Time dependent density functional for open systems

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Ways to find DF:

1. Traditional, variational method:
   - energy $E[\langle 0|n(\boldsymbol{x})|0\rangle]$
   - ground state by minimizing E
   - static density
2. QFT, functional Legendre transformation
   - effective action $\Gamma[\langle 0|n(t,\boldsymbol{x})|0\rangle]$
   - space-time dependence by solving $\dfrac{\delta\Gamma[n]}{\delta n} = 0$
   - a systematic scheme

Three reasons to go beyond the usual framework of QFT:

1. To deal with effective dynamics with open channels (interactions!)
2. To handle IR and collinear contributions
   - QED: cancellation between real and virtual excitations (Bloch-Nordsiek, Yennie et. al, Kinoshita-Lee-Neuenberg)
   - QCD: no cancellation (χSB, confinement)
   - Collective soft modes in gas, fluid and plasma: large but finite contributions
3. Time dependence: PTC theorem
   - Causality (retardation)
   - Irreversibility, hydrodynamics

1. Expectation values
2. CTP
3. Examples
    3.1 Ideal fermi gas: Current-density functional
    3.2 QED: Current-density-EMF functional
4. CTP vs. the usual, amplitude based formalism
5. Summary & Outlook

Expectation values

Mixed states

Renormalization: elimination of degrees of freedom $\Longrightarrow$mixed state

Fock space: $\mathcal{H} = \mathcal{H}_{E<\Lambda} \otimes \mathcal{H}_{E>\Lambda} = \left\{\sum_{nN} c_{nN}|n\rangle \otimes |N\rangle\right\}$

$$
\begin{aligned}
\bar{\mathcal{O}} &= \langle 0|U^\dagger(t)\mathcal{O}U(t)|0\rangle \\
&= \mathrm{Tr}[\mathcal{O}\,\underbrace{U(t)|0\rangle\langle 0|U^\dagger(t)}_{\rho(t)}] \\
&= \sum_{nN}\langle N| \otimes \langle n|\mathcal{O}\rho(t)|n\rangle \otimes |N\rangle \\
&= \sum_{n}\langle 0| \otimes \langle n|\mathcal{O}\rho(t)|n\rangle \otimes |0\rangle + \mathcal{O}(\Lambda^{-1})
\end{aligned}
$$

- negligible effects in renormalizable theories where $\Lambda \to \infty$
- finite contributions in effective theories with finite Λ, like DF

Q: How to find expectation values?

A: Effective theory for the reduced density matrix

Expectation values

Radiation

1. Radiation: localised excited states tend to decay in time
   - $\not T$ in the effective charge dynamics

2. Infinitesimal $\not T$ in H_0:
   - Feynman's $i\epsilon$ in perturbation expansion:
     $H_0|n\rangle = E^{(0)}_n|n\rangle$, $E^{(0)}_n \to E^{(0)}_n - i\epsilon$, $n \geq 1$ in the free Hamiltonian
   - stable ground state: $\Im E^{(0)}_0 = 0$
   - projection onto the true vacuum, $|0\rangle$, by time evolution
   - Wick theorem $\Longrightarrow$matrix elements between the perturbative vacuum using $\langle 0_p|A|0_p\rangle$ only

3. Finite $\not T$ in H:
   - $\not T$: natural line width, expanding spherical waves only
   - irreversibility: spontaneous symmetry breaking by radiation
   - effective charge dynamics: $U^\dagger U \neq 1$
   - loss of time translation invariance of Heisenberg representation
     $$\begin{aligned}\cdots AU(t_A - t_B)B\cdots &= \cdots AU(t_A - t_i)U^{-1}(t_B - t_i)B\cdots \\ &\neq \cdots AU(t_A - t_i)U^\dagger(t_B - t_i)B\cdots\end{aligned}$$
   - loss of IR finiteness, protected by unitarity, a nonlinear identity

Q: How to preserve unitarity together with radiation energy loss?

A: Effective theory for the reduced density matrix
radiation leaves but unitarity is preserved by a mixed state

An environment can be:

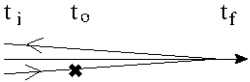
1. Virtual: $M_{env} > \Lambda$
   - localised polarisations: polaron, near field effects
   - closed, conservative effective dynamics
   - slave mixing
     - renormalization of transition amplitudes only
     - effect of mixing found in dressing, eg. form factors
2. Massive real: $M_{env} < \Lambda$
   - open channels: delocalized excitations, radiation
   - open effective dynamics
   - needs CTP for unitarity
3. Soft real: $M_{env} = 0 < \Lambda$
   - as the massive real environment
   - IR divergences suppressed in the mixed state by unitarity

CTP

Expectation values

$$
\begin{aligned}
\bar{\mathcal{O}} &= \langle 0|U^\dagger(t_1)\mathcal{O}U(t_1)|0\rangle \\
&= \mathrm{Tr}[\mathcal{O}U(t_1)|0\rangle\langle 0|U^\dagger(t_1)] \\
&= \mathrm{Tr}[U(t_2)\mathcal{O}U(t_1)|0\rangle\langle 0|U^\dagger(t_1+t_2)] \\
&= \mathrm{Tr}[T[\mathcal{O}_1 U(t_1+t_2)]|0\rangle\langle 0|U^\dagger(t_1+t_2)]
\end{aligned}
$$

Closed Time Path
J. Schwinger



Heisenberg representation: $\bar{\mathcal{O}} = \mathrm{Tr}[\mathcal{O} U \rho(t_i) U^\dagger]$

Independent perturbative construction of U and $U^\dagger$:

Reduplication of the source $\hat{j} = (j^+, j^-)$

the degrees of freedom $\hat{\phi} = (\phi^+, \phi^-)$

Generator functional:

$$
\begin{aligned}
e^{iW[\hat{j}]} &= \mathrm{Tr} T[e^{-i\int dtH + i\int dx j^+ \phi}]\rho(t_i) T^* e^{i\int dtH + i\int d^4x j^- \phi}] \\
&= \int D[\hat{\phi}] e^{iS[\phi^+] - iS^*[\phi^-] + i\int dx \hat{j}\hat{\phi}} \\
&= \int D[\hat{\phi}] e^{iS_{CTP}[\hat{\phi}] + i\int dx \hat{j}\hat{\phi}}
\end{aligned}
$$

Trace $\Longrightarrow$boundary condition at the final time: $\phi(t_f, \boldsymbol{x})^+ = \phi(t_f, \boldsymbol{x})^-$

Closed Time Path $\Longrightarrow$**Open Time Path**

$$\begin{aligned}\rho[\psi^+,\psi^-] &= \langle\psi^+|T[e^{-i\int dtH+i\int dxj^+\phi}]\rho(t_i)T^*e^{i\int dtH+i\int d^4xj^-\phi}]|\psi^-\rangle\\ &= \int D[\hat\phi]e^{iS[\phi^+]-iS^*[\phi^-]} \qquad \text{with } \phi^\pm(t_f,\boldsymbol{x})=\psi^\pm(\boldsymbol{x})\end{aligned}$$

$\nearrow$

contribution of $\hat\phi=(\phi^+,\phi^-)$ to the **density matrix**

$$\bar{\mathcal{O}} = \mathrm{Tr}[T[\mathcal{O}[\phi]U]\rho(t_i)U^\dagger] = \mathcal{O}\left[\frac{\delta}{\delta i j^+}\right]\int D[\hat{\phi}]e^{iS[\phi^+]-iS^*[\phi^-]+i\int dx\hat{j}\hat{\phi}}{}_{|\hat{j}=0}$$
$$= \mathrm{Tr}[U\rho(t_i)T^*[U^\dagger\mathcal{O}[\phi]]] = \mathcal{O}\left[\frac{\delta}{\delta i j^-}\right]\int D[\hat{\phi}]e^{iS[\phi^+]-iS^*[\phi^-]+i\int dx\hat{j}\hat{\phi}}{}_{|\hat{j}=0}$$

Physical variable: $\langle\phi^+\rangle = \langle\phi^-\rangle$, $\phi = \frac{1+\kappa}{2}\phi^+ + \frac{1-\kappa}{2}\phi^-$

Decoherence of ϕ: $\phi^d = \phi^+ - \phi^-$ $\leftarrow$ **quantum fluctuations**

$$j^\pm = \frac{j}{2} \pm \left(j^p + \frac{\kappa}{2}j\right), \quad j^+\phi^+ + j^-\phi^- = j\phi + j^d\phi^d$$

j^p: physical external source with unitary time evolution
j: book-keeping variable, set to zero at the end

κ is arbitrary

$\kappa \neq 0$ is needed for linear EoM
Keldysh: $\kappa = 0$

System: ϕ, environment: χ, $S[\phi,\chi]=S_s[\phi]+S_e[\phi,\chi]$

$$\begin{aligned} e^{iW[\hat{j}]} &= \int D[\hat{\phi}]D[\hat{\chi}]e^{iS[\phi^+,\chi^+]-iS[\phi^-,\chi^-]+i\hat{j}\cdot\hat{\phi}} \\ &= \int D[\hat{\phi}]e^{iS_c[\phi^+]-iS_c[\phi^-]-iS_o[\phi^+,\phi^-]+i\hat{j}\cdot\hat{\phi}} \end{aligned}$$

Unicity: $\frac{\delta^2 S_o[\phi^+,\phi^-]}{\delta\phi^+\delta\phi^-}\neq 0$

- $S_c[\phi]$: virtual environment, renormalizations, closed system
- $S_o[\phi^+,\phi^-]$: real environment, open dynamics, dissipation, etc.

Real (mass-shell) system-environment interactions
mapped into
interactions between the CTP doublets

Ideal fermi gas

System-environment separation without factorization in linear space

Bosonization by operator product expansion:

$$\begin{aligned}
\psi^\dagger\left(x-\frac{y}{2}\right)\psi\left(x+\frac{y}{2}\right) &= \sum_{jn}\frac{(-1)^{j-n}}{(j-n)!n!}\partial^{j-n}\psi^\dagger(x)\partial^n\psi(x)y^j \\
&= \sum_j O_j(x)y^j \\
O_j(x) &= \sum_{0\le n\le j}\frac{(-1)^{j-n}}{(j-n)!n!}\partial^{j-n}\psi^\dagger(x)\partial^n\psi(x)
\end{aligned}$$

Change of variable: $(\psi^\dagger,\psi)\to O_j$

System: $j=0$ density $\rho=\psi^\dagger\psi$

$j=1$: current $\boldsymbol{j}=-\frac{i\hbar}{2m}[\psi^\dagger\boldsymbol{\nabla}\psi-(\boldsymbol{\nabla}\psi^\dagger)\psi]$

Environment: $j\ge 2$

Factorisation is not needed, $\mathcal{H}\neq\mathcal{H}_{sys}\otimes\mathcal{H}_{env}$

Observable: $J^\mu = (\rho, \boldsymbol{j}) = \psi^\dagger K^\mu \psi$

$$
\begin{aligned}
e^{iW[\hat{a}]} &= \mathrm{Tr} T[e^{-i\int dtH + \int dx a^+ J}]\rho(t_i) T^*[e^{i\int dtH + i\int dx a^- J}] \\
&= \int D[\hat{\psi}] D[\hat{\psi}^\dagger] e^{i\hat{\psi}^\dagger[\hat{G}_0^{-1} + \hat{a}K + \frac{\hat{\boldsymbol{a}}^2}{2m}]\hat{\psi}} \\
\hat{G}_0^{-1} &= \begin{pmatrix} G_0^{-1} & 0 \\ 0 & -G_0^{-1*} \end{pmatrix} + G_{\mathrm{BC}}^{-1}, \quad G_0^{-1} = i\partial_t + \frac{\Delta}{2m} + \mu
\end{aligned}
$$

Generator functional:

$$
W[\hat{a}] = -i \mathrm{Tr} \ln \left(\hat{G}_0^{-1} + \hat{a}K + \frac{\hat{\boldsymbol{a}}^2}{2m} \right)
$$

Ideal fermi gas

Nonpolynomial "interaction"

$$\begin{aligned} W[\hat{a}] &= -i\mathrm{Tr}\ln\left(\hat{G}_0^{-1} + \hat{a}K + \frac{\hat{\boldsymbol{a}}^2}{2m}\right) \\ &= -i\mathrm{Tr}\ln\hat{G}_0^{-1} - i\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathrm{Tr}\left[\hat{G}_0\left(\hat{a}K + \frac{\hat{\boldsymbol{a}}^2}{2m}\right)\right]^n \end{aligned}$$

Non-polynomial form:
- connected graphs in arbitrary orders
- no small parameter to truncate
- more troublesome than real interactions

Effective action:

$$\Gamma[\hat{J}] = W[\hat{a}] - \int d^4x\hat{a}(x)\hat{J}(x), \quad \hat{J} = \frac{\delta W[\hat{a}]}{\delta\hat{a}}$$

Interaction vertices in arbitrary high orders:
- J couples to infinitely many normal modes
- independent particle modes are hidden when bosonised quantities are observed
- simplicity returns in bulk observables

Ideal fermi gas

Classical toy model

Free particles: $x = (x_1, \dots, x_N)$, $S[x] = \frac{1}{2}\int dt x D^{-1} x$

Collective coordinate: $y = F(x)$

Effective dynamics for y:

$$S[x, y] = \int dt \left[\frac{1}{2} x D^{-1} x - \lambda(y - F(x))^2\right]$$

$F = \mathcal{O}(x^2) \Longrightarrow S = \mathcal{O}(x^4)$

"Interaction" like correlations in an ideal gas

when diagnosed by density-current (a composite operator)
from wave function to density
normal modes lost

Ideal fermi gas

Generator functional in $\mathcal{O}(\hat{a}^2)$

$$
\begin{aligned}
W[\hat{a}] &= -i\mathrm{Tr}\ln\left(\hat{G}_0^{-1} + \hat{a}K + \frac{\hat{\boldsymbol{a}}^2}{2m}\right) \\
&= \hat{a}\hat{J}_v - \frac{1}{2}\hat{a}\left(\hat{D} - \hat{S}\right)\hat{a} + \mathcal{O}(\hat{a}^3), \quad S^{\sigma\sigma'} = \delta^{\sigma\sigma'}\frac{\hbar^2\rho_0}{m}\begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
J_v^{\sigma\mu} &= -i\mathrm{tr}[\hat{G}_0^{\sigma\sigma}(x,x)K^\mu] = (\rho_0, \mathbf{0}) \\
D_{\mu\mu'}^{\sigma\sigma'}(x,x') &= -i\mathrm{tr}[\hat{G}_0^{\sigma\sigma'}(x,x')K_{\mu'}\hat{G}_0^{\sigma'\sigma}(x',x)K_\mu]
\end{aligned}
$$

Block-structure of CTP propagators:

$$
\begin{aligned}
\hat{G}(x,y) &= \begin{pmatrix} G^{++} & G^{+-} \\ G^{-+} & G^{--} \end{pmatrix} = -i\begin{pmatrix} \langle T[J(x)J(y)]\rangle & \langle J(y)J(x)\rangle \\ \langle J(x)J(y)\rangle & \langle T[J(y)J(x)]\rangle^* \end{pmatrix} \\
&= \begin{pmatrix} G^n & -G^f \\ G^f & -G^n \end{pmatrix} + iG^i\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad G^{\substack{r \\ a}} = G^n \pm G^f
\end{aligned}
$$

Within a time axis: $G^{++} = \langle T[JJ]\rangle = G_F$ Feynman propagator?

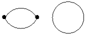
Coupling time axes: $G^{-+} = \langle JJ\rangle = W$ Wightmann function, spectral weight **makes the system open**

Ideal fermi gas

Feynman graphs

Ideal gas: One-loop current Green function

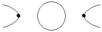
Tr[U ρ_i $U^\dagger$]



$G_F = G^{++} = G^n + iG$

Feynman propagator

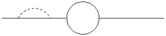
Tr[U ρ_i $U^\dagger$]



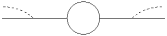
$W = G^{-+} = G^f + iG$

Spectral density

QED: One-loop electron self energy



- virtual state
- localized vacuum polarizations
- leave the system closed
- within a time axis



- real, mass-shell state
- delocalized radiation
- opens the system
- in between the time axes

Lines at t_f: Particles contributing in $Tr[U\rho_i U^\dagger] = \sum_n \langle n|U\rho_i U^\dagger|n\rangle$

Ideal fermi gas

Physical variables, quantum fluctuations

Physical field: $J = \frac{1+\kappa}{2} J^+ + \frac{1-\kappa}{2} J^-$

Quantum fluctuations: $J^d = J^+ - J^-$ ← **decoherence**

Source: $a^\pm = \frac{a}{2} \pm \left(a^p + \frac{\kappa}{2} a\right), \qquad a^+ J^+ + a^- J^- = aJ + a^p J^d$

Effective action:

$$\begin{aligned}\Gamma[\hat{J}] &= W[\hat{a}] - Ja - J^d a^p \\ &= J^d \frac{1}{D^r - S} J_v - \frac{1}{2}(J, J^d) \begin{pmatrix} 0 & \frac{1}{D^a - S} \\ \frac{1}{D^r - S} & \frac{\kappa}{D^r - S}(S - D^n)\frac{1}{D^a - S} \end{pmatrix} \begin{pmatrix} J \\ J^d \end{pmatrix}\end{aligned}$$

Equations of motion:

$$-a = \frac{\delta \Gamma[J, J^d]}{\delta J}, \qquad -a^p = \frac{\delta \Gamma[J, J^d]}{\delta J^d}$$

for $a = 0$:

$$\begin{aligned} 0 &= \frac{1}{D^a - S} J^d \quad \rightarrow \quad J^d = 0 \\ a^p &= \frac{1}{D^r - S}(J - J_v) \end{aligned}$$

Ideal fermi gas

Linearized EoM in the gradient Laurent expansion

Fourier space: $(\rho, \boldsymbol{j}) = J - J_0$, $\boldsymbol{j} = \boldsymbol{j}_T + \boldsymbol{n} j_L$, $\boldsymbol{n} = \frac{\boldsymbol{q}}{|\boldsymbol{q}|}$, $\omega\rho = q j_L$

$$
\begin{aligned}
\frac{n_s m k_F}{4\pi^2} a^0 &= \left(c_0 + c_z \frac{m}{k_F} \frac{i\omega}{q} + \frac{c_{qq}}{k_F^2} q^2 \right) \rho \\
\frac{n_s m k_F}{4\pi^2} \frac{k_F^2}{m^2} \boldsymbol{a}^T &= \left(b_0 + b_z \frac{m}{k_F} \frac{i\omega}{q} + \frac{b_{qq}}{k_F^2} q^2 \right) \boldsymbol{j}^T
\end{aligned}
$$

$$
\begin{array}{ccc}
\uparrow & \uparrow & \uparrow \\
D^n & D^f & D^n
\end{array}
$$

$\nearrow$ absent in finite volume, spontaneous $\not{T}$

Loop-integral:

$$
\begin{aligned}
c_0 &= -\frac{1}{2}, \quad c_z = -\frac{\pi}{4}, \quad c_{zz} = \frac{\pi^2 - 8}{8}, \quad c_{qq} = -\frac{1}{24} \\
b_0 &= -\frac{3}{2}, \quad b_z = -\frac{9\pi}{4}, \quad b_{zz} = \frac{4\pi^2 - 27}{6}, \quad b_{qq} = -\frac{3}{8}
\end{aligned}
$$

Ideal fermi gas
Hydrodynamics

Real space, $a = 0$:

$$\int d^3y \frac{\partial_t \rho(\boldsymbol{y})}{(\boldsymbol{x}-\boldsymbol{y})^2} = \frac{2\pi^2 k_F}{m b_z}\left(c_0 - \frac{c_{qq}}{k_F^2}\Delta\right)\rho(\boldsymbol{x}) + \mathcal{O}(\partial_t^2) + \mathcal{O}(\Delta^2) + \mathcal{O}(J^2)$$

$$\int d^3y \frac{\partial_t \boldsymbol{j}(\boldsymbol{y})}{(\boldsymbol{x}-\boldsymbol{y})^2} = \frac{2\pi^2 k_F}{m b_z}\left(b_0 - \frac{b_{qq}}{k_F^2}\Delta\right)\boldsymbol{j}(\boldsymbol{x}) - \boldsymbol{\nabla}\Phi(\boldsymbol{x}) + \cdots$$

$\uparrow$ D^f $\quad$ $\uparrow$ D^n $\quad$ $\uparrow$ $\mathcal{O}(\hbar)$ $\quad$ $\uparrow$ "Pressure"

$$\Phi(\boldsymbol{x}) = \frac{k_F^2}{c_z b_z m^2}\left(c_0 b_0 - \frac{c_0 b_{qq} + b_0 c_{qq}}{k_F^2}\Delta_x\right)\int d^3y \frac{\rho(\boldsymbol{y})}{(\boldsymbol{x}-\boldsymbol{y})^2}$$

- No need of thermodynamics to close
- Irreversibility (spontaneous breakdown of time reversal invariance)
- Dissipative hydrodynamics (no gap for particle-hole excitations)
- Sub-classical fields (not decohered)

QED

One-loop approximation

$$
\begin{aligned}
e^{iW[a,j]} &= \int D[\hat{\psi}]D[\hat{\bar{\psi}}]D[\hat{A}]e^{i\hat{\bar{\psi}}[\hat{G}_0^{-1}+\hat{\not a}-e\sigma\hat{\not A}]\hat{\psi}+\frac{i}{2}\hat{A}\hat{D}_0^{-1}\hat{A}+i(\hat{n}+\hat{j})\hat{A}} \\
&= \int D[\hat{A}]e^{\mathrm{Tr}\ln\hat{G}_0^{-1}[\hat{a}]+\frac{i}{2}\hat{A}\hat{D}^{-1}[\hat{a}]\hat{A}+i\hat{j}\hat{A}+\mathcal{O}(\hat{A}^3)}
\end{aligned}
$$

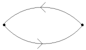
with $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Electron propagator:

$$
\hat{G}_0[\hat{a}] = \frac{1}{\hat{G}_0^{-1} + \hat{\not a}}
$$

Photon propagator :

$$
\begin{aligned}
\hat{D}[\hat{a}] &= \frac{1}{\hat{D}_0^{-1} - \hat{\Pi}[\hat{a}]} \\
\hat{\Pi}[\hat{a}] &= e^2\sigma\hat{G}[\hat{a}]\sigma \\
G_{(\sigma x\mu),(\sigma' y\nu)}[a] &= -i\mathrm{tr}(G_0^{\sigma'\sigma}{}_{yx}[a]\gamma_\mu G_0^{\sigma\sigma'}{}_{xy}[a]\gamma_\nu)
\end{aligned}
$$



QED

One-loop, quadratic current-density-EMF functional

$$e^{iW[a,j]} = \int D[\hat{A}]e^{\mathrm{Tr}\ln\hat{G}_0^{-1}[\hat{a}]+\frac{i}{2}\hat{A}\hat{D}^{-1}[\hat{a}]\hat{A}+i\hat{j}\hat{A}+\mathcal{O}(\hat{A}^3)}$$

$$W[\hat{a},\hat{j}] = -i\mathrm{Tr}\ln\hat{G}_0^{-1}[\hat{a}]+\frac{i}{2}\mathrm{Tr}\ln\hat{D}^{-1}[\hat{a}]-\frac{1}{2}(\hat{j}+e\hat{a}\hat{G}\sigma)\hat{D}[\hat{a}](\hat{j}+e\sigma\hat{G}\hat{a})$$

$$= -\frac{1}{2}\,(\hat{a},\hat{j})\begin{pmatrix}\hat{G} & e\hat{G}\sigma\hat{D}_0\\ e\hat{D}_0\sigma\hat{G} & \hat{D}\end{pmatrix}\begin{pmatrix}\hat{a}\\ \hat{j}\end{pmatrix},\quad \hat{G}=\hat{G}[0],\quad \hat{D}=\hat{D}[0]$$

Effective action: $W[\hat{a},\hat{j}]=\Gamma[\hat{J},\hat{A}]+\hat{a}\hat{J}+\hat{j}\hat{A}$, $\hat{J}=\frac{\delta W}{\delta\hat{a}}$, $\hat{A}=\frac{\delta W}{\delta\hat{j}}$

$$\Gamma[\hat{J},\hat{A}]=\frac{1}{2}\hat{J}\hat{G}^{-1}\hat{J}+\frac{1}{2}\hat{A}\hat{D}_0^{-1}\hat{A}-e\hat{A}\sigma\hat{J}$$

$\uparrow$

ideal gas

- Higher orders by FRG
- EoM (inverse Legendre transformation): $\frac{\delta\Gamma}{\delta\hat{J}}=-\hat{a}$, $\frac{\delta\Gamma}{\delta\hat{A}}=-\hat{j}$
- MHD

$$\Gamma[\hat{J},\hat{A}] = \frac{1}{2}\hat{J}\hat{G}^{-1}\hat{J} + \frac{1}{2}\hat{A}\hat{D}_0^{-1}\hat{A} - e\hat{A}\sigma\hat{J}$$

EoM for the EM field: $\hat{A} = e\hat{D}_0\sigma\hat{J}$

$$\Gamma[\hat{J}] = \frac{1}{2}\hat{J}\hat{G}_t^{-1}\hat{J}, \quad \hat{G}_t^{-1} = \hat{G}^{-1} - e^2\sigma\hat{D}_0\sigma$$

Physical fields:

$$\Gamma[J,J^d] = \Re W[a,a^p] - Ja - J^d a^p = \frac{1}{2}JG_t^{a-1}J^d + \frac{1}{2}J^d G_t^{r-1}J$$

EoM: $a^p = -G_t^{r-1}J$ $\leftarrow$ **Retardation** $\nearrow$

CTP vs. the usual, amplitude based formalism

EMF functional by Schwinger-Dyson partial resummation

When is $U|0\rangle = |0\rangle \Longrightarrow \mathrm{Tr}T[UA]|0\rangle\langle 0|U^\dagger] = \langle 0|T[UA]|0\rangle$ correct?

Usual formalism:

$$\Gamma[A] = \frac{1}{2}AD^{-1}A, \quad D^{-1} = D_0^{-1} - \Pi, \quad \Pi = e^2 G$$

CTP:

$$\begin{aligned}
\Gamma[\hat{A}] &= \frac{1}{2}\hat{A}\hat{D}^{-1}\hat{A}, \quad \hat{D}^{-1} = \hat{D}_0^{-1} - \hat{\Pi}, \quad \hat{\Pi} = e^2\sigma\hat{G}\sigma \\
\hat{\Pi} &= \sigma\begin{pmatrix} \Pi^n + i\Pi^i & -\Pi^f + i\Pi^i \\ \Pi^f + i\Pi^i & -\Pi^n + i\Pi^i \end{pmatrix}\sigma, \quad \Pi^{\substack{r\\a}} = \Pi^n \pm \Pi^f \\
\hat{D}_0^{-1} &= \sigma\begin{pmatrix} D_0^{-1n} + iD_0^{-1i} & -D_0^{-1f} + iD_0^{-1i} \\ D_0^{-1f} + iD_0^{-1i} & -D_0^{-1n} + iD_0^{-1i} \end{pmatrix}\sigma, D_0^{-1\substack{r\\a}} = D_0^{-1n} \pm D_0^{-1f} \\
D^{\substack{r\\a}} &= \frac{1}{D_0^{-1\substack{r\\a}} - \Pi^{\substack{r\\a}}}, \qquad D^i = -D^r(D_0^{-1i} - \Pi^i)D^a
\end{aligned}$$

as if Feynman rules held for the S-D resummed retarded propagator

CTP vs. the usual, amplitude based formalism

Separation of the block-diagonal, single time axis part

Self energy:

$$
\begin{aligned}
i\hat{G}(x,y) &= \begin{pmatrix} \langle T[J(x)J(y)]\rangle & \langle J(y)J(x)\rangle \\ \langle J(x)J(y)\rangle & \langle T[J(y)J(x)]\rangle^* \end{pmatrix} = i\begin{pmatrix} G_F & W^{tr} \\ W & -G_F^* \end{pmatrix} \\
\hat{\Pi} &= e^2\sigma\hat{G}\sigma = \hat{\Pi}_d + \hat{\Pi}_{nd} = e^2\begin{pmatrix} G_F & 0 \\ 0 & G_F^* \end{pmatrix} - e^2\begin{pmatrix} 0 & W^{tr} \\ W & 0 \end{pmatrix}
\end{aligned}
$$

Full propagator:

$$\hat{D} = \frac{1}{\hat{D}_0^{-1} - \hat{\Pi}} = \frac{1}{\hat{D}_d^{-1} - \hat{\Pi}_{nd}}$$

Block diagonal propagator:

$$\hat{D}_d = \begin{pmatrix} D_F & 0 \\ 0 & -D_F^* \end{pmatrix}, \quad D_F = \frac{1}{D_{0F}^{-1} - e^2 G_F}$$

as close as one can get to the usual, amplitude based formalism

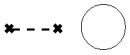
CTP vs. the usual, amplitude based formalism

Exclusive vs. inclusive contributions

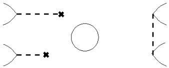
$$\hat{D} = \begin{pmatrix} D_F & e^2|D_F|^2(W_G^f - iW_G^i) \\ e^2|D_F|^2(W_G^f + iW_G^i) & -D_F^* \end{pmatrix} \frac{1}{1 - |D_F|^2(W_G^{f2} + W_G^{i2})}$$

$\nearrow$ S-D resummation

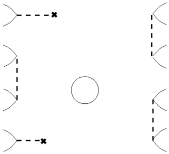
D_F: exclusive, $\langle 0| \ldots |0\rangle$



$D_F Y^* Y$: inclusive
$\langle (e^- e^+)^2| \ldots |(e^- e^+)^2\rangle$



$D_F (Y^* Y)^2$: inclusive
$\langle (e^- e^+)^3| \ldots |(e^- e^+)^3\rangle$

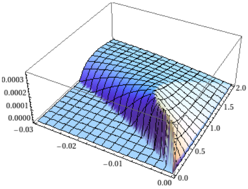


Inclusive contributions: $D^{++} \neq D_F$ (open chanels$\Longrightarrow$interference terms)

CTP vs. the usual, amplitude based formalism

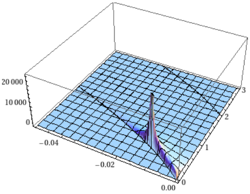
Approximative cancellation between soft real and virtual exciations

For $k_F = 0.02m$, on the $(\frac{\omega}{k_F}, \frac{|\boldsymbol{q}|}{k_F})$ plane:

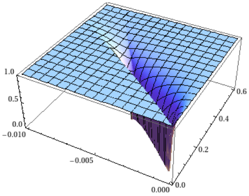


Spectral function for electron-hole pairs:

$\langle JJ \rangle = iG^{-+}$



Spectral function for photons:
$\langle AA \rangle = iD^{-+}$
Peak where zero sound and plasmon meet



$\frac{1}{1-|D_F|^2(W_G^{f2}+W_G^{i2})} = \frac{D^{++}}{D_F} = \frac{D^{-+}}{D^{-+}_{excl}}$

Suppression by real excitations

Soft excitations (zero sound):

- partial cancellation between real and virtual soft modes
  - $|D^{++}| < |D_F|$
  - full spectral weight < exclusive spectral weight
- electron propagator: qualitatively similar

Summary & Outlook

DF by functional Legendre transformation in CTP:

1. Problems with mixed states and open channels in usual QFT
2. Need of density matrix $\Longrightarrow$CTP
3. CTP
   3.1 Time dependence
   3.2 Causality (retardation) ?
   3.3 Irreversibility as spontaneous $\not{T}$
4. Non-trivial density-current dynamics for ideal gas
5. Schwinger-Dyson resummed propagators in electron gas
   5.1 Differences between CTP and the usual, amplitude based formalism
   5.2 Partial cancellations among real and virtual degenerate excitations
   An extension of the Kinoshita-Lee-Nauenberg theorem?
   5.3 No cancellation in the usual formalism
   $\Longrightarrow$S-D resummation is consistent in CTP only

Promising feature: nonperturbative contributions by FRG
cf. S. Kemler, J. Braun, poster