QCD Critical Point: A Theoretical View

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Historical Theoretical View of the QCD Critical Pt

- Expectations for a proper (first order) phase transition
  - From Reinhard Stock's talk on Monday:

Cabibbo and Parisi 1975

Volume 59B, number 1 PHYSICS LETTERS 13 October 1975

EXPONENTIAL HADRONIC SPECTRUM AND QUARK LIBERATION

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Istituto di Fisica, Universitá di Roma,
Istituto Nazionale di Fisica Nucleare, Sezione di Rome, Italy

G. PARISI
Istituto Nazionale di Fisica Nucleare, Frascati, Italy

Received 9 June 1975

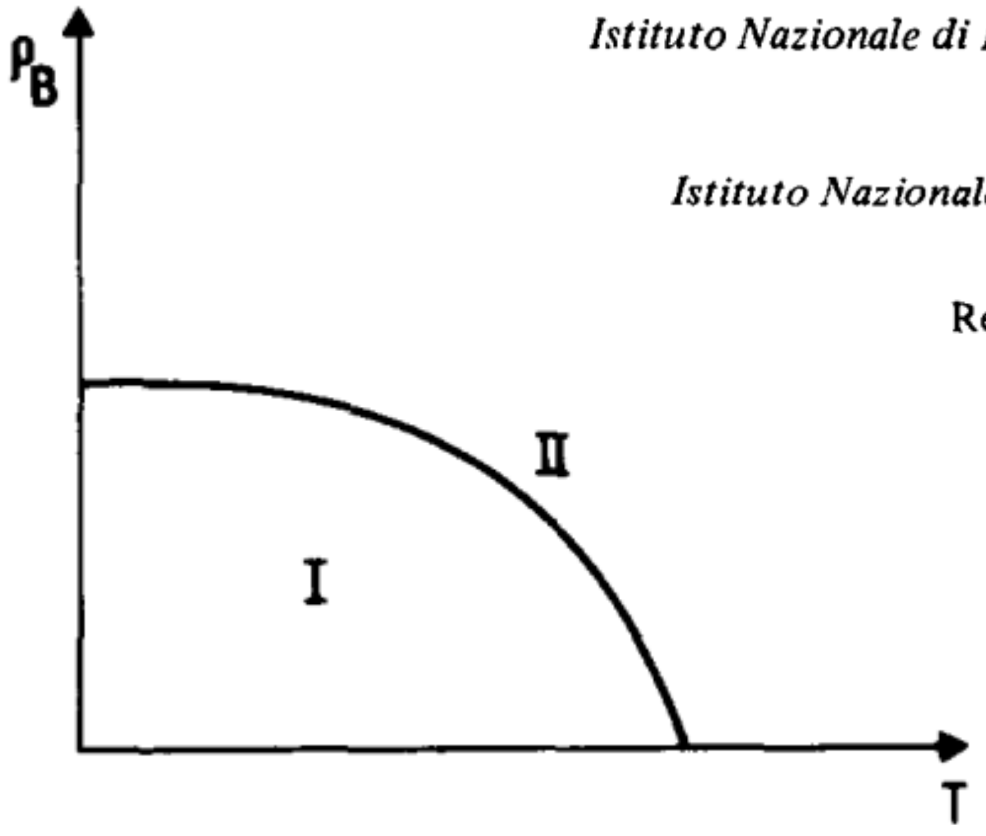

Fig. 1. Schematic phase diagram of hadronic matter. ρ_B is the density of baryonic number. Quarks are confined in phase I and unconfined in phase II.

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- Theoretical efforts on the nature of the QCD phase transition

  - Early efforts

[10] Robert D. Pisarski and Frank Wilczek. Remarks on the chiral phase transition in chromodynamics. *Phys. Rev.*, D29:338–341, 1984.

[11] T. Celik, J. Engels, and H. Satz. The order of the deconfinement transition in su(3) yang- mills theory. *Phys. Lett.*, B125:411–414, 1983.

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[19] T. Herpay, A. Patkos, Zs. Szep, and P. Szepfalusy. Mapping the boundary of the first order finite temperature restoration of chiral symmetry in the (m(pi) - m(k))-plane with a linear sigma model. *Phys. Rev.*, D71:125017 [15 pages], 2005.

  - Physical point: Aoki et al (2006)

Rapid crossover!

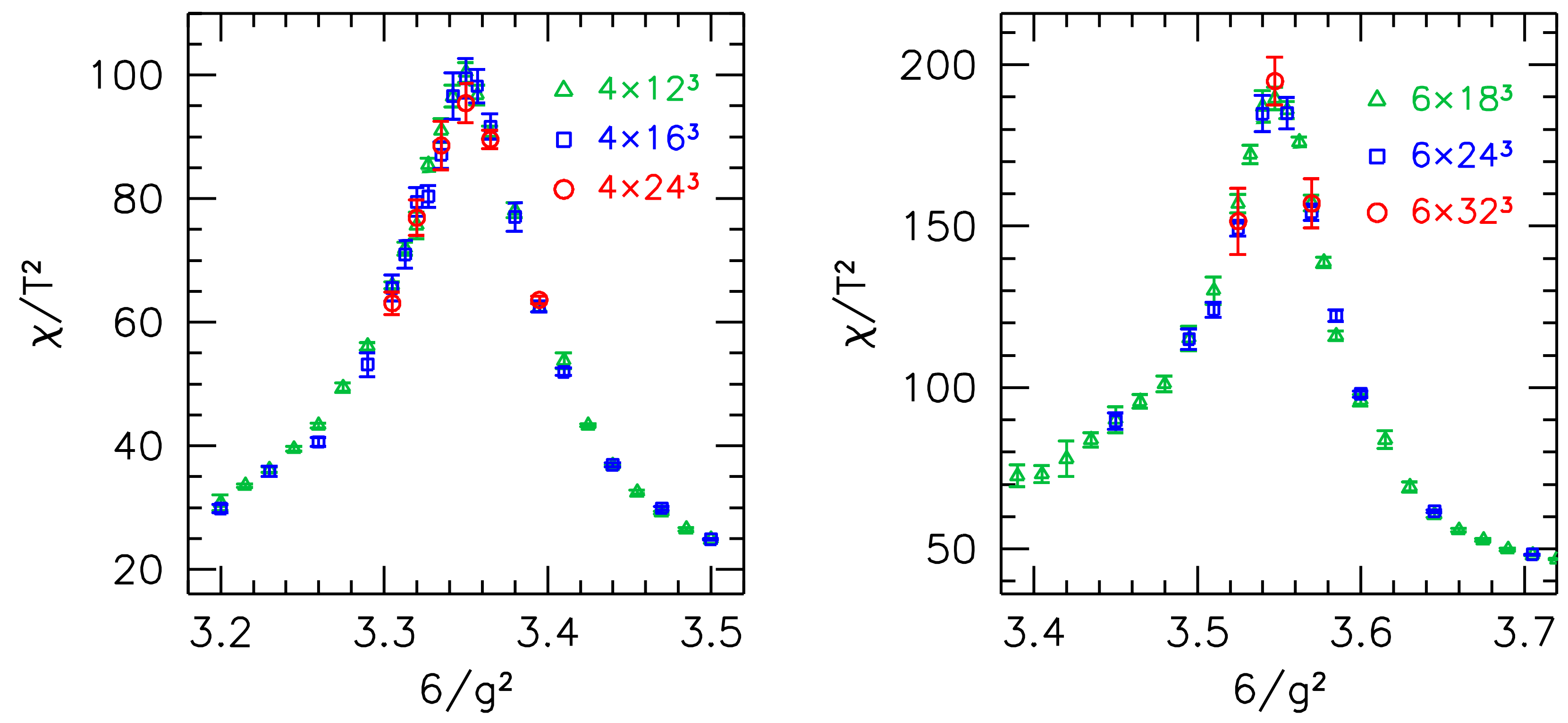

Figure 1: Susceptibilities for the light quarks for N_t=4 (left panel) and for N_t=6 (right panel) as a function of $6/g^2$, where g is the gauge coupling (T grows with $6/g^2$). The largest volume is eight times bigger than the smallest one, so a first-order phase transition would predict a susceptibility peak that is eight times higher (for a second-order phase transition the increase would be somewhat less, but still dramatic). Instead of such a significant change we do not observe any volume dependence. Error bars are s.e.m.

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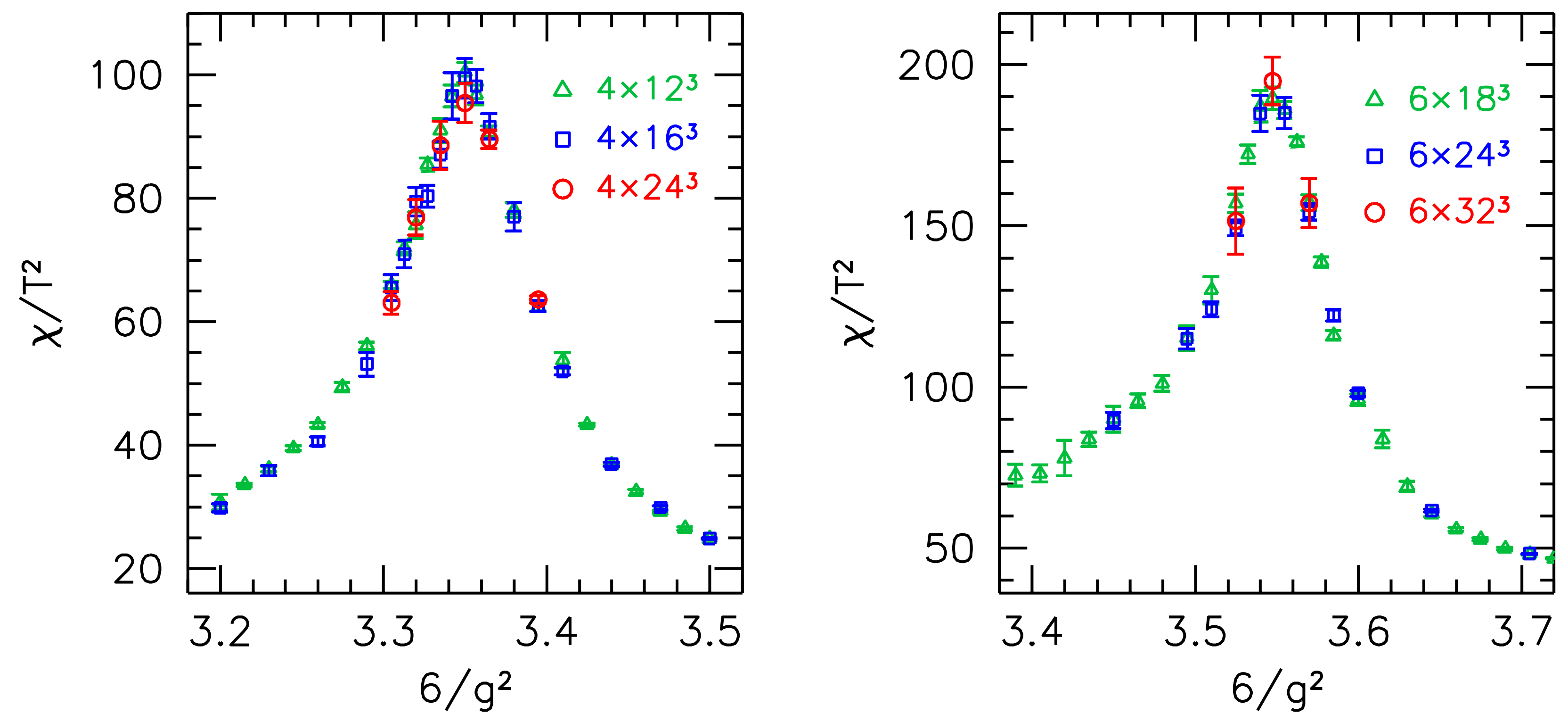

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What about finite density?

Historical Theoretical View of the QCD Critical Pt

- Change in the order of the transition → critical point: enter universality classes
  - Static: 3D Ising - Rajagopal & Wilczek, Nucl.Phys.B (1993)
  - Dynamic: Model H - Son & Stephanov, Phys.Rev.D (2004)
  - Scaling equation of state of 3D Ising model - Guida & Zinn-Justin, Nucl.Phys.B 489 (1997) based Josephson-Schofield (1969) parametric equation of state

Exponent	Definition
α	$C \propto (T - T_c)^{-\alpha}$
β	$M \propto (T_c - T)^{\beta}$
γ	$\chi \propto (T - T_c)^{-\gamma}$
δ	$M \propto h^{1/\delta}$
ν	$\xi \propto (T - T_c)^{-\nu}$
η	$\Gamma(n) \propto \lvert n \rvert^{2-d-\eta}$

Historical Theoretical View of the QCD Critical Pt

- Fluctuations serve as critical signal (diverging ξ):
  - M. Stephanov, K. Rajagopal and E. Shuryak, PRL (1998)
  - From Mesut Arslandok's talk on Friday:

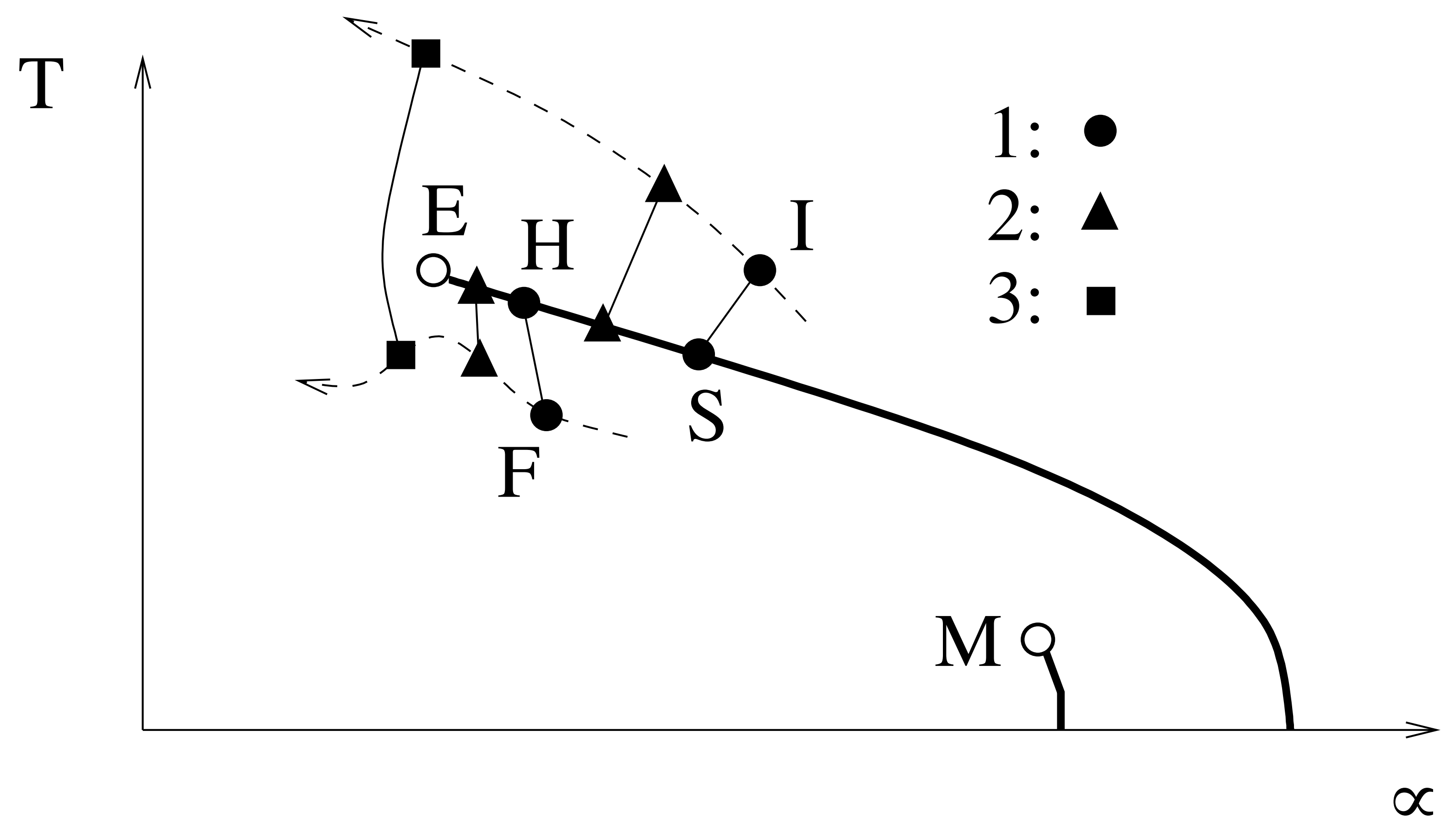
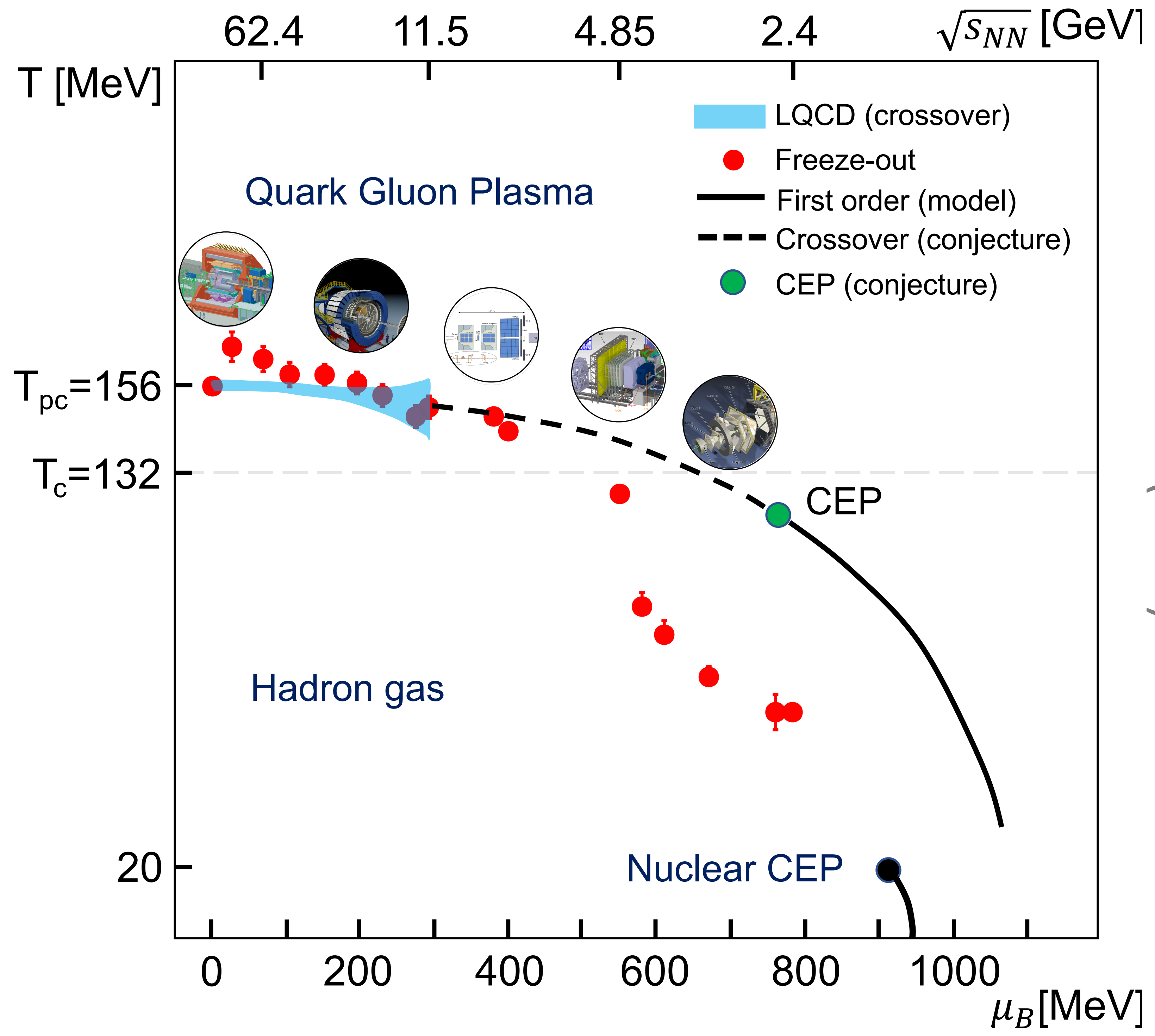


FIG. 2. Schematic examples of three possible trajectories for three values of x on the phase diagram of QCD (see. Fig. 1). The points I, S, H and F on different trajectories are marked with different symbols. The dashed lines show the locations of the initial, I, and final, F, points as x is increased in the direction shown by the arrows.



P. B.-Munzinger, A. Rustamov, J. Stachel
EPJC 83 (2023) 1125

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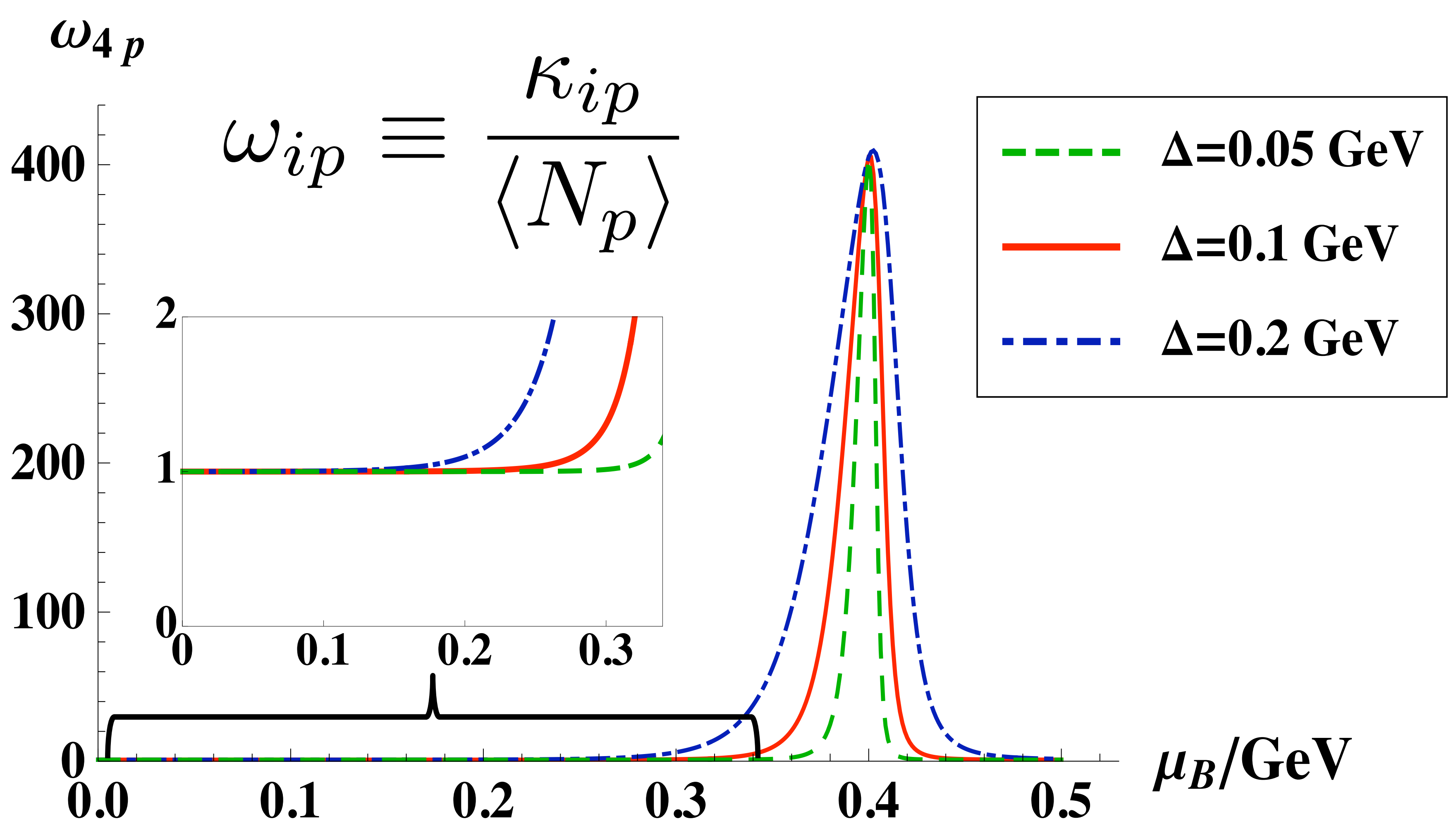
  - …
  - M. Stephanov, PRL (2009) & PRL (2011)

$$\kappa_2 = \langle \sigma^2 \rangle = VT\xi^2$$

$$\kappa_3 = \langle \sigma^3 \rangle = 2\lambda_3 VT^2 \xi^6 = 2\tilde{\lambda}_3 VT^{3/2}\xi^{9/2}$$

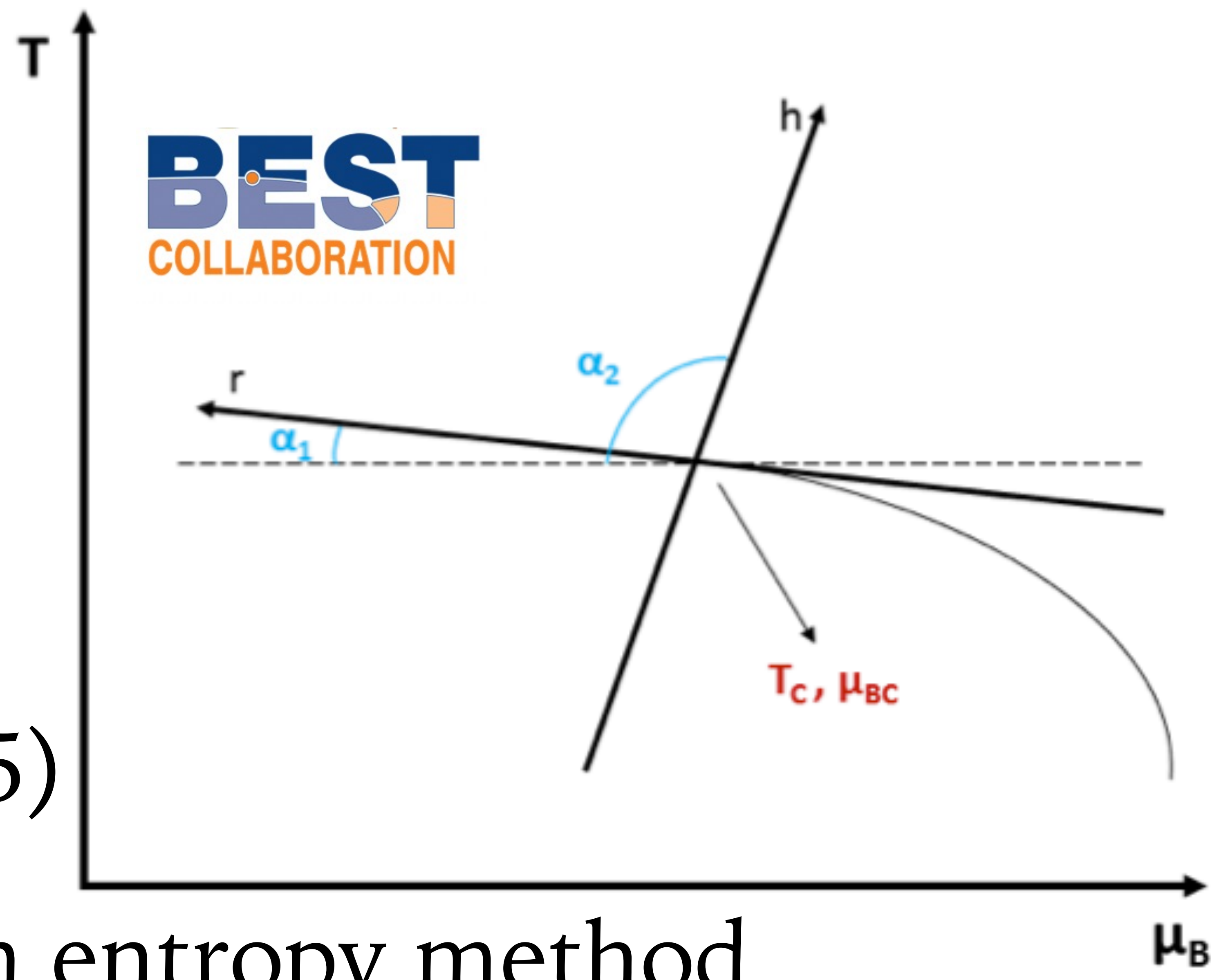
$$\kappa_4 = \langle \sigma^4 \rangle_c \equiv \langle \sigma^4 \rangle - 3\langle \sigma^2 \rangle^2$$
$$= 6VT^3[2(\lambda_3\xi)^2 - \lambda_4]\xi^8$$
$$= 6VT^2[2\tilde{\lambda}_3^2 - \tilde{\lambda}_4]\xi^7$$

- Relate to experimental observables: C. Athanasiou, K. Rajagopal, M. Stephanov, PRD (2010)



Ingredients to Study Critical Point Effect

- Need an equation of state with critical features: utilize the BEST EoS mapping between the 3D Ising model and QCD
  - P. Parotto et al, PRC (2020),
  - $\langle n_S = 0 \rangle$, $\dfrac{n_Q}{n_B} = 0.4$: J. M. Karthein et al, EPJ+ (2021)
  - Higher μ_B: M. Kahangirwe et al, PRD (2024)
  - First order: J.M Karthein, V. Koch, C. Ratti, PRD (2025)



- Need a method of relating to particle correlators: maximum entropy method
  - M. Pradeep & M. Stephanov, PRL (2023)

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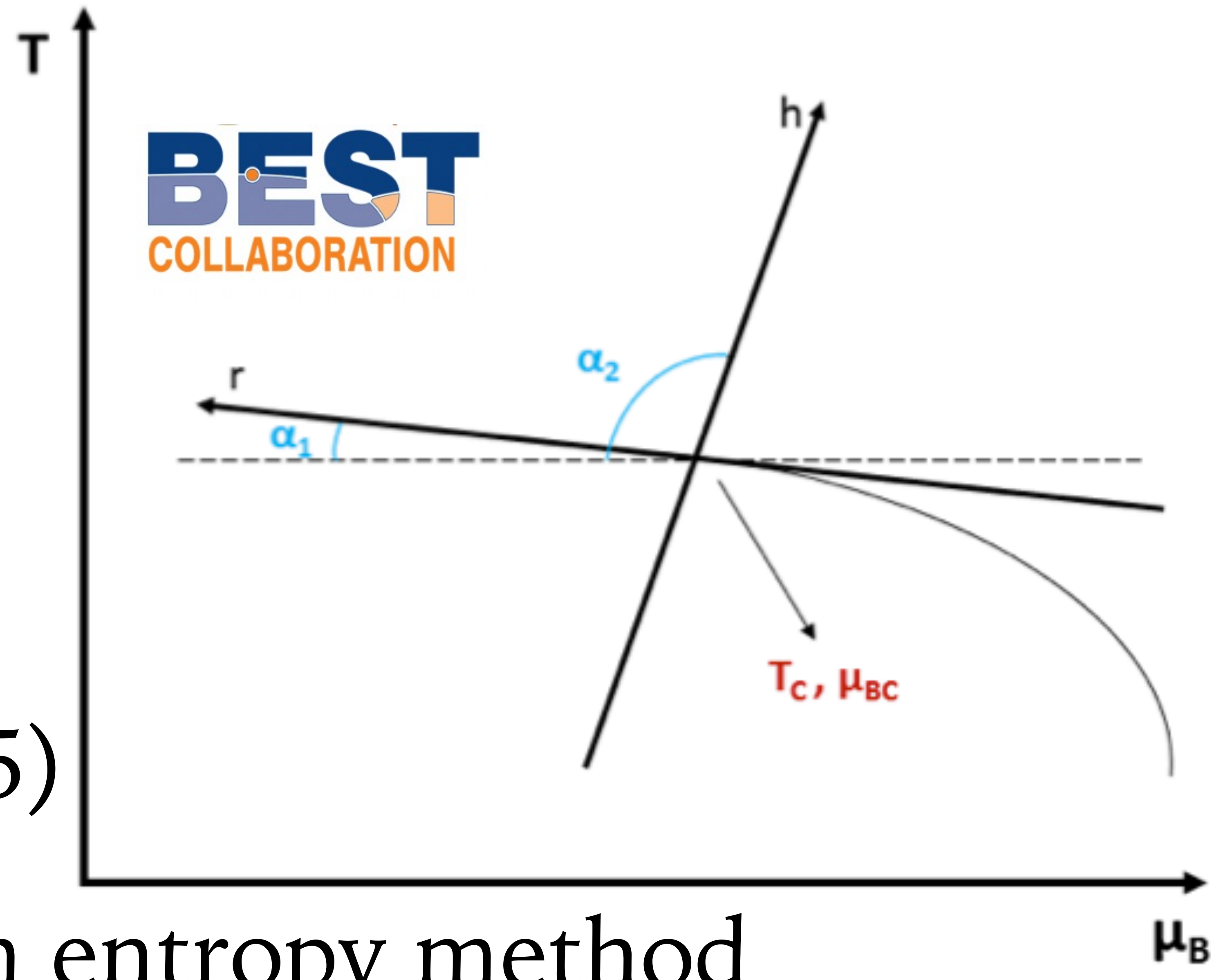

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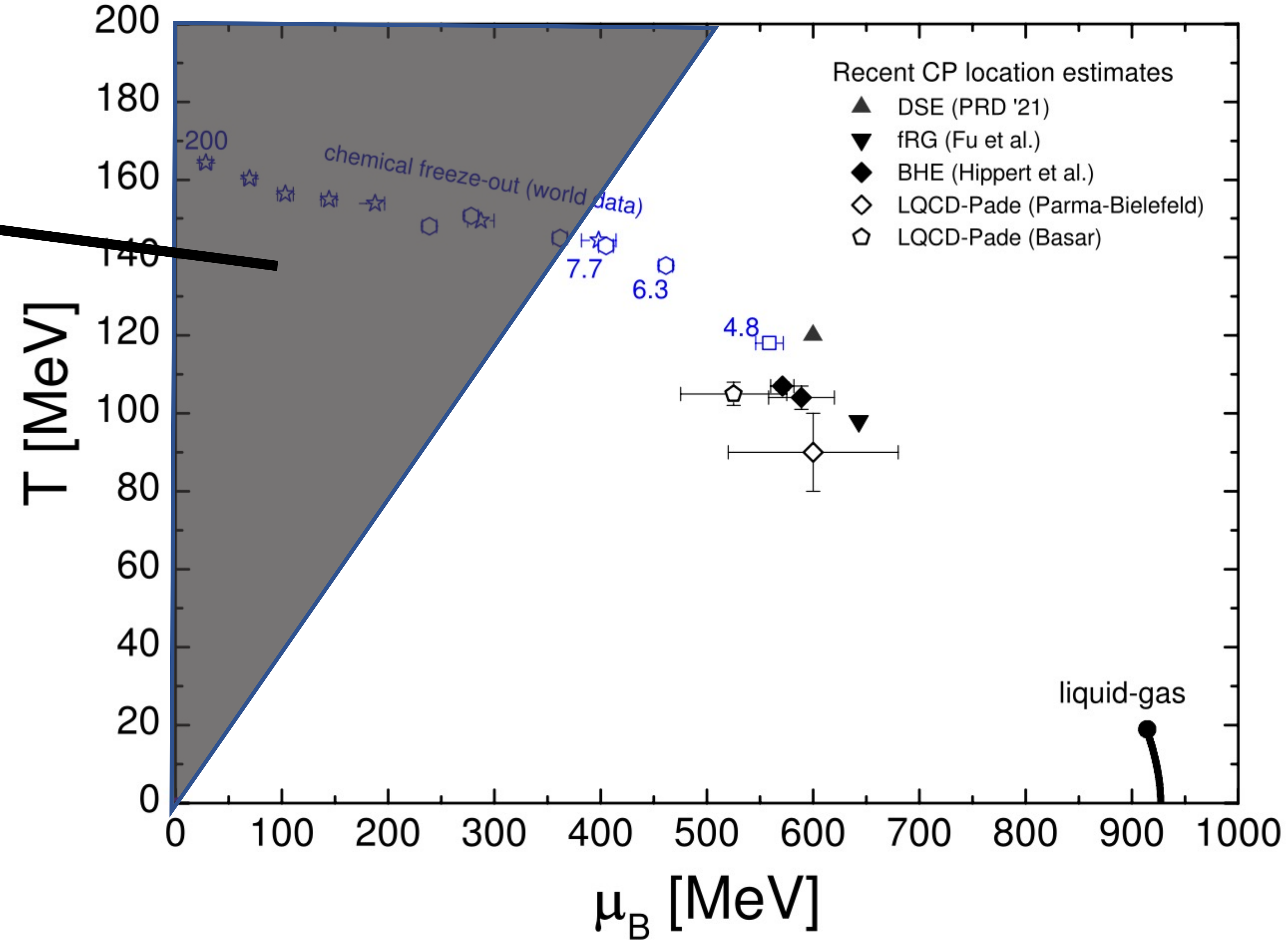
Also need dynamics! T. Schaefer, M. Singh talks at QM25; Hydro+: M. Stephanov and Y. Yin PRD (2018) & K. Rajagopal, G. Ridgway et al PRD (2020) & M. Pradeep, K. Rajagopal et al PRD (2022)

I. Selected recent efforts on limits of critical point location from theory

Limits on Critical Point as of QM 2023

- Lattice calculations then limited exclusions strictly to the expansion parameter
    - From Volodymyr Vovchenko's talk at QM 2023

Critical point disfavored



No sign of criticality for $\mu_B/T \lesssim 2-3$

Limit from Curvature of Chiral Phase Transition

- HotQCD collaboration: estimate from the curvature of the chiral phase transition line and limit on the critical temperature from A. Halasz et al, PRD (1998)
  - From Jishnu Goswami's talk on Wednesday:

The parametrization of the pseudo-critical line of QCD :

$T_{pc}(\mu_B) = T_{pc,0}\left[1 - \kappa_2 \hat{\mu}_B^2 + \kappa_4 \hat{\mu}_B^4\right]$

$T_{pc,0} = (156.5 \pm 1.5)$ MeV, $\kappa_2 = 0.012(4), \kappa_4 = 0.000(4)$

The CEP most likely will exist below, $T < 132$ MeV.

[Halasz et al,arXiv:hep-ph/9804290]

$$\mu_B = \frac{T_{pc}(\mu_B)}{\sqrt{0.012}}\sqrt{1 - \frac{T_{pc}(\mu_B)}{156.5\ \text{MeV}}}$$

$T_{pc} \rightarrow 156.5$ MeV ; $\mu_B \rightarrow 0$

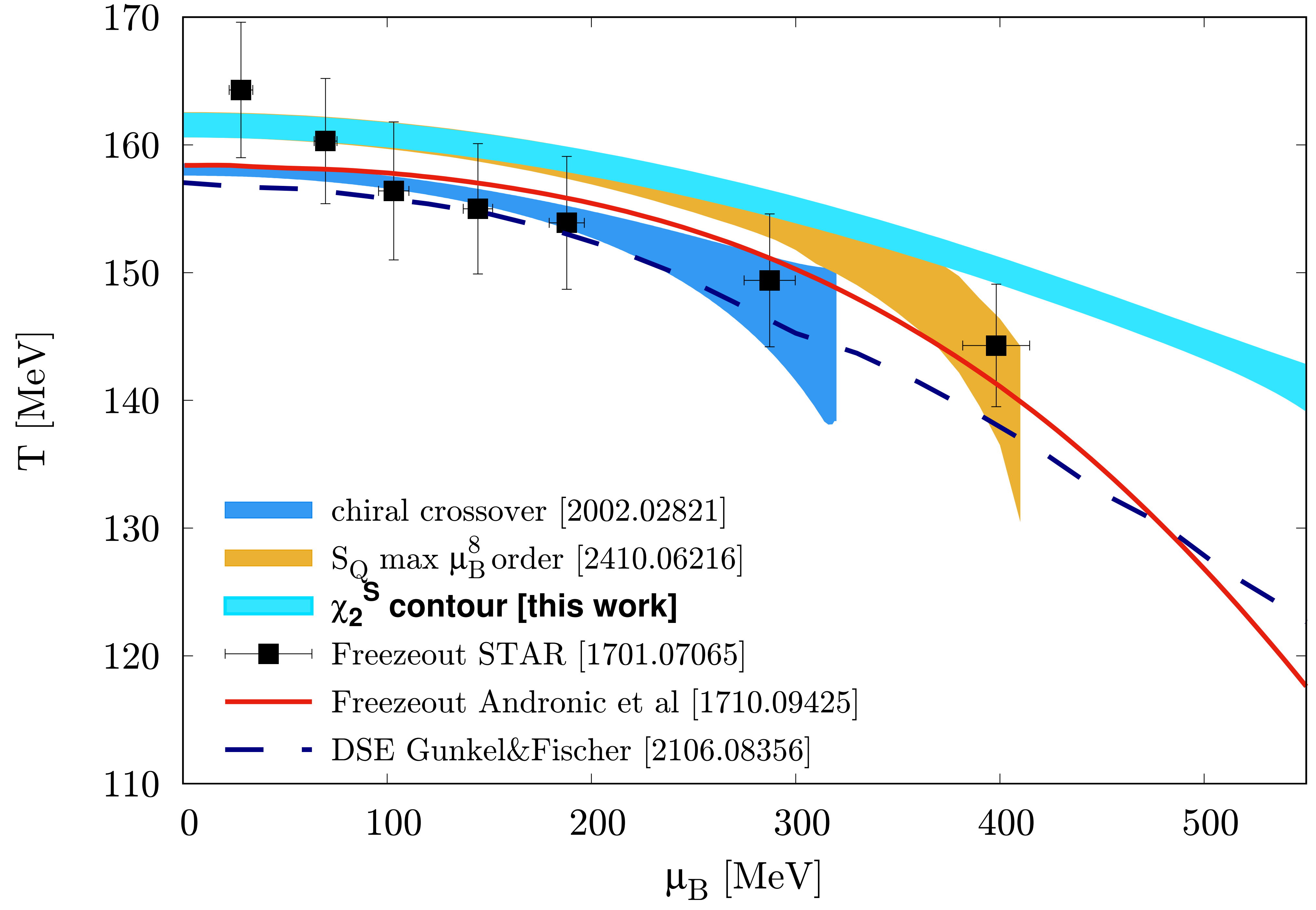
$T_{pc} \rightarrow 132$ MeV ; $\mu_B \rightarrow 470$ MeV

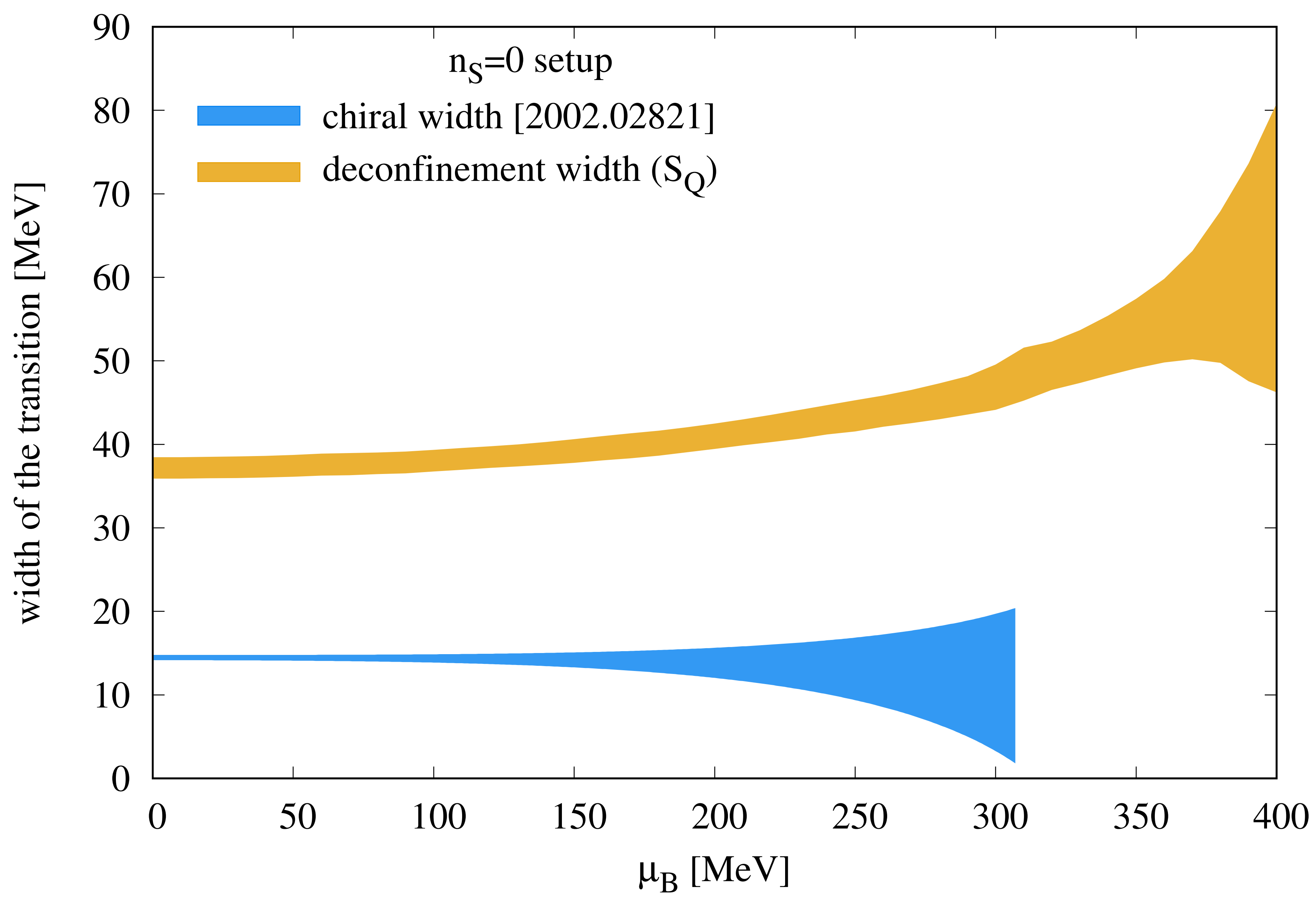
$T_{pc} \rightarrow 0$ MeV ; $\mu_B \rightarrow 0$

No sign of criticality for $T > 132$ MeV and $\mu_B < 470$ MeV

Deconfinement & Strangeness for Phase Boundary

- Wuppertal-Budapest collaboration: obtain transition lines from the maximum of the static quark entropy and the strangeness susceptibilities
  - From Paolo Parotto's poster and Chik Him Wong's talk on Wednesday:



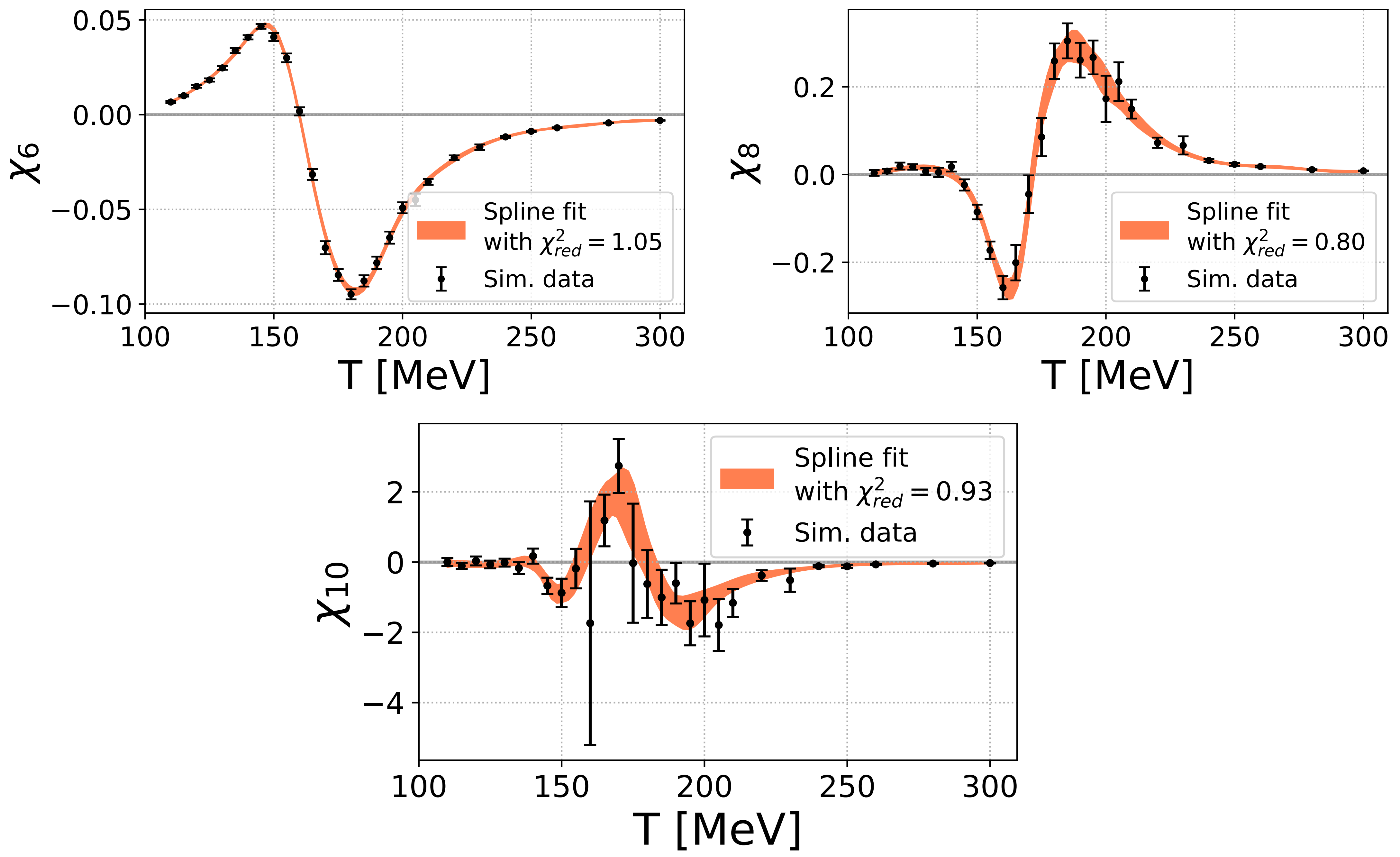


No sign of criticality for $\mu_B < 400$ MeV (S_Q max) or as large as 600 MeV (χ_S^2)

High Statistics for Higher Order Fluctuations

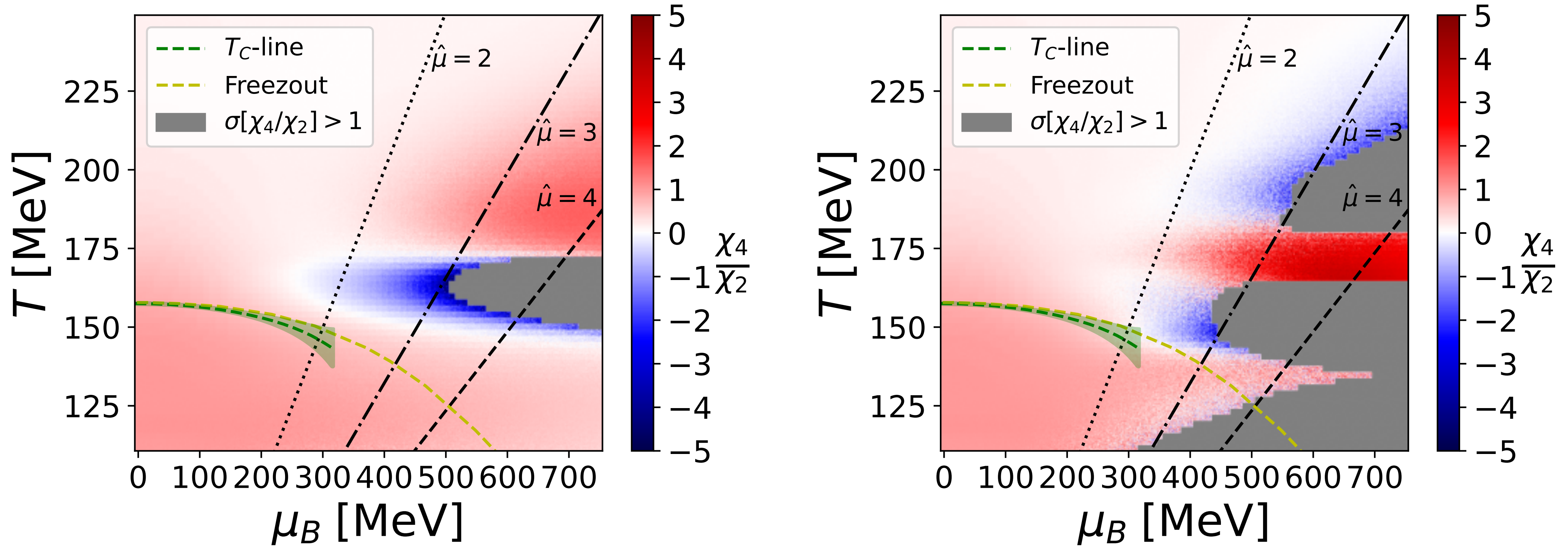
- Wuppertal-Budapest collaboration: brute force lattice method with extreme statistics to obtain higher order baryon cumulants (Taylor coefficients)
  - From Alexander Adam's talk on Thursday:

- Volume : $16^3 \times 8$
- ≈ 2 million configs. per T at $\mu = 0$
- 2+1 flavours with 4 HEX smearing
- Simulated at physical quark mass



Motivation: Stephanov, M.A. [0809.3450]

$$\left[\frac{\chi_4}{\chi_2}\right](\mu, T) = \frac{\chi_4 + \frac{\chi_6}{2!}\hat{\mu}^2 + \frac{\chi_8}{4!}\hat{\mu}^4\left(+\frac{\chi_{10}}{6!}\hat{\mu}^6\right)}{\chi_2 + \frac{\chi_4}{2!}\hat{\mu}^2 + \frac{\chi_6}{4!}\hat{\mu}^4 + \frac{\chi_8}{6!}\hat{\mu}^6\left(+\frac{\chi_{10}}{8!}\hat{\mu}^8\right)}$$

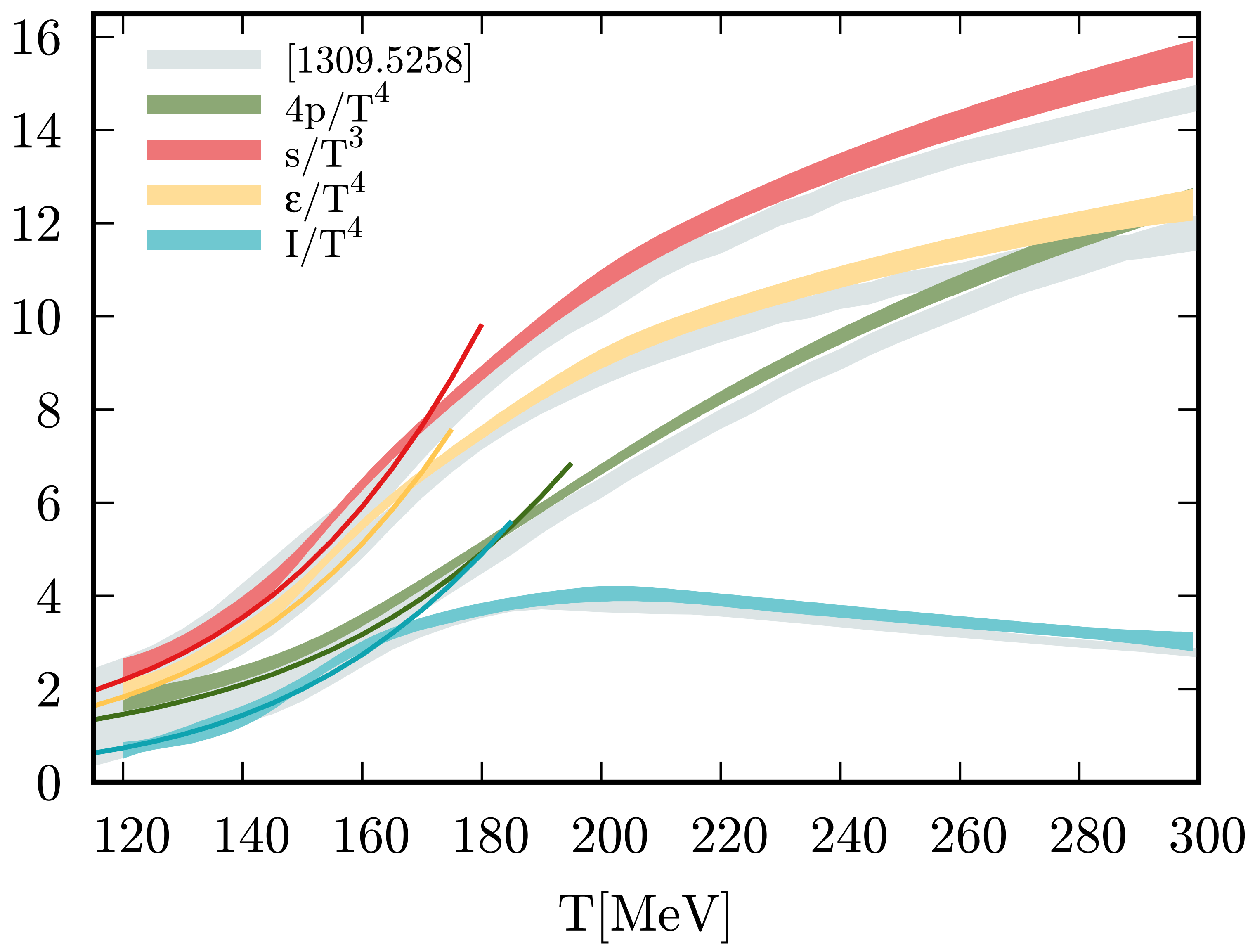


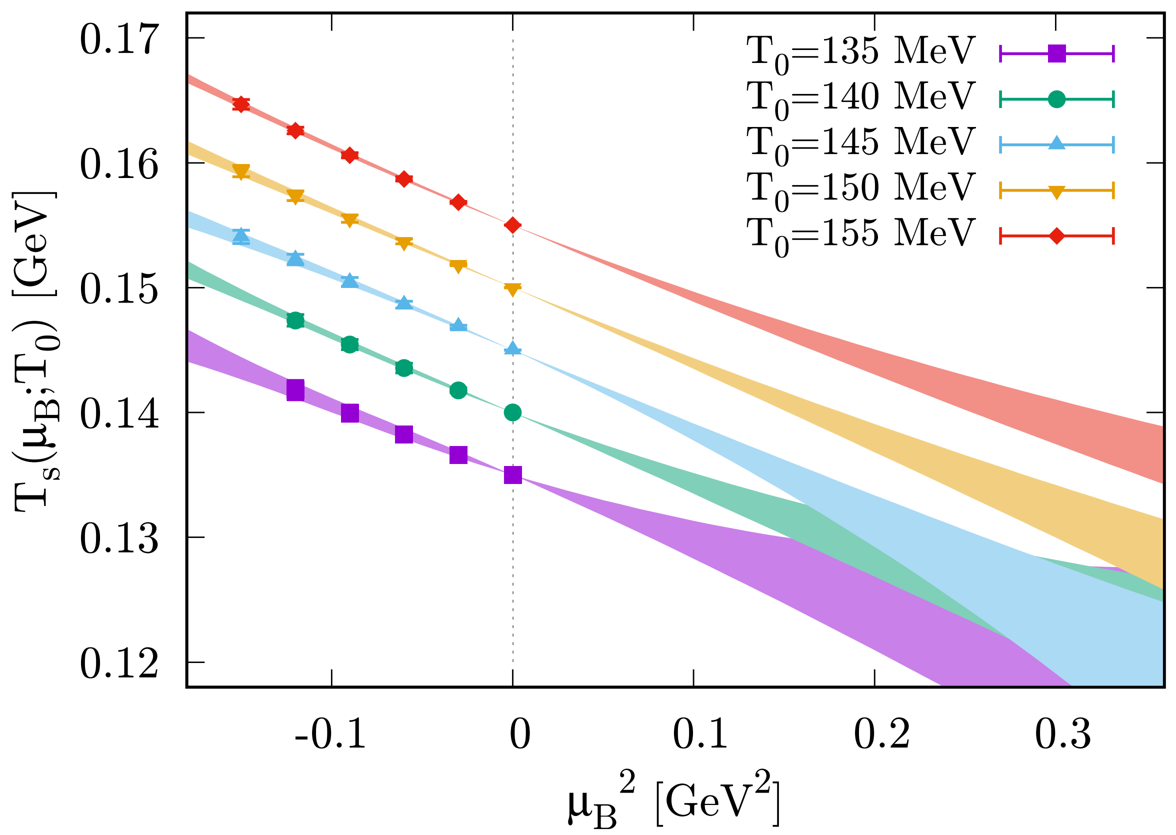
using up to χ_8 using up to χ_{10}

No sign of criticality observed

Increased Precision Equation of State

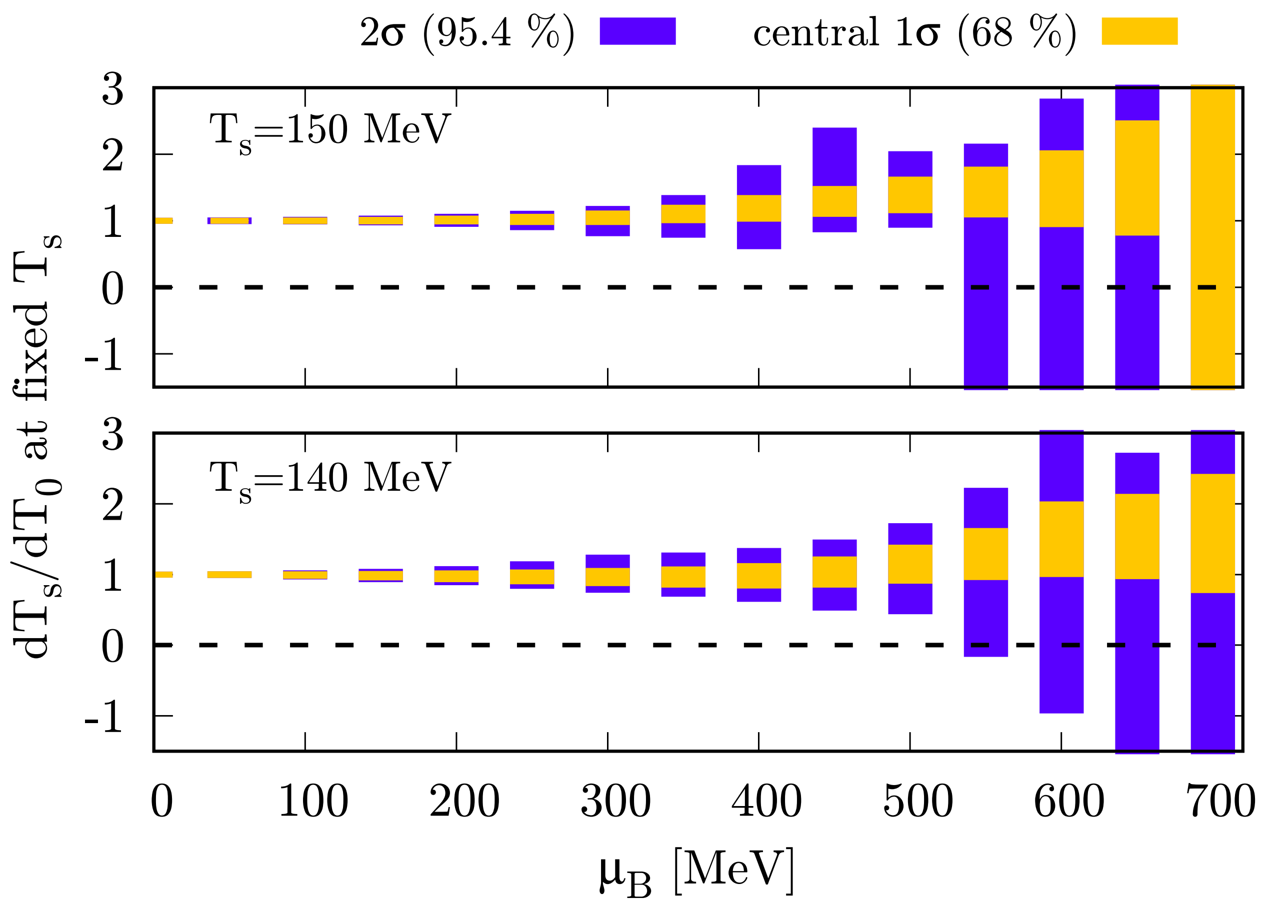
- Wuppertal-Budapest collaboration: increased precision on the EoS → extract information via extrapolation to real, finite μ_B
  - From Jana's Guenther's talk on Thursday:





$$T_s(\mu_B^2, T_0) = \frac{T_0 + a\mu_B^2}{1 + b\mu_B^2}$$

At the critical point T_S v. T_0 becomes flat: $\dfrac{dT_S}{dT_0} = 0$



Contours of constant entropy obtained from fixing entropy $s(T_0)$ and determining $T_S(\mu_B^2, T_0)$:

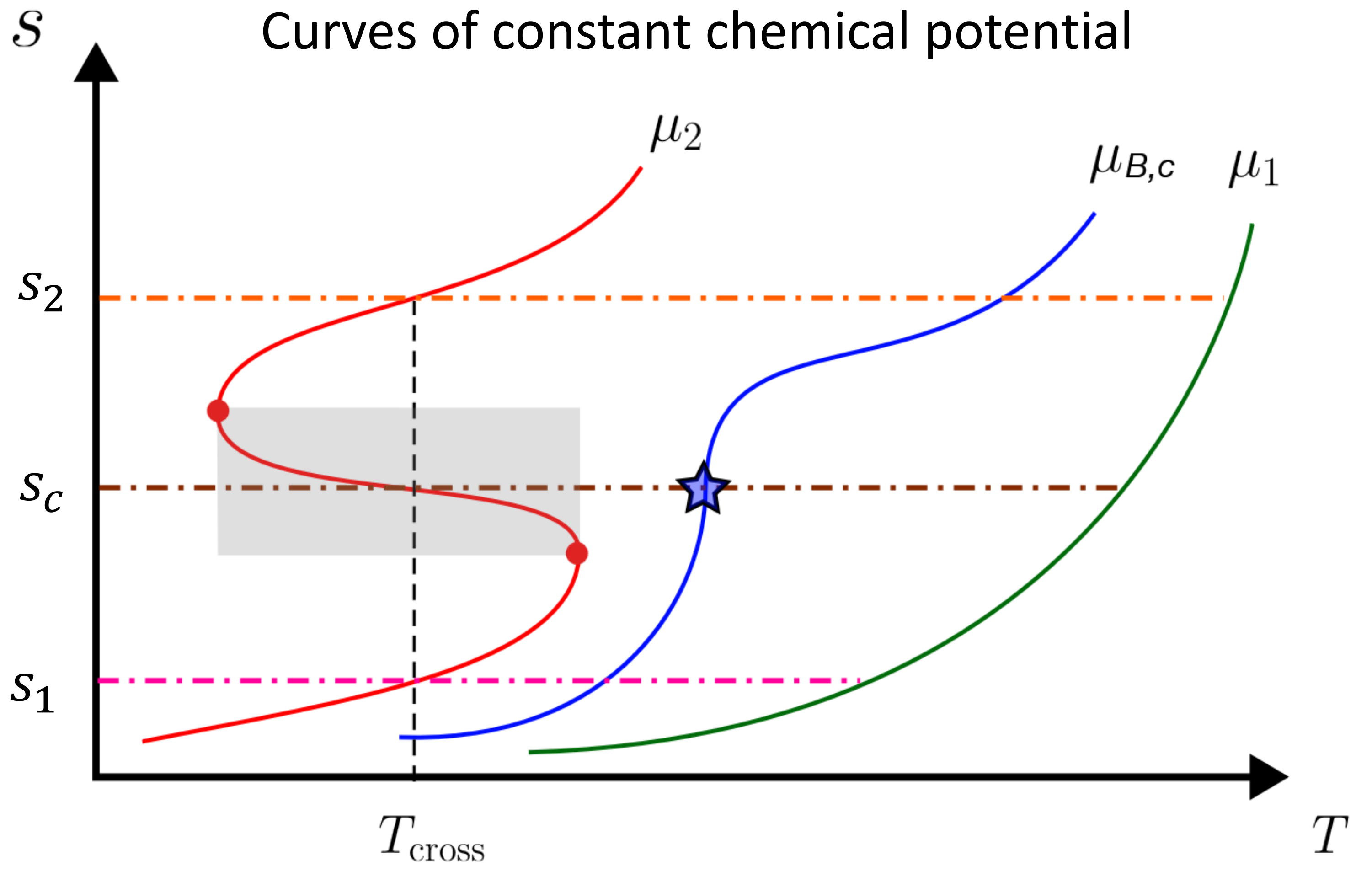
No sign of criticality for $\mu_B < 450$ MeV

II. Selected recent efforts on estimating critical point location from theory

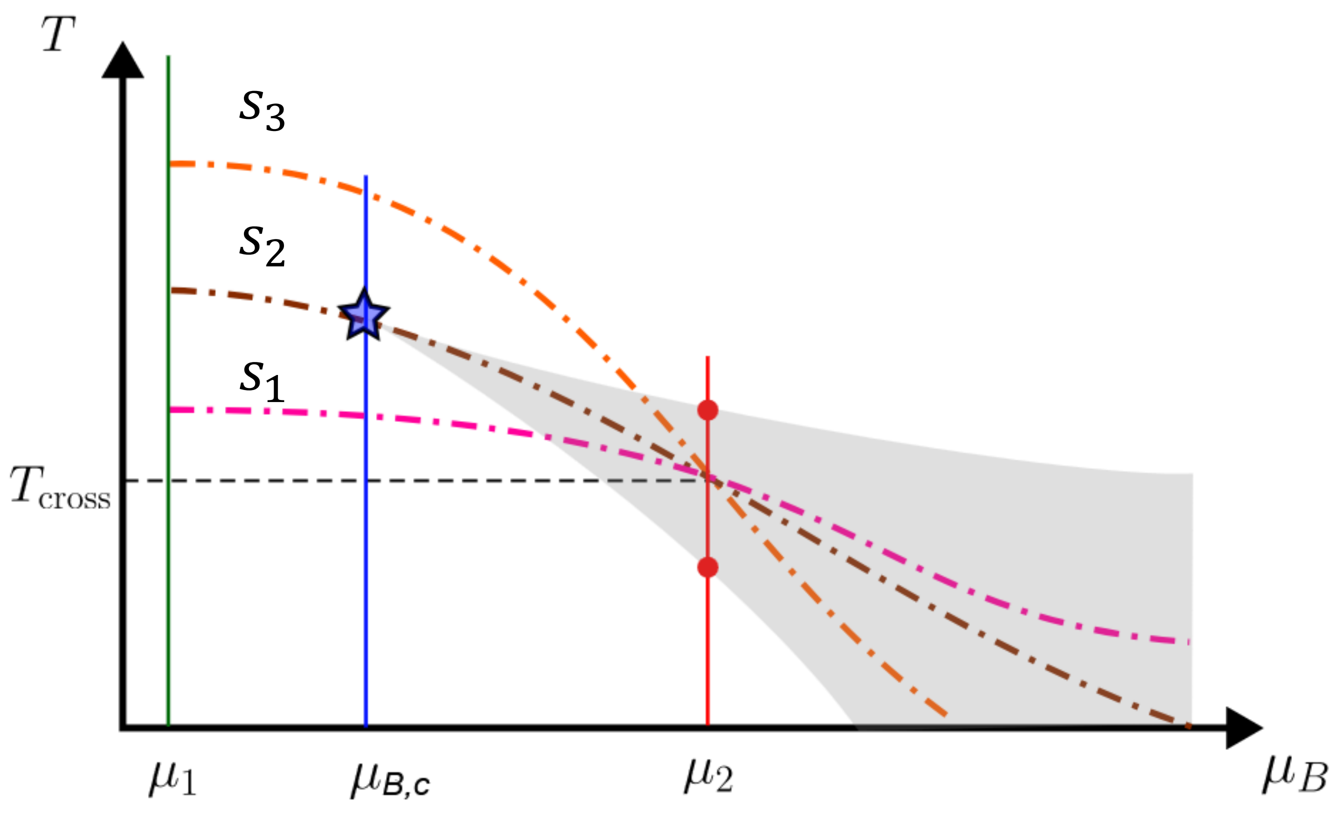
Extrapolation from Constant Entropy Contours

- Estimate critical point location from spinodal features of first order transition
  - From Hitansh Shah's talk on Wednesday:

Curves of constant chemical potential



Constant entropy density contours in T vs μ_B plane



The power series expansion for these entropy density contours

$$T_s(\mu_B;T_0) \sim T_0 + \sum_n^N \alpha_{2n}(T_0)\frac{\mu_B^{2n}}{(2n)!} + O(\mu_B^{2(N+1)})$$

$$\alpha_{2n}(T_0) = \left(\frac{\partial^{2n}T}{\partial\mu_B^{2n}}\right)_s\Big|_{T=T_0,\mu_B=0}$$

We truncate the expansion up to 2nd order in μ_B which gives:

$$T_s(T_0;\mu_B) = T_0 + \frac{\alpha_2(T_0)\mu_B^2}{2!}$$

$$\alpha_2(T_0) = \left(\frac{\partial^2 T}{\partial\mu_B^2}\right)_s\Big|_{T=T_0,\mu_B=0} = -\frac{2T_0\chi_2^B(T_0) + T_0^2\chi_2^{B'}(T_0)}{s'(T_0)}\Big|_{\mu_B=0}$$

- Estimates from lattice parametrization and spline fit with collision energy range:

Lattice parametrization: $\mu_{B,c} = 602 \pm 62$ MeV, $T_c = 114 \pm 7$ MeV

Spline fit: $\mu_{B,c} = 556 \pm 50$ MeV, $T_c = 119 \pm 5$ MeV

We find the collision energy $\sqrt{s_{NN}}$ from the range $4-6$ GeV to be in the closest vicinity of the CP.

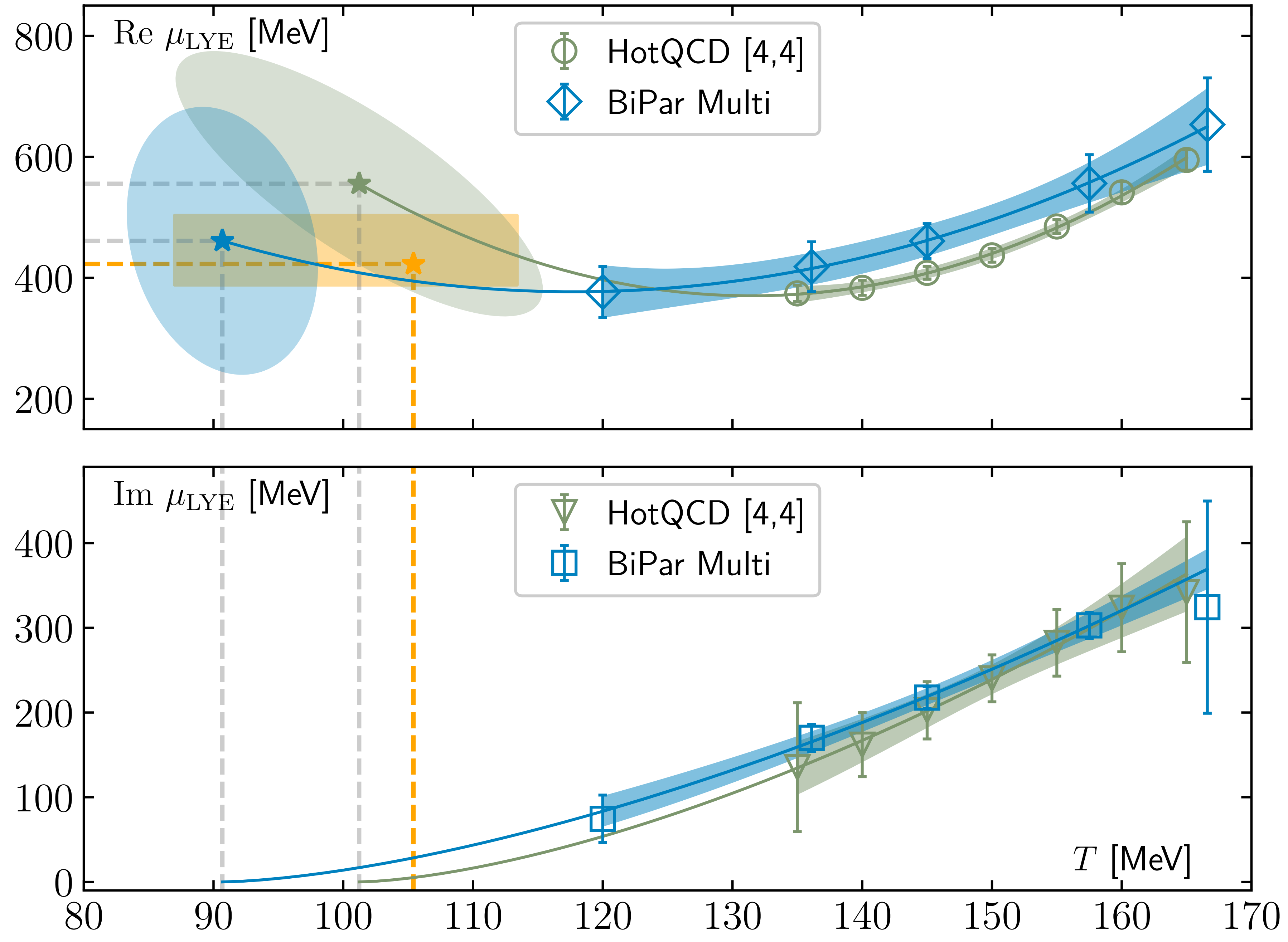
Also applied to holographic model fit to lattice with critical point around ($T_c = 103$ MeV, $\mu_{B,c} = 599$ MeV)

Estimate from Lee Yang Edge Singularity

- Lee Yang edge singularity: zeroes of the partition function appear at imaginary values of μ_B due to a branch cut along the real h axis with location from universality:

$$z_{LY} = \frac{t}{h^{1/\beta\delta}} = |z_c| e^{i\frac{\pi}{2\beta\delta}}$$

- Critical point observed where branch cut pinches the real axis
    - From Christian Schmidt's poster:

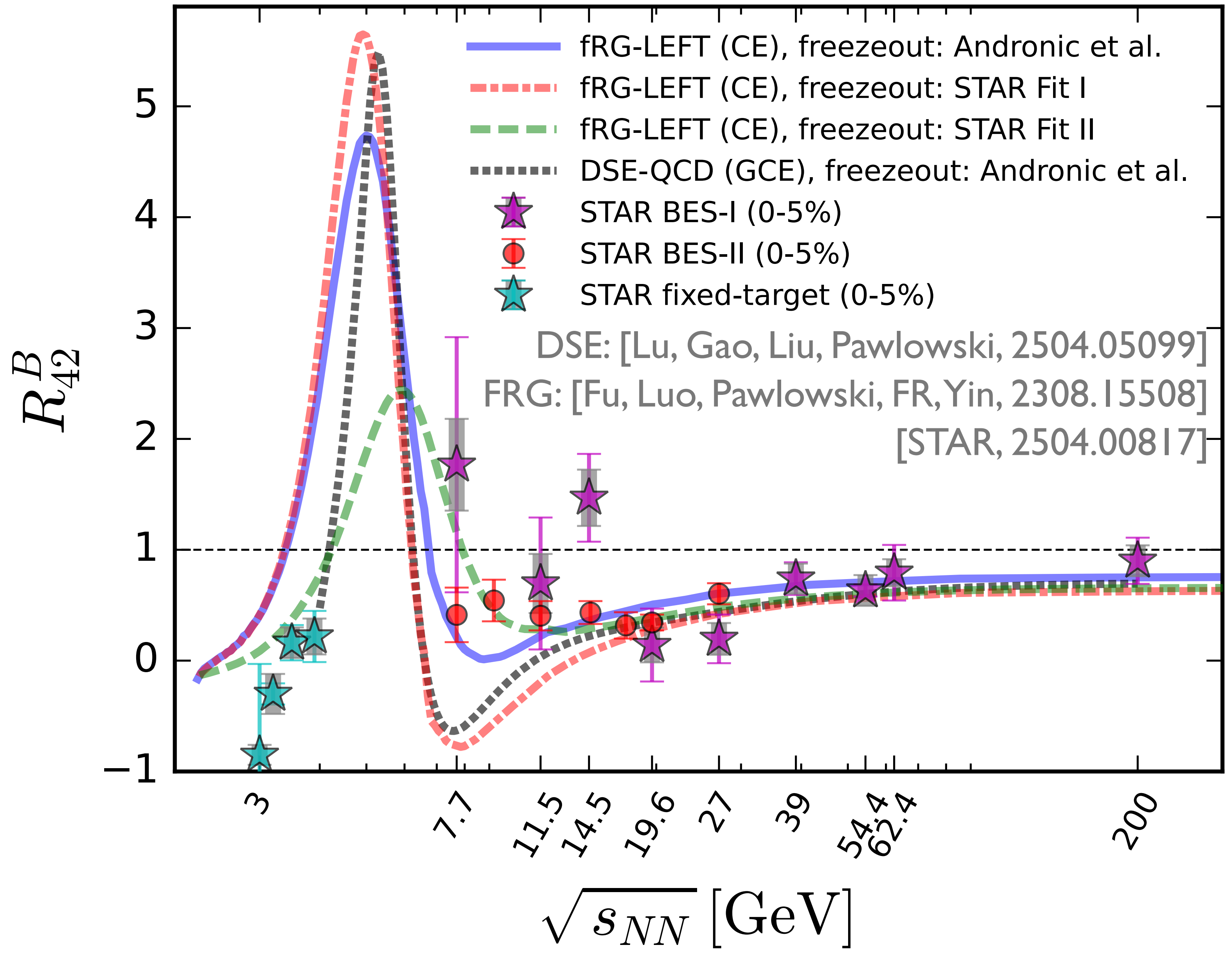


See also: Basar PRC (2024)

Estimates from Functional QCD Methods

- Dyson-Schwinger (DSE) approach and functional renormalization group (FRG) methods in agreement with different truncations
  - From Fabian Rennecke's talk:

net-baryon fluctuations in QCD vs net-protons from STAR



apples to half-apples comparison! [Vovchenko, QM2023]
qualitative features matter here!

- direct calculations: non-monotonicity at low beam-energies
- **no signs of critical scaling** seen along freeze-out

→ **direct signal of narrowed chiral crossover; CEP location encoded in peak height**

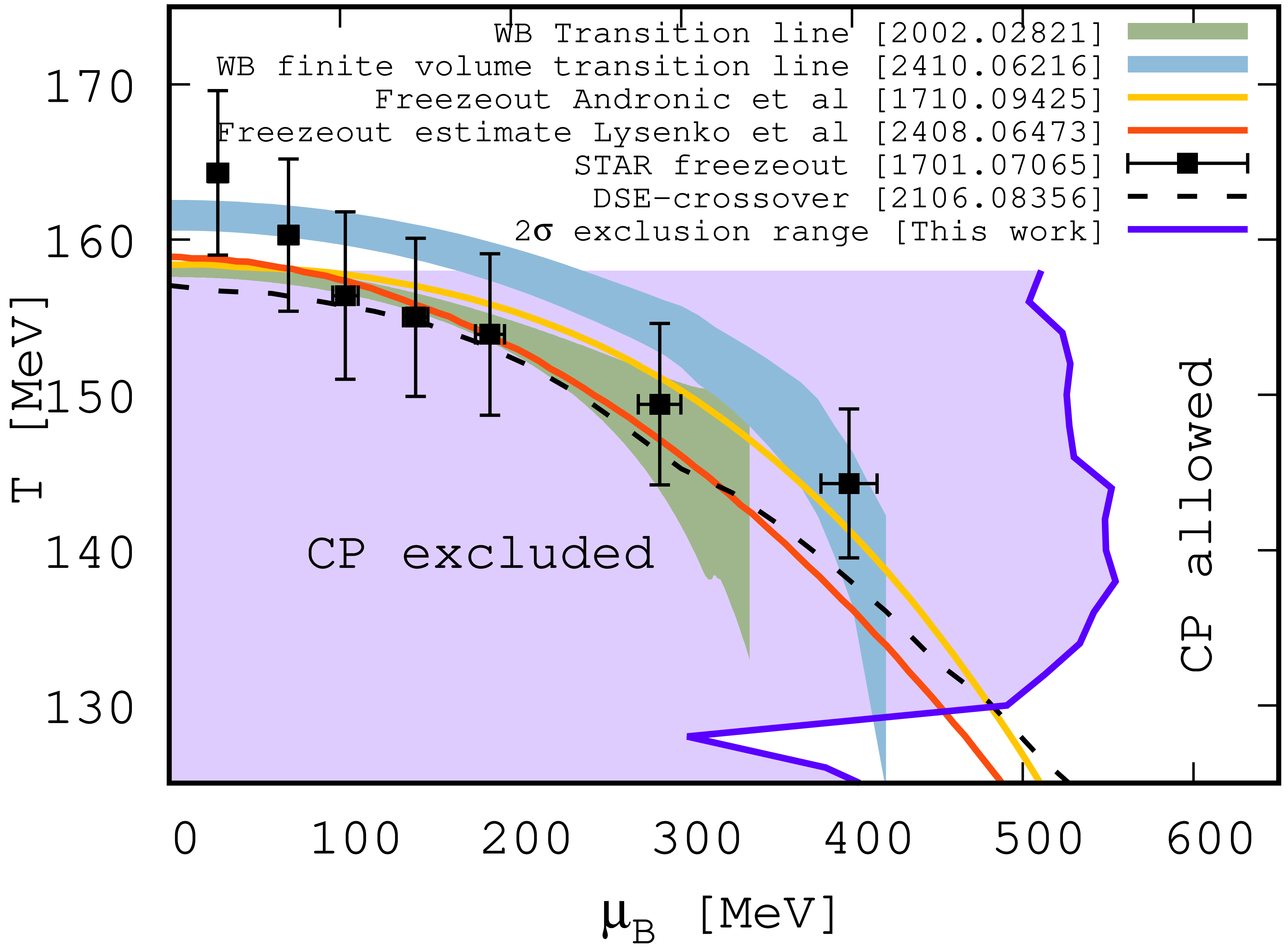
[Fu, Luo, Pawlowski, FR, Yin, 2308.15508]

→ **data between** $\sqrt{s_{NN}} = 4 - 8\,\mathrm{GeV}$ **will be crucial!**

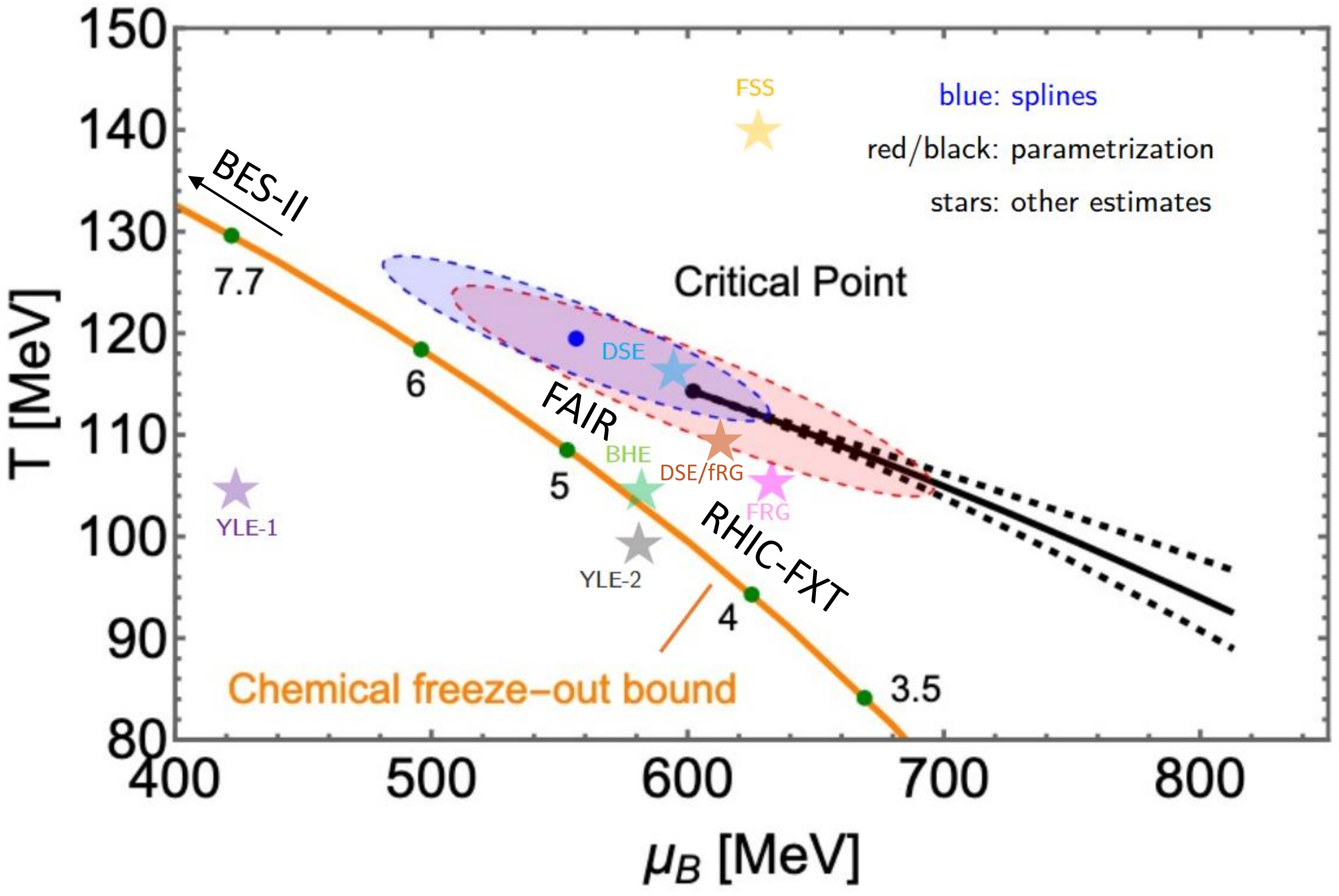
Focus of next talk by Jan Pawlowski

Conclusions

- ➤ Many groups/methods converging on expectations for critical point location!
- ➤ Lattice limits: $T \lesssim 130$ MeV and $\mu_B \gtrsim 450$ MeV



- ➤ Theory estimates: $T_c \sim 100$ MeV and $\mu_B \sim 600$ MeV



YLE-1: D.A. Clarke et al, arXiv:2405.10196
YLE-2: G. Basar, PRC 110, 015203 (2024)
BHE: M. Hippert et al., PRD 110, 094006 (2024)
FRG: W-J. Fu et al., PRD 101, 054032 (2020)
DSE: P.J. Gunkel et al., PRD 104, 052202 (2021)
DSE/fRG: Gao, Pawlowski., PLB 820, 136584 (2021)
FSS: A. Sorensen et al., arXiv:2405.10278