# Recent progress on lattice QCD results

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(HotQCD collaboration)

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GOETHE UNIVERSITÄT FRANKFURT AM MAIN

CRC-TR 211 Strong-interaction matter under extreme conditions

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HGS-HIRe for FAIR Helmholtz Graduate School for Hadron and Ion Research

# Conserved charge fluctuations in (2+1)-flavor QCD

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In QCD with two light $(u, d)$ and one strange flavor $(s)$ , pressure is expressed via a Taylor expansion as,

$$\frac{P}{T^4} = \sum_{i,j,k=0}^{\infty} \frac{\chi_{ijk}^{BQS}}{i!j!k!} \hat{\mu}_B^i \hat{\mu}_Q^j \hat{\mu}_S^k .$$

Generalized susceptibilities at vanishing chemical potentials

ALICE: $T_{cf} = 156.5(1.5)$ MeV,

A. Andronic et al, Nature volume 561, pages, 321–330 (2018)

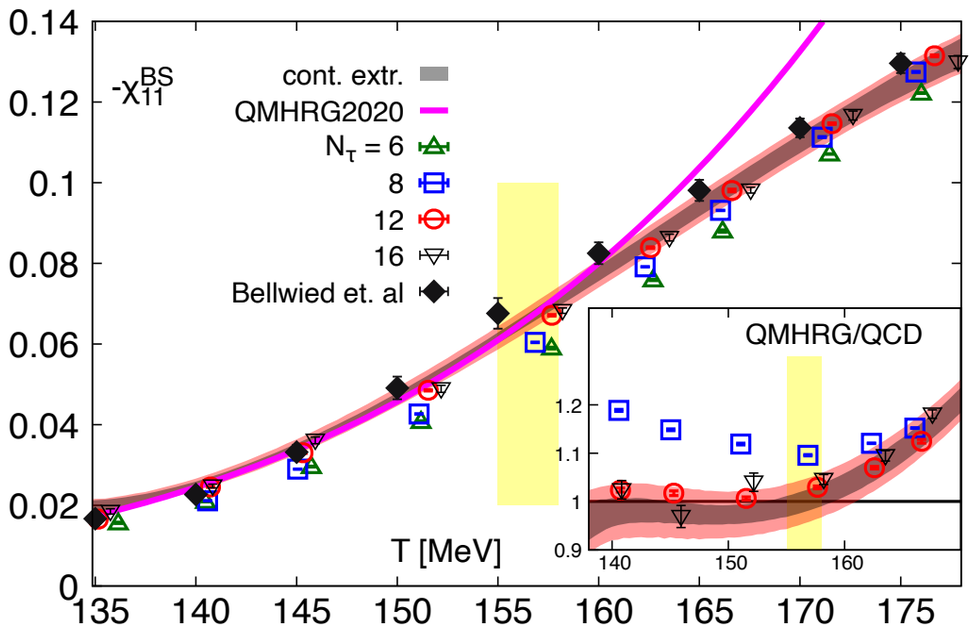
The condition satisfy : $n_S = 0$ (strangeness neutral) and $n_Q/n_B = 0.4$

We satisfy this two conditions order by order:

$$\mu_Q = q_1 \mu_B + q_3 \mu_B^3 + \ldots .$$

$$\mu_S = s_1 \mu_B + s_3 \mu_B^3 + \ldots .$$

# Revisiting second-order cumulants

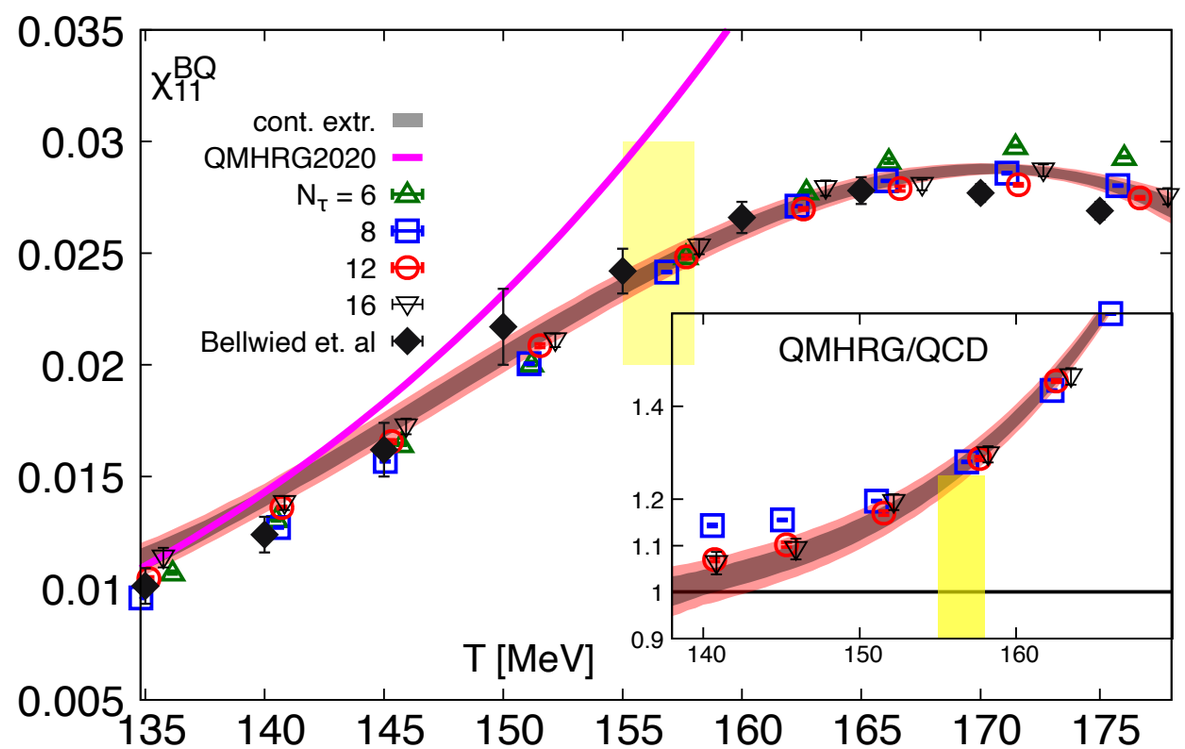



D. Bollweg et al, 2107.10011 [hep-lat]

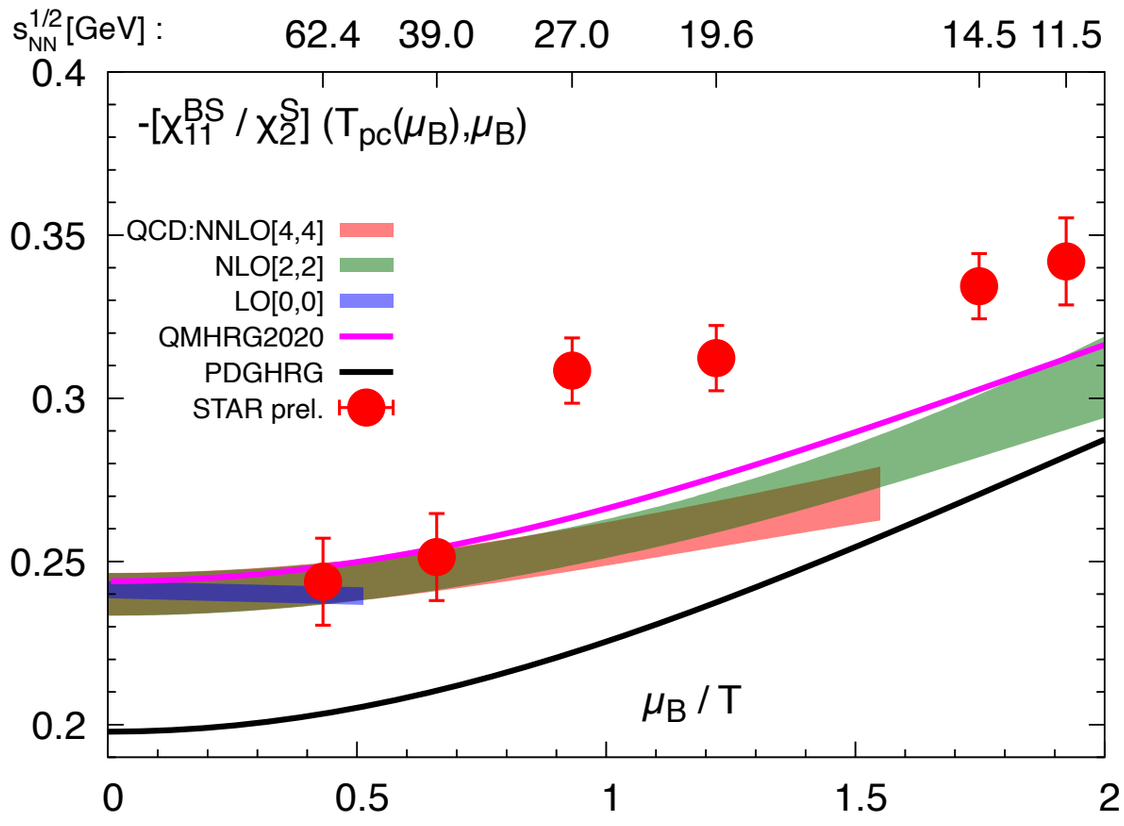
- Good agreement between QCD and HRG for $T < 155$ MeV.
- $\dfrac{-d\chi_{11}^{BS}}{dT}$ differ notably for $T > 155$ MeV between QCD and HRG.

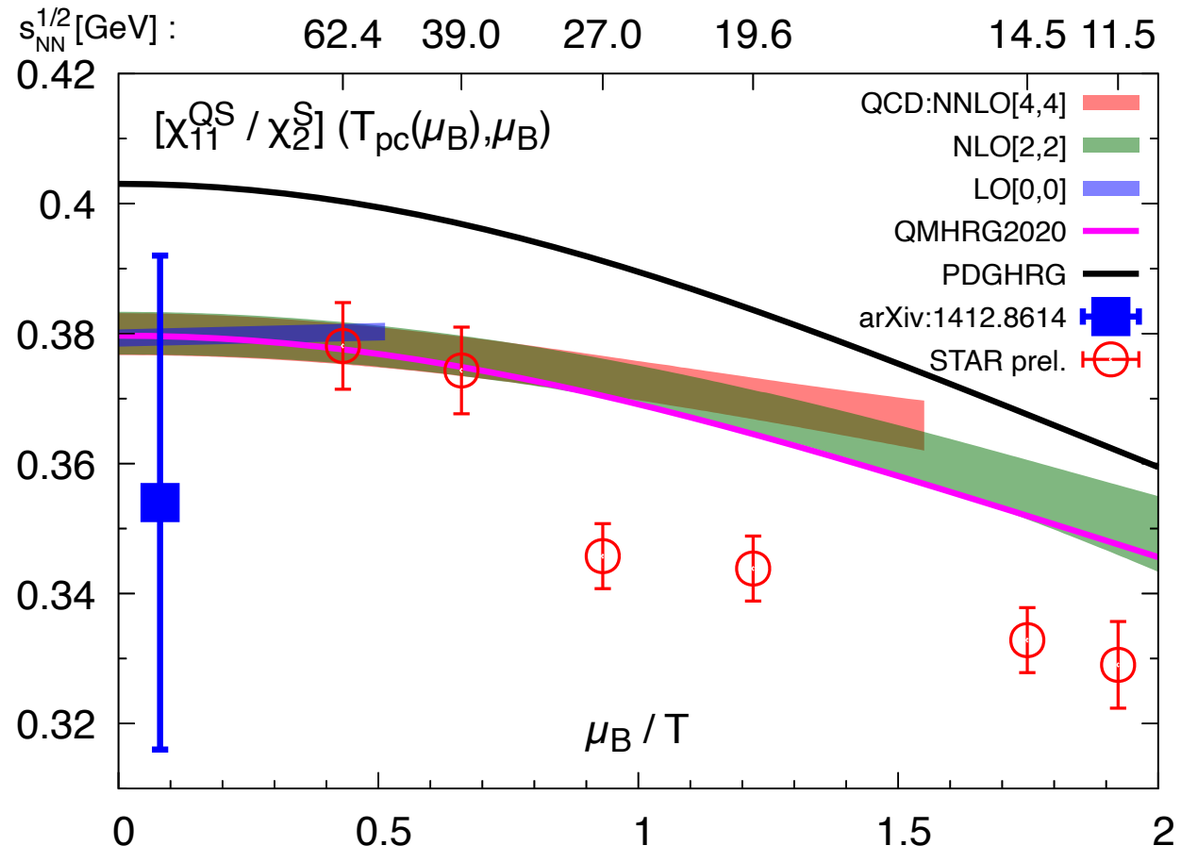
- HRG already disagrees with QCD, $T < 145$ MeV.
- $\dfrac{d\chi_{11}^{BQ}}{dT}$ is very different $T > 155$ MeV for QCD and HRG.
- $T$-derivatives of a second order cumulants are like 4-th order cumulants $\left(\dfrac{\partial}{\partial T} \simeq \dfrac{\partial^2}{\partial \mu_B^2}\right)$ thus deviations are expected to show up.

for, $\vec{\mu} = 0$, $\chi_2^S = 2\chi_{11}^{QS} - \chi_{11}^{BS}$,
$\chi_2^B = 2\chi_{11}^{BQ} - \chi_{11}^{BS}$



# Baryon-strangeness correlations

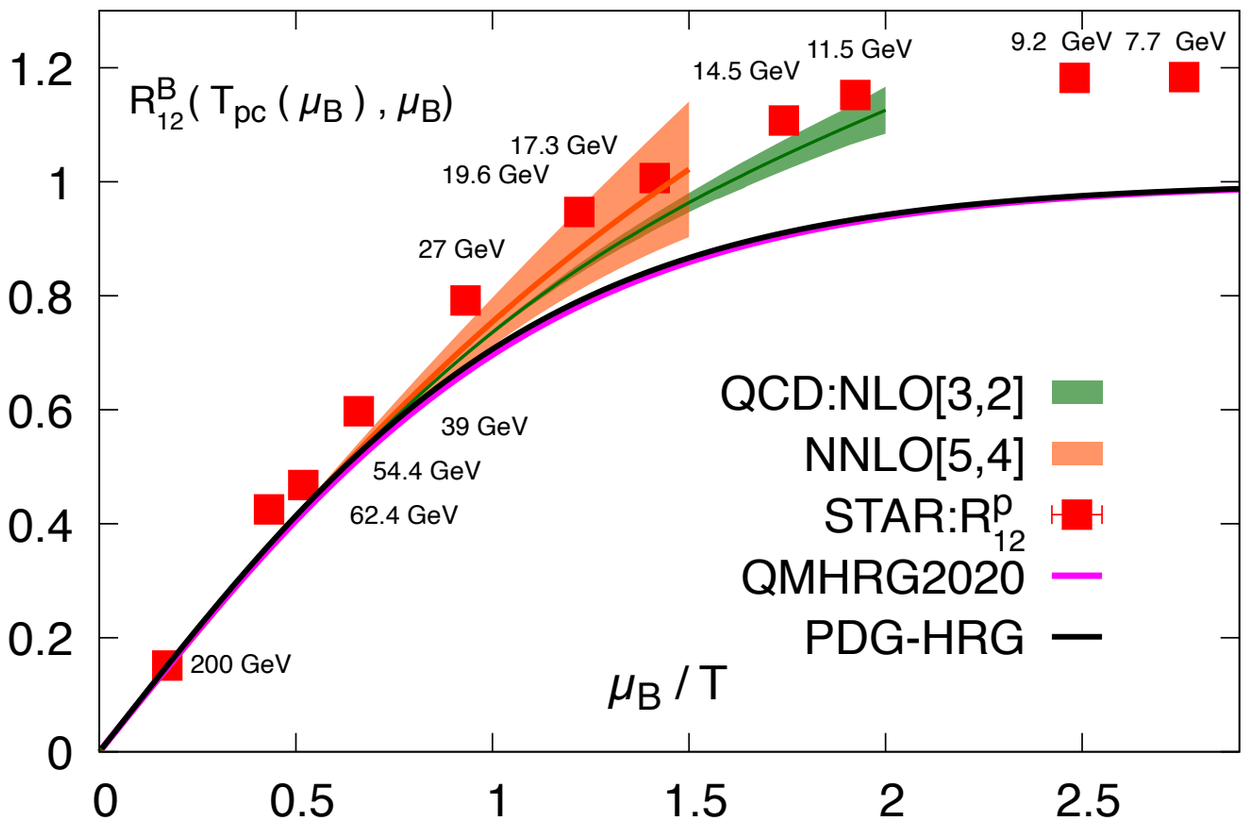



- Significant differences between QCD and Prelim. STAR results for $\sqrt{s_{NN}} \leq 27$ GeV.
- QCD sum rule : $2\frac{\chi_{11}^{QS}(T,\vec{\mu})}{\chi_2^S(T,\vec{\mu})} - \frac{\chi_{11}^{BS}(T,\vec{\mu})}{\chi_2^S(T,\vec{\mu})} = 1 + \frac{\Delta^{BQS}(T,\vec{\mu})}{\chi_2^S(T,\vec{\mu})}$ ; $\vec{\mu} = (\mu_B, \mu_Q, \mu_S)$
- The Lattice results agree more closely with the QMHRG2020 predictions.

**HotQCD 2024 : *Phys.Rev.D* 110 (2024); STAR results : CPOD2024, QM2025**

# Lower order Baryon number fluctuations

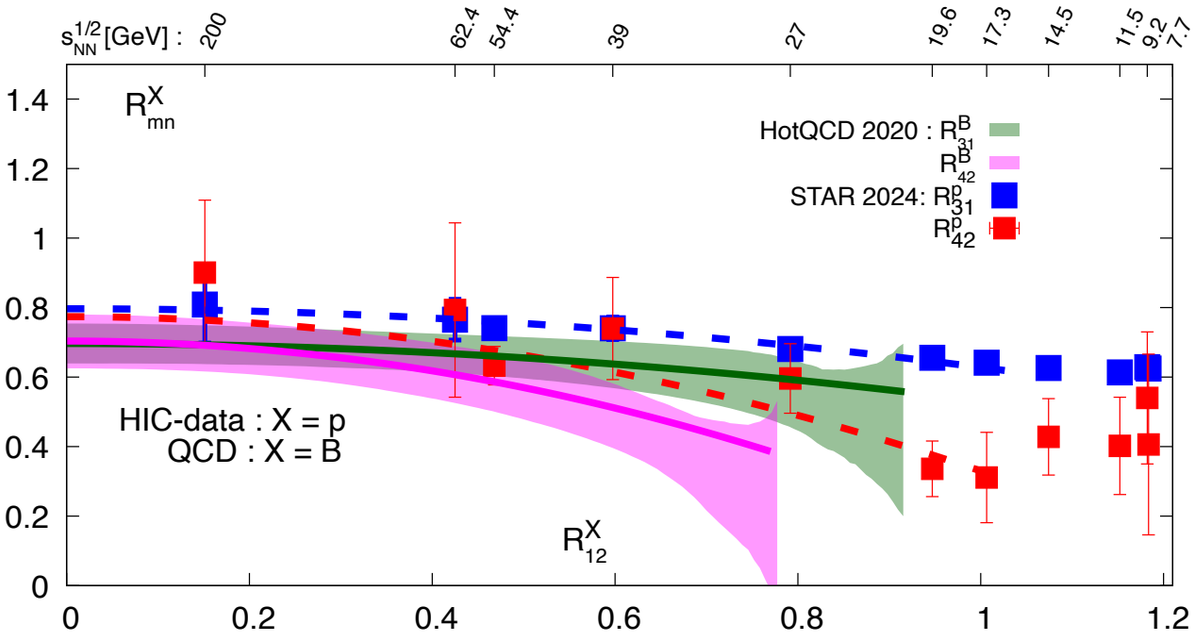


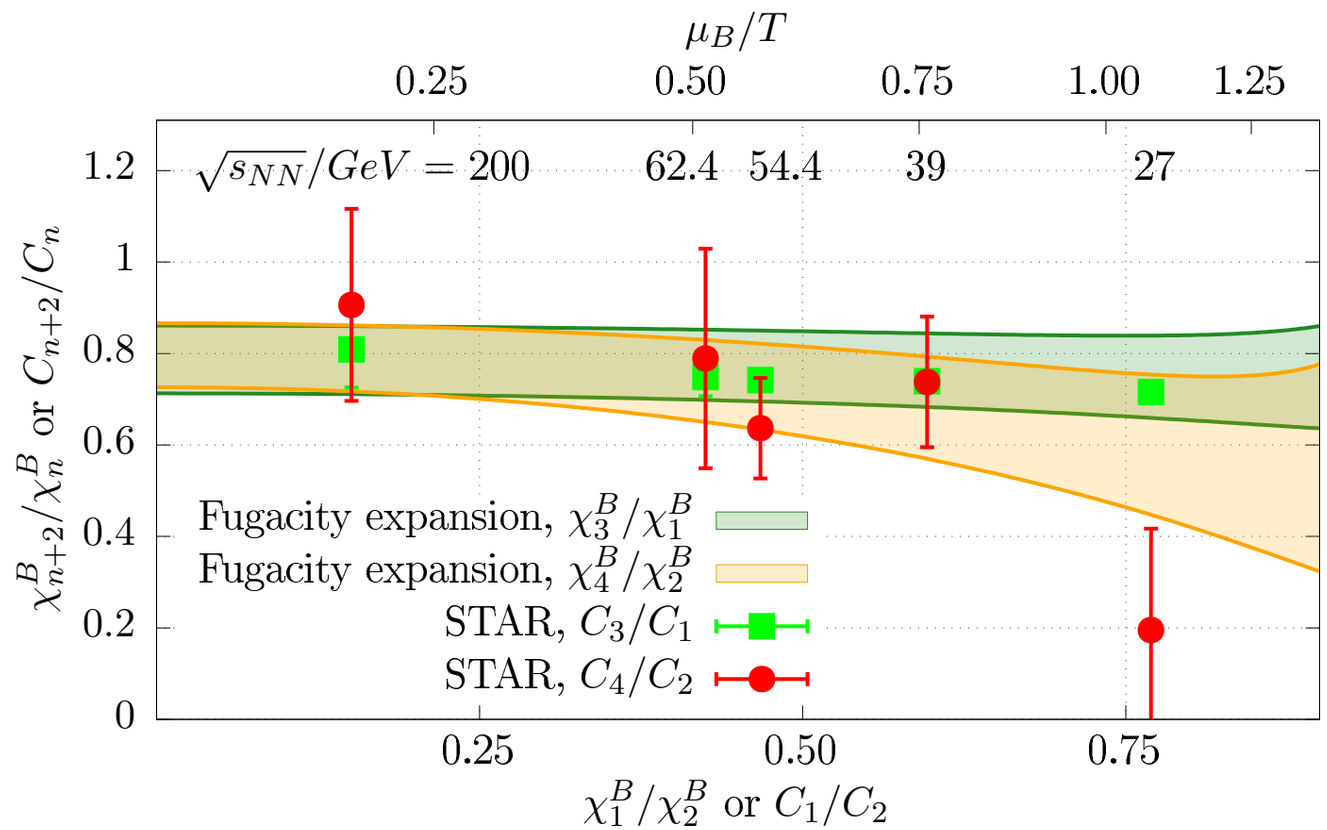

- In good agreement with the STAR data down to $\sqrt{s_{NN}} \simeq 11.5$ GeV.
- $R^p_{12}$ becomes larger than unity for $\sqrt{s_{NN}} \leq 17.3$ GeV.
- Consistent with QCD but not consistent with non interacting HRG.

$$R^B_{12} = \frac{\chi^B_1(T,\hat{\mu}_B)}{\chi^B_2(T,\hat{\mu}_B)} = \hat{\mu}_B\left(1 + s_1\frac{\chi^{BS}_{11}}{\chi^B_2} + q_1\frac{\chi^{QS}_{11}}{\chi^B_2}\right) + \mathcal{O}(\hat{\mu}^3_B)$$

**HotQCD 2024 : Phys.Rev.D 110 (2024); STAR results : CPOD2024**

# Comparison of STAR data with $R^B_{42} = \kappa_B \sigma^2_B$ at the pseudo critical line of QCD




**A. Bazavov et al, 2001.08530 [hep-lat] ; Karsch, Goswami XQCD2024, STAR results CPOD 2024, QM2025.**

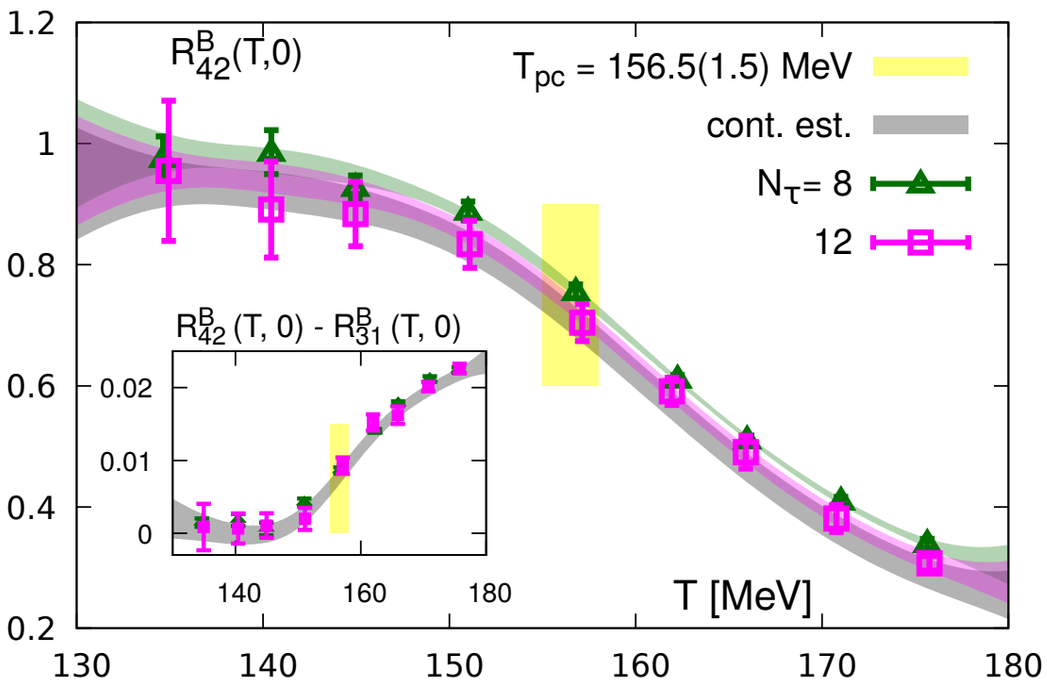
**Rene Bellwied et al, arXiv:2102.06625 [hep-lat]**

- $R^B_{31} = S_B \sigma^3_B$ and $R^B_{42} = \kappa_B \sigma^2_B$ ratios of net baryon-number fluctuations on the pseudo-critical line of QCD.

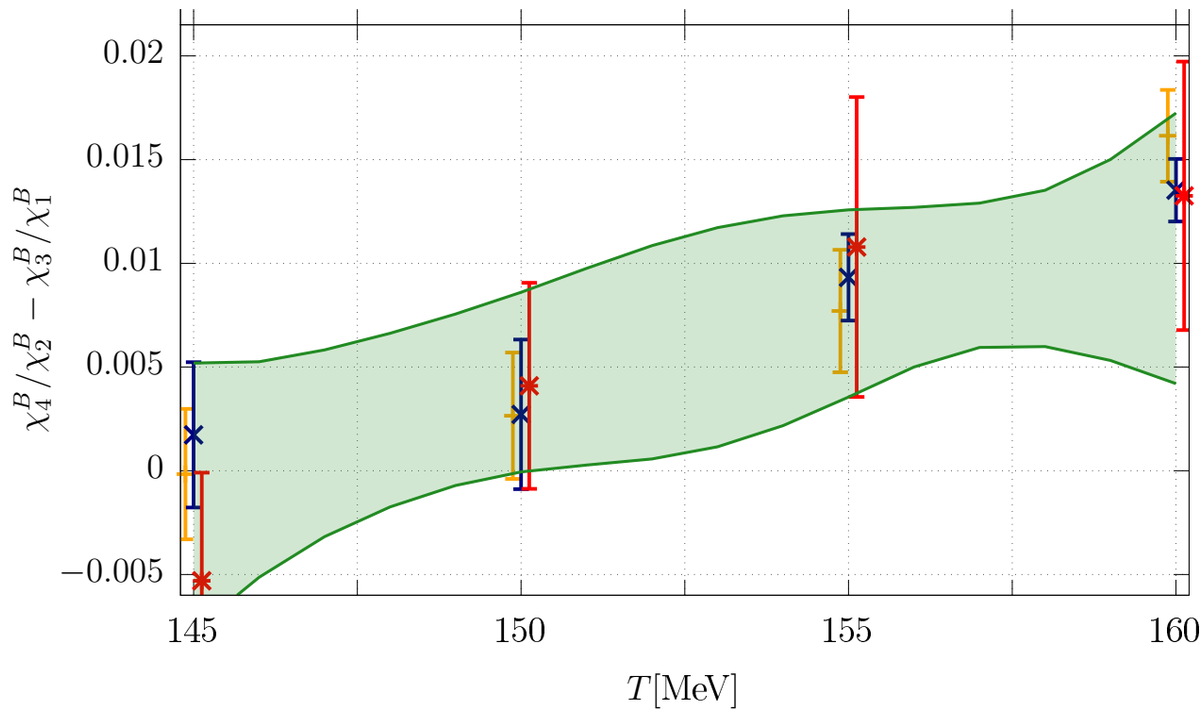
# A closer look at $\frac{\chi_4^B}{\chi_2^B}$ at vanishing chemical potential

- $R_{31}^B = S_B\sigma_B^3 = \frac{\left(\chi_4^B + s_1\chi_{31}^{BS} + q_1\chi_{31}^{BQ}\right)\mu_B}{\left(\chi_2^B + s_1\chi_{11}^{BS} + q_1\chi_{11}^{BQ}\right)\mu_B} \simeq \frac{\chi_4^B}{\chi_2^B}$

- $R_{42}^B = \kappa_B\sigma_B^2 = \frac{\chi_4^B}{\chi_2^B}$




A. Bazavov et al, 2001.08530 [hep-lat]




Rene Bellwied et al, arXiv:2102.06625 [hep-lat]

- $R_{42}^B = \kappa_B\sigma_B^2 = \chi_4^B/\chi_2^B$, of baryon number relevant for ALICE energies and STAR beam energy, $\sqrt{s_{NN}} = 200$ GeV .

QCD : $R_{42}^B = \frac{\chi_4^B}{\chi_2^B}(\mu_B = 0) = 0.70(8)$, $R_{31}^B(\mu_B = 0) \simeq 0.69$

STAR data : $R_{42}^B = 0.90(2)()$ ; $R_{31}^B(\mu_B = 0) \simeq 0.80$

# Algebraic Estimation of the Critical End Point (CEP)

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The natural parametrization of the pseudo-critical line of QCD :

$$\frac{T_{pc}(\mu_B)}{T_{pc,0}} = \left[1 - \kappa_2 \left(\frac{\mu_B}{T_{pc}(\mu_B)}\right)^2 - \kappa_4 \left(\frac{\mu_B}{T_{pc}(\mu_B)}\right)^4\right]$$

$T_{pc,0} = (156.5 \pm 1.5)$ MeV, $\kappa_2 = 0.012(4)$, $\kappa_4 = 0.000(4)$

The CEP most likely will exist below, $T < 132$ MeV.

**[Halasz et al,arXiv:hep-ph/9804290]**

$$\mu_B = \frac{T_{pc}(\mu_B)}{\sqrt{0.012}}\sqrt{1 - \frac{T_{pc}(\mu_B)}{156.5 \text{ MeV}}}$$

$T_{pc} \to 156.5$ MeV ; $\mu_B \to 0$

$T_{pc} \to 132$ MeV ; $\mu_B \to 470$ MeV

$T_{pc} \to 0$ MeV ; $\mu_B \to 0$

If we truncate at $O(\mu_B^2)$, the pseudo critical line '**turns around**' at:

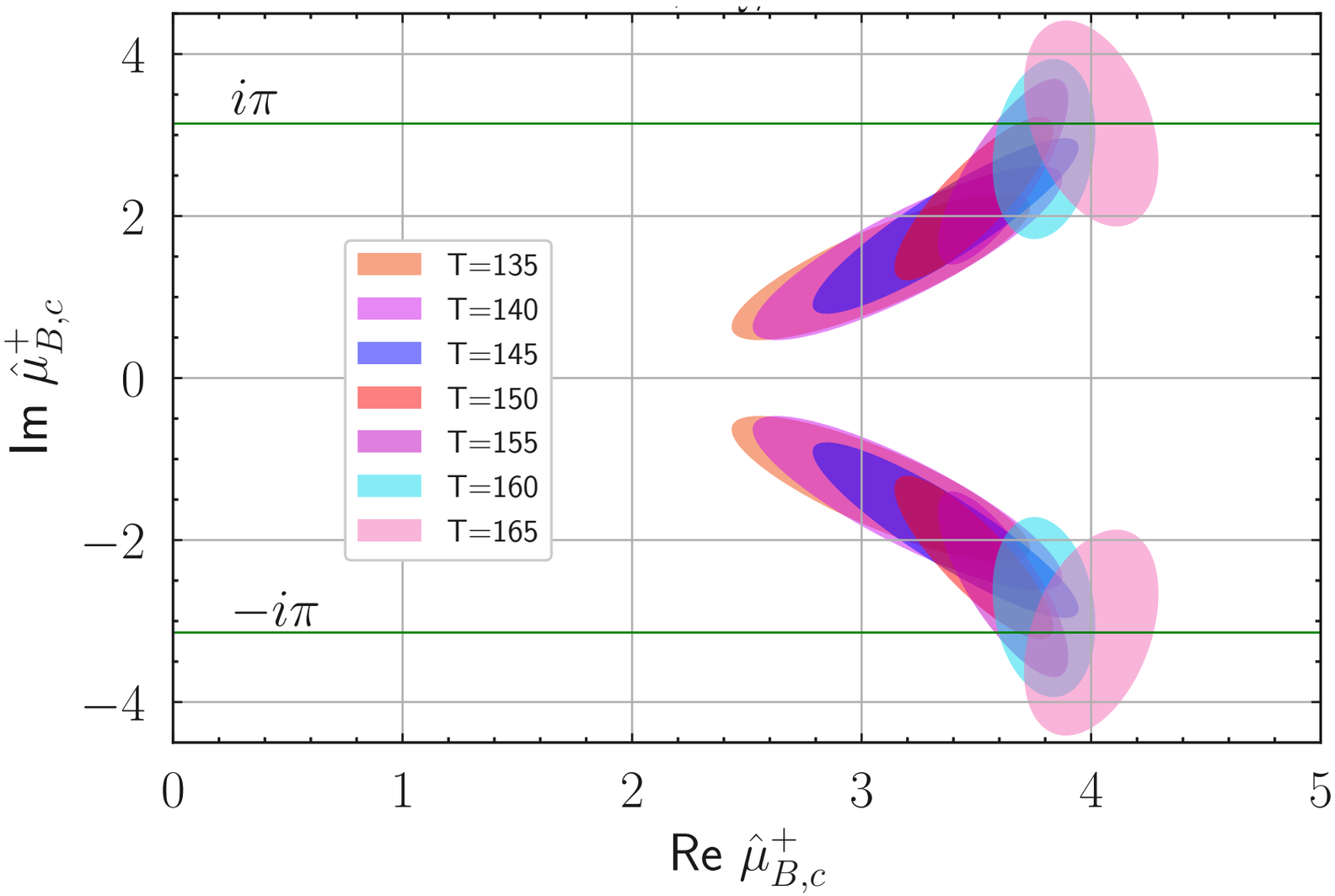
$(T_{pc}, \mu_B) \approx (104.3 \text{ MeV}, 550 \text{ MeV})$ [ **A limitation !!]**

The excluded region of the QCD critical point : $T_{pc}(\mu_B) > 132$ MeV, $\mu_B < 470$ MeV

No CEP in the BES II in the collider mode !! (Most likely)

**Consistent with : S Borsanyi et al, arXiv:2502.10267 [hep-lat] ; D A. Clarke et al, 2405.10196 [hep-lat] ; D. Bollweg et al, Phys.Rev.D 105 (2022) 7, 074511 ; Phys.Rev.D 108 (2023) 1, 014510 ; J .Goswami, Acta Phys.Polon.Supp. 16 (2023) 1, 1-A76 ;**

# Pade approximations




Singularity of the pressure series using a [4,4] padè constructed from 8th order Taylor series

Expectation :

$T_{CEP} < T_{chiral}$

$(T_{chiral} \sim 132 \text{ MeV})$

**H.T. Ding *et al*, *Phys.Rev.Lett.* 123 (2019) 6, 062002**

Bound for CEP :

$T^{CEP} < 132 \text{ MeV}, \hat{\mu}_B/T \geq 2.5$

**D. Bollweg et. al (HotQCD collaboration),Phys.Rev.D 105 (2022) 7, 074511, J. Goswami et. al (HotQCD collaboration), Acta Phys.Polon.Supp. 16 (2023) 1, 76**

- Lee-yang theorem: Singularity in the real axis is a hint for a critical point.

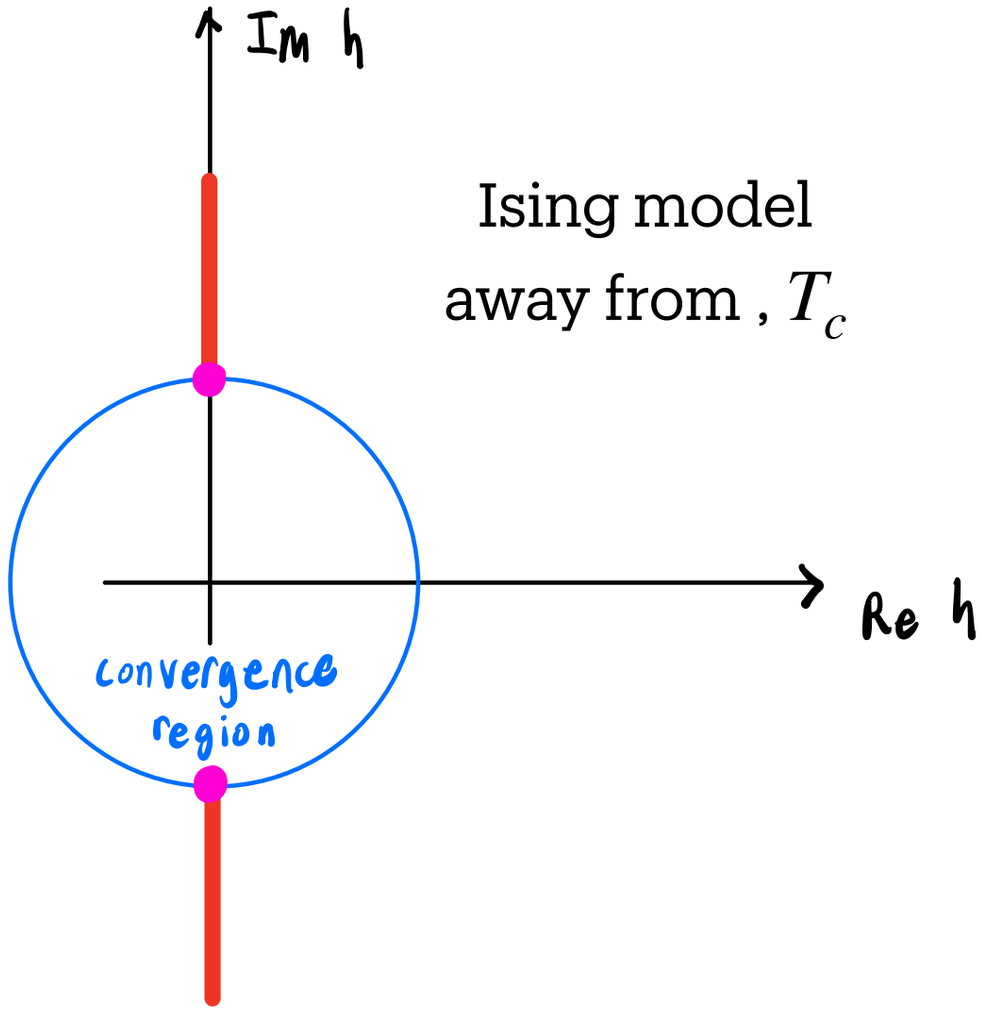
# LeeYang Edge singularities (LYEs)

LeeYang Edge (LYEs) : The singularities closest to real axis away from the critical temperature.

Position of the LYEs:

$$z_c = |z_c| e^{\pm i\pi/2\beta\delta}$$

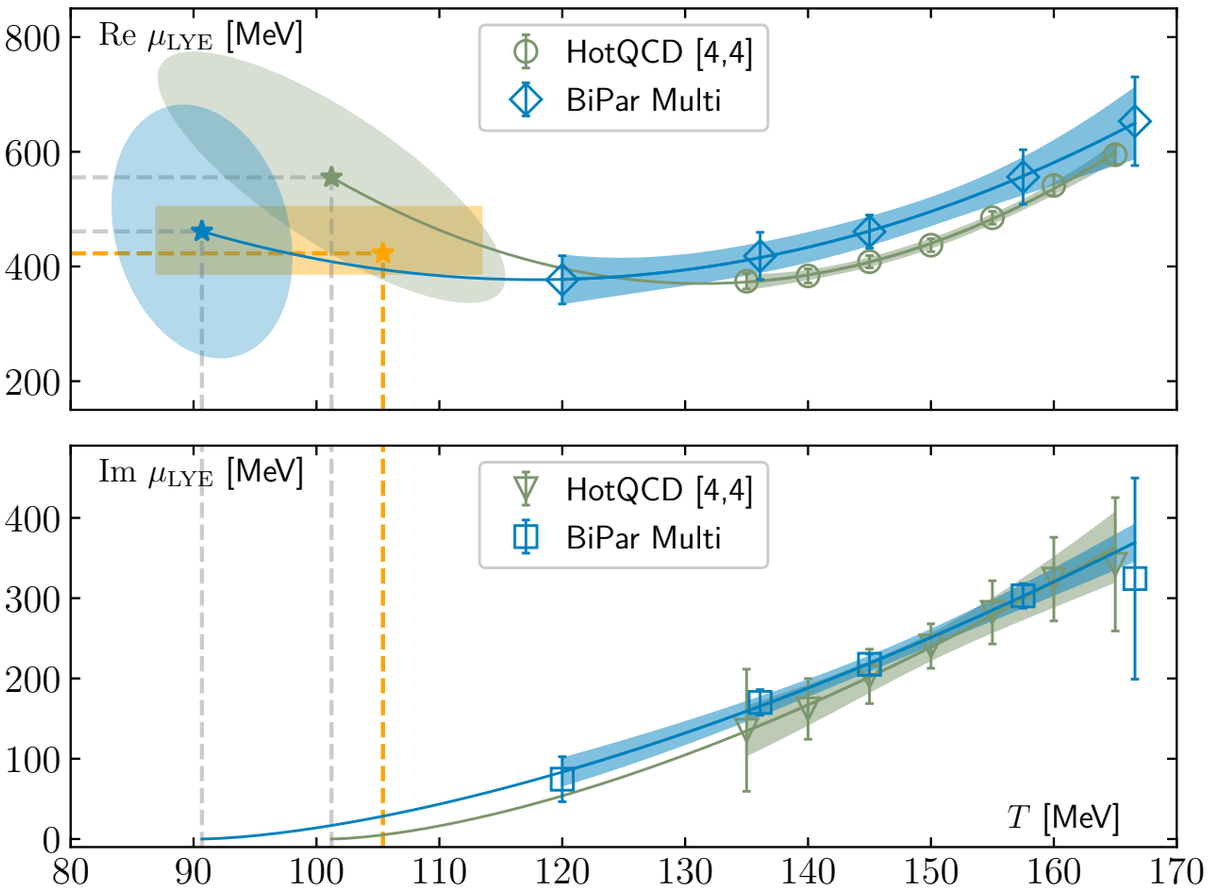
$z = t/h^{1/\beta\delta}$, $\beta, \delta$ are critical exponents



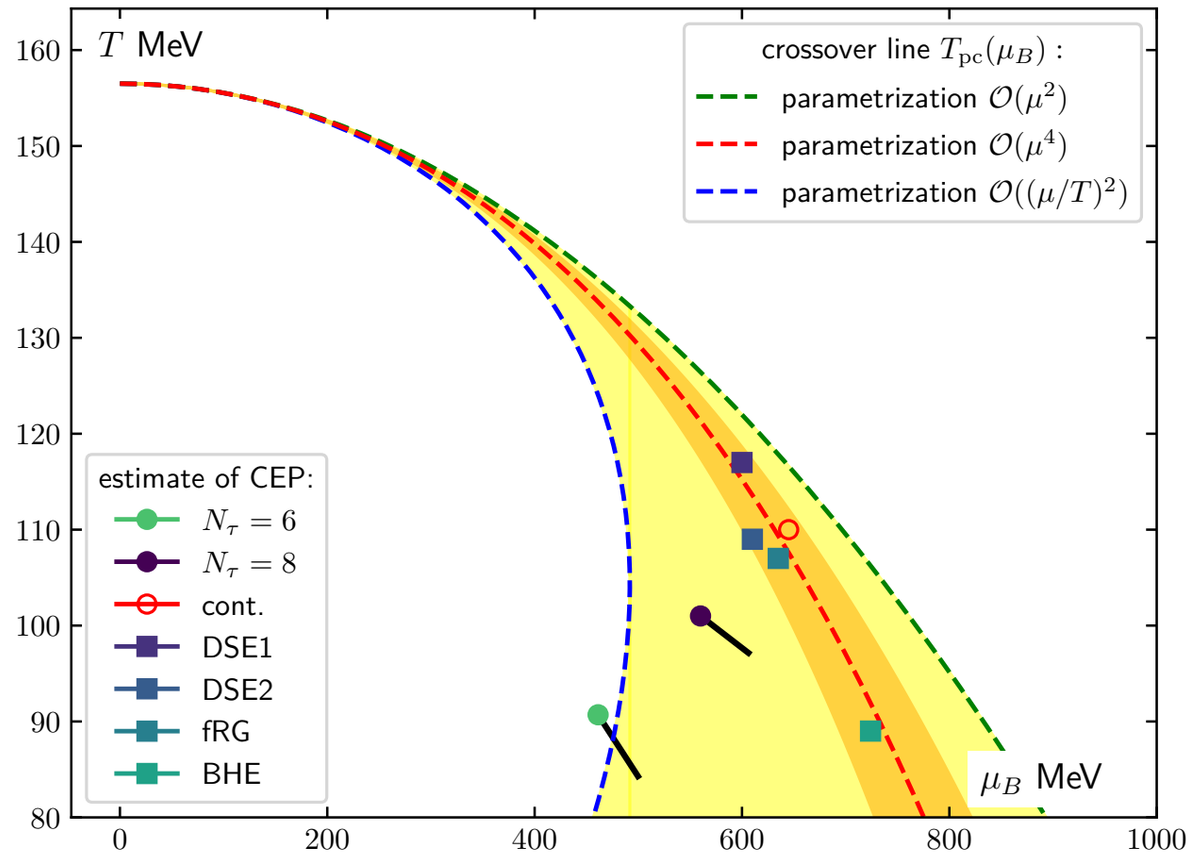

**Extended analyticity conjecture: LYEs are the nearest singularity to the origin.**

P Fonseca and A Zamolodchikov, J. Stat. Phys. 110, 527–590 (2003)

# Estimation of the QCD CEP using pade approximations




David A. Clarke et al, 2405.10196 [hep-lat]




Christian Schmidt, 2504.00629 [hep-lat]

$$T_{CEP} = 105^{+8}_{-18}\ \text{MeV and}\ \mu_{\rm B_{CEP}} \sim 422^{+80}_{-35}\ \text{MeV}$$

The natural parametrization of the pseudo-critical line of QCD :

$$\frac{T_{pc}(\mu_B)}{T_{pc,0}} = \left[1 - \kappa_2\left(\frac{\mu_B}{T_{pc}(\mu_B)}\right)^2 - \kappa_4\left(\frac{\mu_B}{T_{pc}(\mu_B)}\right)^4 + \ldots\right]$$

For larger $\mu_B$ convergence of $T_{pc}(\mu_B)$ is important i.e. higher order in expansion terms needed

# Outlook: isothermal compressibility

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_{T,\vec{N}} = \frac{1}{n_B}\left(\frac{\partial n_B}{\partial p}\right)_{T,N_B,N_Q,N_S} = \frac{2}{n_B^2}(\chi_{11}^{BQ} - \chi_{11}^{BS}\chi_{11}^{QS}/\chi_2^S), \text{ For, } n_S = 0, \; ; n_Q/n_B = 0.5$$

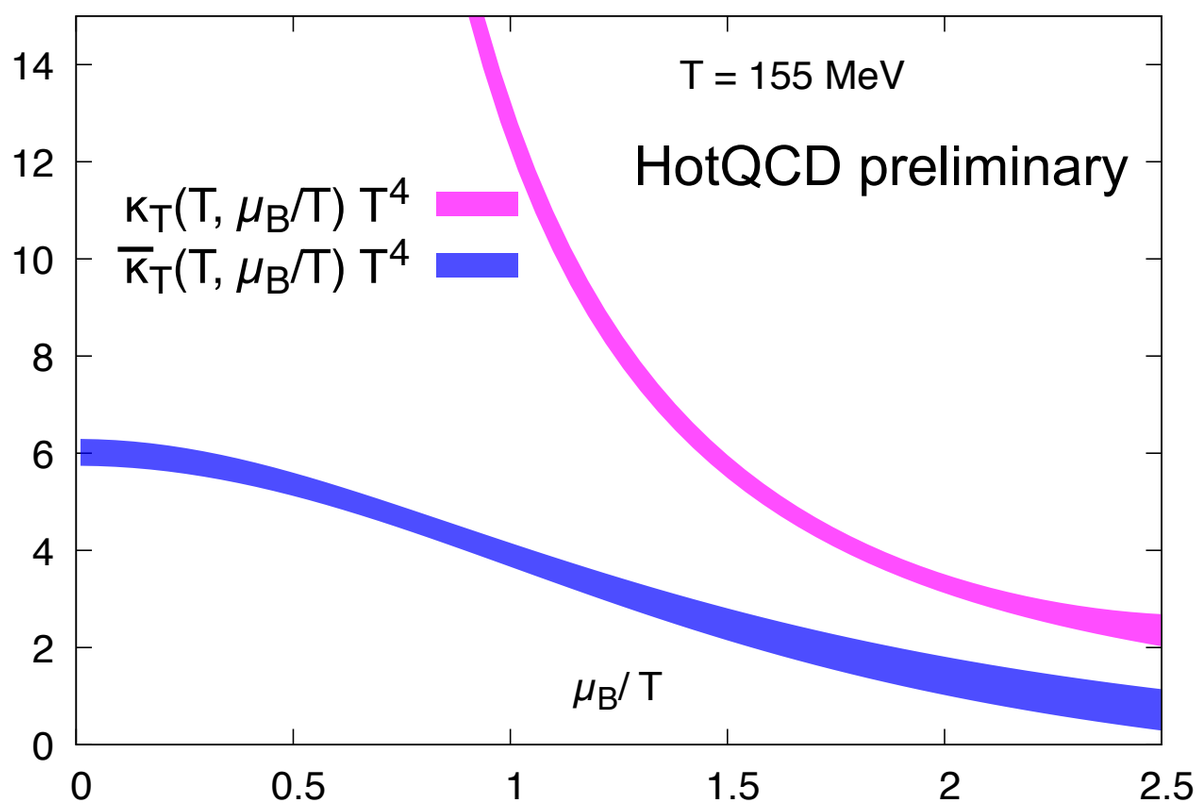
$$(\mu_B, 0, \mu_S)$$

**Stefan Floerchinger et al, Phys. Rev. C 92, 064906 (2015)**

$n_B \to 0$ for ALICE energies.

Possible lattice QCD definition which can be measure and compare at ALICE energies ::

$$\bar{\kappa}_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_{T,\vec{N}} = \frac{1}{\chi_2^B}\left(\frac{\partial \chi_2^B}{\partial p}\right)_{T,X_2^B,N_Q,N_S} \qquad X_2^B = V\chi_2^B$$




"Isothermal compressibility should stay finite except for the first order transition region including the critical point." Statistical Mechanics, Kerson Huang,

**S Mrowczynski et al, Phys.Lett. B430 (1998) 9-14**
**ALICE Collaboration, Eur. Phys. J. C 81 (2021) 1012**

# Summary

CRC-TR 211

**Constraints on QCD Critical Endpoint (CEP) :**

- $T_{pc}(\mu_B) < 132$ MeV, $\mu_{\rm B} > 470$ MeV (using $O((\mu_B/T_{pc}(\mu_B))^2)$ parameterization)
- For precise CEP estimation at $\mu_B > 550$ MeV, higher-order terms required.
- Padé analysis (**HotQCD & Bielefeld-Parma**) identifies singularities approaching the real $\mu_B$-axis. $T_{CEP} = 105^{+8}_{-18}$ MeV and $\mu_{\rm B_{CEP}} \sim 422^{+80}_{-35}$ MeV

**Comparison with Experimental Results (STAR & ALICE)**

- STAR results of $\dfrac{\chi_{11}^{BS}(T,\hat{\mu}_B)}{\chi_2^{S}(T,\hat{\mu}_B)}$ doesn't agree with QCD results for $\sqrt{s_{NN}} \leq 27\text{GeV}$.
- $R_{12}^B > 1$ at low beam energies: consistent with lattice QCD, but incompatible with non-interacting HRG models.
- $R_{42}^B = \dfrac{\chi_4^B}{\chi_2^B}(\mu_B = 0) = 0.70(8)$, Current ALICE results do not (yet??) agree with Lattice QCD calculations $T_{pc} = 156.5(1.5)$ MeV.

*Thank you for your attention!!*

# Chemical potential on the lattice

Partion function for (2+1)-flavor QCD,

$$\mathcal{Z}_{QCD} = \int \mathcal{D}U \, \det[M(m_u, \mu_u)] \det[M(m_d, \mu_d)] \, \det[M(m_s, \mu_s)] \, e^{-S_G(U)}$$

Continuum prescription, Divergence for the free fermion case in second order susceptibility: $\chi_2 \sim 1/a^2$

The prescription for chemical potential on the lattice,

$$(1 \pm \gamma_4) U_{\pm 4}(x) \rightarrow (1 \pm \gamma_4) e^{\pm \hat{\mu}} U_{\pm 4}(x)$$

P. Hasenfratz, F. Karsch, Phys.Lett.B 125 (1983) 308-310
R. V. Gavai,Phys. Rev. D 32, 519

No additional divergences appear in the interacting theory.

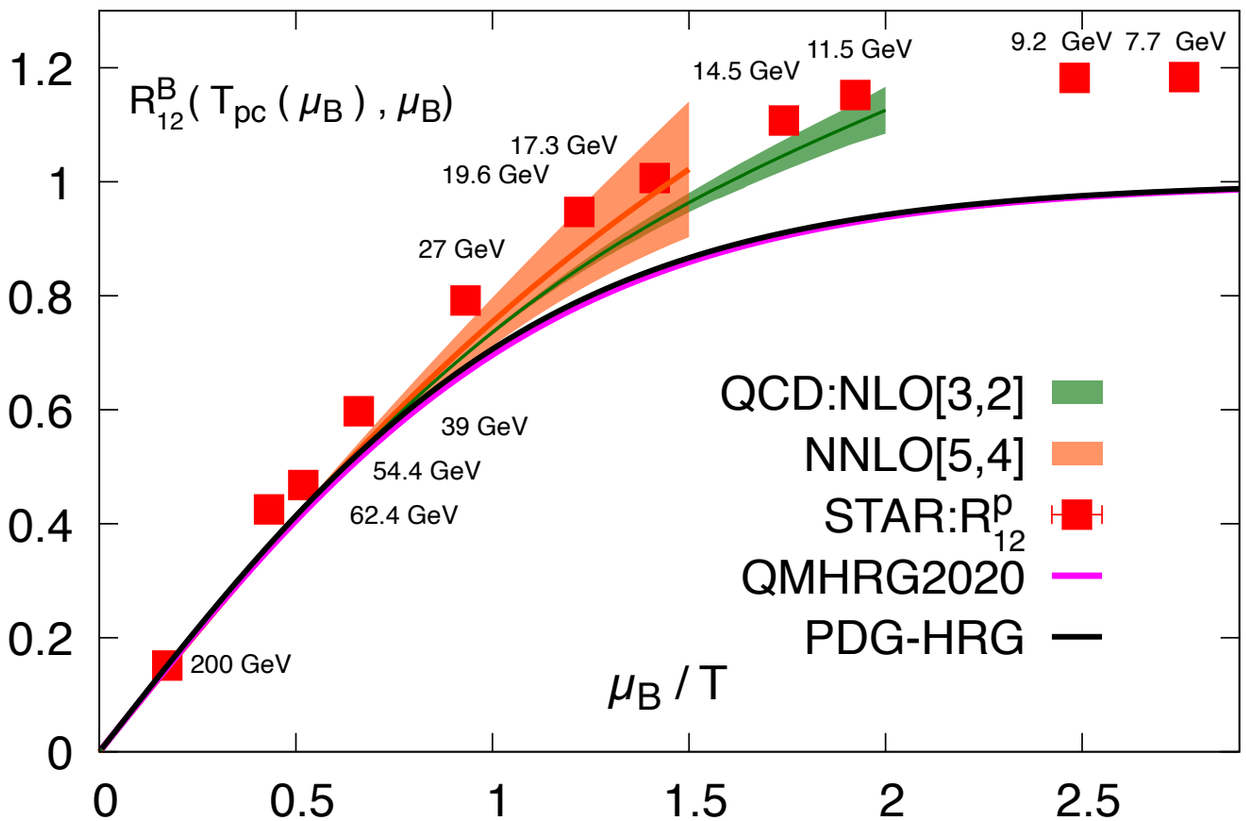
Steven Gottlieb, W. Liu, D. Toussaint, R. L. Renken, and R. L. Sugar, Phys. Rev. Lett. 59, 2247.
Rajiv V. Gavai, Sayantan Sharma, *Phys.Lett.B* 749 (2015) 8-13

Sign problem for, $\mu_f \neq 0, \, f = \{u, d, s\}$. We use Taylor expansions.

Rajiv V Gavai, Sourendu Gupta, Phys. Rev. D **71**, 114014

Saumen Datta, Rajiv V. Gavai, Sourendu Gupta, arXiv:1210.6784 **[hep-lat]**

# Baryon number fluctuations




- In good agreement with the STAR data down to $\sqrt{s_{NN}} \simeq 11.5$ GeV.
- $R^p_{12}$ becomes larger than unity for $\sqrt{s_{NN}} \simeq 17.3$ GeV.
- Consistent with QCD but not consistent with non interacting HRG.

$$R^B_{12} = \frac{\chi^B_1(T,\hat{\mu}_B)}{\chi^B_2(T,\hat{\mu}_B)} = \hat{\mu}_B\left(1 + s_1\frac{\chi^{BS}_{11}}{\chi^B_2} + q_1\frac{\chi^{QS}_{11}}{\chi^B_2}\right) + \mathcal{O}(\hat{\mu}^3_B)$$

# LYEs from universal scaling functions

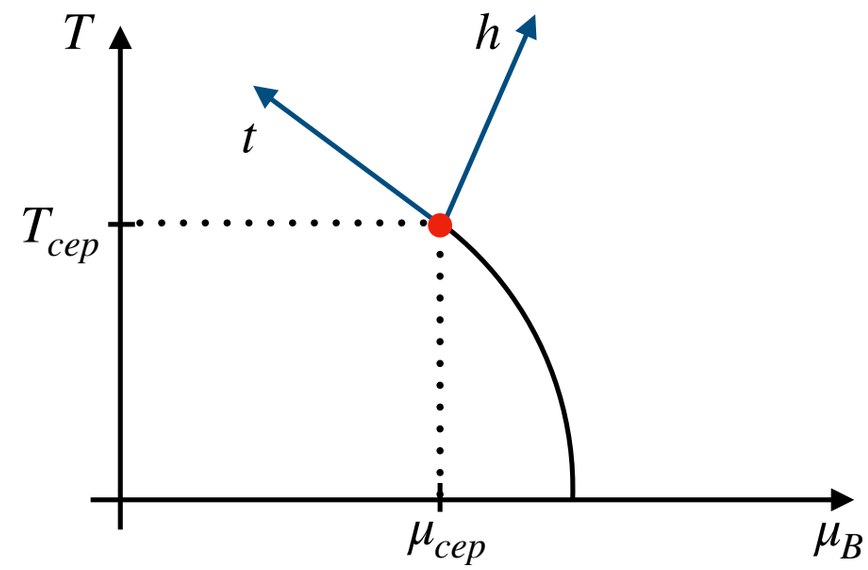
**The universal scaling function for $2^{nd}$ order phase transition, $t = (T - T_c)/T_c$ ,**

$$t \to 0,\ h \to 0$$

$$f = b^{-d} f_s(b^{y_t} t, b^{y_h} h) \ + \ regular$$

$$f \sim h^{\frac{2-\alpha}{\beta}} f_s(z) + \ regular,\ z \to t/h^{1\beta\delta}$$

**Lee Yang Edge singularities (LYEs) can be obtained by solving, $z = z_c = |z_c| e^{\pm i\pi/2\beta\delta}$**




**A linear mapping,**

**For CEP** $Z(2)$**,**

$$t = \alpha_t(T - T_{cep}) + \beta_t(\mu_B - \mu_{cep})$$

$$h = \alpha_h(T - T_{cep}) + \beta_h(\mu_B - \mu_{cep})$$

**Stephanov, Phys. Rev. D, 73.9, 094508 (2006)**

**LYEs with many unknown parameters,**

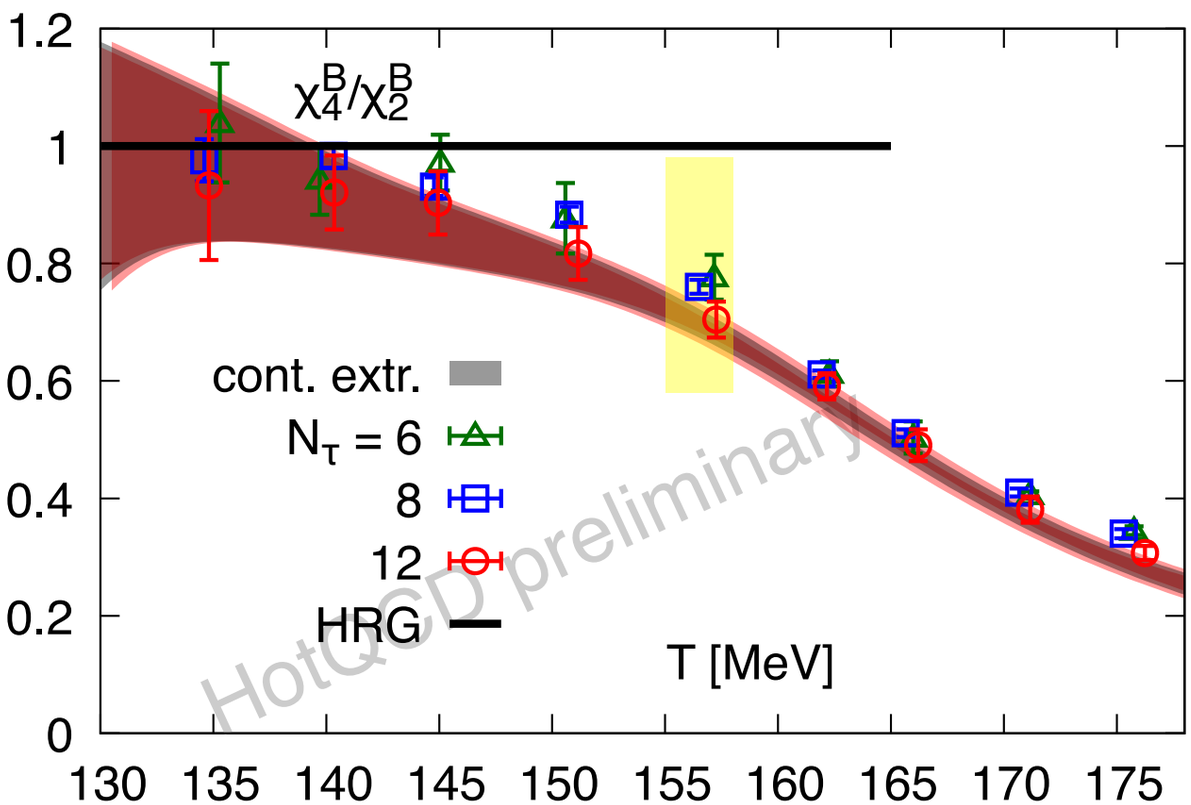
$$\mu_{LY} = \mu_{cep} - c_1(T - T_{cep}) + ic_2|z_c|^{-\beta\delta}(T - T_{cep})^{\beta\delta}$$

**Real part : linear in T  Imaginary part : Power law in T**

**At, $\mu_{LY} \to \mu_{cep}$, imaginary part vanishes.**

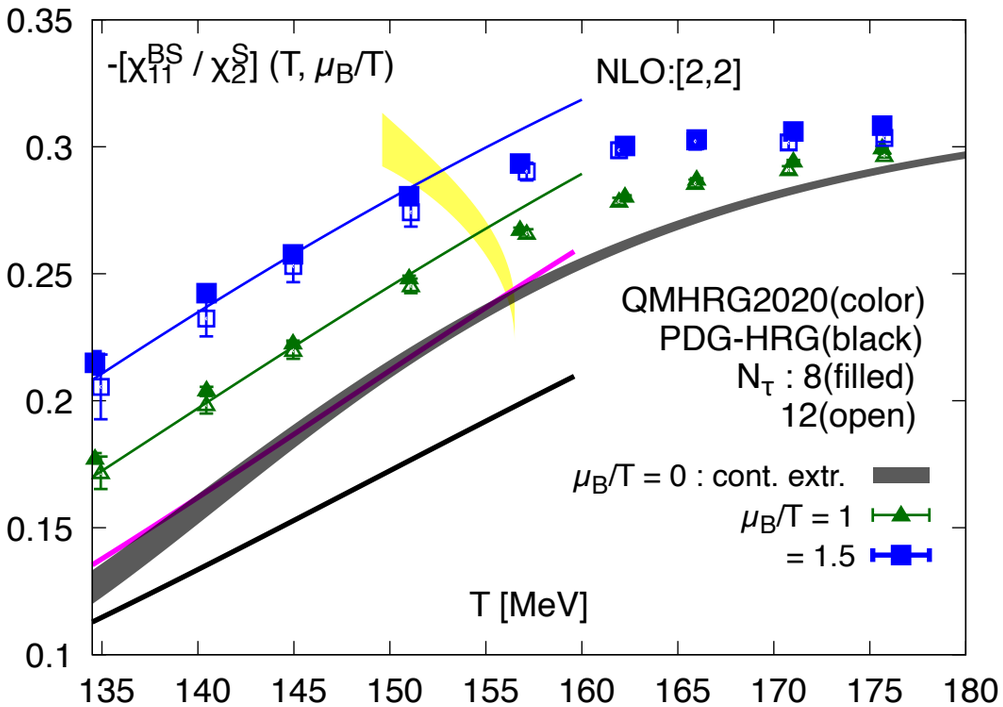
# Higher order baryon number fluctuation at vanishing chemical potential

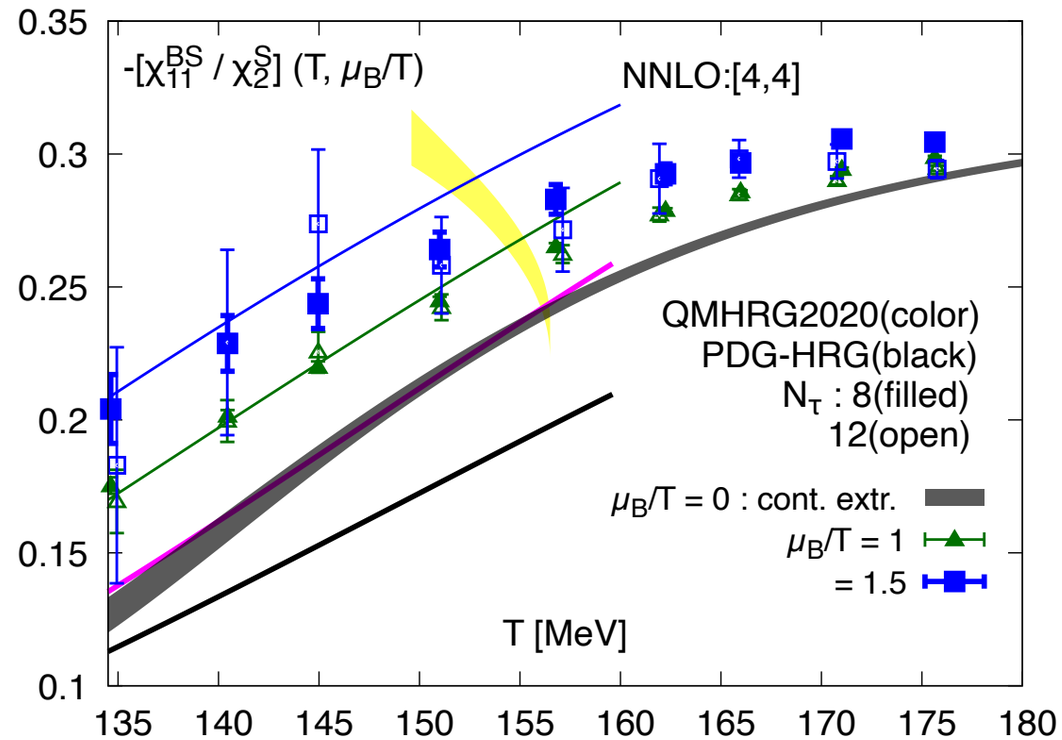
- A recap at vanishing chemical potential :




- $R_{42}^B = \kappa_B \sigma_B^2 = \chi_4^B/\chi_2^B$, of baryon number relevant for ALICE energies and STAR beam energy, $\sqrt{s_{NN}} = 200$ GeV .
- $$R_{42}^B = \frac{\chi_4^B}{\chi_2^B}(\mu_B = 0) = 0.70(8)$$
- 30 % smaller than the HRG calculations at the pseudo-critical temperature of QCD.

# Baryon strangeness correlations in (2+1)-flavor QCD with HISQ fermions




D. Bollweg et al, Phys.Rev.D 110 (2024) 5, 054519

The pseudo critical line from lattice QCD,

$$T_{pc}(\mu_B) = T_{pc,0}\left[1 - \kappa_2\hat{\mu}_B^2 + \kappa_4\hat{\mu}_B^4\right]$$

A. Bazavov et al. (HotQCD), Phys. Lett. B 795, 15 (2019);
B. S. Borsanyi et al, Phys. Rev. Lett. 125, 052001 (2020)

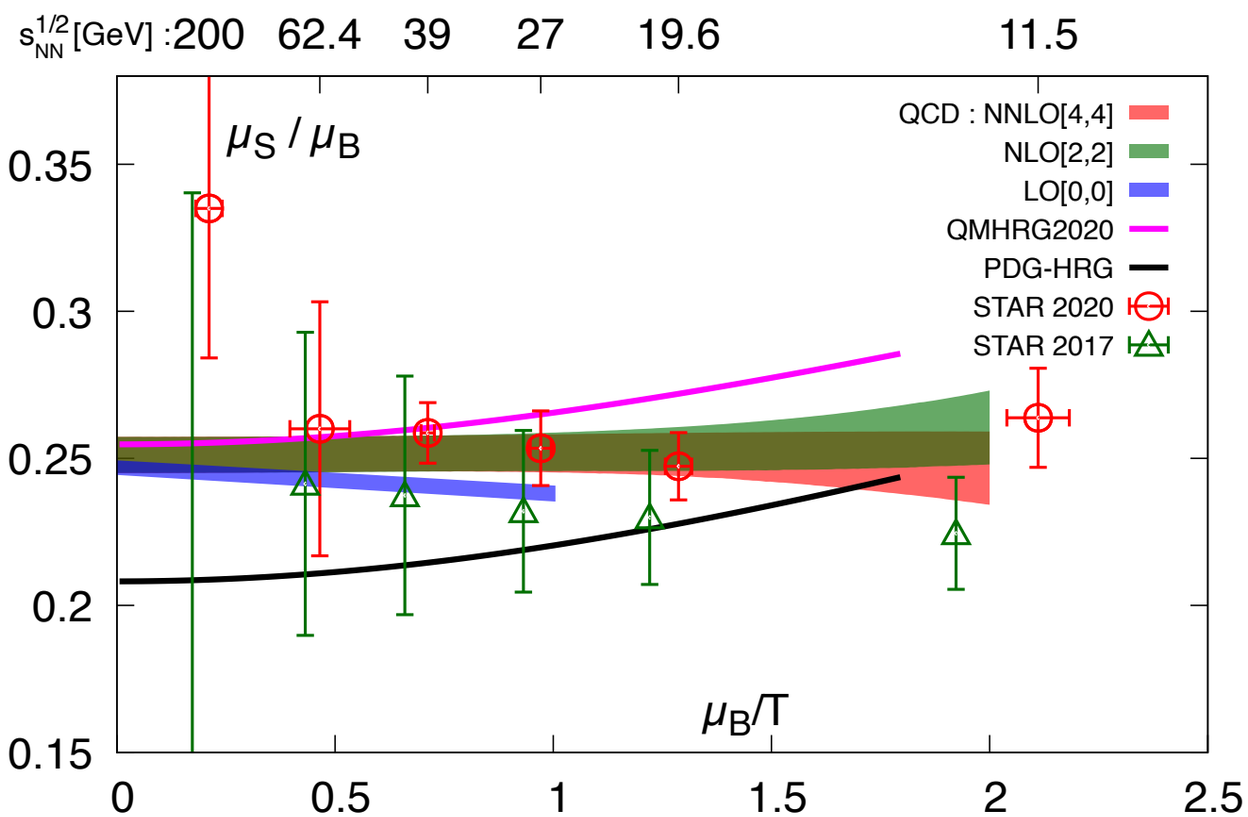
$$T_{pc,0} = (156.5 \pm 1.5)\ \text{MeV} \quad \text{and} \quad \kappa_2 = 0.012(4)$$

$$\frac{\chi_{11}^{BS}(T,\mu_B/T)}{\chi_2^S(T,\mu_B/T)} = \frac{\chi_{11}^{BS}(T,0) + ()\hat{\mu}_B^2 + ()\hat{\mu}_B^4 + \ldots}{\chi_2^S(T,0) + ()\hat{\mu}_B^2 + ()\hat{\mu}_B^4 + \ldots}$$

# Ratio of strangeness and baryon chemical potential

$$\mu_Q = q_1\mu_B + q_3\mu_B^3 + \ldots \qquad \mu_S = s_1\mu_B + s_3\mu_B^3 + \ldots$$

$$\left.\frac{\mu_S}{\mu_B}\right|_{LO} = -\frac{\chi_{11}^{BS}(T,0)}{\chi_2^S(T,0)} - \frac{\chi_{11}^{QS}(T,0)}{\chi_2^S(T,0)} q_1 + \mathcal{O}(\mu_B^2)$$




- QCD and STAR results are in good agreement for almost all the beam energies.
- The Lattice results agree more closely with the QMHRG2020 predictions.

J. Adam et al. (STAR), Phys. Rev. C 102, 034909 (2020);
L. Adamczyk et al. (STAR), Phys. Rev. C 96, 044904 (2017)