$$\bar{v}_s(p)\gamma_5\zeta_{\mu}^{-}\gamma^{\mu}v_r(p) = -2m\,\boldsymbol{\zeta}_*^{-} \cdot \boldsymbol{\sigma}_{rs}.$$

The new spin distribution function

The new spin distribution function

Note that, $\exp\left(\gamma_5 \gamma_\mu a_\pm^\mu\right) =$

$$\cosh\sqrt{-a_\pm^2}\left[1 + \frac{\gamma_5 \gamma_\mu a_\pm^\mu}{\sqrt{-a_\pm^2}}\tanh\sqrt{-a_\pm^2}\right].$$

Where, $(a_\pm^2 < 0)$.

The new spin distribution function

Note that, $\exp(\gamma_5 \gamma_\mu a_\pm^\mu) =$

$$\cosh\sqrt{-a_\pm^2}\left[1 + \frac{\gamma_5\gamma_\mu a_\pm^\mu}{\sqrt{-a_\pm^2}}\tanh\sqrt{-a_\pm^2}\right].$$

Where, $(a_\pm^2 < 0)$.

And

$$\bar{u}_r(p)\gamma_5\zeta_\mu^+\gamma^\mu u_s(p) = 2m\,\boldsymbol{\zeta}_*^+ \cdot \boldsymbol{\sigma}_{rs},$$
$$\bar{v}_s(p)\gamma_5\zeta_\mu^-\gamma^\mu v_r(p) = -2m\,\boldsymbol{\zeta}_*^- \cdot \boldsymbol{\sigma}_{rs}.$$

The new spin distribution function

Note that, $\exp\left(\gamma_5\gamma_\mu a_\pm^\mu\right) =$

$$\cosh\sqrt{-a_\pm^2}\left[1 + \frac{\gamma_5\gamma_\mu a_\pm^\mu}{\sqrt{-a_\pm^2}}\tanh\sqrt{-a_\pm^2}\right].$$

Where, $(a_\pm^2 < 0)$.

And

$$\bar{u}_r(p)\gamma_5\zeta_\mu^+\gamma^\mu u_s(p) = 2m\,\boldsymbol{\zeta}_*^+ \cdot \boldsymbol{\sigma}_{rs},$$
$$\bar{v}_s(p)\gamma_5\zeta_\mu^-\gamma^\mu v_r(p) = -2m\,\boldsymbol{\zeta}_*^- \cdot \boldsymbol{\sigma}_{rs}.$$

$$X_s^\pm(x, p) = \exp\left[\pm\xi(x) - \beta_\mu(x)p^\mu + \gamma_5\gamma_\mu a^\mu\right].$$

The new spin distribution function

Note that, $\exp\left(\gamma_5 \gamma_\mu a_\pm^\mu\right) =$

$$\cosh\sqrt{-a_\pm^2}\left[1 + \frac{\gamma_5 \gamma_\mu a_\pm^\mu}{\sqrt{-a_\pm^2}} \tanh\sqrt{-a_\pm^2}\right].$$

Where, $(a_\pm^2 < 0)$.

And

$$\bar{u}_r(p)\gamma_5 \zeta_\mu^+ \gamma^\mu u_s(p) = 2m\, \boldsymbol{\zeta}_*^+ \cdot \boldsymbol{\sigma}_{rs},$$
$$\bar{v}_s(p)\gamma_5 \zeta_\mu^- \gamma^\mu v_r(p) = -2m\, \boldsymbol{\zeta}_*^- \cdot \boldsymbol{\sigma}_{rs}.$$

$$X_s^\pm(x, p) = \exp\left[\pm\xi(x) - \beta_\mu(x) p^\mu + \gamma_5 \gamma_\mu a^\mu\right].$$

Where, $\zeta_\pm^\mu = \frac{a_\pm^\mu}{\sqrt{-a_\pm^2}} \tanh\sqrt{-a_\pm^2}$. (compatible with $0 \le |\boldsymbol{\zeta}_*^\pm| \le 1$)

The new spin distribution function

Note that, $\exp\left(\gamma_5\gamma_\mu a_\pm^\mu\right) =$

$$\cosh\sqrt{-a_\pm^2}\left[1 + \frac{\gamma_5\gamma_\mu a_\pm^\mu}{\sqrt{-a_\pm^2}}\tanh\sqrt{-a_\pm^2}\right].$$

Where, $(a_\pm^2 < 0)$.

And

$$\bar{u}_r(p)\gamma_5\zeta_\mu^+\gamma^\mu u_s(p) = 2m\,\boldsymbol{\zeta}_*^+ \cdot \boldsymbol{\sigma}_{rs},$$
$$\bar{v}_s(p)\gamma_5\zeta_\mu^-\gamma^\mu v_r(p) = -2m\,\boldsymbol{\zeta}_*^- \cdot \boldsymbol{\sigma}_{rs}.$$

$$X_s^\pm(x, p) = \exp\left[\pm\xi(x) - \beta_\mu(x)p^\mu + \gamma_5\gamma_\mu a^\mu\right].$$

Where, $\zeta_\pm^\mu = \frac{a_\pm^\mu}{\sqrt{-a_\pm^2}}\tanh\sqrt{-a_\pm^2}.$ (compatible with $0 \leq |\boldsymbol{\zeta}_*^\pm| \leq 1$)

The spin Lagrange multiplier enters as

$$a_\mu(x, p) = -\frac{1}{2m}\tilde{\omega}_{\mu\nu}(x)p^\nu,$$

Where, $\tilde{\omega}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\omega^{\alpha\beta}$

‘Spin’ thermodynamics

‘Spin’ thermodynamics

$$N^\mu(x) = 2\int \mathrm{d}P\, p^\mu [f_0^+(x,p) - f_0^-(x,p)] \cosh\sqrt{-a^2}.$$

$$\mathcal{N}^\mu = 2\int \mathrm{d}P\, p^\mu [f_0^+(x,p) + f_0^-(x,p)] \cosh\sqrt{-a^2}.$$

$$T^{\mu\nu} = 2\int \mathrm{d}P\, p^\mu p^\nu [f_0^+(x,p) + f_0^-(x,p)] \cosh\sqrt{-a^2}.$$

$$S^{\lambda,\mu\nu} = \frac{1}{m}\int \mathrm{d}P\, p^\lambda\, [f_0^+(x,p) + f_0^-(x,p)]\, \frac{\sinh\sqrt{-a^2}}{\sqrt{-a^2}}\, \epsilon^{\mu\nu}{}_{\rho\sigma} a^\rho p^\sigma.$$

‘Spin’ thermodynamics

$$N^{\mu}(x) = 2\int \mathrm{d}P\, p^{\mu}[f_0^{+}(x,p) - f_0^{-}(x,p)]\cosh\sqrt{-a^2}.$$

$$\mathcal{N}^{\mu} = 2\int \mathrm{d}P\, p^{\mu}[f_0^{+}(x,p) + f_0^{-}(x,p)]\cosh\sqrt{-a^2}.$$

$$T^{\mu\nu} = 2\int \mathrm{d}P\, p^{\mu}p^{\nu}[f_0^{+}(x,p) + f_0^{-}(x,p)]\cosh\sqrt{-a^2}.$$

$$S^{\lambda,\mu\nu} = \frac{1}{m}\int \mathrm{d}P\, p^{\lambda}\,[f_0^{+}(x,p) + f_0^{-}(x,p)]\,\frac{\sinh\sqrt{-a^2}}{\sqrt{-a^2}}\,\epsilon^{\mu\nu}{}_{\rho\sigma}a^{\rho}p^{\sigma}.$$

With entropy defined as, $S^{\mu}(x) = -\frac{1}{2}\int \mathrm{d}P\, p^{\mu}\{\mathrm{tr}_4\,[X_s^{+}\,(\ln X_s^{+} - 1)] + \mathrm{tr}_4\,[X_s^{-}\,(\ln X_s^{-} - 1)]\}.$

[Florkowski et al. PRD 97, 116017 (2018)]

'Spin' thermodynamics

$$N^\mu(x) = 2\int \mathrm{d}P\, p^\mu[f_0^+(x,p) - f_0^-(x,p)]\cosh\sqrt{-a^2}.$$

$$\mathcal{N}^\mu = 2\int \mathrm{d}P\, p^\mu[f_0^+(x,p) + f_0^-(x,p)]\cosh\sqrt{-a^2}.$$

$$T^{\mu\nu} = 2\int \mathrm{d}P\, p^\mu p^\nu[f_0^+(x,p) + f_0^-(x,p)]\cosh\sqrt{-a^2}.$$

$$S^{\lambda,\mu\nu} = \frac{1}{m}\int \mathrm{d}P\, p^\lambda\,[f_0^+(x,p) + f_0^-(x,p)]\,\frac{\sinh\sqrt{-a^2}}{\sqrt{-a^2}}\,\epsilon^{\mu\nu}{}_{\rho\sigma}a^\rho p^\sigma.$$

With entropy defined as, $S^\mu(x) = -\frac{1}{2}\int \mathrm{d}P\, p^\mu\{\mathrm{tr}_4\,[X_s^+\,(\ln X_s^+ - 1)] + \mathrm{tr}_4\,[X_s^-\,(\ln X_s^- - 1)]\}.$

[Florkowski et al. PRD 97, 116017 (2018)]

Satisfies, $dS^\mu = -\xi dN^\mu + \beta_\lambda dT^{\lambda\mu} - \frac{1}{2}\omega_{\alpha\beta}dS^{\mu,\alpha\beta}$, (First law of spin thermodynamics)

‘Spin’ thermodynamics

$$N^{\mu}(x) = 2\int \mathrm{d}P\, p^{\mu}[f_0^+(x,p) - f_0^-(x,p)]\cosh\sqrt{-a^2}.$$

$$\mathcal{N}^{\mu} = 2\int \mathrm{d}P\, p^{\mu}[f_0^+(x,p) + f_0^-(x,p)]\cosh\sqrt{-a^2}.$$

$$T^{\mu\nu} = 2\int \mathrm{d}P\, p^{\mu}p^{\nu}[f_0^+(x,p) + f_0^-(x,p)]\cosh\sqrt{-a^2}.$$

$$S^{\lambda,\mu\nu} = \frac{1}{m}\int \mathrm{d}P\, p^{\lambda}\,[f_0^+(x,p) + f_0^-(x,p)]\,\frac{\sinh\sqrt{-a^2}}{\sqrt{-a^2}}\,\epsilon^{\mu\nu}{}_{\rho\sigma}a^{\rho}p^{\sigma}.$$

With entropy defined as, $S^{\mu}(x) = -\frac{1}{2}\int \mathrm{d}P\, p^{\mu}\{\mathrm{tr}_4\,[X_s^+\,(\ln X_s^+ - 1)] + \mathrm{tr}_4\,[X_s^-\,(\ln X_s^- - 1)]\}$.

[Florkowski et al. PRD 97, 116017 (2018)]

Satisfies, $dS^{\mu} = -\xi dN^{\mu} + \beta_{\lambda}dT^{\lambda\mu} - \frac{1}{2}\omega_{\alpha\beta}dS^{\mu,\alpha\beta}$, (First law of spin thermodynamics)

$d\mathcal{N}^{\mu} = N^{\mu}d\xi - T^{\lambda\mu}d\beta_{\lambda} + \frac{1}{2}S^{\mu,\alpha\beta}d\omega_{\alpha\beta}$, (Gibbs-Duhem relation for spin thermodynamics)

‘Spin’ thermodynamics

$$N^{\mu}(x) = 2\int \mathrm{d}P\, p^{\mu}[f_0^{+}(x,p) - f_0^{-}(x,p)]\cosh\sqrt{-a^2}.$$

$$\mathcal{N}^{\mu} = 2\int \mathrm{d}P\, p^{\mu}[f_0^{+}(x,p) + f_0^{-}(x,p)]\cosh\sqrt{-a^2}.$$

$$T^{\mu\nu} = 2\int \mathrm{d}P\, p^{\mu}p^{\nu}[f_0^{+}(x,p) + f_0^{-}(x,p)]\cosh\sqrt{-a^2}.$$

$$S^{\lambda,\mu\nu} = \frac{1}{m}\int \mathrm{d}P\, p^{\lambda}\,[f_0^{+}(x,p) + f_0^{-}(x,p)]\,\frac{\sinh\sqrt{-a^2}}{\sqrt{-a^2}}\,\epsilon^{\mu\nu}{}_{\rho\sigma}a^{\rho}p^{\sigma}.$$

With entropy defined as, $S^{\mu}(x) = -\frac{1}{2}\int \mathrm{d}P\, p^{\mu}\{\mathrm{tr}_4\,[X_s^{+}\,(\ln X_s^{+} - 1)] + \mathrm{tr}_4\,[X_s^{-}\,(\ln X_s^{-} - 1)]\}.$

[Florkowski et al. PRD 97, 116017 (2018)]

Satisfies, $dS^{\mu} = -\xi dN^{\mu} + \beta_{\lambda} dT^{\lambda\mu} - \frac{1}{2}\omega_{\alpha\beta} dS^{\mu,\alpha\beta},$ (First law of spin thermodynamics)

$d\mathcal{N}^{\mu} = N^{\mu} d\xi - T^{\lambda\mu} d\beta_{\lambda} + \frac{1}{2} S^{\mu,\alpha\beta} d\omega_{\alpha\beta},$ (Gibbs-Duhem relation for spin thermodynamics)

Consequently,

$$S^{\mu} = T^{\mu\alpha}\beta_{\alpha} - \frac{1}{2}\omega_{\alpha\beta}S^{\mu,\alpha\beta} - \xi N^{\mu} + \mathcal{N}^{\mu},$$

[Florkowski, Hontarenko, PRL 134 (2025)]

Divergence type theories – a simple approach to verify causality

Divergence type theories – a simple approach to verify causality

- Recent work has explored the compatibility of (classical) spin hydrodynamics with a divergence type theory framework. *[N. Abboud, L. Gavassino, R. Singh, E. Speranza, arXiv: 2506.19786]*

Divergence type theories – a simple approach to verify causality

- Recent work has explored the compatibility of (classical) spin hydrodynamics with a divergence type theory framework. *[N. Abboud, L. Gavassino, R. Singh, E. Speranza, arXiv: 2506.19786]*
- We checked our quantum spin hydrodynamical theory in a similar framework,

Divergence type theories – a simple approach to verify causality

- Recent work has explored the compatibility of (classical) spin hydrodynamics with a divergence type theory framework. *[N. Abboud, L. Gavassino, R. Singh, E. Speranza, arXiv: 2506.19786]*
- We checked our quantum spin hydrodynamical theory in a similar framework,

$$dS^{\mu} = -\xi dN^{\mu} + \beta_{\lambda} dT^{\lambda\mu} - \frac{1}{2}\omega_{\alpha\beta} dS^{\mu,\alpha\beta},$$

(First law of spin thermodynamics)

Divergence type theories – a simple approach to verify causality

- Recent work has explored the compatibility of (classical) spin hydrodynamics with a divergence type theory framework. *[N. Abboud, L. Gavassino, R. Singh, E. Speranza, arXiv: 2506.19786]*
- We checked our quantum spin hydrodynamical theory in a similar framework,

$$dS^\mu = -\xi dN^\mu + \beta_\lambda dT^{\lambda\mu} - \frac{1}{2}\omega_{\alpha\beta} dS^{\mu,\alpha\beta},$$

(First law of spin thermodynamics)

→

The fields are $\zeta_A \equiv (\zeta, \zeta_\mu, \zeta_{\mu\nu})$,
with $\zeta \equiv \xi$, $\zeta_\mu \equiv -\beta_\mu$ and, $\zeta_{\mu\nu} \equiv \omega_{\mu\nu}$

Divergence type theories – a simple approach to verify causality

- Recent work has explored the compatibility of (classical) spin hydrodynamics with a divergence type theory framework. *[N. Abboud, L. Gavassino, R. Singh, E. Speranza, arXiv: 2506.19786]*
- We checked our quantum spin hydrodynamical theory in a similar framework,

$dS^\mu = -\xi dN^\mu + \beta_\lambda dT^{\lambda\mu} - \frac{1}{2}\omega_{\alpha\beta} dS^{\mu,\alpha\beta}$,
(First law of spin thermodynamics)

→

The fields are $\zeta_A \equiv (\zeta, \zeta_\mu, \zeta_{\mu\nu})$,
with $\zeta \equiv \xi$, $\zeta_\mu \equiv -\beta_\mu$ and, $\zeta_{\mu\nu} \equiv \omega_{\mu\nu}$

Generating function
χ

Divergence type theories – a simple approach to verify causality

- Recent work has explored the compatibility of (classical) spin hydrodynamics with a divergence type theory framework. [N. Abboud, L. Gavassino, R. Singh, E. Speranza, arXiv: 2506.19786]
- We checked our quantum spin hydrodynamical theory in a similar framework,

$$dS^{\mu} = -\xi dN^{\mu} + \beta_{\lambda} dT^{\lambda\mu} - \frac{1}{2}\omega_{\alpha\beta} dS^{\mu,\alpha\beta},$$

(First law of spin thermodynamics)

→

The fields are $\zeta_A \equiv (\zeta, \zeta_\mu, \zeta_{\mu\nu})$, with $\zeta \equiv \xi$, $\zeta_\mu \equiv -\beta_\mu$ and, $\zeta_{\mu\nu} \equiv \omega_{\mu\nu}$

Generating function χ

↓

$$N^{\lambda A} = \left(\frac{\partial \mathcal{N}^{\lambda}}{\partial \zeta_A}\right) = -\left(\frac{\partial^2 \chi}{\partial \zeta_A \partial \beta_\lambda}\right)$$

(Conserved currents)

Divergence type theories – a simple approach to verify causality

- Recent work has explored the compatibility of (classical) spin hydrodynamics with a divergence type theory framework. [N. Abboud, L. Gavassino, R. Singh, E. Speranza, arXiv: 2506.19786]
- We checked our quantum spin hydrodynamical theory in a similar framework,

$$dS^{\mu} = -\xi dN^{\mu} + \beta_{\lambda} dT^{\lambda\mu} - \frac{1}{2}\omega_{\alpha\beta} dS^{\mu,\alpha\beta},$$

(First law of spin thermodynamics)

→

The fields are $\zeta_A \equiv (\zeta, \zeta_\mu, \zeta_{\mu\nu})$,
with $\zeta \equiv \xi$, $\zeta_\mu \equiv -\beta_\mu$ and, $\zeta_{\mu\nu} \equiv \omega_{\mu\nu}$

Generating function χ

$$\mathcal{N}^{\lambda} = \frac{\partial \chi}{\partial \zeta_\lambda} = -\frac{\partial \chi}{\partial \beta_\lambda}$$

(Particle current)

$$N^{\lambda A} = \left(\frac{\partial \mathcal{N}^{\lambda}}{\partial \zeta_A}\right) = -\left(\frac{\partial^2 \chi}{\partial \zeta_A \partial \beta_\lambda}\right)$$

(Conserved currents)

Divergence type theories – a simple approach to verify causality

- Recent work has explored the compatibility of (classical) spin hydrodynamics with a divergence type theory framework. *[N. Abboud, L. Gavassino, R. Singh, E. Speranza, arXiv: 2506.19786]*
- We checked our quantum spin hydrodynamical theory in a similar framework,

$$dS^\mu = -\xi dN^\mu + \beta_\lambda dT^{\lambda\mu} - \frac{1}{2}\omega_{\alpha\beta} dS^{\mu,\alpha\beta},$$

(First law of spin thermodynamics)

The fields are $\zeta_A \equiv (\zeta, \zeta_\mu, \zeta_{\mu\nu})$, with $\zeta \equiv \xi$, $\zeta_\mu \equiv -\beta_\mu$ and, $\zeta_{\mu\nu} \equiv \omega_{\mu\nu}$

$$M^{\lambda AB} = \left(\frac{\partial N^{\lambda A}}{\partial \zeta_B}\right) = -\left(\frac{\partial^3 \chi}{\partial \zeta_B \partial \zeta_A \partial \beta_\lambda}\right).$$

('Test' quantity)

Generating function χ

$$\mathcal{N}^\lambda = \frac{\partial \chi}{\partial \zeta_\lambda} = -\frac{\partial \chi}{\partial \beta_\lambda}$$

(Particle current)

$$N^{\lambda A} = \left(\frac{\partial \mathcal{N}^\lambda}{\partial \zeta_A}\right) = -\left(\frac{\partial^2 \chi}{\partial \zeta_A \partial \beta_\lambda}\right)$$

(Conserved currents)

The new quantum approach is causal and stable

The new quantum approach is causal and stable

We identify the generating function as,

$$\chi = 2\int \mathrm{d}P\, [f_0^+(x,p) + f_0^-(x,p)]\cosh\sqrt{-a^2}$$

The new quantum approach is causal and stable

We identify the generating function as,

$$\chi = 2\int \mathrm{d}P\,[f_0^+(x,p) + f_0^-(x,p)]\cosh\sqrt{-a^2}$$

The theory is non-linearly causal and symmetric hyperbolic if and only if the ‘test’ quantity *[R. Geroch, L. Lindblom, Annals Phys. 207 (1991) 394-416]*

$$M^{\lambda AB} Z_A Z_B$$

The new quantum approach is causal and stable

We identify the generating function as,

$$\chi = 2\int \mathrm{d}P\,[f_0^+(x,p) + f_0^-(x,p)]\cosh\sqrt{-a^2}$$

The theory is non-linearly causal and symmetric hyperbolic if and only if the ‘test’ quantity *[R. Geroch, L. Lindblom, Annals Phys. 207 (1991) 394-416]*

$$M^{\lambda AB} Z_A Z_B$$

is non spacelike and future directed for all real Z_A, Z_B where, $Z_A = (Z, Z_\mu, Z_{\mu\nu})$

The new quantum approach is causal and stable

We identify the generating function as,

$$\chi = 2\int \mathrm{d}P\,[f_0^+(x,p) + f_0^-(x,p)]\cosh\sqrt{-a^2}$$

The theory is non-linearly causal and symmetric hyperbolic if and only if the ‘test’ quantity *[R. Geroch, L. Lindblom, Annals Phys. 207 (1991) 394-416]*

$$M^{\lambda AB} Z_A Z_B$$

is non spacelike and future directed for all real Z_A, Z_B where, $Z_A = (Z, Z_\mu, Z_{\mu\nu})$

Please note that,

$$M^{\lambda AB} Z_A Z_B = \left(Z\frac{\partial}{\partial \xi} - Z_\mu \frac{\partial}{\partial \beta_\mu} + \frac{Z_{\mu\nu}}{2}\frac{\partial}{\partial \omega_{\mu\nu}} \right)^2 \mathcal{N}^\lambda.$$

The new quantum approach is causal and stable

The new quantum approach is causal and stable

$$M^{\lambda AB} Z_A Z_B = 2\int d\,P p^{\lambda}\Bigg[\Bigg\{(Z + Z\cdot p)^2 + 2(Z + Z\cdot p)\left(\frac{\tilde{Z}^{\rho\sigma}a_\rho p_\sigma}{2m\sqrt{-a^2}}\right)\tanh\sqrt{-a^2} + \left(\frac{\tilde{Z}^{\rho\sigma}a_\rho p_\sigma}{2m\sqrt{-a^2}}\right)^2\Bigg\} f_0^+ cosh\sqrt{-a^2}$$

$$+\Bigg\{(-Z + Z\cdot p)^2 + 2(-Z + Z\cdot p)\left(\frac{\tilde{Z}^{\rho\sigma}a_\rho p_\sigma}{2m\sqrt{-a^2}}\right)\tanh\sqrt{-a^2} + \left(\frac{\tilde{Z}^{\rho\sigma}a_\rho p_\sigma}{2m\sqrt{-a^2}}\right)^2\Bigg\} f_0^- cosh\sqrt{-a^2}$$

$$+\frac{\tilde{Z}^{\alpha\gamma}p_\gamma}{2m}\frac{\tilde{Z}^{\rho\sigma}p_\sigma}{2m}\left\{-\eta_{\alpha\rho} - \frac{a_\alpha a_\rho}{\left(\sqrt{-a^2}\right)^2}\right\}\frac{\sinh\sqrt{-a^2}}{\sqrt{-a^2}}(f_0^+ + f_0^-)\Bigg]$$

The new quantum approach is causal and stable

$$M^{\lambda AB} Z_A Z_B = 2 \int d\,P p^{\lambda} \Bigg[\left\{ (Z + Z\cdot p)^2 + 2(Z + Z\cdot p)\left(\frac{\tilde{Z}^{\rho\sigma} a_\rho p_\sigma}{2m\sqrt{-a^2}}\right) \tanh\sqrt{-a^2} + \left(\frac{\tilde{Z}^{\rho\sigma} a_\rho p_\sigma}{2m\sqrt{-a^2}}\right)^2 \right\} f_0^+ cosh\sqrt{-a^2}$$

$$+ \left\{ (-Z + Z\cdot p)^2 + 2(-Z + Z\cdot p)\left(\frac{\tilde{Z}^{\rho\sigma} a_\rho p_\sigma}{2m\sqrt{-a^2}}\right) \tanh\sqrt{-a^2} + \left(\frac{\tilde{Z}^{\rho\sigma} a_\rho p_\sigma}{2m\sqrt{-a^2}}\right)^2 \right\} f_0^- cosh\sqrt{-a^2}$$

$$+ \frac{\tilde{Z}^{\alpha\gamma} p_\gamma}{2m} \frac{\tilde{Z}^{\rho\sigma} p_\sigma}{2m} \left\{ -\eta_{\alpha\rho} - \frac{a_\alpha a_\rho}{\left(\sqrt{-a^2}\right)^2} \right\} \frac{\sinh\sqrt{-a^2}}{\sqrt{-a^2}} (f_0^+ + f_0^-) \Bigg]$$

One may check that the quantity in [] brackets is positive in the particle rest frame

The new quantum approach is causal and stable

$$M^{\lambda AB} Z_A Z_B = 2\int d\,P p^{\lambda}\Bigg[\left\{(Z+Z\cdot p)^2 + 2(Z+Z\cdot p)\left(\frac{\tilde{Z}^{\rho\sigma}a_\rho p_\sigma}{2m\sqrt{-a^2}}\right)\tanh\sqrt{-a^2} + \left(\frac{\tilde{Z}^{\rho\sigma}a_\rho p_\sigma}{2m\sqrt{-a^2}}\right)^2\right\} f_0^+ cosh\sqrt{-a^2}$$

$$+\left\{(-Z+Z\cdot p)^2 + 2(-Z+Z\cdot p)\left(\frac{\tilde{Z}^{\rho\sigma}a_\rho p_\sigma}{2m\sqrt{-a^2}}\right)\tanh\sqrt{-a^2} + \left(\frac{\tilde{Z}^{\rho\sigma}a_\rho p_\sigma}{2m\sqrt{-a^2}}\right)^2\right\} f_0^- cosh\sqrt{-a^2}$$

$$+\frac{\tilde{Z}^{\alpha\gamma}p_\gamma}{2m}\frac{\tilde{Z}^{\rho\sigma}p_\sigma}{2m}\left\{-\eta_{\alpha\rho} - \frac{a_\alpha a_\rho}{\left(\sqrt{-a^2}\right)^2}\right\}\frac{\sinh\sqrt{-a^2}}{\sqrt{-a^2}}\left(f_0^+ + f_0^-\right)\Bigg]$$

One may check that the quantity in [] brackets is positive in the particle rest frame

Hence, our theory is non linearly causal and symmetric hyperbolic.

Many more aspects of the theory

Many more aspects of the theory

- When do the integrals of the current converge?

Many more aspects of the theory

- When do the integrals of the current converge?
- In the $\mu = 0$ limit, the criterion looks like this

*[Z. Drogosz, W. Florkowski, V. Mykhaylova, Phys. Rev. D **112**, L051901]*

$$\frac{1}{2}\sqrt{\boldsymbol{b}'^2 + \boldsymbol{e}'^2 + 2|\boldsymbol{e}' \times \boldsymbol{b}'|} < \frac{m}{T}$$

Many more aspects of the theory

- When do the integrals of the current converge?
- In the $\mu = 0$ limit, the criterion looks like this

*[Z. Drogosz, W. Florkowski, V. Mykhaylova, Phys. Rev. D **112**, L051901]*

$$\frac{1}{2}\sqrt{\boldsymbol{b}'^2 + \boldsymbol{e}'^2 + 2|\boldsymbol{e}' \times \boldsymbol{b}'|} < \frac{m}{T}$$

- More details will be given in the next talk.

Many more aspects of the theory

- When do the integrals of the current converge?
- In the $\mu = 0$ limit, the criterion looks like this

  *[Z. Drogosz, W. Florkowski, V. Mykhaylova, Phys. Rev. D **112**, L051901]*

$$\frac{1}{2}\sqrt{\boldsymbol{b}'^2 + \boldsymbol{e}'^2 + 2|\boldsymbol{e}' \times \boldsymbol{b}'|} < \frac{m}{T}$$

- More details will be given in the next talk.
- Does this quantum spin distribution reduce to the classical theory in certain limits?

Many more aspects of the theory

- When do the integrals of the current converge?
- In the $\mu = 0$ limit, the criterion looks like this

  *[Z. Drogosz, W. Florkowski, V. Mykhaylova, Phys. Rev. D **112**, L051901]*

$$\frac{1}{2}\sqrt{\boldsymbol{b}'^2 + \boldsymbol{e}'^2 + 2|\boldsymbol{e}' \times \boldsymbol{b}'|} < \frac{m}{T}$$

- More details will be given in the next talk.
- Does this quantum spin distribution reduce to the classical theory in certain limits?
- Refer the work done by my colleague : *[Zbigniew Drogosz, arXiv:2509.06014 [hep-ph]]*

Conclusions

Conclusions

- Local equilibrium Wigner function for spin-1/2 particles with proper normalization of the mean polarization vector was obtained.

Conclusions

- Local equilibrium Wigner function for spin-1/2 particles with proper normalization of the mean polarization vector was obtained.
- We found an agreement of the generalized spin thermodynamics that was only seen before in the classical case.

Conclusions

- Local equilibrium Wigner function for spin-1/2 particles with proper normalization of the mean polarization vector was obtained.
- We found an agreement of the generalized spin thermodynamics that was only seen before in the classical case.
- The new distribution function was tested against a divergence type framework and showed to be non-linearly causal and symmetric hyperbolic

Conclusions

- Local equilibrium Wigner function for spin-1/2 particles with proper normalization of the mean polarization vector was obtained.
- We found an agreement of the generalized spin thermodynamics that was only seen before in the classical case.
- The new distribution function was tested against a divergence type framework and showed to be non-linearly causal and symmetric hyperbolic

Thank you for your attention

$$\boldsymbol{b}_{*\cdot} = \frac{1}{m}\left[E_p\,\boldsymbol{b} - \boldsymbol{p}\times\boldsymbol{e} - \frac{\boldsymbol{p}\cdot\boldsymbol{b}}{E_p+m}\boldsymbol{p}\right],$$

The perfect spinfluid

Nick Abboud,[1,*] Lorenzo Gavassino,[2,†] Rajeev Singh,[3,‡] and Enrico Speranza[4,5,§]

$$E^{\lambda} = \frac{1}{2} M^{\lambda AB} \, \delta\zeta_A \, \delta\zeta_B + \mathcal{O}(\epsilon^3),$$

If the information current is spacelike, the information flows at a speed faster than light leading to acausality in the quadratic deviations in the field