Theoretical background

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A crash course on Galactic dynamics + Hands-on tutorial on Galpy orbit modeling

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Bibliography

- Jo Bovy's Online graduate textbook (2023): Dynamics and astrophysics of galaxies
- Binney & Merrifield (1998): “Galactic Astronomy”
- Binney & Tremaine (2008): “Galactic Dynamics”
- Eugene Vasiliev’s Modern Galactic dynamics in the era of plentiful data (2020) workshop

NGC 7773 (Credit: HST)

NGC 5679 & Arp 274 (Credit: HST)

M100 (Credit: HST)

Outline

- **Theoretical background**
  - Gravity and potentials
  - Lagrangian and Hamiltonian formalisms
  - Conserved quantities (e.g., energy, ang momentum)
  - Orbits in spherical and disk potentials
- **Observational background**
  - The Milky Way
  - Surveys
  - Streams
  - The accretion history of the Milky Way
  - The effects of satellite accretion
- **Galpy tutorial**
  - The basics: installation and getting to know the package
  - Generating orbits
  - The effect of the Large Magellanic Cloud
  - Comparing and modifying potentials

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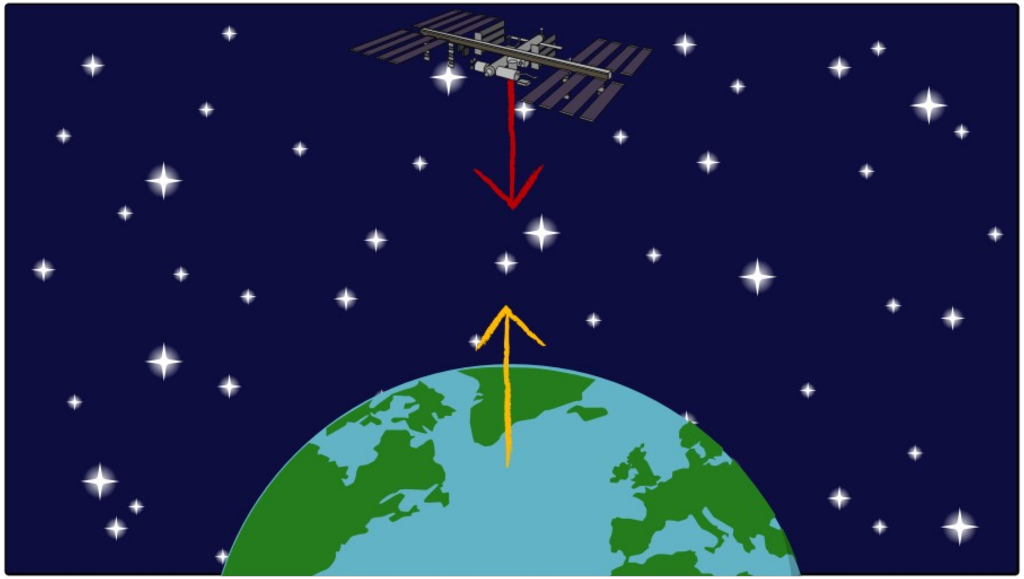
Matter and the gravitational field

Masses under the influence of gravity (long range force)

$$\Phi(\vec{x}) = \sum_{i=0}^{N} \frac{-GM_i}{|\vec{x} - \vec{x}_i|} \qquad \vec{F}(\vec{x}) = -m \nabla \Phi(\vec{x})$$



Credit: Storyboard.com

Matter and the gravitational field

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$$\vec{F}(\vec{x}) = -m \nabla \Phi(\vec{x})$$

$$\nabla^2 \Phi(\vec{x}) = 4\pi G \rho$$

$$\vec{F}(r) = -\frac{GMm}{r^2} \hat{r}$$

The gravitational potential for a set of N point masses is the sum of the potential for the individual point masses. Therefore:

$$\vec{F}(\vec{x}) = \sum_{i=0}^{N} \frac{-GM_i m}{|\vec{x} - \vec{x}_i|^3} (\vec{x} - \vec{x}_i)$$

In galactic dynamics:

- relativity is neglected;
- cosmological expansion is neglected;
- potential is negative and tends to zero at infinity.

Matter and the gravitational field

The mass of galaxies as a sum of discrete constituents:

- Stars,
- Dark matter particles,
- ISM

Their distribution is rather uniform

Mass density can be approximated as a smooth function (→ the potential and the force are also smooth)

M101 (Credit: ESA & NASA)

NGC 660 (Credit: Gemini Obs. & AURA)

Circular velocity

- For a circular orbit, the centripetal acceleration is balanced by the gravitational field

Circular velocity at radius r

$$a_r = -\frac{v^2}{r} = -\frac{GM(<r)}{r^2}$$

$$v_c^2 = \frac{GM(<r)}{r}$$

Connection to rotation curves and dark matter:
Flat rotation curve → vc constant with r implies linear relation between mass and r

For reference, for the Sun

$$R_0 \approx 8\,kpc\,,\, v_c \approx 220\,km\,s^{-1}$$

$$M(<8\,kpc) \approx 9 \times 10^{10} M_0$$

Dynamical time (or crossing time)

- Time required to cross a dynamical system

$$t_{dyn} = \frac{2\pi r}{v_c} = \sqrt{\frac{3\pi}{G<\rho>}}$$

Depends on the average density!

- short dynamical timescales in the Galactic center and long in the halo
- For the sun, t_{dyn} ~ 225 Myr (and is close to 1 Gyr at 50 kpc)

Since the age of the Universe is ~13 Gyr, and galaxies are ~10 Gyr old, they are dynamically young!

Examples of potentials

Point mass

$$\Phi(r) = -\frac{GM}{r}$$

$$v_c(r) = \sqrt{\frac{GM}{r}}$$

$$t_{dyn}(r) = \frac{2\pi r}{v_c}$$

e.g., Solar system (Keplerian orbits), supermassive black holes

Homogeneous sphere (radius R)

$$\Phi(r) = \begin{cases} \frac{2\pi G\rho_0}{3}\left(r^2 - 3R^2\right), & (r < R) \\ \frac{4\pi G\rho_0 R^3}{3r}, & (r \geq R) \end{cases}$$

$$v_c(r) = \sqrt{\frac{4\pi G\rho_0}{3}} r$$

$$t_{dyn} = \sqrt{\frac{3\pi}{G\rho_0}}$$

t_{dyn} valid for circular and radial orbits (harmonic oscillator)

Examples of potentials

Plummer sphere

(Plummer 1911)

$$\Phi(r) = -\frac{GM}{\sqrt{r^2+b^2}}$$

b: Plummer scale length

Isochrone potential

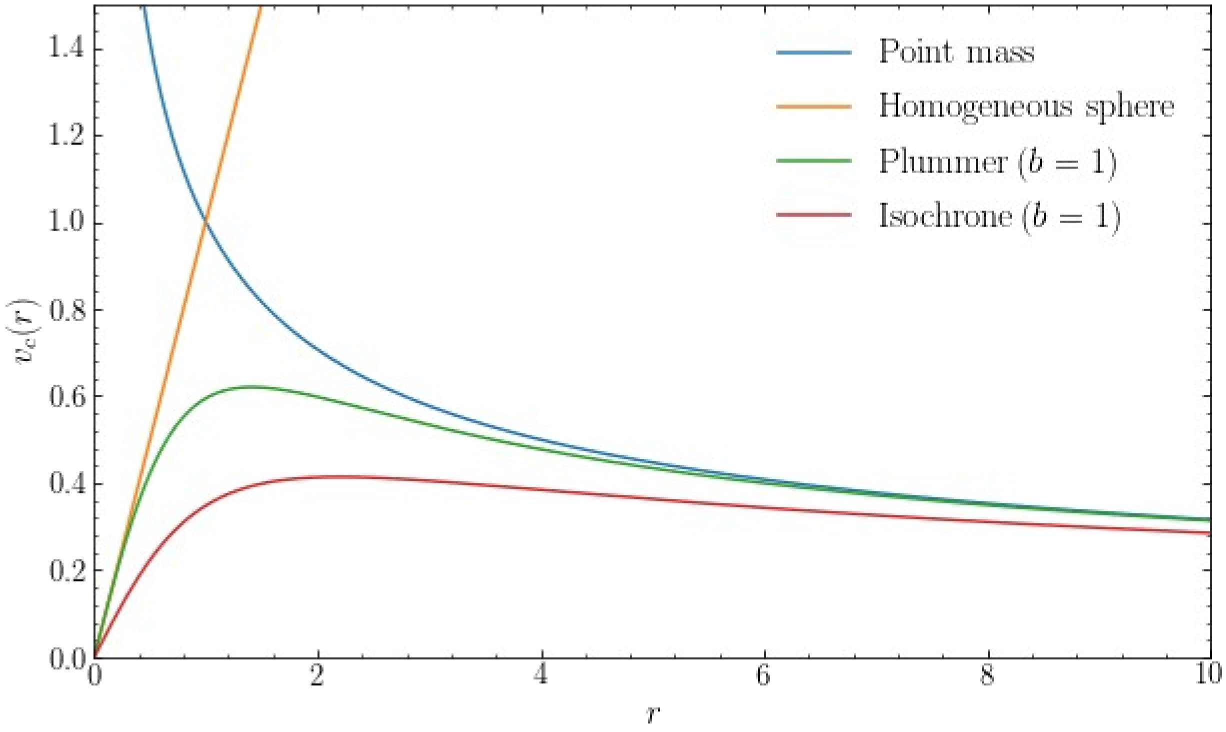
(Heron 1959)

$$\Phi(r) = -\frac{GM}{b+\sqrt{r^2+b^2}}$$

$$v_c(r) = \sqrt{\frac{GM\,r^2}{(b+a)^2\,a}} \qquad a = \sqrt{r^2+b^2}$$

e.g., dark matter halos, small dwarf galaxies,
originally used for globular clusters too

Examples of potentials



Important power-law potentials

$$\rho(r) = \frac{\rho_0 a^{\alpha}}{r^{\alpha}(1+r/a)^{\beta-\alpha}}$$

$$\rho(r) = \begin{cases} \rho_0 \left(\frac{a}{r}\right)^{\alpha}, & r \ll a, \\ \rho_0 \left(\frac{a}{r}\right)^{\beta}, & r \gg a. \end{cases}$$

Hernquist model

(Hernquist 1990)

$\alpha = 1 \quad \beta = 4$

$$\Phi(r) = -\frac{4\pi G \rho_0 a^2}{\sqrt{2(1+r/a)}} = -\frac{GM}{r+a}$$

e.g., dark matter halos, elliptical galaxies, galactic bulges

Navarro-Frenk-White (NFW) model

(Navarro et al. 1997)

$\alpha = 1 \quad \beta = 3$

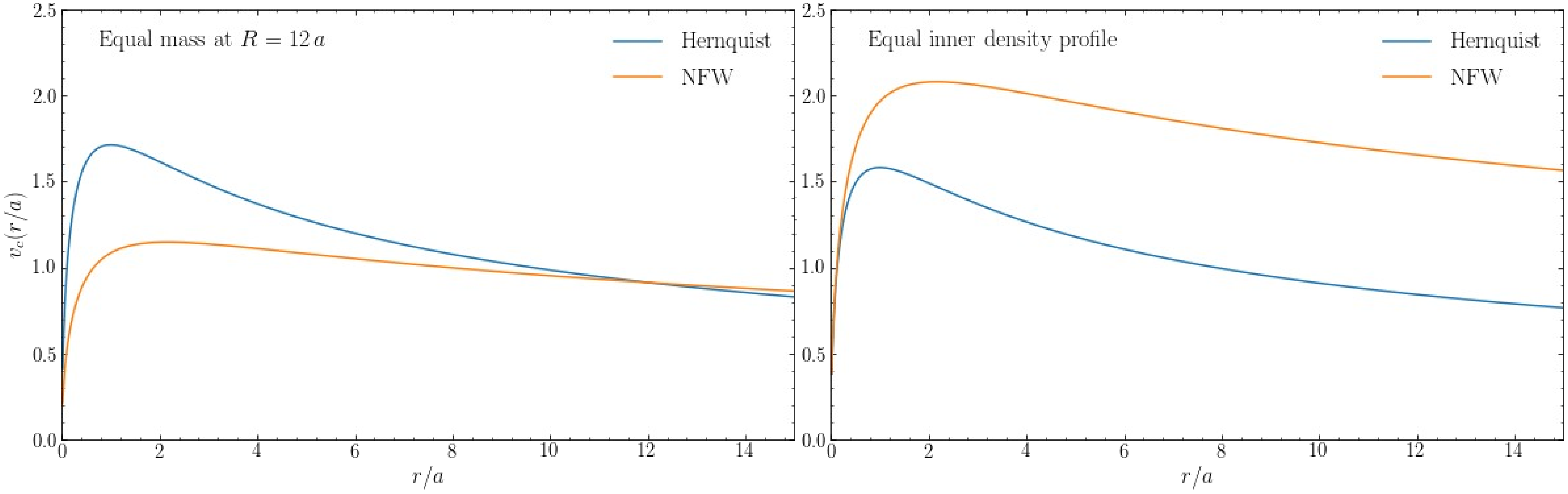
$$\Phi(r) = -4\pi G \rho_0 a^3 \ln(1+r/a)/r$$

e.g., dark matter halos in cosmological simulations

Important power-law potentials

Same enclosed mass at r = 12 a

Normalized so that both have the same inner profile



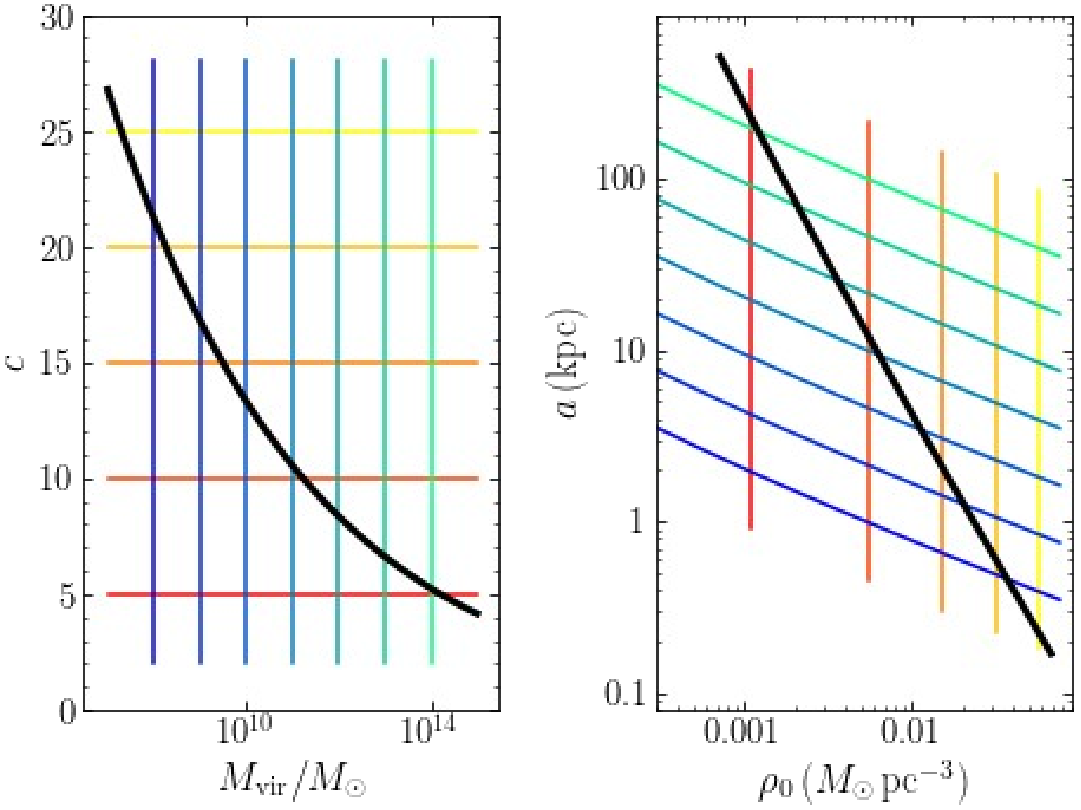
Parametrization of NFW potentials

- Virial mass and concentration parameter
- Virial radius: radius within which the mean density is a factor Δ_v times the Universe's critical density
- Typically, $\Delta_v = 200$ → r_{200}, m_{200}

$$\rho(r) = \frac{\Delta_v \rho_{crit}}{3} \frac{c^3}{f(c)} \qquad f(c) = \ln(1+c) - \frac{c}{1+c} \qquad c = \frac{r_{vir}}{a}$$

Concentration-mass relation

$$\log_{10} c = 0.905 - 0.101 \log_{10}\left(\frac{M_{vir}}{10^{12} h^{-1} M_0}\right)$$



Notions of Classical Mechanics

- Classical mechanics: motions of bodies under the influence of “classical” forces (gravity, electromagnetic)
- The basic concepts are contained in Newton’s laws of motion

$$\vec{p} = m\vec{v} \qquad \vec{F} = \dot{\vec{p}}$$

- A very important quantity in the study of Galactic orbits is the angular momentum

$$\vec{L} = \vec{x} \times \vec{p} \qquad \dot{\vec{L}} = x \times \vec{F}$$

Torque

If the torque (or one of its components) is zero, the angular momentum (or one of its components) is conserved

Notions of Classical Mechanics

- Work and energy

$$W_{ij} = \int_{x_i}^{x_j} \vec{F} \cdot d\vec{x} = T_j - T_i$$

If the force is independent of time, the force is conservative

Kinetic energy

$$T = \frac{m|v|^2}{2}$$

Potential energy (of a body in a gravitational field)

$$V = m\Phi$$

Total energy

$$E = T + V$$

Energy and escape velocity

- The energy E is conserved when the force is conservative (independent of time)
- Escape velocity: minumum velocity necessary to “escape” the gravitational field, from a given position to infinity

$$v_{esc} = \sqrt{2(\Phi_{inf} - \Phi(x))} \qquad v_{esc}(r = 8\,kpc) = \sqrt{2\left(\frac{GM(r < 8\,kpc)}{8\,kpc}\right)}$$

Under the correct assumptions, this equation provides strong constraints on the mass of the Milky Way!

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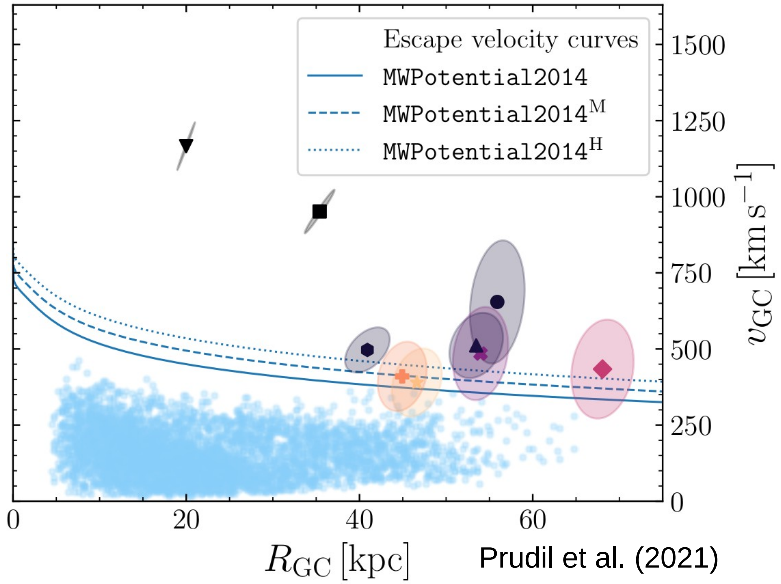

Prudil et al. (2021)

Lagrangian formalism

- Newtonian approach is familiar and intuitive, but more powerful frameworks exist
- Hamilton's principle: The motion of a system from t1 to t2 is such that the action integral has an extremal value

Action integral

$$S = \int_{t_1}^{t_2} L(\vec{x}, \dot{\vec{x}}, t)\, dt = \int_{t_1}^{t_2} T - V\, dt$$

Euler-Lagrange

Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

Lagrangian formalism

- If a generalized coordinate qj does not appear in L, the associated momentum component is conserved
- In a well-chosen coordinate frame, L can reveal a system's conserved quantities

Generalized momentum

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \qquad \dot{p}_j = \frac{\partial L}{\partial q_j}$$

Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

Hamiltonian formalism

- Hamiltonian:

$$H(\vec{q},\vec{p},t)=\dot{\vec{q}}\,\vec{p}-L(\vec{q},\dot{\vec{q}},t)$$

- Hamilton equations treat the coordinates q and the momentum p on the same footing

$$\dot{\vec{q}}=\frac{\partial H}{\partial \vec{p}} \qquad \dot{\vec{p}}=-\frac{\partial H}{\partial \vec{q}} \qquad \frac{\partial H}{\partial t}=-\frac{\partial L}{\partial t}$$

Instead of a set of second-order differential equations for q, we now have a twice as large set of first-order differential equations for q and p!

Hamiltonian formalism

- Hamiltonian: $H(\vec{q},\vec{p},t)=\dot{\vec{q}}\,\vec{p}-L(\vec{q},\dot{\vec{q}},t)$

q → **configuration space**
(q, p) → **phase space**

- If the potential does not depend on the velocities,

H = E = T + V

Hamiltonian formalism

- Canonical transformations and generating function

$$\dot{\vec{q}}\,\vec{p} - H = \dot{\vec{q}}\,'\,\vec{p}\,' - K + \frac{dF}{dt}$$

- If generating function S(q, p', t) such that Hamiltonian K = 0

$$H\left(\vec{q}, \frac{\partial S}{\partial \vec{q}}, t\right) + \frac{\partial S}{\partial t} = 0$$

Hamilton-Jacobi equation

If $\frac{\partial H}{\partial t} = 0$ and $S(\vec{q}, \vec{p}\,', t) = W(\vec{q}, \vec{p}\,') - Et$

$$H\left(\vec{q}, \frac{\partial W}{\partial \vec{q}}\right) = E$$

Action-angle variables

If S(q) along a dynamical trajectory can be written as a sum over functions of a single component, the solution of the Hamilton-Jacobi equation can be defined based on N quantities

$$J_i = \frac{1}{2\pi} \oint dq_i \frac{\partial W_{q_i}(q_i, \vec{C})}{\partial q_i}$$

The transformed configuration coordinates are now

$$\vec{\theta} = \sum_i \frac{\partial W_i(q_i, \vec{J})}{\partial J_i}$$

And the time evolution of the system is given by

$$\dot{\vec{\theta}} = \frac{\partial H(\vec{J})}{\partial \vec{J}} = constant \qquad \dot{\vec{J}} = -\frac{\partial H(\vec{J})}{\partial \vec{\theta}} = 0$$

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"Angle" variables: θ_i

"Action" variables: J_i

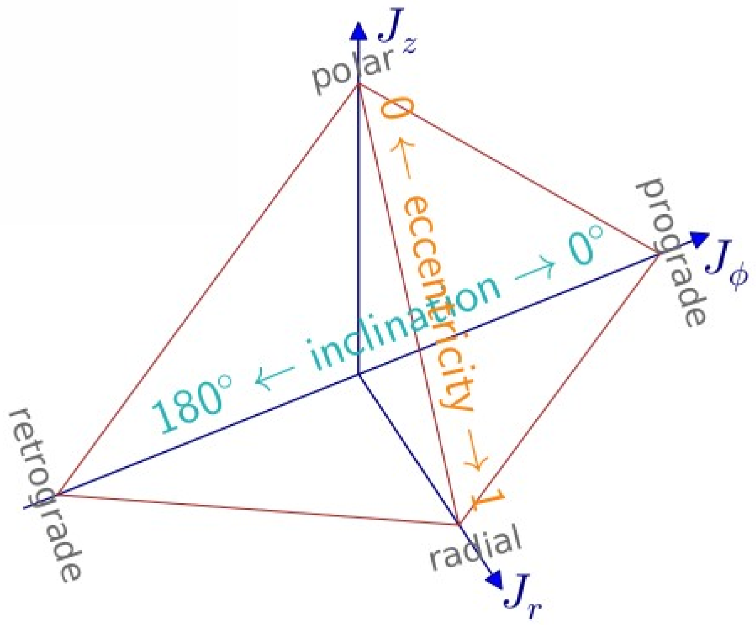
Action-angle variables

- Describe the extent of oscillations in each dimension
- Natural description of motion (angles change linearly with time)
- Canonical coordinates → the 6D phase-space volume element is

$$d^3x\, d^3v = d^3J\, d^3\theta$$

- Actions are adiabatic invariants (conserved under slow variation of potential)
- Efficient methods for conversion between {x,v} and {J,θ} exist

$$\dot{\vec{\theta}} = \frac{\partial H(\vec{J})}{\partial \vec{J}} = constant \qquad \dot{\vec{J}} = -\frac{\partial H(\vec{J})}{\partial \vec{\theta}} = 0$$

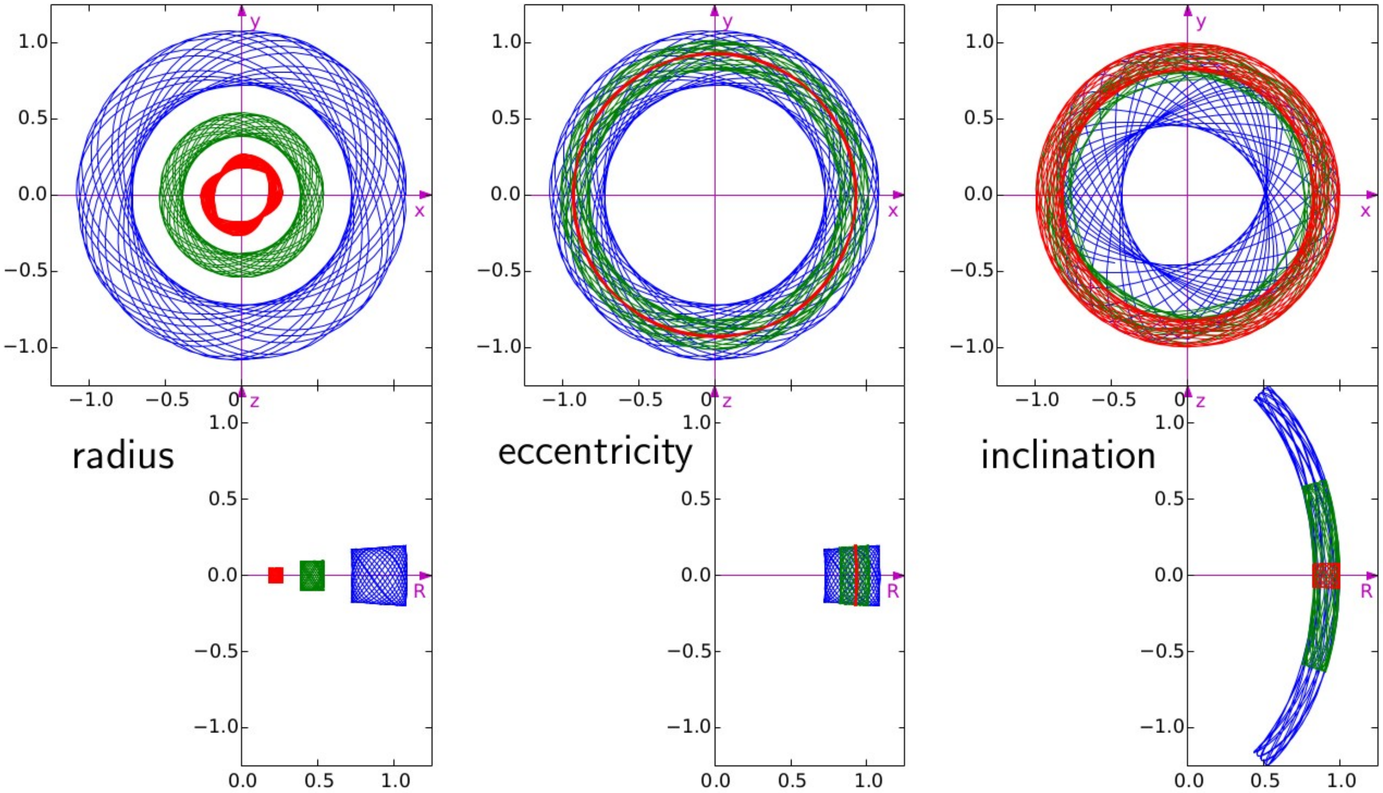


“Angle” variables: θ_i

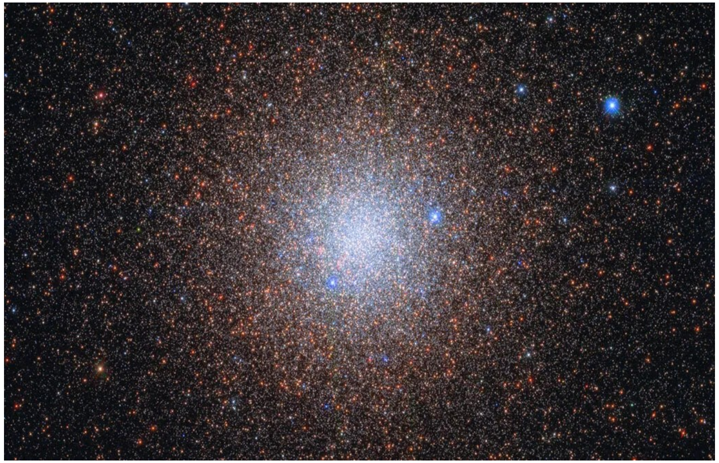
“Action” variables: J_i

Study of orbits and integrals of motion

How large is the variety of orbits?



Orbits in spherical mass distributions



Credit: Sci.news (left), ESO 461-036 NASA/ESA/Hubble (center), NGC 6441 Hubble/ESA and NASA (right)

Orbits in spherical mass distributions

- “Orbit”: The trajectories that bodies travel on under the influence of gravity
- The acceleration in a gravitational field is independent of the mass of the body

$$m\ddot{\vec{x}}=m\vec{g}(\vec{x})$$

- If the mass of the body does not affect the gravitational field, the entire orbit is independent of mass (“test particles”)

Orbits in spherical mass distributions

- Because the orbital trajectory is determined by a second-order differential equation, an orbit is fully determined by its initial phase-space coordinate (for test particles only)
- For spherically symmetric potentials, the (specific) angular momentum vector is conserved

$$\vec{L} = \vec{r} \times \dot{\vec{r}}$$

$\rightarrow$ the orbit is constrained to a plane, which can be described in polar coordinates r, ψ

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$\rightarrow$ the orbit is constrained to a plane, which can be described in polar coordinates r, ψ $\rightarrow$ $\dot{\psi}=\frac{L}{r^2}$ Kepler's second law

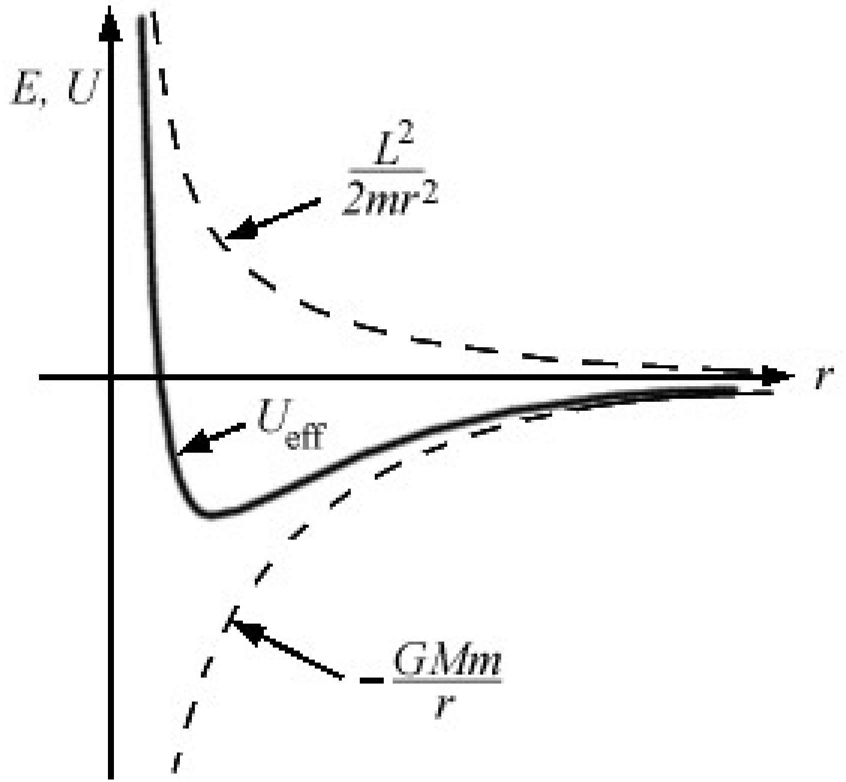
General properties of orbits in spherical potentials

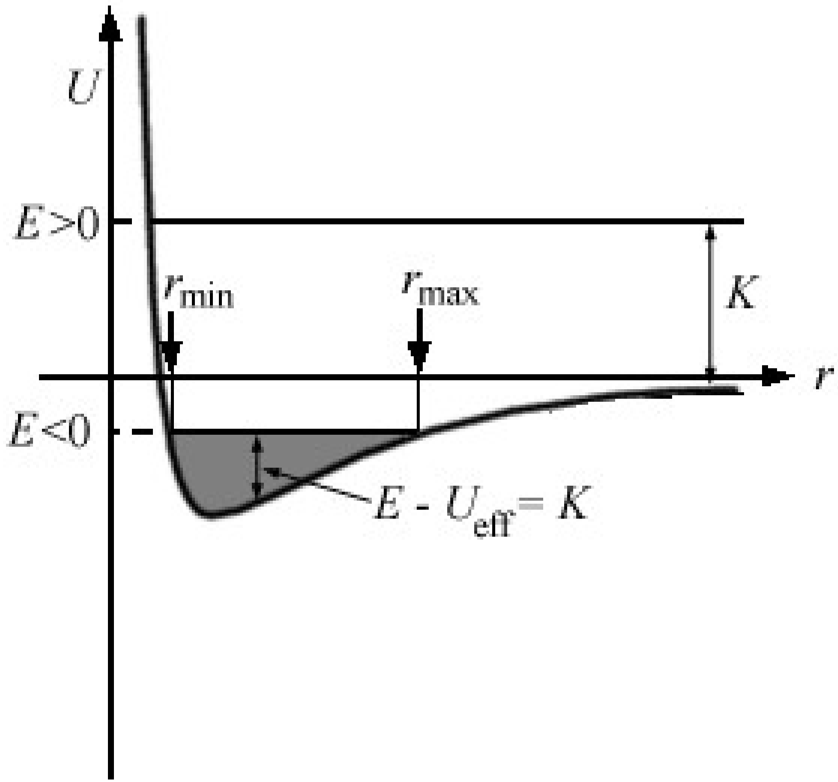
- The specific energy of an orbital in a spherical potential is

$$E = \frac{\dot{r}^2}{2} + \frac{L^2}{2r^2} + \phi(r)$$

where $\phi_{eff}(r) = \frac{L^2}{2r^2} + \phi(r)$ is the effective potential

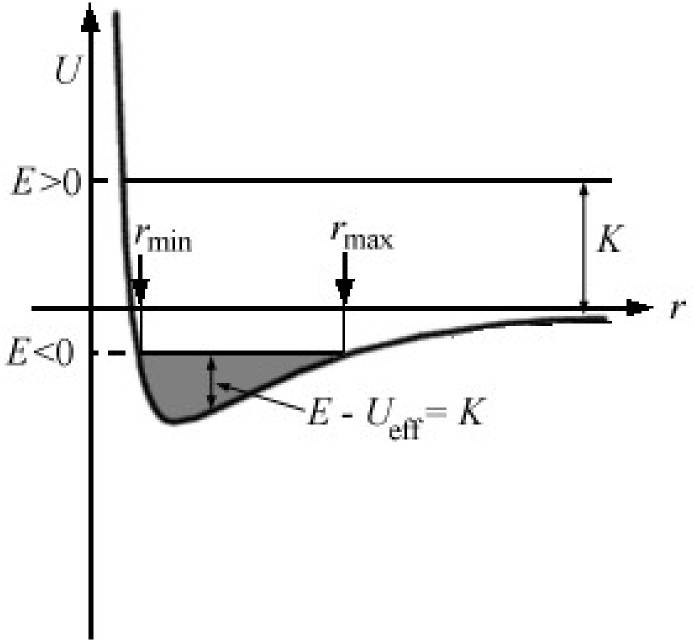
General properties of orbits in spherical potentials





General properties of orbits in spherical potentials

- Hyperbolic orbit: $E > 0$
- Parabolic orbit ($\rightarrow$ escape velocity!): $E = 0$
- Elliptical orbit: $E < 0$
- Circular orbit: $E = \phi_{eff,min}(r)$



General properties of orbits in spherical potentials

- Pericenter and apocenter

$$\dot{r}^2 = 2\left|E - \phi(r)\right| - \frac{L^2}{r^2} = 0$$

- Orbital eccentricity

$$e = \frac{r_a - r_p}{r_a + r_p}$$

e = 0 implies circular orbit

e $\rightarrow$ 1 implies unbount orbit

General properties of orbits in spherical potentials

- Radial period: $T_r = 2\int_{r_p}^{r_a} dt$

- Rotational (azimuthal) period: $T_\psi = \frac{2\pi}{|\Delta\psi|} T_r$

Orbits in homogeneous sphere

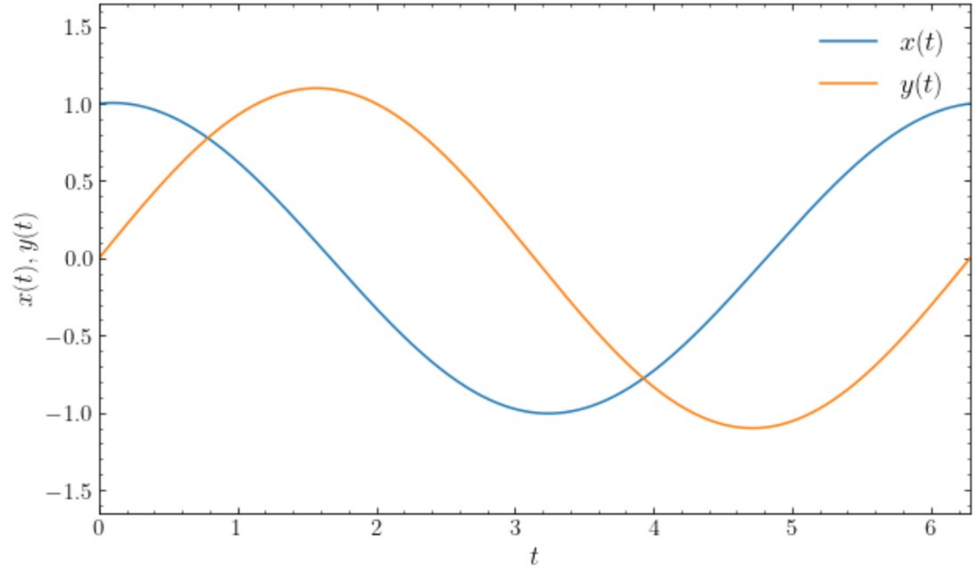
Orbits at r < R

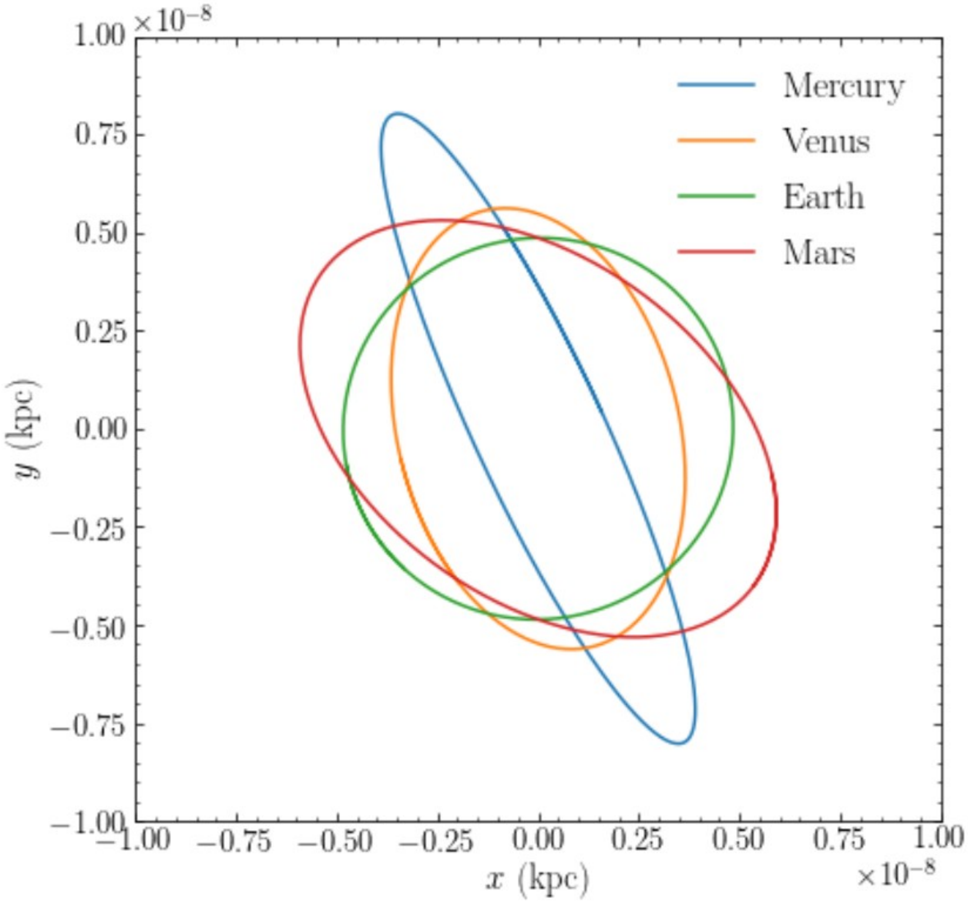
$$\Phi(r)=\frac{1}{2}\omega^2 r^2$$

$$\omega^2=4\pi G\rho_0/3$$

$$x(t)=a\cos(\omega t+\psi_x)$$

$$y(t)=b\cos(\omega t+\psi_y)$$





Orbits in homogeneous sphere

- Orbits at r < R

$$\Phi(r) = \frac{1}{2}\omega^2 r^2 \qquad \omega^2 = 4\pi G\rho_0/3 \quad \longrightarrow \quad \begin{aligned} x(t) &= a\cos(\omega t + \psi_x) \\ y(t) &= b\cos(\omega t + \psi_y) \end{aligned}$$

- Ellipses with the origin at r=0
- Four constants: a, b, orientation i, and initial position

Also: $T_\psi = 2T_r = \frac{2\pi}{\omega}$ $\longrightarrow$ the period of every orbit is the same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Orbits in Kepler potential

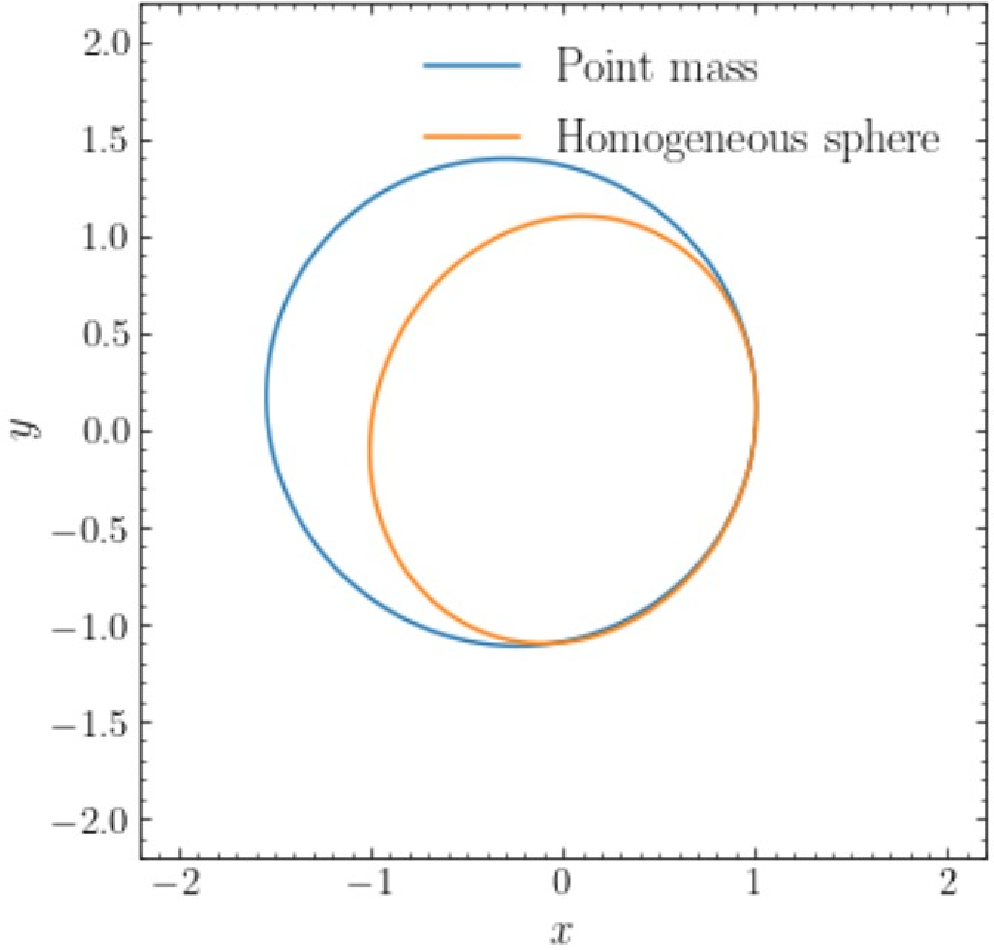
- Slightly more complicated than the homogeneous sphere

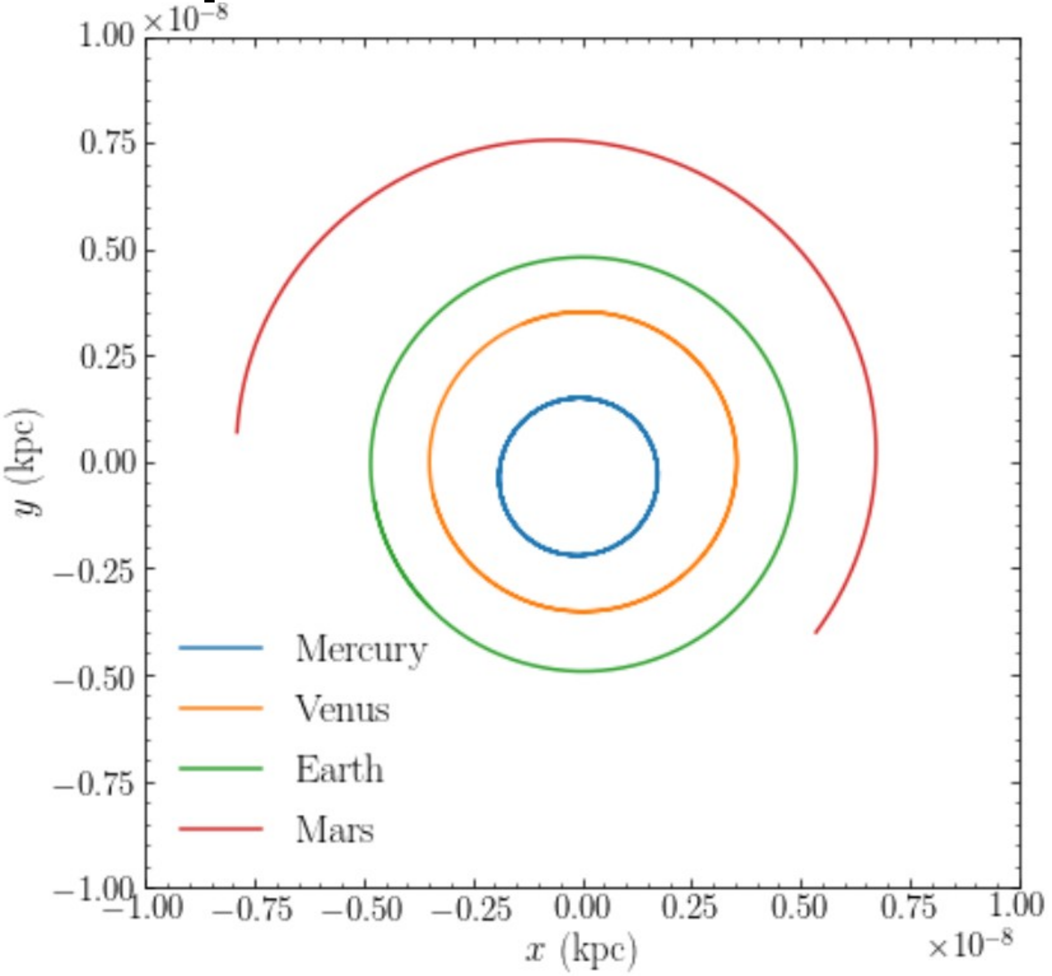
$$r(\psi)=\frac{1}{C\cos(\psi-\psi_0)+GM/L^2}$$

- The orbit only depends on a cosine
- This solution represents an ellipse with the origin at one focus, with:

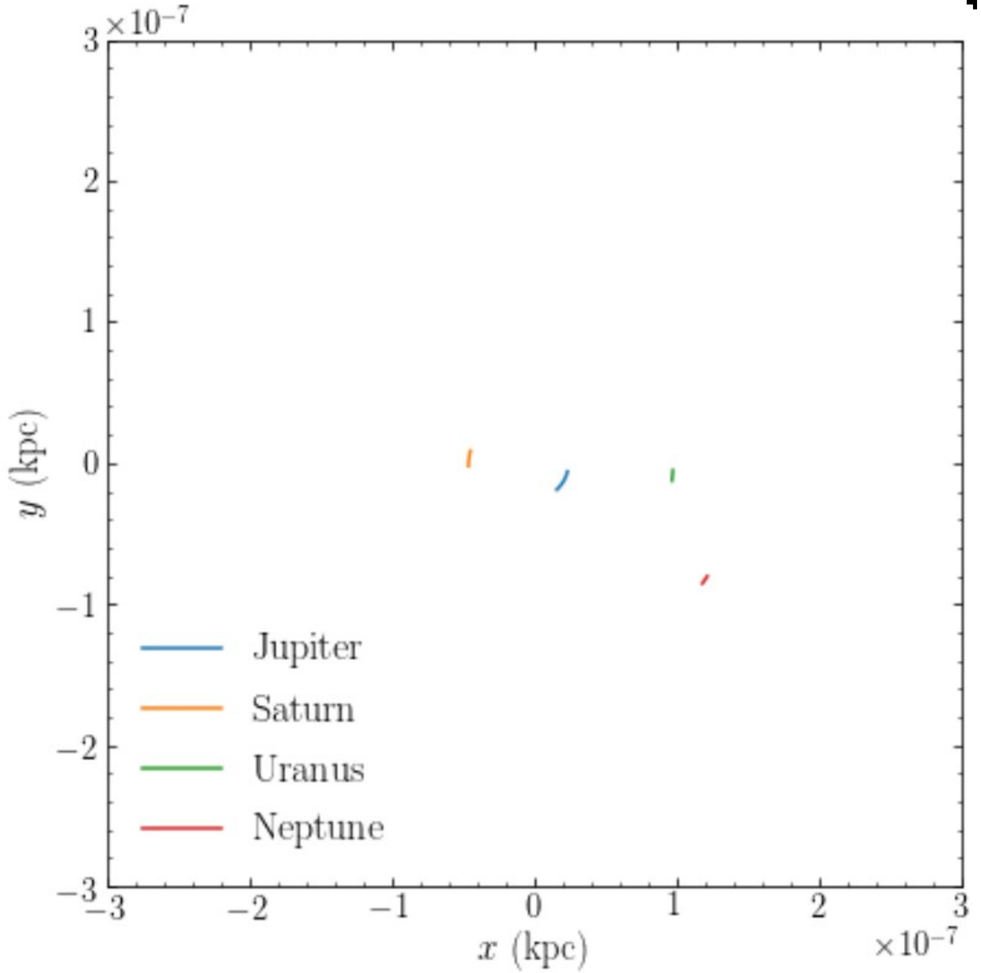
$$a=\frac{L^2}{GM(1-e^2)} \qquad e=\frac{CL^2}{GM} \qquad \begin{aligned} r_p&=a(1-e)\\ r_a&=a(1+e)\end{aligned} \qquad T_r=T_\psi=\frac{2\pi a^{3/2}}{\sqrt{GM}}$$

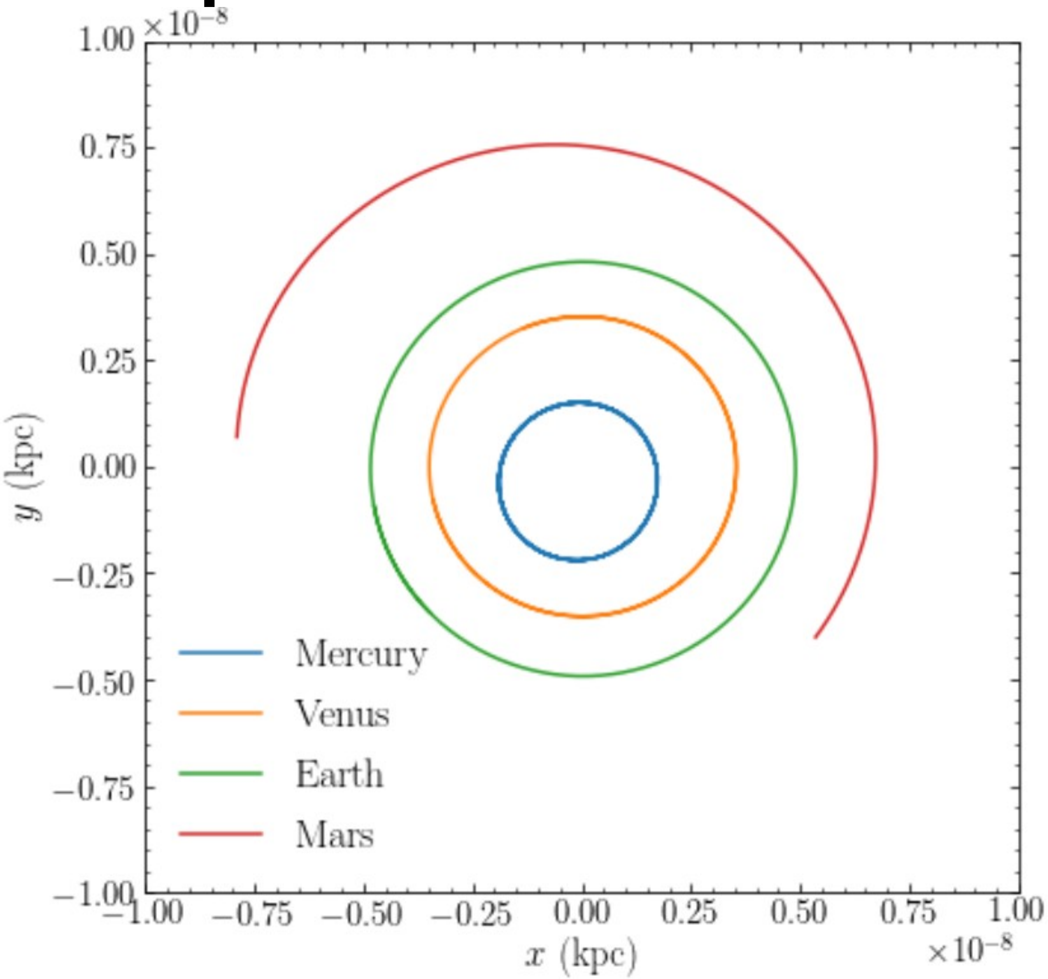
Orbits in Kepler potential





Orbits in Kepler potential



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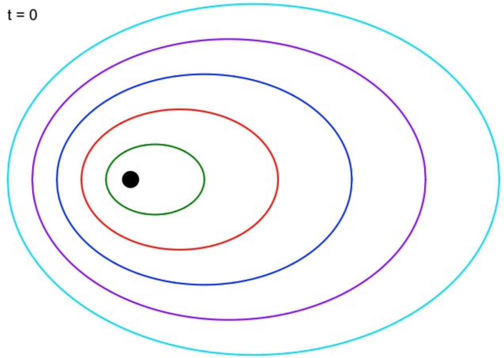
Orbits in other spherical potentials

- Isochrone potential: same orbital period as for Kepler's potential

$$T_r = \frac{\pi}{\sqrt{2}} \frac{GM}{\sqrt{-E^3}} \quad \text{and} \quad \frac{1}{2} \leqslant \frac{T_r}{T_\psi} \leqslant 1$$

- Orbits in galactic potentials are typically not closed (only in homogeneous sphere or Kepler do orbits close!)



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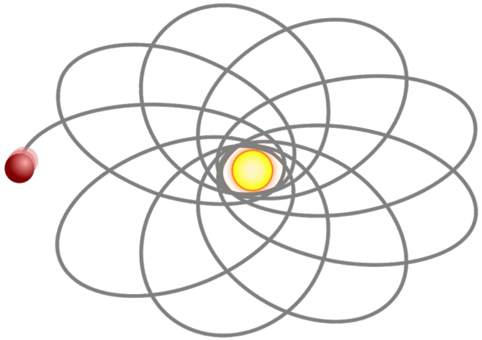
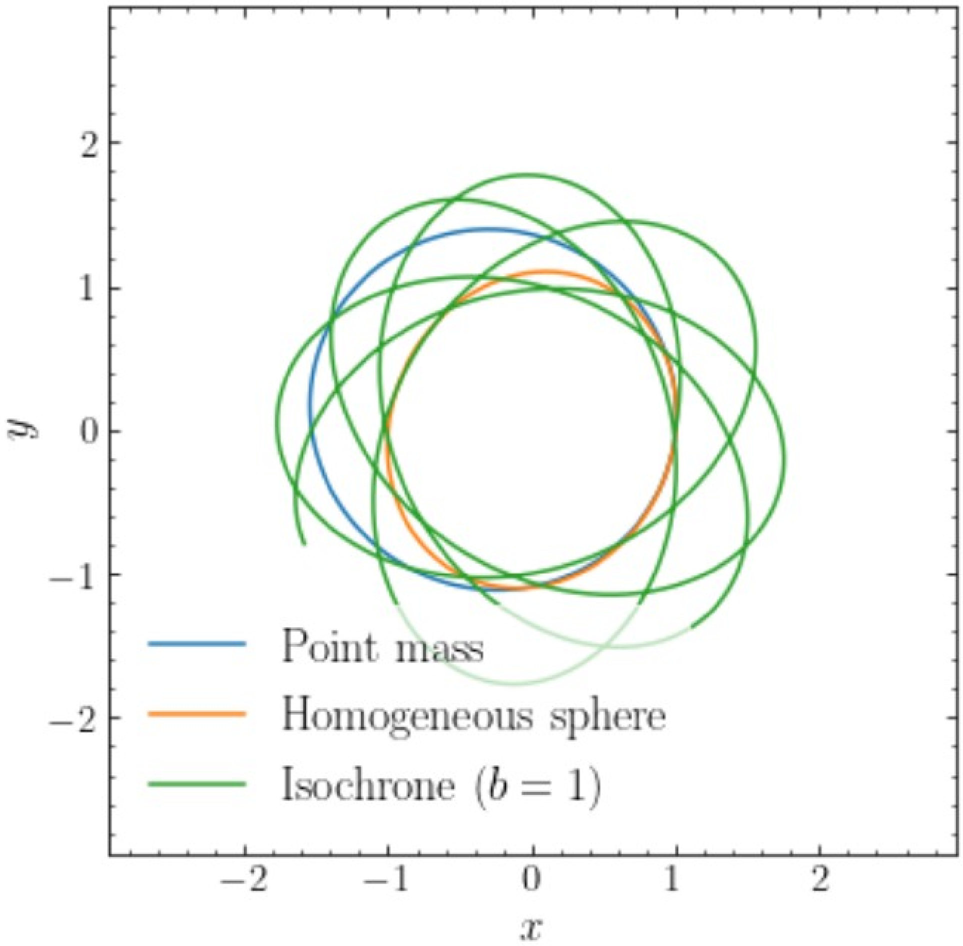
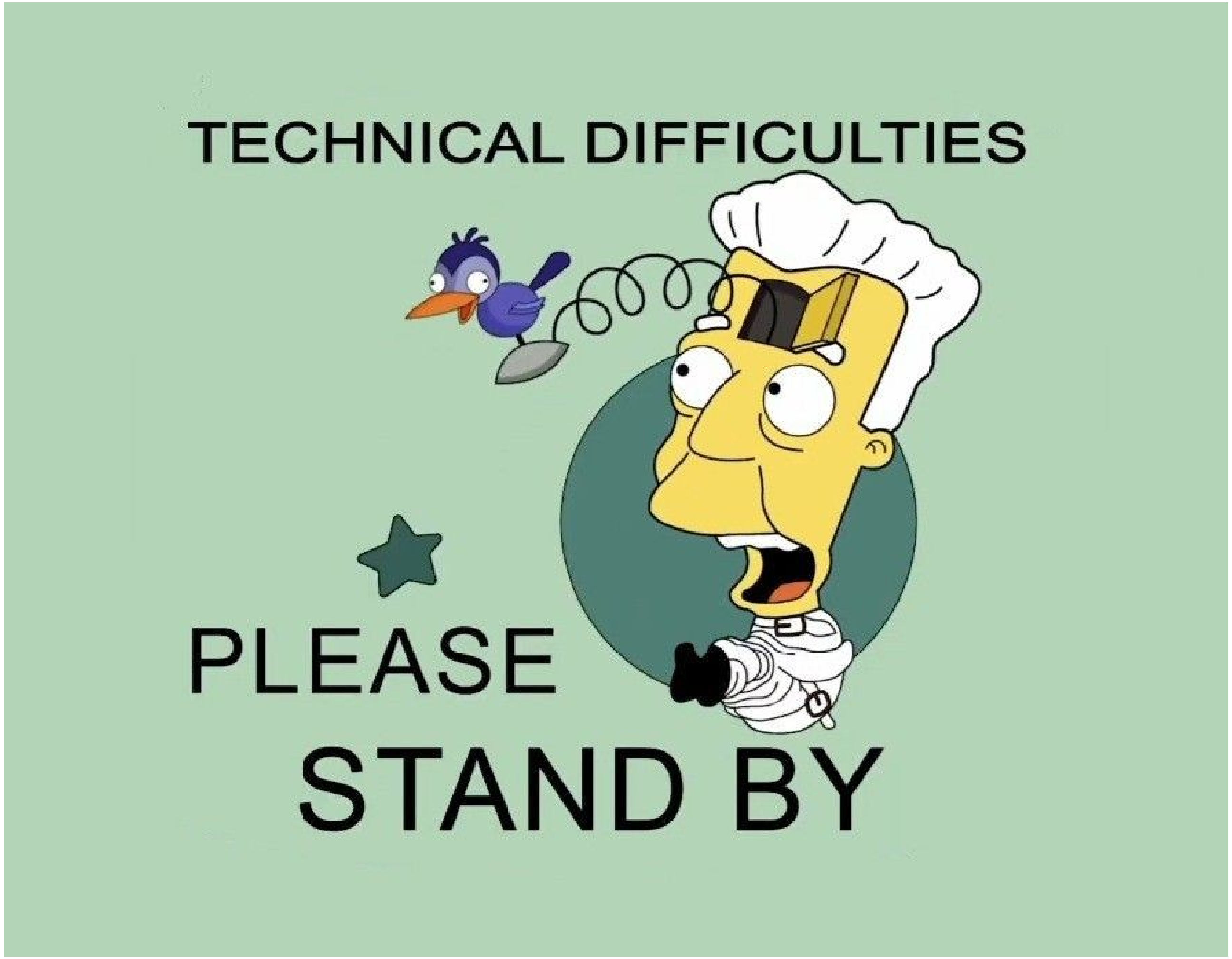


Orbits in other spherical potentials



Orbits instead form **rosettes** and eventually fill the entire space between the peri/apo center (as allowed by E and L)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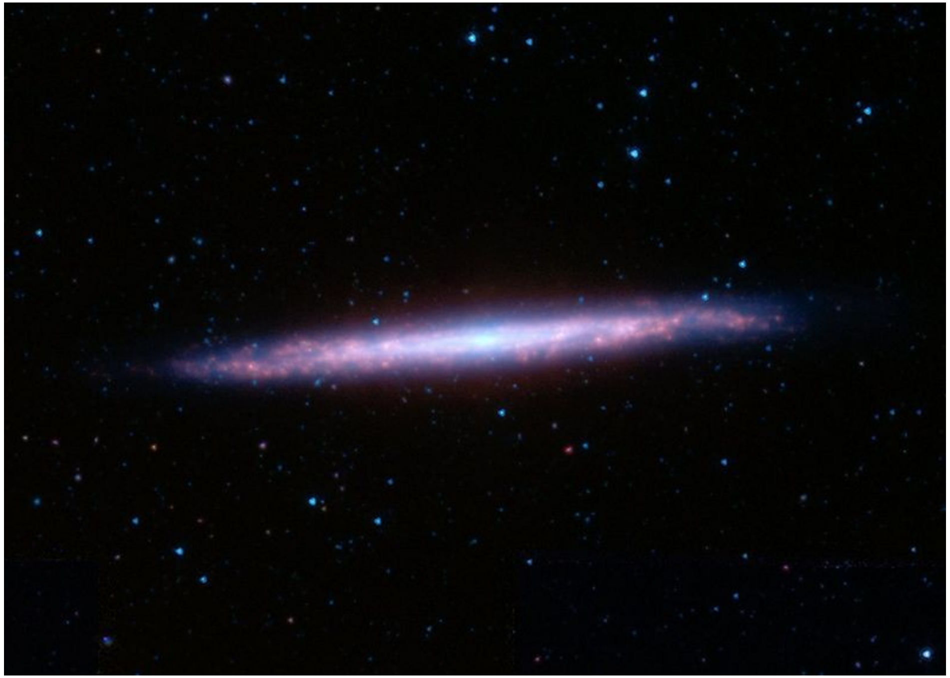
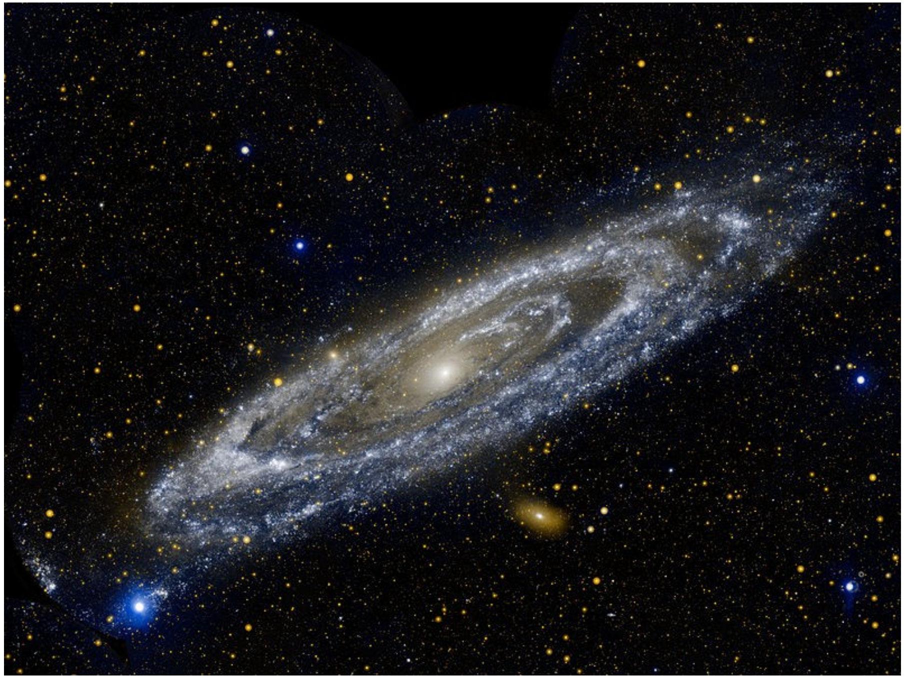
Gravitation in galactic disks



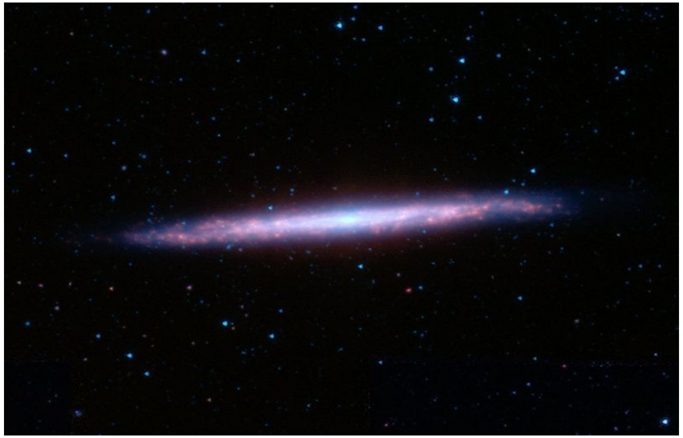


Credit: NGC 5907: NASA/JPL-Caltech/M.L.N. Ashby (Harvard-Smithsonian CfA); M31: NASA/JPL-Caltech

Gravitation in galactic disks

- Axisymmetric systems: symmetric w.r.t. rotations around the axis perpendicular to the disk → cylindrical coordinates!

$$\rho(R,\phi,z)=\rho(R,z) \qquad \Phi(R,\phi,z)=\Phi(R,z)$$

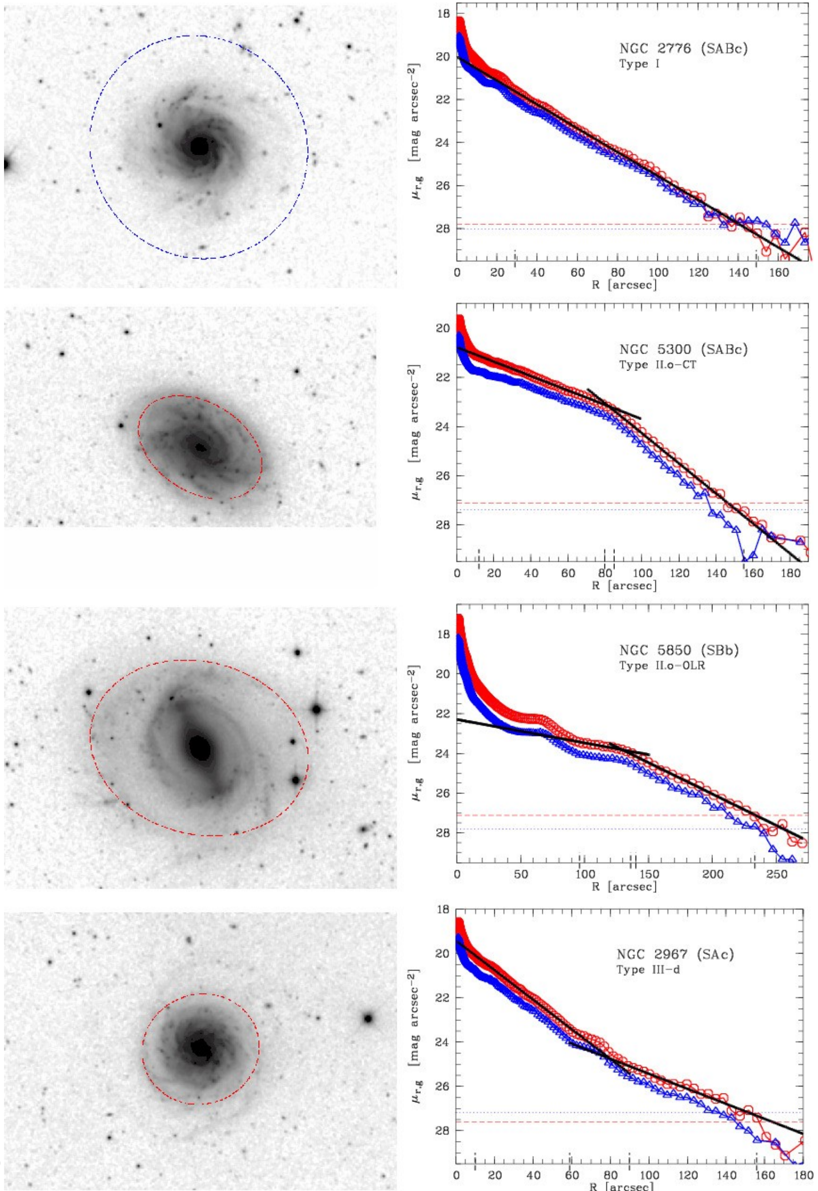


Credit: NGC 5907: NASA/JPL-Caltech/M.L.N. Ashby (Harvard-Smithsonian CfA); M31: NASA/JPL-Caltech

Gravitation in galactic disks

The radial surface-brightness of galactic disks decays exponentially

$$I(R) = I_0 \exp(-R/h_R)$$



Credit: Pohlen & Trujillo (2006)

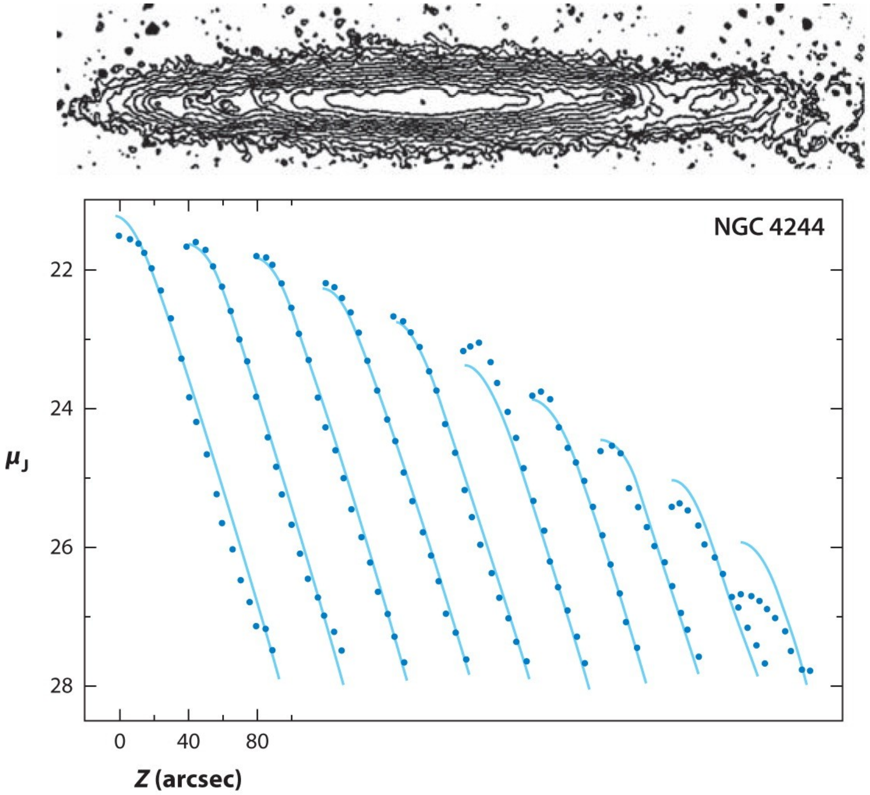
Gravitation in galactic disks

Vertical profile

$$sech^2\left(\frac{z}{2h_z}\right)=4\left[\exp\left(\frac{z}{2h_z}\right)-\exp\left(-\frac{z}{2h_z}\right)\right]^{-2}$$

Double exponential disk

$$n(R,z)\propto\exp\left(-\frac{R}{h_r}-\frac{|z|}{h_z}\right)$$



Credit: van der Kruit PC, Freeman KC, 2011, ARAA 49:301-71

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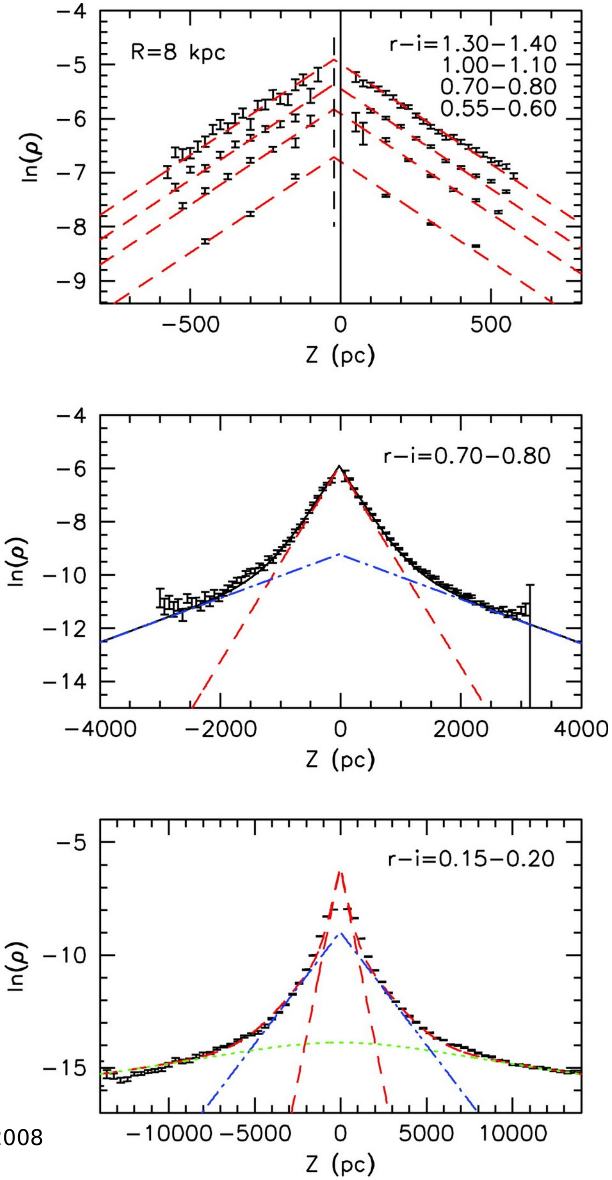


Credit: Juric et al. 2008

The Kuzmin model

- A simple flattened axisymmetric potential

$$\Phi(R, z) = -\frac{GM}{\sqrt{R^2 + (|z| + a)^2}}$$

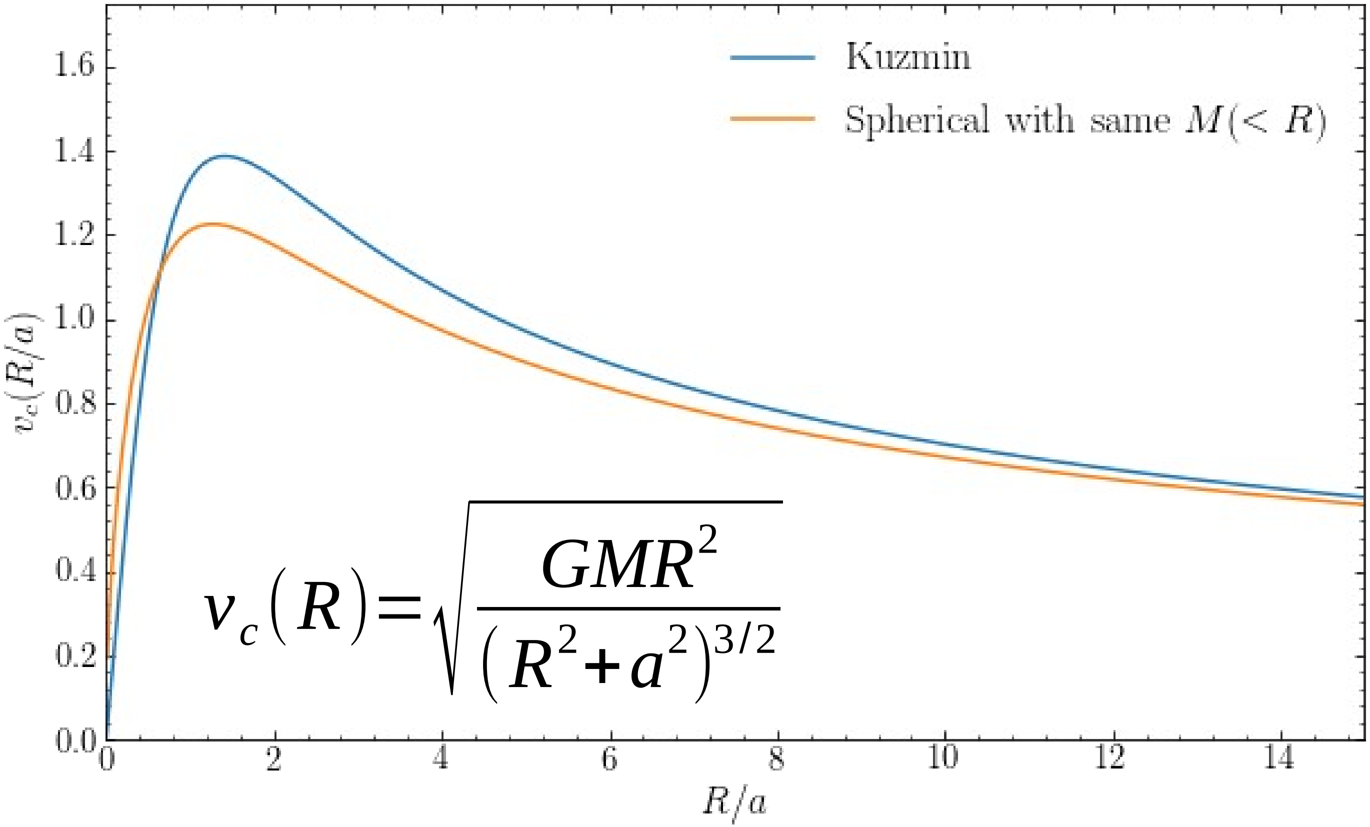
- Surface density

$$\Sigma(R) = \frac{Ma}{2\pi} \frac{1}{(R^2 + a^2)^{3/2}}$$

- Rotation curve

$$v_c(R) = \sqrt{\frac{GMR^2}{(R^2 + a^2)^{3/2}}}$$

The Kuzmin model



$$v_c(R) = \sqrt{\frac{GMR^2}{(R^2 + a^2)^{3/2}}}$$

The Miyamoto-Nagai model

- Thickened disk
- Akin to isochrone (spherical) potential

$$\Phi(R,z) = -\frac{GM}{\sqrt{R^2 + \left(\sqrt{z^2+b^2}+a\right)^2}}$$

- Mass density

Ugly

The Miyamoto-Nagai model

- Thickened disk
- Akin to isochrone (spherical) potential

$$\Phi(R,z) = -\frac{GM}{\sqrt{R^2 + \left(\sqrt{z^2+b^2}+a\right)^2}}$$

- Mass density

$$\rho(R,z) = \frac{b^2 M}{4\pi} \frac{aR^2 + \left(3\sqrt{z^2+b^2}+a\right)\left(\sqrt{z^2+b^2}+a\right)^2}{\left(R^2+\left(\sqrt{z^2+b^2}+a\right)^2\right)^{5/2}\left(z^2+b^2\right)^{3/2}}$$

The Miyamoto-Nagai model

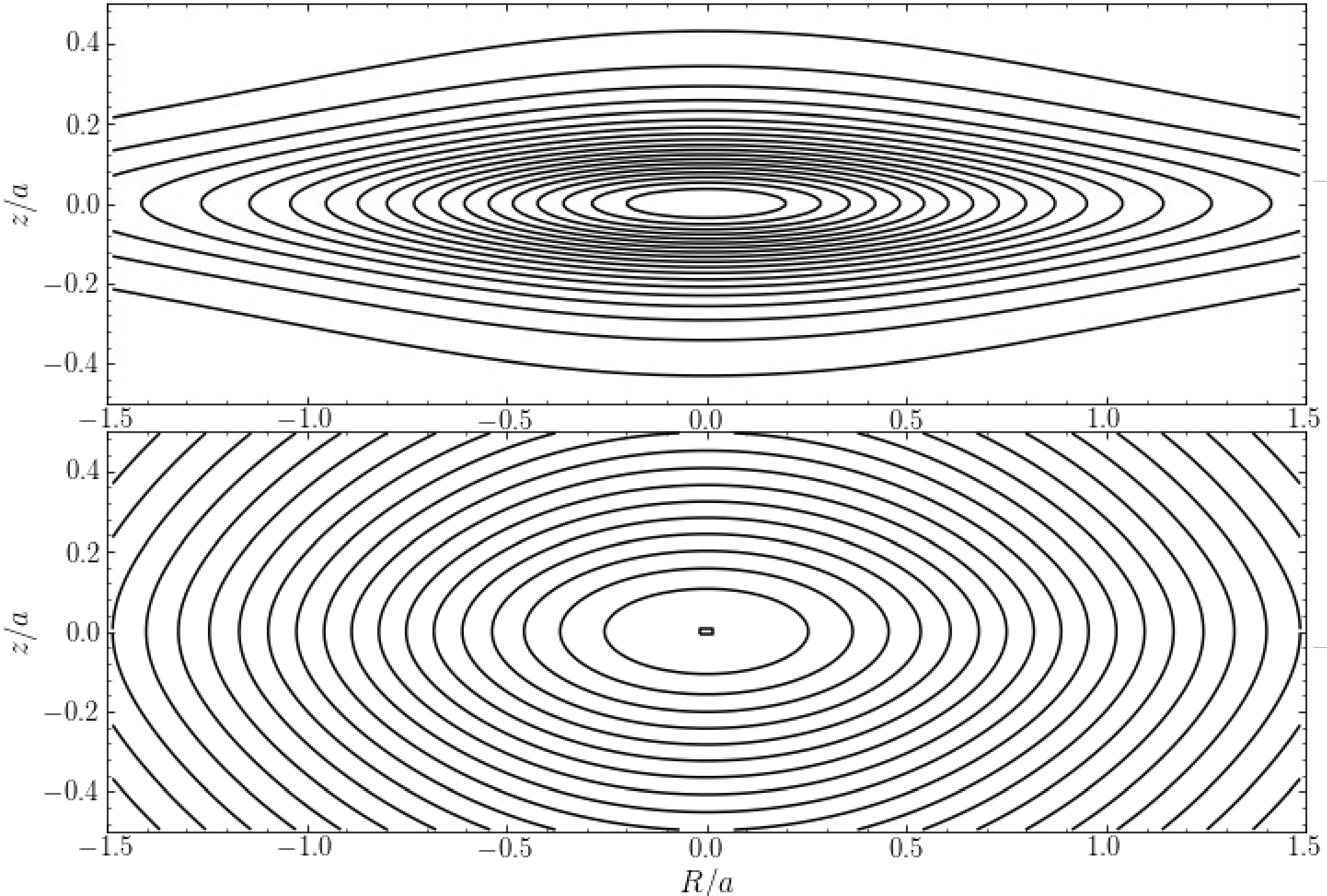
- Thickened disk
- Akin to isochrone (spherical) potential

$$\Phi(R,z) = -\frac{GM}{\sqrt{R^2 + \left(\sqrt{z^2+b^2}+a\right)^2}}$$

- Mass density

I told you

The Miyamoto-Nagai model

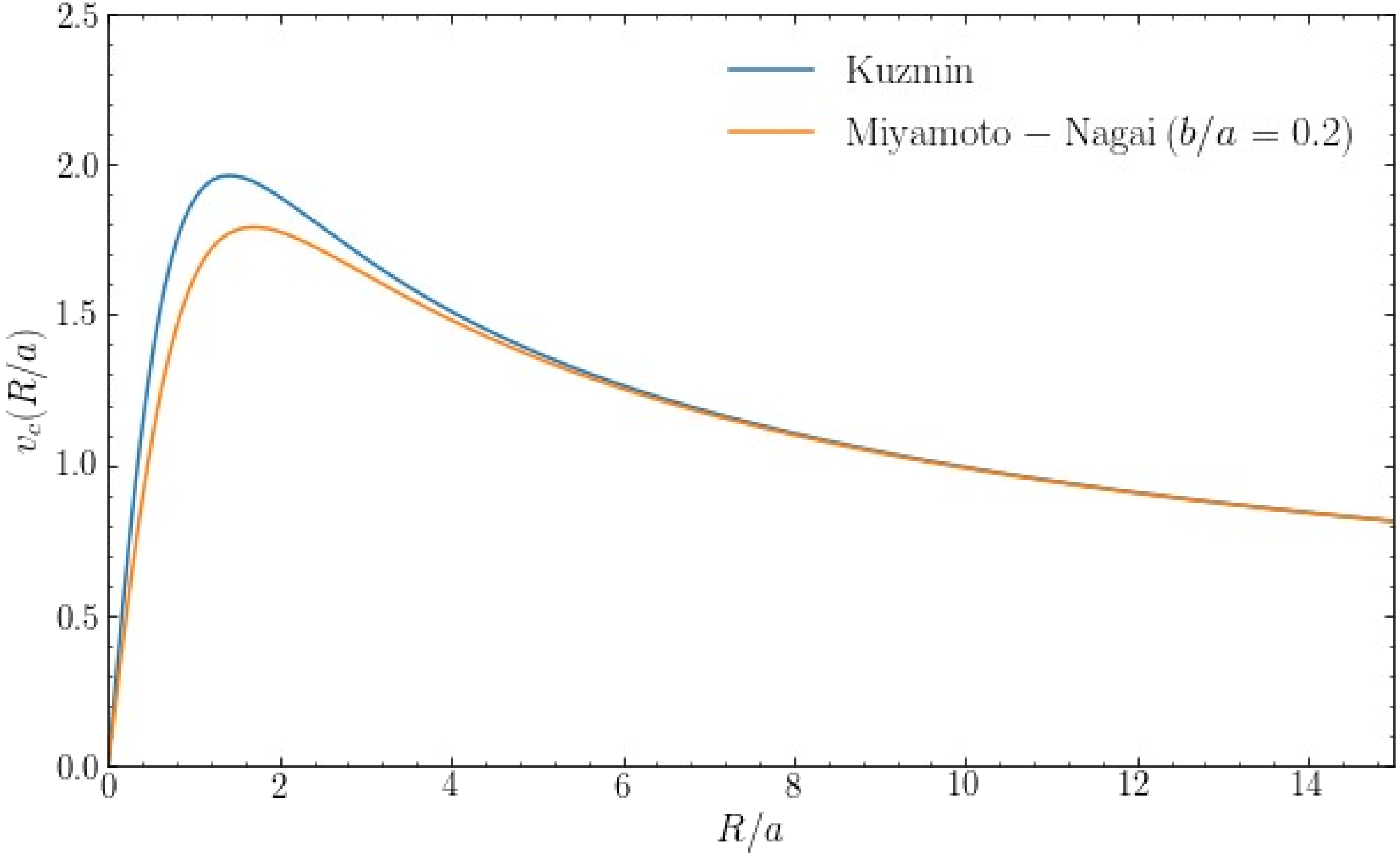


Isodensity surfaces

are more flattened than

equipotential surfaces

The Miyamoto-Nagai model



Orbits in disks

- Lagrangian in cylindrical coordinates

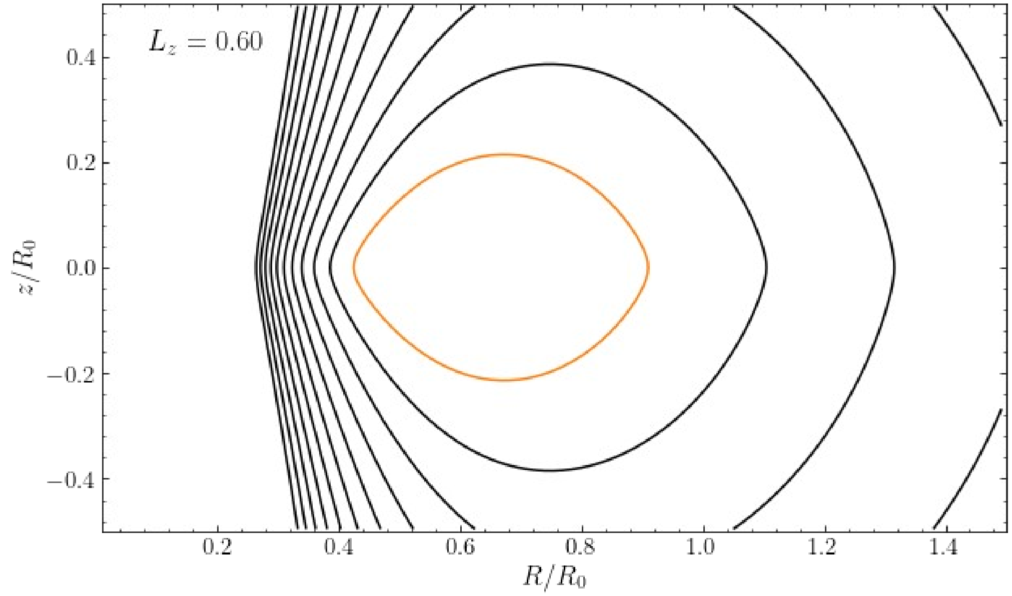
$$\mathcal{L}(R,\phi,z,\dot{R},\dot{\phi},\dot{z})=\frac{m}{2}(\dot{R}^2+[R\dot{\phi}]^2+\dot{z}^2)-m\Phi(R,z)$$

- $p_\phi=\frac{\partial\mathcal{L}}{\partial\dot{\phi}}=mR^2\dot{\phi}=mL_z \rightarrow$ constant! (is conserved)

- Both E and L_z are integrals of motion

Orbits in disks

$$H_{eff}(R, z, p_R, p_z; L_z) = \frac{1}{2m}(p_R{}^2 + p_z{}^2) + m\,\Phi_{eff}(R, z; L_z)$$



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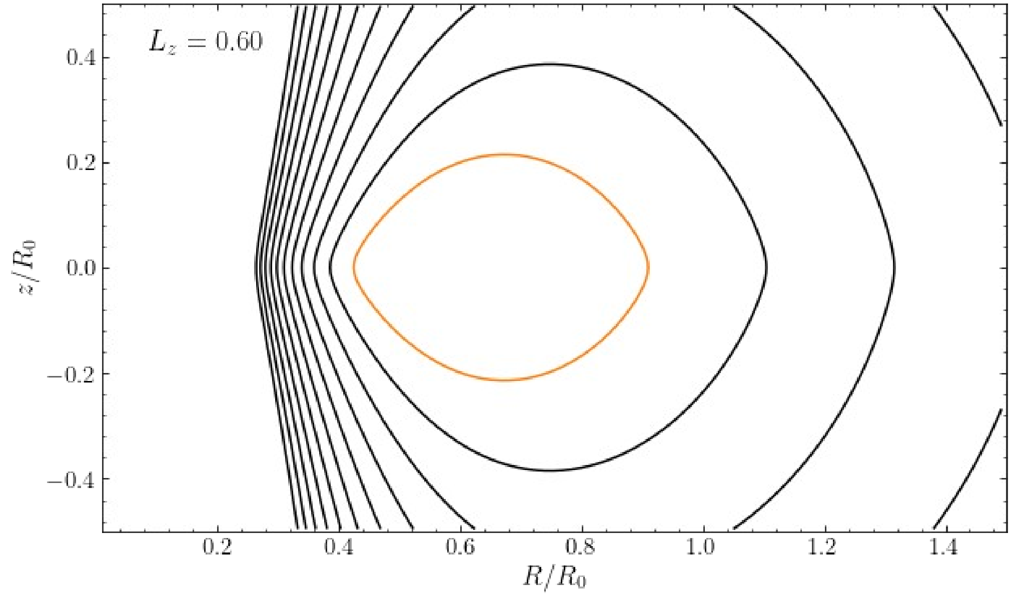
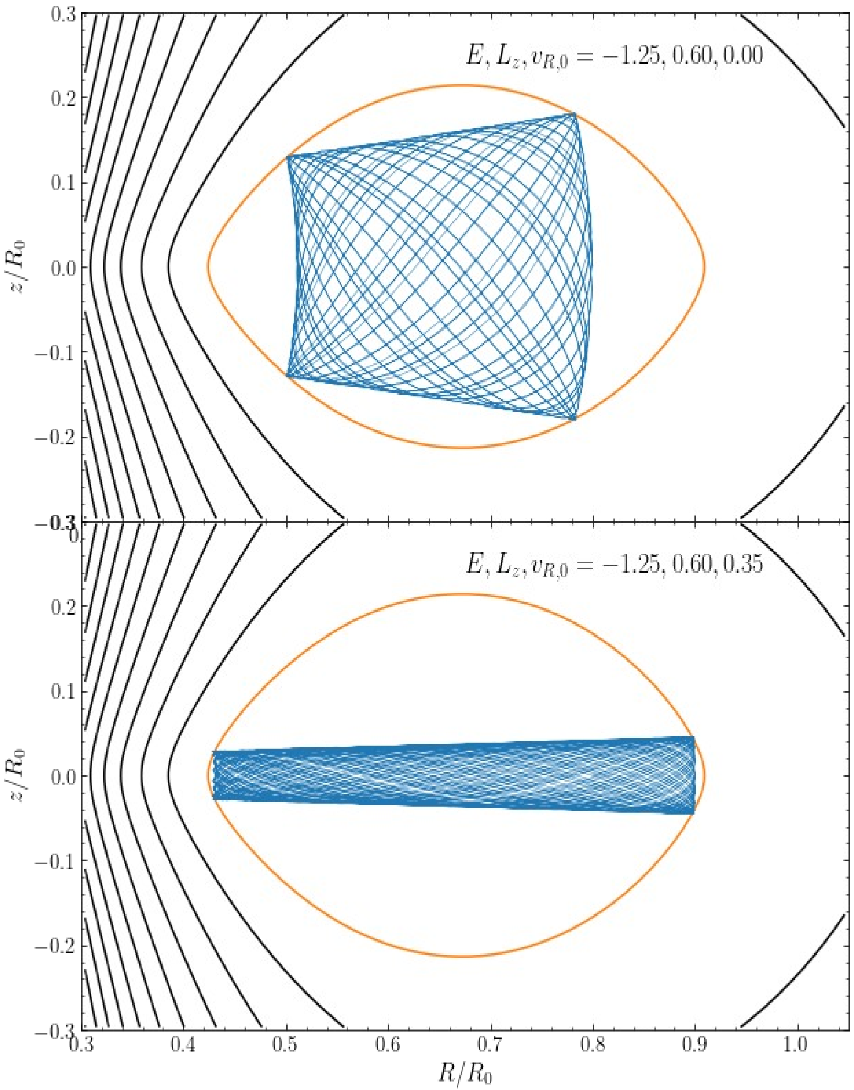
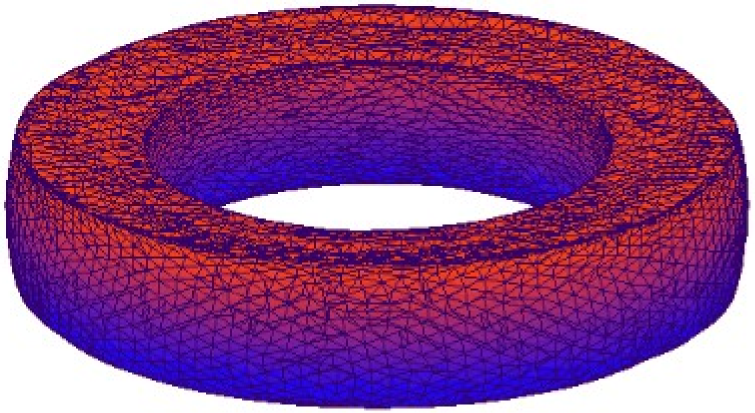


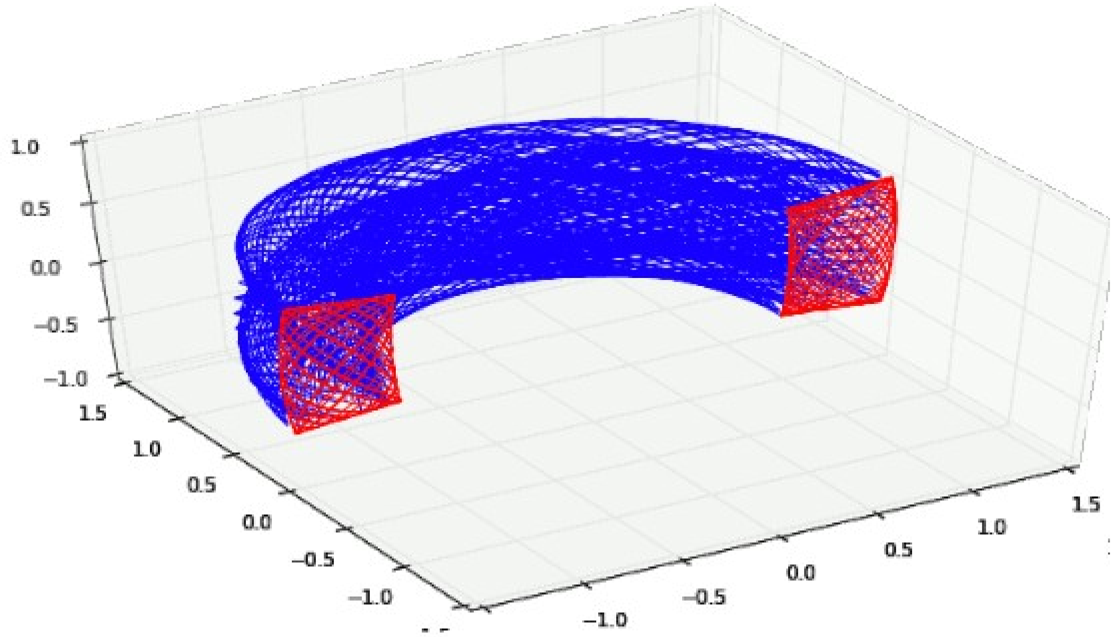


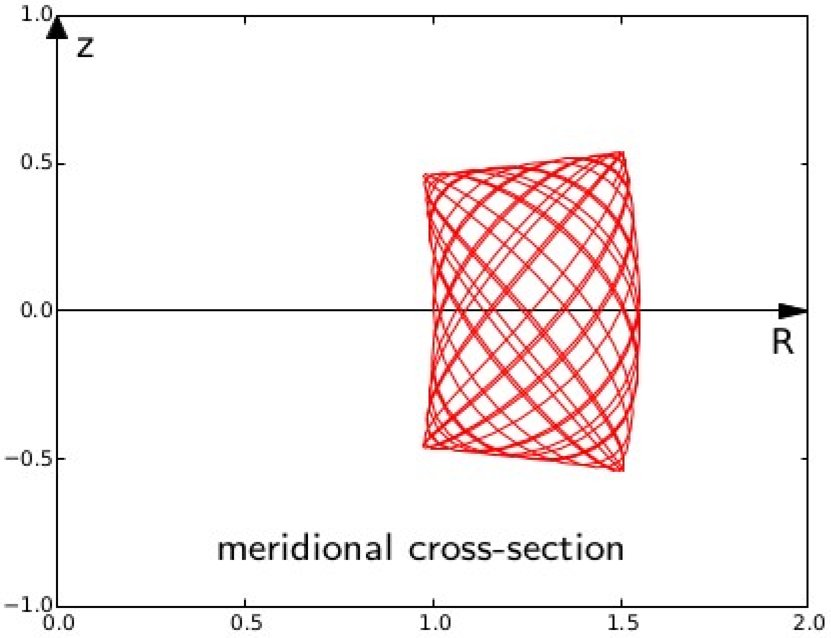
Orbits in disks



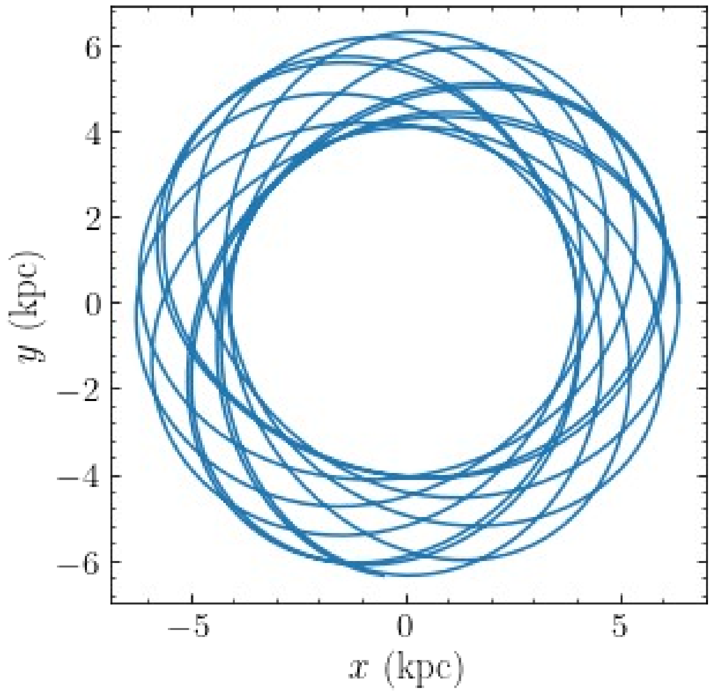
$$H_{eff}(R,z,p_R,p_z;L_z)=\frac{1}{2m}(p_R{}^2+p_z{}^2)+m\,\Phi_{eff}(R,z;L_z)$$

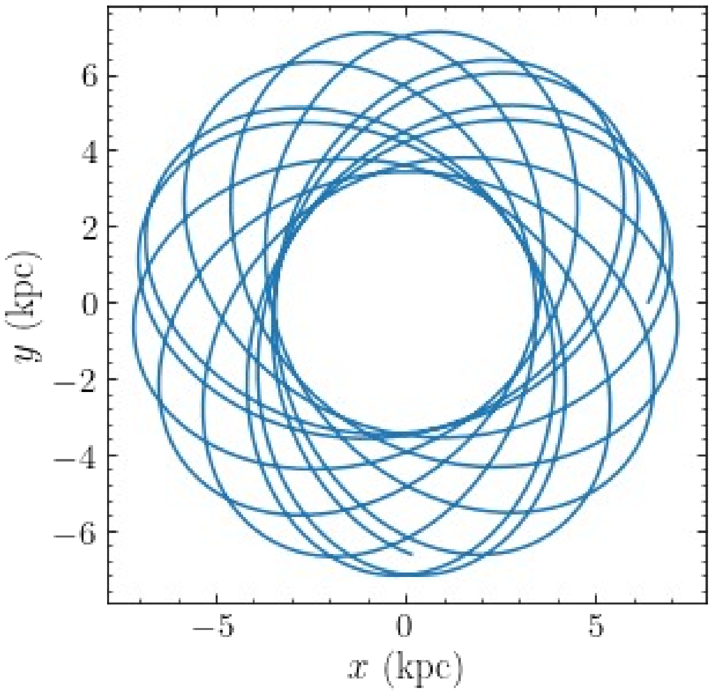
“Tube” orbits





Orbits in disks





Equilibria of collisionless systems

- One of the most useful assumptions of galactic dynamics (→ inference of mass and orbital distributions)
- Non-collisional effects are required to drive stellar systems to an equilibrium state (e.g., violent relaxation and phase-mixing)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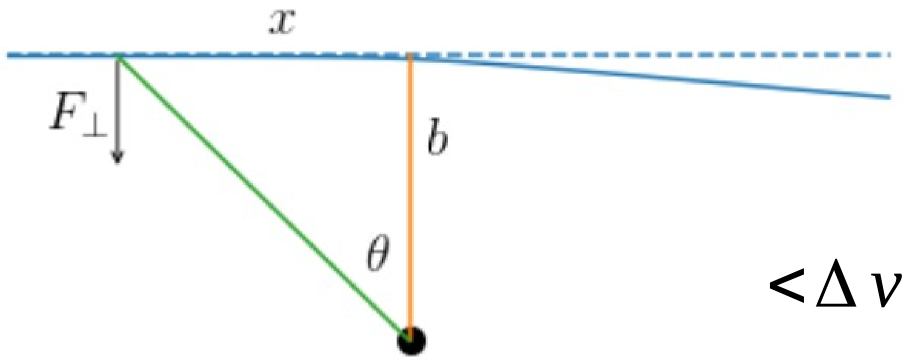
Collisionless vs collisional dynamics

- Are the strong gravitational interactions between stars important drivers of the evolution of galaxies?
- Relaxation time: time over which the combined effect of many close encounters has changed a star’s velocity by 100%

Two-body relaxation

The impulse approximation



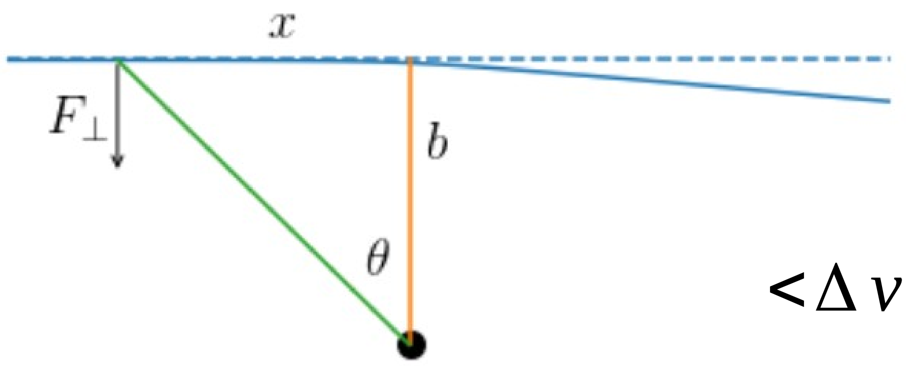
$$m\,\delta v_y = \int_{-\infty}^{\infty} F_y\,dt$$

$$<\Delta v^2> m \approx 8N\left(\frac{Gm}{Rv}\right)^2 \ln\Lambda \qquad \Lambda = \frac{b_{max}}{b_{min}}$$

$$t_{relax} = n_{relax} \times t_{cross} \approx \frac{N}{8\ln N}\frac{2\pi R}{v} = \frac{N}{8\ln N} t_{dyn}$$

Two-body relaxation

The impulse approximation



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For galaxies, $t_{relax} \approx 10^{10}\,Myr$!!! In globular clusters, $t_{relax} \approx 1\,Gyr$

The virial theorem

- Scalar virial theorem for self gravitating systems

$$\sum_{i=1}^{N} m_i \vec{v}_i \cdot \vec{v}_i = \sum_{i=1}^{N} \sum_{j<i} \frac{G m_i m_j}{|\vec{x}_i - \vec{x}_j|}$$

- When generalized:

$$2K + W = 0 \qquad\qquad 2E - W = 0$$

The collisionless Boltzmann equation

- It is fundamental to describe collisionless systems
- A distribution function f is used to represent the number of bodies in a small phase-space volume (at a given time), under the influence of gravity
- N-body distribution function:

$$f^{(N)}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_N, t) = \prod_{i=1}^{N} f(\vec{w}_i, t)$$

$$\vec{w}_i \equiv (\vec{x}_i, \vec{v}_i)$$

The collisionless Boltzmann equation

$$\frac{\partial f(\vec{q},\vec{p},t)}{\partial t}+\dot{\vec{q}}\frac{\partial f(\vec{q},\vec{p},t)}{\partial q}+\dot{\vec{p}}\frac{\partial f(\vec{q},\vec{p},t)}{\partial p}=0$$

- It describes the evolution of *any* collisionless system (not only those in equilibrium)
- It is used to describe the evolution of a subset of bodies orbiting in a general smooth mass distribution that they do not contribute much mass to
- It is the equation solved by N-body simulations
- For equilibrium collisionless systems $f(\vec{q},\vec{p},t)=f(\vec{q},\vec{p})$

The collisionless Boltzmann equation

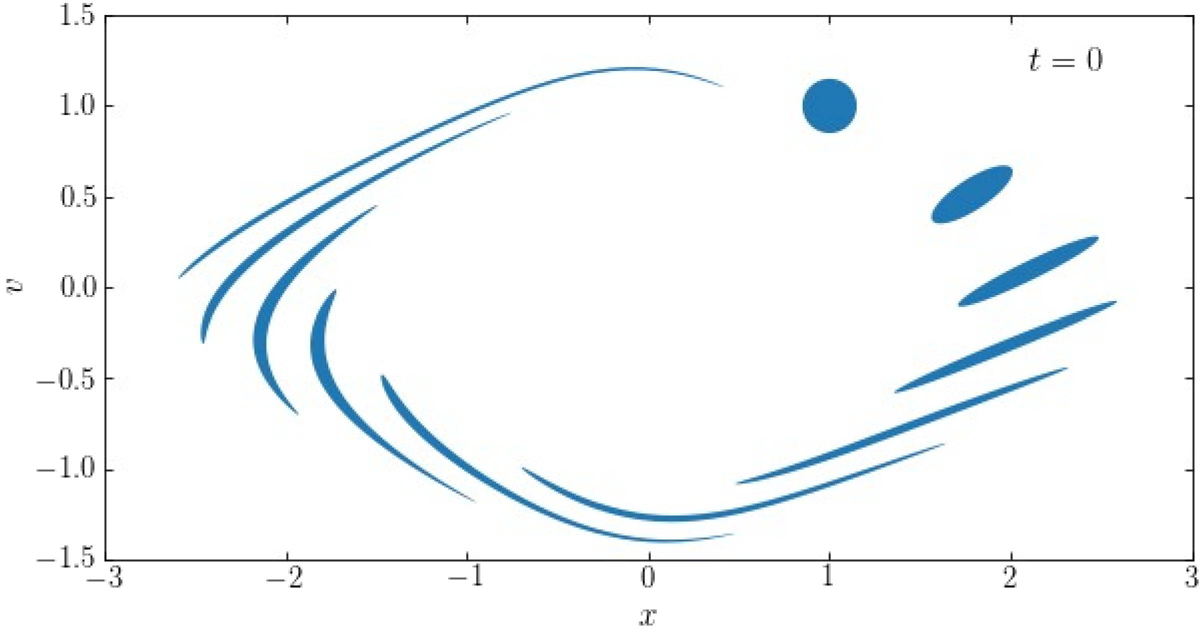
- How does the phase-space density f evolve along the orbit of a body? **Liouville’s theorem**

$$\frac{df(\vec{x},\vec{v})}{dt}=\frac{\partial f(\vec{x},\vec{v})}{\partial t}+\dot{\vec{x}}\frac{\partial f(\vec{x},\vec{v})}{\partial x}+\dot{\vec{v}}\frac{\partial f(\vec{x},\vec{v})}{\partial v}=0$$

→ the phase-space density is conserved along orbital trajectories (key property of Hamiltonian systems!)

The collisionless Boltzmann equation

- How does the phase-space density f evolve along the orbit of a body? **Liouville's theorem**



Jeans equations

- The Jeans equations involve moments of the distribution function

$$\int d\vec{v}\,\frac{\partial f(\vec{x},\vec{v},t)}{\partial t}+\int d\vec{v}\,\dot{\vec{x}}\,\frac{\partial f(\vec{x},\vec{v},t)}{\partial x}-\frac{\partial \Phi}{\partial x}\int d\vec{v}\,\frac{\partial f(\vec{x},\vec{v},t)}{\partial v}=0$$

- Spatial number density $\nu(\vec{x})=\int d\vec{v}\,f(\vec{x},\vec{v})$
- Velocity dispersion tensor

$$\sigma_{i,j}(\vec{x})=\frac{1}{\nu(\vec{x})}\int d\vec{v}\,(v_i-<v_i>)(v_j-<v_j>)\,f(\vec{x},\vec{v})$$

Jeans equations

- Velocity dispersion tensor

$$\sigma_{i,j}(\vec{x}) = \frac{1}{\nu(\vec{x})} \int d\vec{v} \,(v_i - <v_i>)(v_j - <v_j>) f(\vec{x},\vec{v})$$

- Velocity anisotropy parameter (for spherical systems)

$$\beta \equiv 1 - \frac{\sigma_\theta^2 + \sigma_\phi^2}{2\sigma_r^2}$$

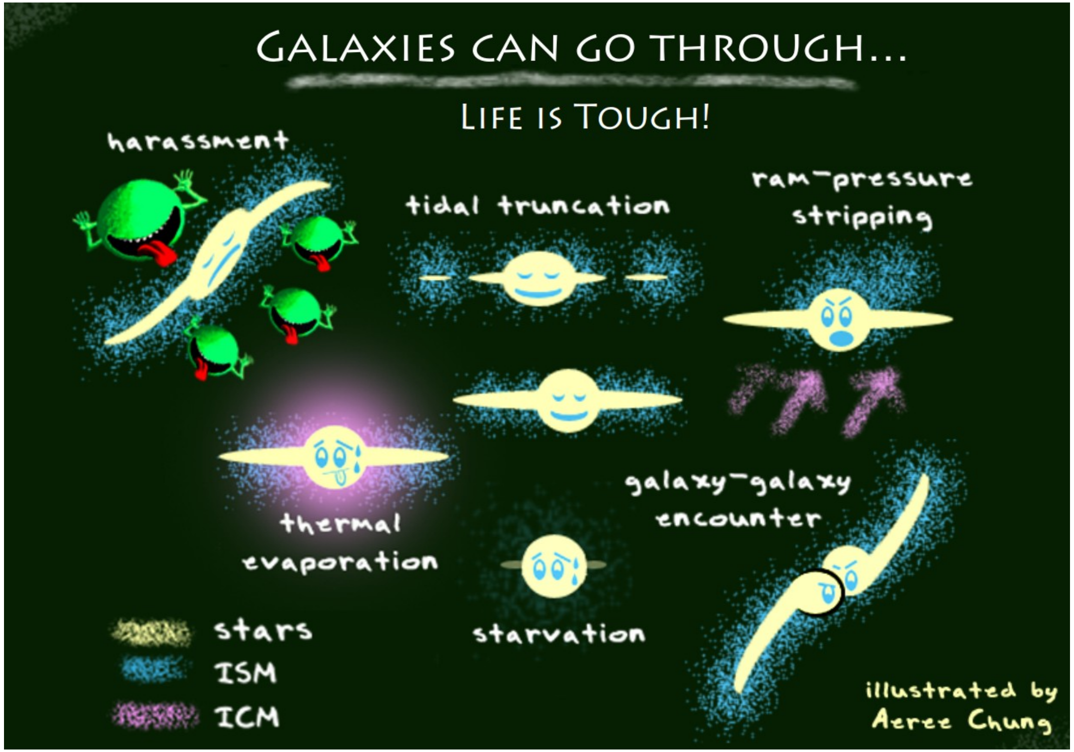
$\beta = 0$ isotropic

$\beta \rightarrow 1$ Radially biased

$\beta \rightarrow -\infty$ Tangentially biased

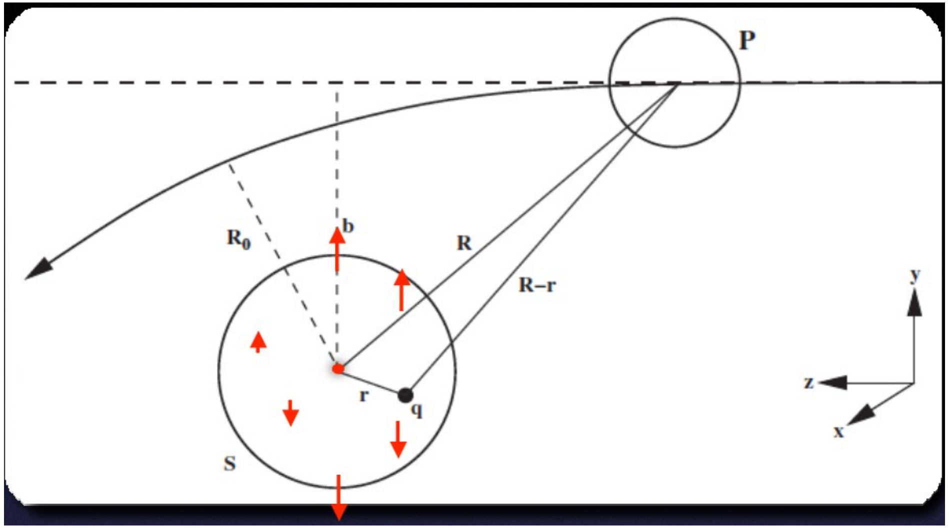
Environmental galaxy evolution

For galaxies in clusters (high density)



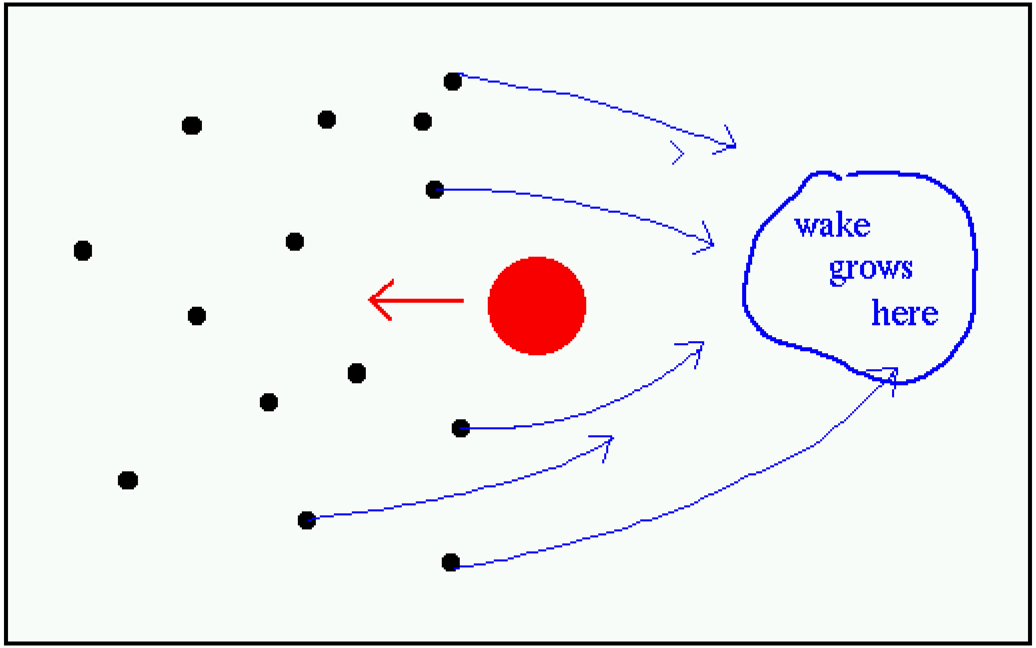
Galaxy interactions

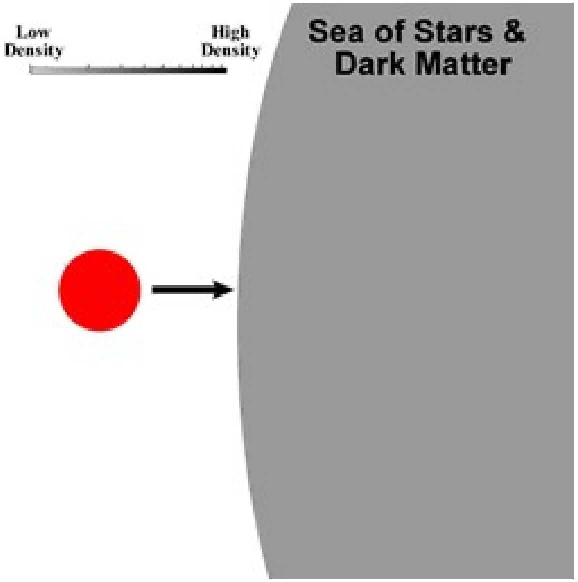
- Galaxy harassment
- Tidal stripping
- Dynamical friction



Dynamical friction

Orbital energy of satellite galaxy (and dark matter subhaloes)transferred to the dark matter particles that make up the host halo





Dynamical friction

- Dynamical friction: Consider the case of a satellite galaxy or a globular cluster with mass **m** moving on a circular orbit with radius **R** and velocity **Vc** around a massive galaxy, i.e., through a background of bodies with mass **M**
- Assumption: velocity dispersion is constant and isotropic
- Dynamical friction timescale (associated with the loss of angular momentum):

$$t_{df} \sim \frac{R_c^2 V_c}{Gm \ln \Lambda} \qquad \Lambda = \frac{R_c V_c^2}{G(m+M)}$$