Facultad de Ciencias
UNAM

Explicit Regularization and medium effects in the NJL model

International School of Nuclear Physics, 46th course

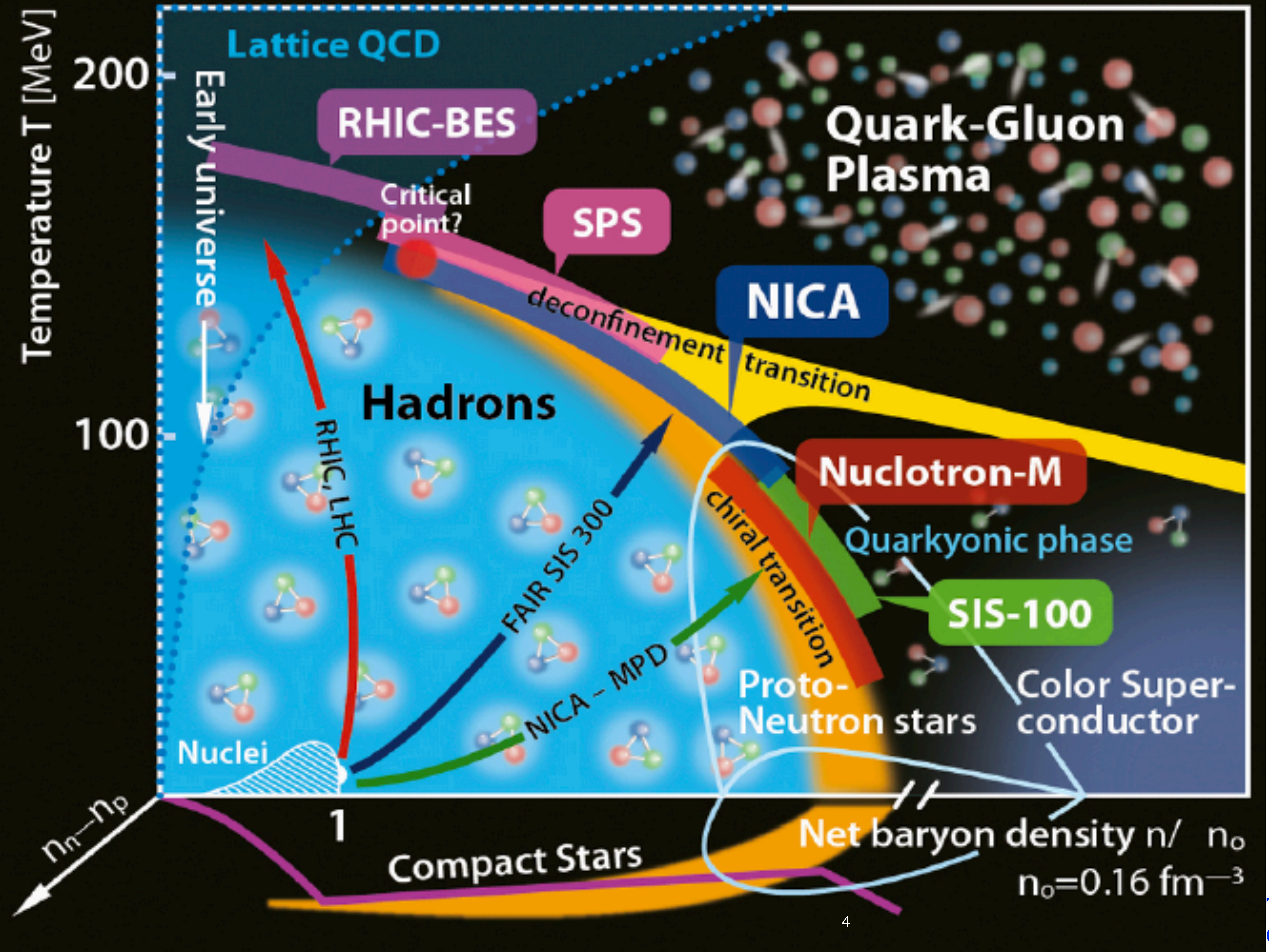
Bernardo Andrade in collaboration with A. Sánchez - September 2025




Hadronic Matter in extreme conditions

- Temperature
  Dominguez PRD **86**, 034030
- Magnetic Field
  Skokov IJMPA 2009 24:31, 5925-5932
- Baryonic Density
  Askawa Nucl. Phys. A 504 (1989) 668
- Angular Momentum
  Becattini PRC.77.024906
- Isospin Son PRL .86.592

- Big Bang
- Heavy Ion Collisions
- Neutron Stars



QCD Phase Diagram

- It mostly shows our ignorance.
- Has theoretical predictions
- Can have many other axes
  - eB
  - I_3
  - L

Taken from Compact Stars in the QCD Phase Diagram VI

Two chiral phases

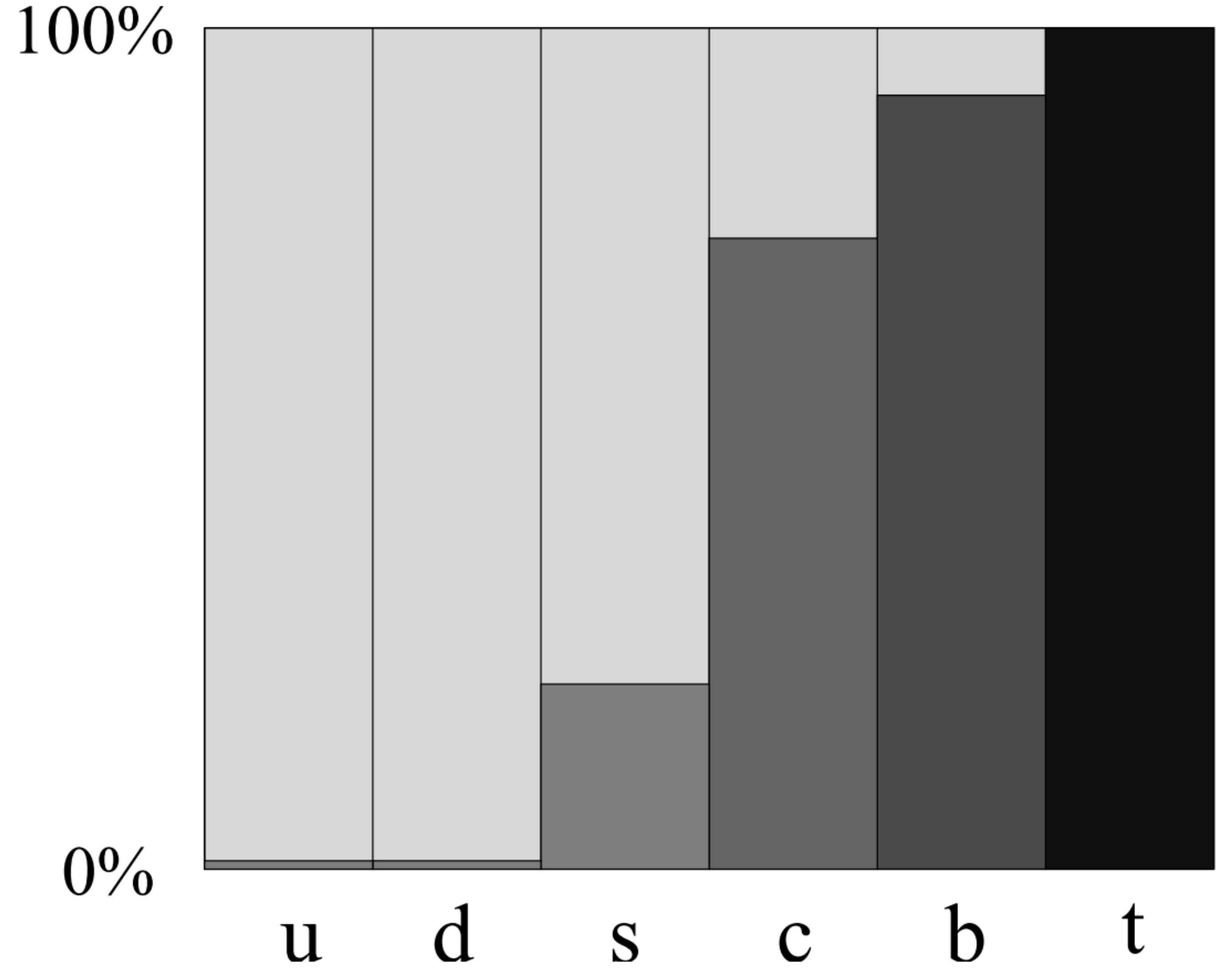
Restored - Broken

- Dynamically generated mass (Not from the Higgs Mechanism)
    - But from QCD vacuum phenomena — condensates, instantons, gluons

Eur. Phys. J. A (2023) 59:252

- Order parameter
    - Quark condensate

Int.J.Mod.Phys.A24:5925-5932,2009



10.1103/RevModPhys.81.1031

Ways to study

- Lattice QCD

  Aoki, Endödi 10.1038/nature05120

- Dyson-Schwinger Equations

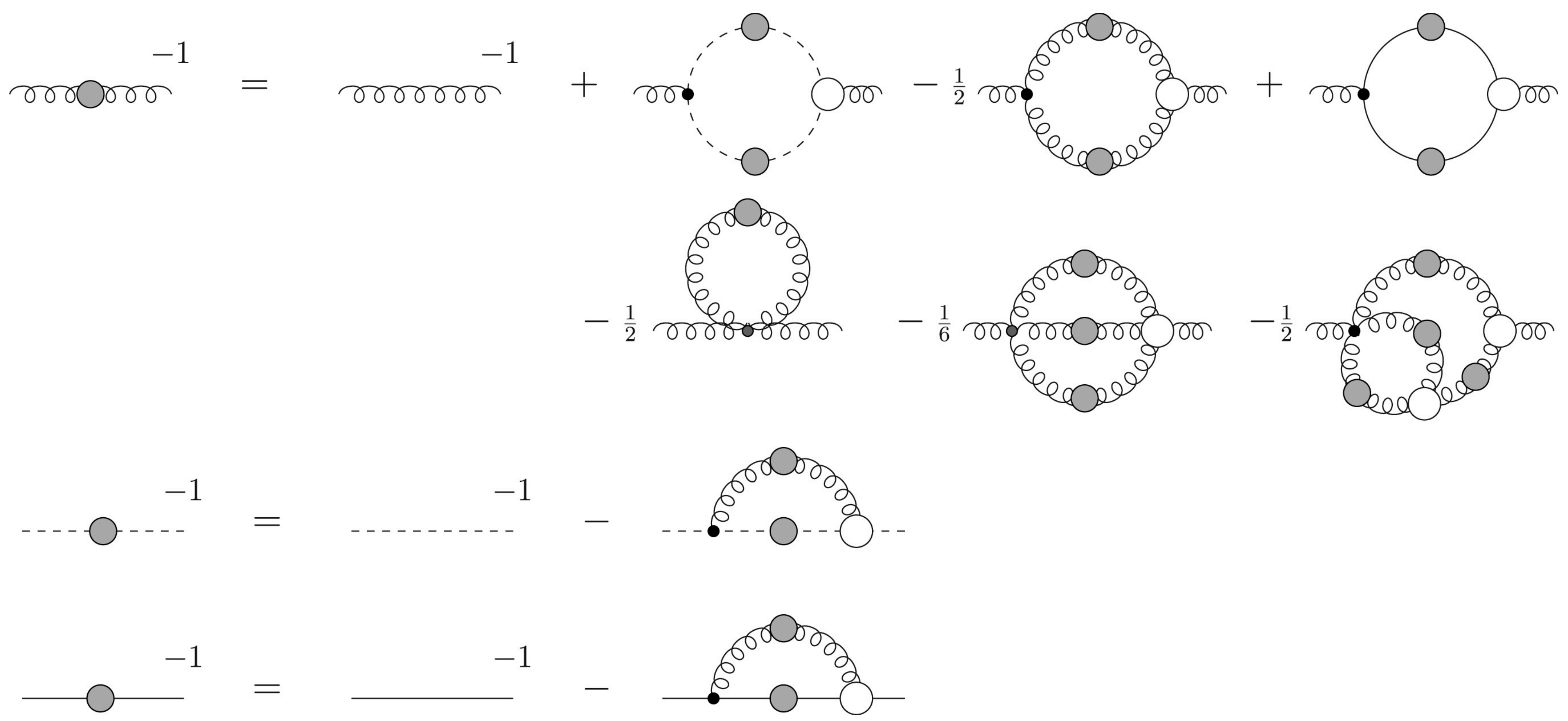
  Fischer 10.1016j.ppnp.2019.01.002

- Effective Field Theories

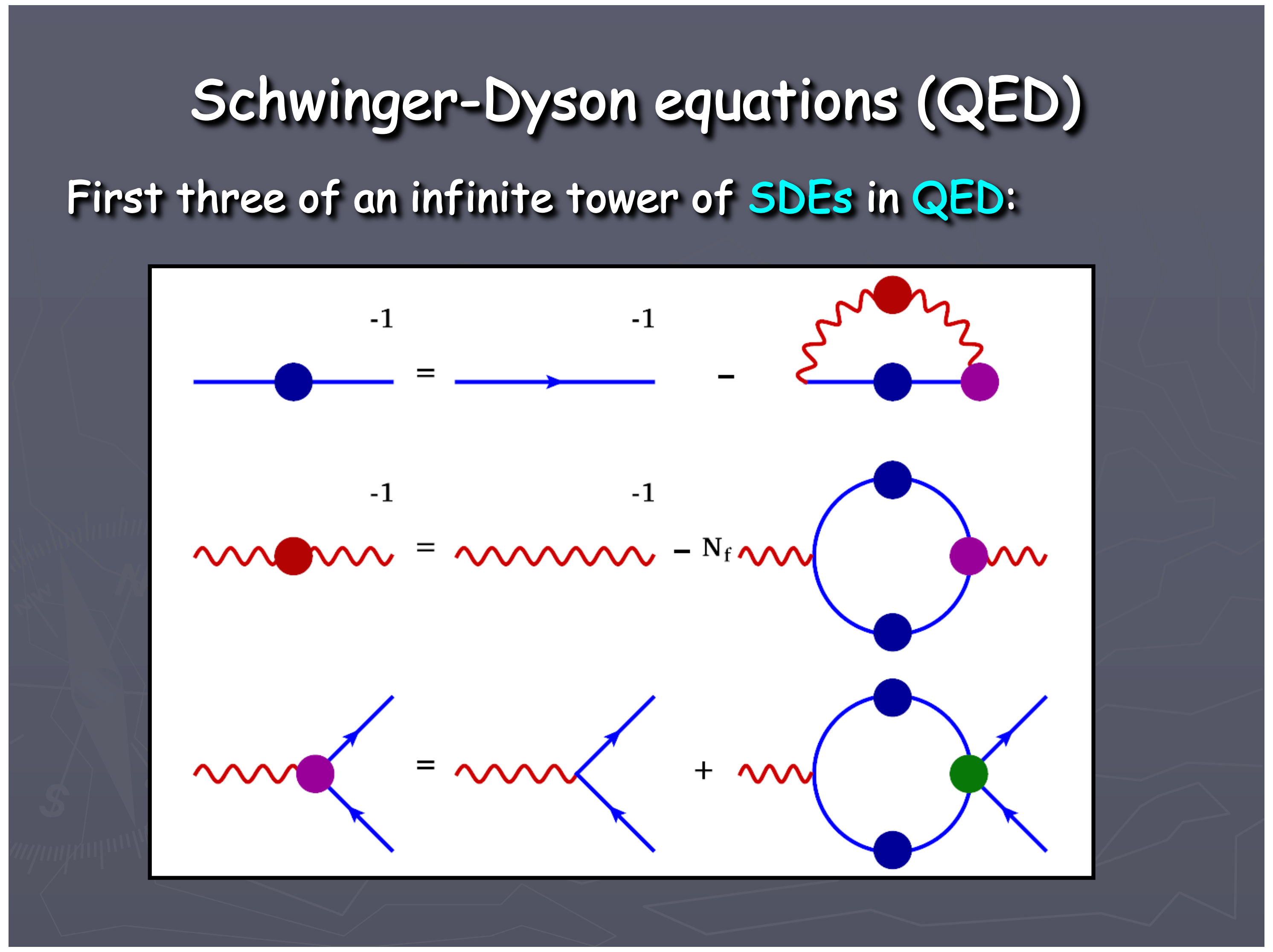
  - Nambu - Jona-Lasinio

    Buballa2005 10.1016/j.physrep.2004.11.004

    Klevansky 10.1103/RevModPhys.64.649



Fischer 10.1016j.ppnp.2019.01.002



Bashir Hugs 2023

Going into NJL

Lagrangian Density

- Contact Interaction

$$\mathscr{L}_{NJL} = \bar{q}(i\gamma^{\mu}\partial_{\mu} - m)q + G\left((\bar{q}q)^2 + (\bar{q}i\gamma_5\tau^a q)^2\right)$$

- BCS - like ground state
  - Dynamically breaks symmetry with quark condensation
- $G = [Energy]^{-2} \rightarrow$ Non renormalizable

Going into NJL

Properties

- Mean Field Approximation → Linearize the interaction

$$\mathscr{L}_{NJL} = \bar{\psi}(i\gamma_\mu \partial^\mu - m)\psi - \bar{\psi}\sigma\psi - \frac{1}{4G}\sigma^2$$

- $M = m + \sigma, \ \sigma = -2G < \bar{\psi}\psi >$

- When divergences occur, associated quantities become dependent on a cutoff scale Λ.

- Parameters must be fixed to reproduce physical quantities

  - Set, $G, \ \Lambda, \ m$ to fit $m_\pi, \ f_{pi}, \ M$

Divergences

A first example: Effective Potential

- Effective Potential

$$V_{\text{eff}}(\sigma) = V_{\text{tree}}(\sigma) + V^{(1)}(\sigma)$$
$$= \frac{\sigma^2}{4G} + i\Omega^{-1} \ln\det\left[-\frac{\delta^2 \mathscr{L}}{\delta\bar{\psi}\delta\psi}\right]$$

- In a convenient base, the second term is diagonal

$$V^{(1)} = tr \int \frac{d^4 p_E}{(2\pi)^4} \ln(p_E^2 + M^2)$$

  - Power counting is sufficient to identify the divergence

Divergences

A first example

$$V^{(1)} = tr \int \frac{d^4 p_E}{(2\pi)^4} \ln(p_E^2 + M^2) = -2N_c N_f \int_{REG} \frac{d^4 p_E}{(2\pi)^4} \ln(p_E^2 + M^2)$$

- A regularization scheme must be implemented
  - Sharp Cutoff
  - Pauli-Villars
  - Dimensional Regularization
  - Form Factor

How to regulate; schemes

Cutoff

- Cutoff - restrict the region of integration
  - 4D $$V^{(1)} = -2N_cN_f \int d\Omega_3 \int_0^{\Lambda_p} \frac{dp_E}{(2\pi)^4} p_E^3 \ln(p_E^2 + M^2)$$

Integration of p_4

  - 3D $$V^{(1)} = -2N_cN_f \int d\Omega_2 \int_0^{\Lambda_p} \frac{dp_E}{(2\pi)^4} p_E^2 \sqrt{p_E^2 + M^2}$$

Use of identity
$$\ln(x) = -\int \frac{ds}{s} e^{-sx}$$

  - Proper Time $$V^{(1)} = \frac{2N_cN_f}{(2\pi)^4} \int_{1/\Lambda_s^2}^{\infty} \frac{ds}{s} \left(\sqrt{\frac{\pi}{s}}\right)^4 e^{-sM^2}$$

Switching infinites

$p \to s$

- When using $\ln(x) = -\int \frac{ds}{s} e^{-sx}$, a new divergence appears when $s \to 0$
  - Cutoff on the nature of the logarithm
  - The procedure of this scheme involves $\int_{-\infty}^{\infty} dp\, f(p)$
  - The original cutoff Λ_p is no more, it gets switched for Λ_s
  - What is the meaning of Λ_s?

Switching infinites

Is this valid?

- Λ_p can be interpreted as the energy for which the model predictions are accurate.
  - Λ_s comes from a mathematical identity.
  - We totally discarded the divergence on p in favor of the divergence on s.
  - Let's see if there is a way to preserve the divergence on momentum.

Another Representation

Can we still capture the essence of proper time?

- A quick revision in the literature shows that

$$\ln x = \int_0^\infty \frac{ds}{s}\left(e^{-s} - e^{-sx}\right)$$

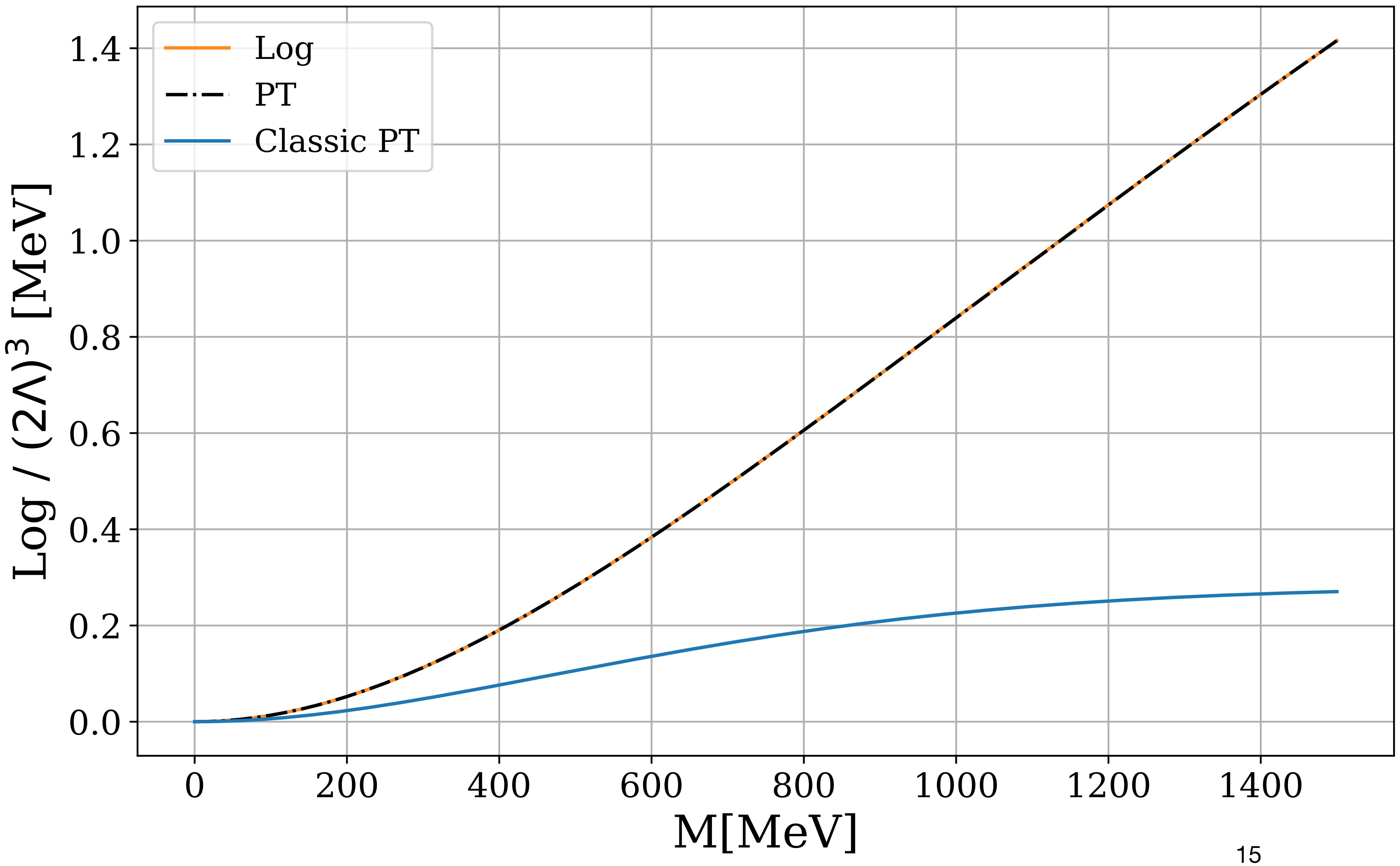
- Plugging into the effective potential, we can compute the momentum integrals

$$V_{eff} = \frac{1}{4G}\sigma^2 - \frac{N_c N_f}{8\pi^2}\int_0^\infty \frac{ds}{s}\left((2\Lambda)^4 e^{-s} - \frac{e^{-sM^2} Erf(\sqrt{s}\Lambda)^4}{s^2}\right)$$

Another Representation

How does this hold up?

- Lets compare the three representations



- $$\int_{-\Lambda_p}^{\Lambda_p} \frac{d^4p}{(2\pi)^4} \ln\left(p_4^2 + \omega^2 + M^2\right)$$ —> Log
- $$-\int_{-\Lambda_p}^{\Lambda_p} \frac{dp_1 dp_2 dp_3 dp_4}{(2\pi)^4} \int_{1/\Lambda_s^2}^{\infty} \frac{ds}{s} e^{-s\left(p_4^2 + \omega^2 + M^2\right)}$$ —> Classic PT
- $$\int_{-\Lambda_p}^{\Lambda_p} \frac{dp_1 dp_2 dp_3 dp_4}{(2\pi)^4} \int_0^{\infty} \frac{ds}{s} \left(e^{-s} - e^{-s\left(p_4^2 + \omega^2 + M^2\right)}\right)$$ —> PT

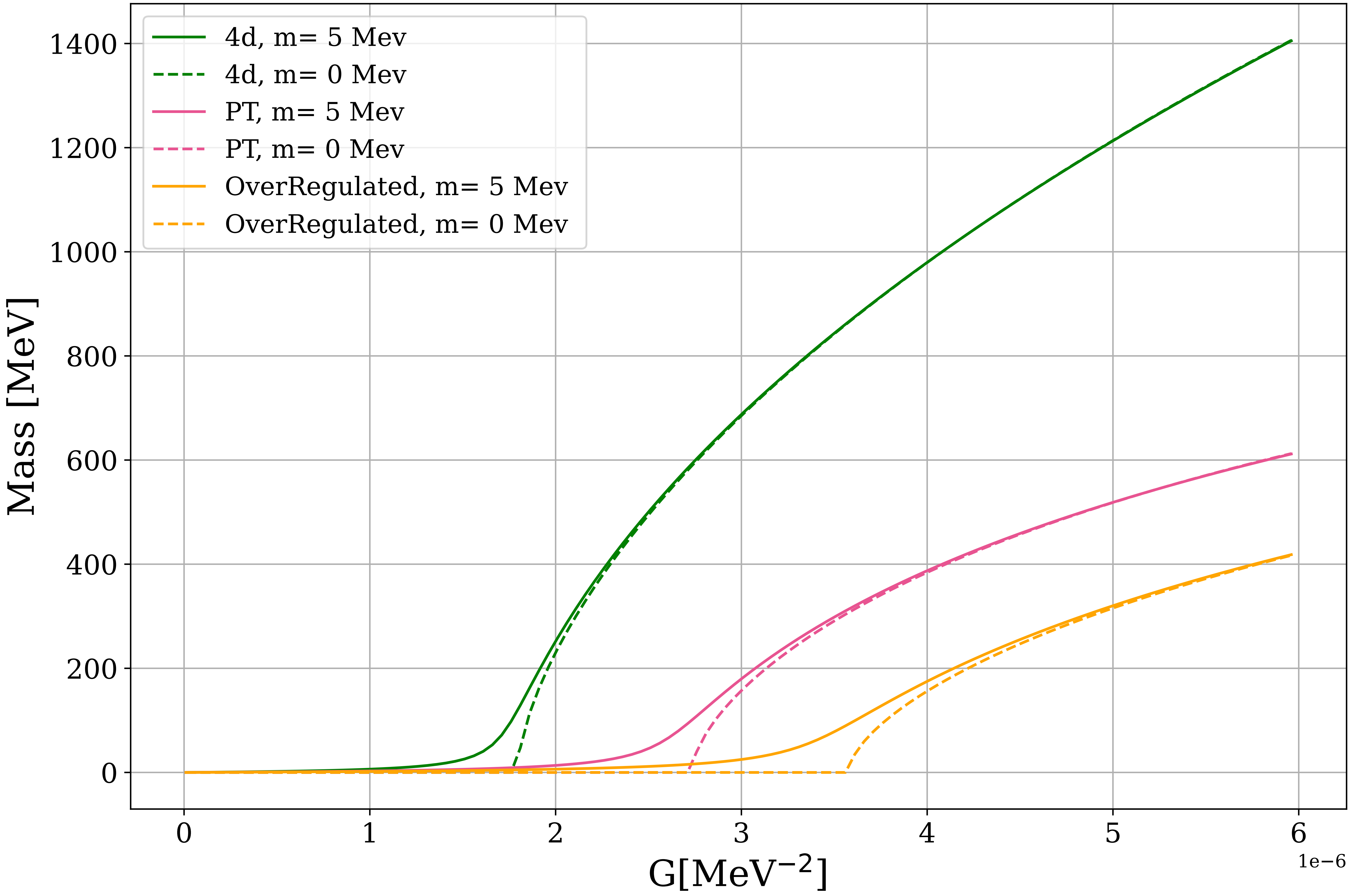
Effects of cutoffs

Gap equations

$$\frac{\partial V_{eff}}{\partial \sigma} = 0$$

	Λ_4	Λ	Λ_s
4D	1027	1027	∞
3D	∞	665	∞
PT	∞	∞	1097
Over	1027	1027	1097

Data from Nuclear Physics B, 896:682–715



Choosing how to integrate

Let's rethink our approach

- In the proper time representation, cartesian coordinates are the easiest to compute!
    - (But they are not the only ones)
    - Using cartesian coordinates, the integration is gaussian
    - What would happen if other geometries in p are used?

Geometry matters

Physical Scenario

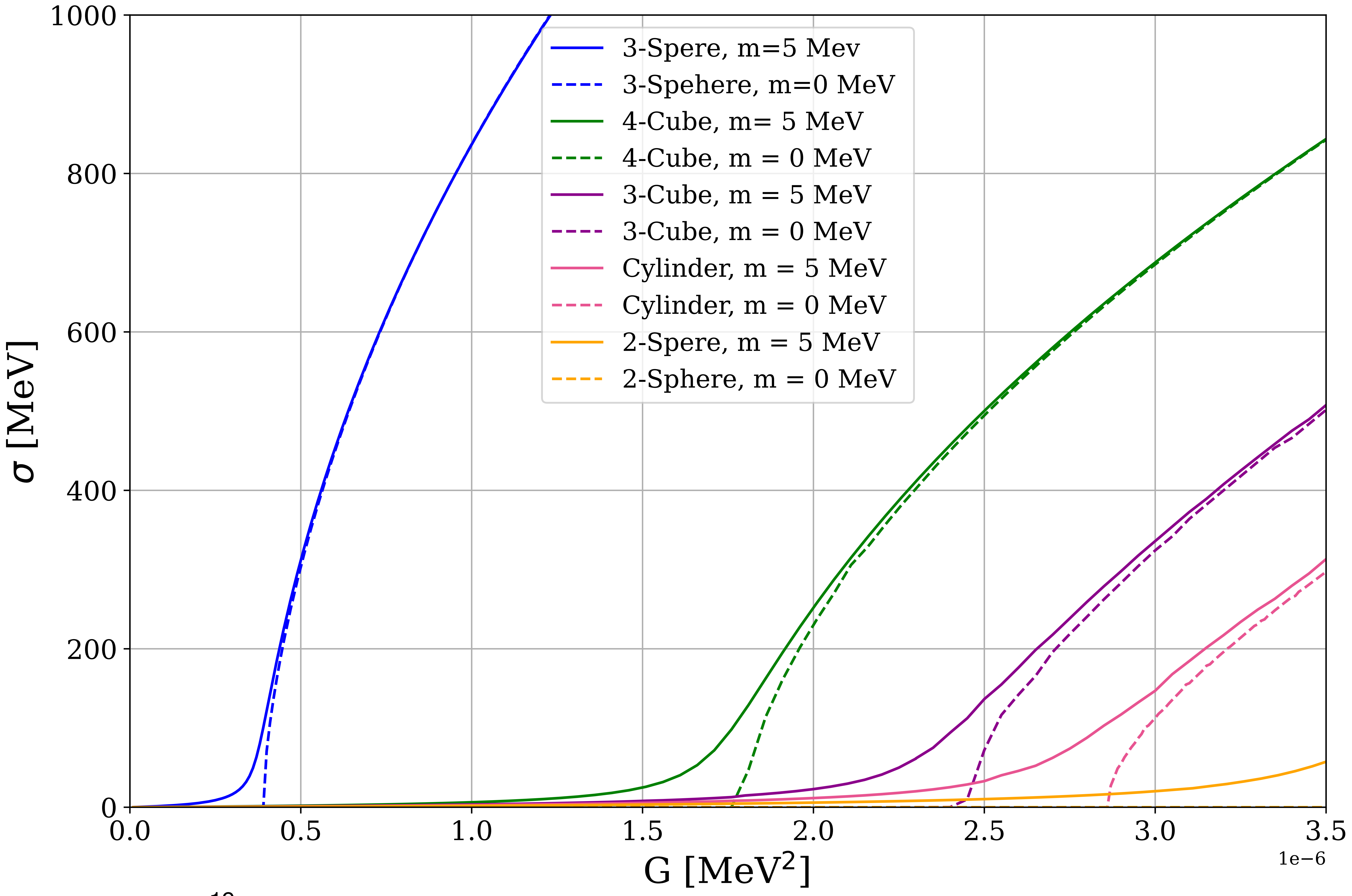
- Different geometries involve different scenarios that every scenario (Landau Level, Matsubara Frequency and normal mode) is taken into account

Volume Element	p^μ Domain
$p^3 dp\ d\Omega_3$	$p \in [0, \Lambda_{4sphere}]$
$dp_4 dp_1 dp_2 dp_3$	$p_i \in [-\Lambda_{4cube}, \Lambda_{4cube}]$
$dp_4 dp_1 dp_2 dp_3$	$p_4 \in [-\Lambda_4, \Lambda_4]$, $p \in [-\Lambda_{3cube}, \Lambda_{3cube}]$
$dp_4\ p^2 dp\ d\Omega_2$	$p_4 \in [-\Lambda_4, \Lambda_4]$, $p \in [0, \Lambda_{3sphere}]$
$dp_4 dp_3\ pdp\ d\Omega_1$	$p_4 \in [-\Lambda_4, \Lambda_4]$, $p \in [0, \Lambda_{pol}]$,$p_3 \in [-\Lambda, \Lambda]$

Geometry matters

They behave like cutoffs

	Λ_4	Λ	Λ_z
4 Cube	—	1027	—
3-Sphere	—	1027	—
3 Cube	∞	665	—
2 sphere	∞	665	—
Cylinder	∞	665	665



Geometry matters

Symmetry breaking is different

- The symmetry breaking depends on the geometry, not only on the volume

    - If one renormalizes, the divergence is an aspect of little matter

        - We do not have access to counterterms, we have to get the most of the non-divergent quantity .

Temperature

What if we have enough time…

- We introduced a new representation

- To include Temperature the Matsubara formalism must hold

$$p_0 \to \omega_n = \pi T(2n+1) \qquad \int dp_0 f(p_0) \to T \sum_{n=-\infty}^{\infty} f(\omega_n)$$

- This assumes thermalization

- And the difference must hold term to term

$$\Omega = \frac{1}{4G}\sigma^2 - \frac{2N_cN_f}{(2\pi)^3}\sum_n \int_0^\infty \frac{ds}{s}\left((2\Lambda)^3 e^{-s} - \pi^{3/2}\frac{e^{-sM^2}Erf(\sqrt{s}\Lambda)^3}{s^{3/2}}e^{-s\omega_n^2}\right)$$

Temperature

First Approach

- The sum is diverges because of the first term
  - We can take the first N terms
  - We loose the vacuum-medium separation
    - This is not good
      - If there is not a clear separation, the medium contribution induces unphysical phenomena

Allen PRD .92.074041 Tavares PRD.109.016011
Lopes arXiv:2507.14343 Duarte PRD.99.016005

Temperature

Separating the medium

- A way we can do this is using the Euler-MacLaurin Formula

$$f(x) = \int_0^\infty \frac{ds}{s}\left((2\Lambda)^3 e^{-s} - \left(\frac{\pi}{s}\right)^{3/2} Erf(\sqrt{s}\Lambda)^3 e^{-sM^2} e^{-s(\pi^2T^2)(2x+1)^2}\right)$$

$$\sum_{n=a}^{b} f(n) = \int_a^b f(x)\ dx + \frac{1}{2}\big(f(a)+f(b)\big) + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}\big(f^{(2k-1)}(b) - f^{(2k-1)}(a)\big) + R_m$$

Vacuum
Divergent!

Medium
Well Behaved

$$\Omega = \Omega_{tree} + \Omega^{(1)}_{vacuum}(\Lambda) + \Omega^{(1)}_{medium}(\Lambda)$$

Temperature

Separating the medium

- The medium still depends on the cutoff, although we can still reproduce some of the physics.

- This cutoff dependance disappears when taking the gap equation or other derivatives so the limit $\Lambda \to \infty$ can be taken.

Temperature

Comparison

$$\Omega = \frac{1}{4G}\sigma^2 - \frac{2N_cN_f}{(2\pi)^3}\sum_n \int_0^\infty \frac{ds}{s}\left((2\Lambda)^3 e^{-s} - \pi^{3/2}\frac{e^{-sM^2}Erf(\sqrt{s}\Lambda)^3}{s^{3/2}}e^{-s\omega_n^2}\right)$$

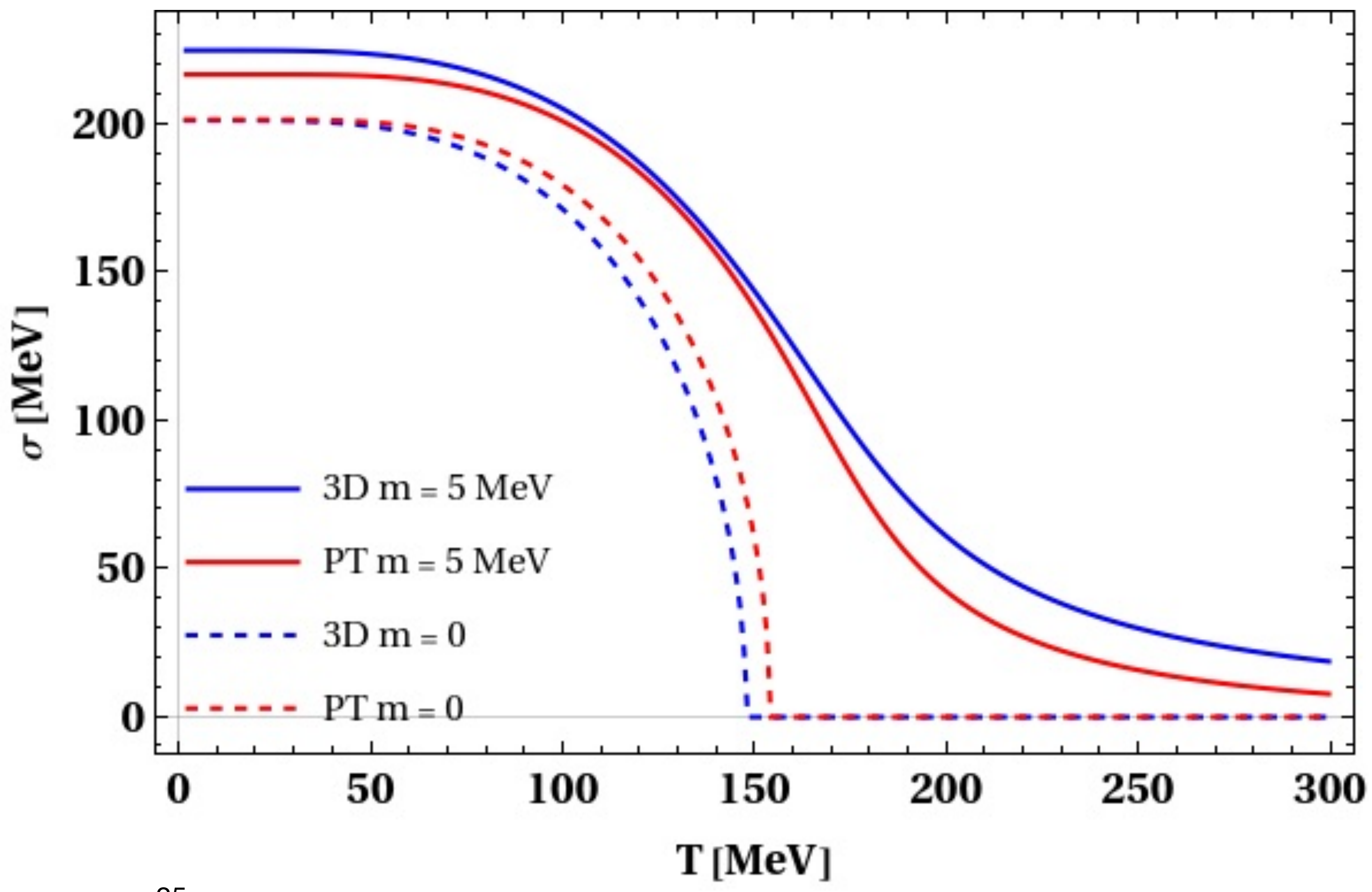
$$\Omega = \frac{\sigma^2}{4G} + \frac{N_cN_fT}{4\pi^{3/2}}\int_{1/\Lambda_s^2}^\infty \frac{ds}{s^{5/2}}e^{-sM^2}\theta_2\left(0, e^{-4\pi^2T^2s}\right).$$

Euler-Maclaurin
At sufficiently high N

The behavior is the same, and T_c varies a little

$G_{PT} = 3.14 \times 10^{-6} MeV^{-2}$ $\Lambda_{PT} = 1097 MeV$

$G_{3D} = 2.72 \times 10^{-6} MeV^{-2}$ $\Lambda_{3D} = 665 MeV$



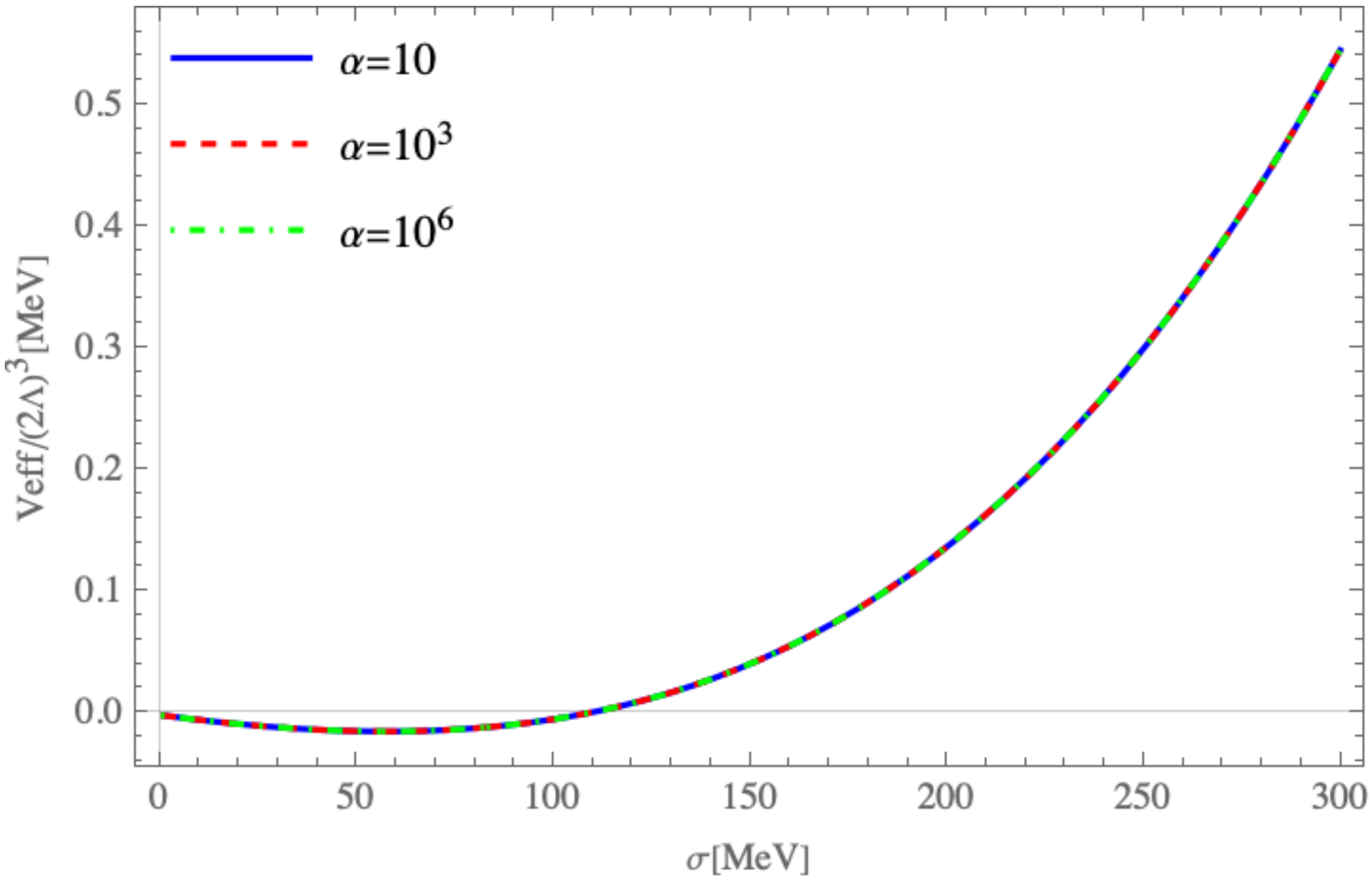
Summary and Conclusions

- We showed the effects of interchanging divergences and when working on a nonrenormalizable model.
  - We also showed that the regularization scheme takes a toll on the parameters.
- There is also a strong dependence on the way we integrate, so the geometry and the physical scenario gain more importance.

Thank you!
¡Gracias!
Grazie!

Backup

Dependence of a dimension full parameter



$$\int_{-\Lambda_p}^{\Lambda_p} \frac{dp_1 dp_2 dp_3 dp_4}{(2\pi)^4} \int_0^\infty \frac{ds}{s} \left(e^{-s\alpha^2} - e^{-s\left(p_4^2 + \omega^2 + M^2\right)} \right)$$