K-Matrix

Applications, Examples and Practical Aspects

Lectures at the "School on Concepts of Modern Amplitude Analysis Techniques"

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Dynamical Functions are Complicated

Search for resonance enhancements
is a major tool in meson spectroscopy

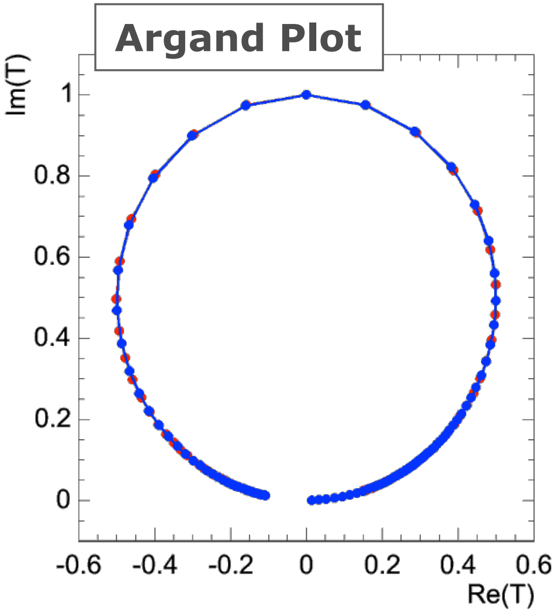
The Breit-Wigner Formula was derived
for a single resonance
appearing in a single channel

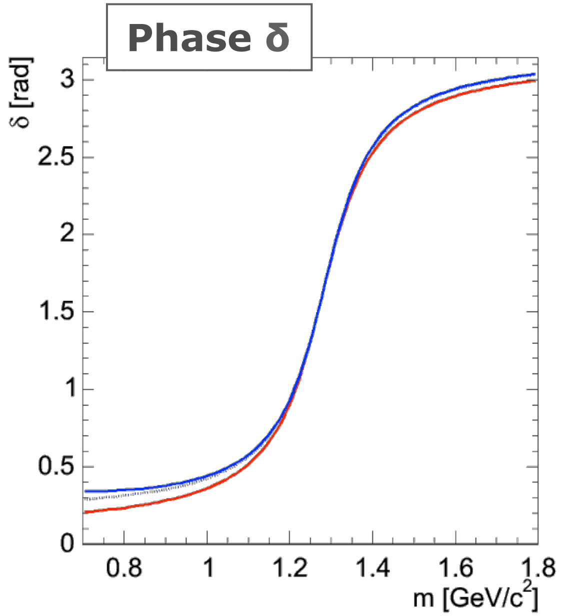
But: Nature is more complicated

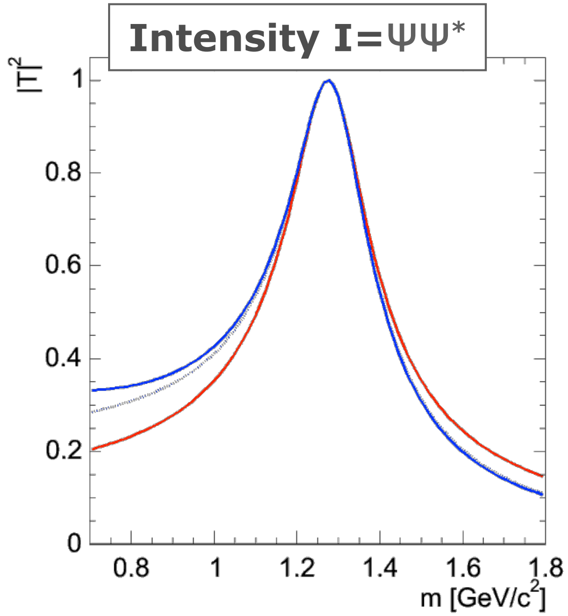
- Resonances decay into several channels
- Several resonances appear within the same channel
- Thresholds distort line shapes due to available phase space

A more general approach is needed
for a detailed understanding (see last lecture!)

Relativistic Breit-Wigner







By migrating from Schrödinger's equation (non-relativistic) to Klein-Gordon's equation (relativistic) the energy term changes different energy-momentum relation $E=p^2/m$ vs. $E^2=m^2c^4+p^2c^2$

The propagators change to s_R-s from m_R-m

$$T(s) = \frac{\gamma}{s_r - s - \iota\frac{2q\gamma}{\sqrt{s}}} = \frac{\Gamma}{m_r^2 - m^2 - \iota\rho m_0 \Gamma}$$

Barrier Factors - Introduction

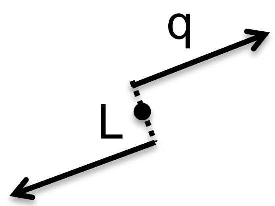
At low energies, near thresholds $\Gamma_r \propto q^{2l+1} = \rho q^{2l}$

but is not valid far away from thresholds -- otherwise the width would explode and the integral of the Breit-Wigner diverges

It reflects the non-zero size of the object

Need more realistic centrifugal barriers

known as Blatt-Weisskopf damping factors



We start with the semi-classical impact parameter

$$b = [L(L+1)]^{\frac{1}{2}}/q$$

and use the approximation for the stationary solution of the radial differential equation

$$\frac{\partial^2}{\partial\rho^2} U_l^n \rho \simeq \left(\frac{b_n^2}{r^2} - 1\right) U_l^n \rho \qquad U_l^n \rho \overset{r>R}{\simeq} iC_n \rho h_l^{(1)}(\rho) \sim C_n e^{\iota\left(\rho - \frac{1}{2}L\pi\right)}$$

with

$$\left[H_l^n(R/b)\right]^{-1} \equiv \rho^2 |h_l^{(1)}(\rho)|^2$$ we obtain

$$\Gamma_n(q_n) = \Gamma_n^0 \frac{\frac{q_n}{m} H_l^n(R/b_n)}{\frac{q_n^0}{m} H_l^n(R/b_n^0)}$$

Blatt-Weisskopf Barrier Factors

The energy dependence is usually parameterized in terms of spherical Hankel-Functions

$$j_l(x) \equiv \frac{\pi}{2x}^{\frac{1}{2}} J_{1+\frac{1}{2}}(x)$$

$$n_l(x) \equiv \frac{\pi}{2x}^{\frac{1}{2}} N_{1+\frac{1}{2}}(x)$$

$$h_l^{(1,2)}(x) \equiv \frac{\pi}{2x}^{\frac{1}{2}} \left[J_{1+\frac{1}{2}}(x) \pm N_{1+\frac{1}{2}}(x) \right]$$

$$h_0^{(1)}(x) = \frac{e^{\iota x}}{\iota x}$$

$$h_1^{(1)}(x) = \frac{-e^{\iota x}\left(1+\frac{\iota}{x}\right)}{x}$$

$$h_2^{(1)}(x) = \iota \frac{e^{\iota x}\left(1+\frac{3\iota}{x}-\frac{3}{x^2}\right)}{x}$$

we define $F_l(q)$ with the following features

$$F_l(q) \overset{x=\frac{q}{q_{scale}}}{=} \sqrt{\frac{|h_l^{(1)}(x)|^2}{|h_l^{(1)}(x=1)|^2}}$$

$$F_l(q) \overset{q \to q_{scale}}{=} 1$$

$$F_l(q) \overset{q \to 0}{=} q^l$$

Main problem is the choice of the scale parameter $q_R = q_{scale}$

Blatt-Weisskopf Barrier Factors (l=0 to 3)

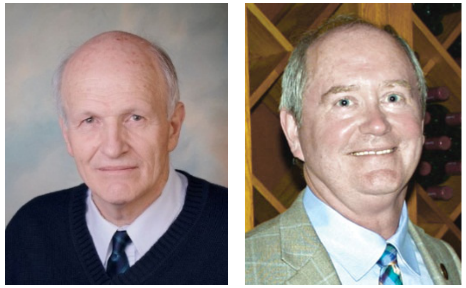
$$F_0(x) = 1$$

$$F_1(x) = \sqrt{\frac{x}{x+1}}$$

$$F_2(x) = \sqrt{\frac{13x^2}{(x-3)^2+9x}}$$

$$F_3(x) = \sqrt{\frac{277x^3}{x(x-15)^2+9(2x-5)^2}}$$

$$B_l(q, q_R) = \frac{F_l(q)}{F_l(q_R)}$$

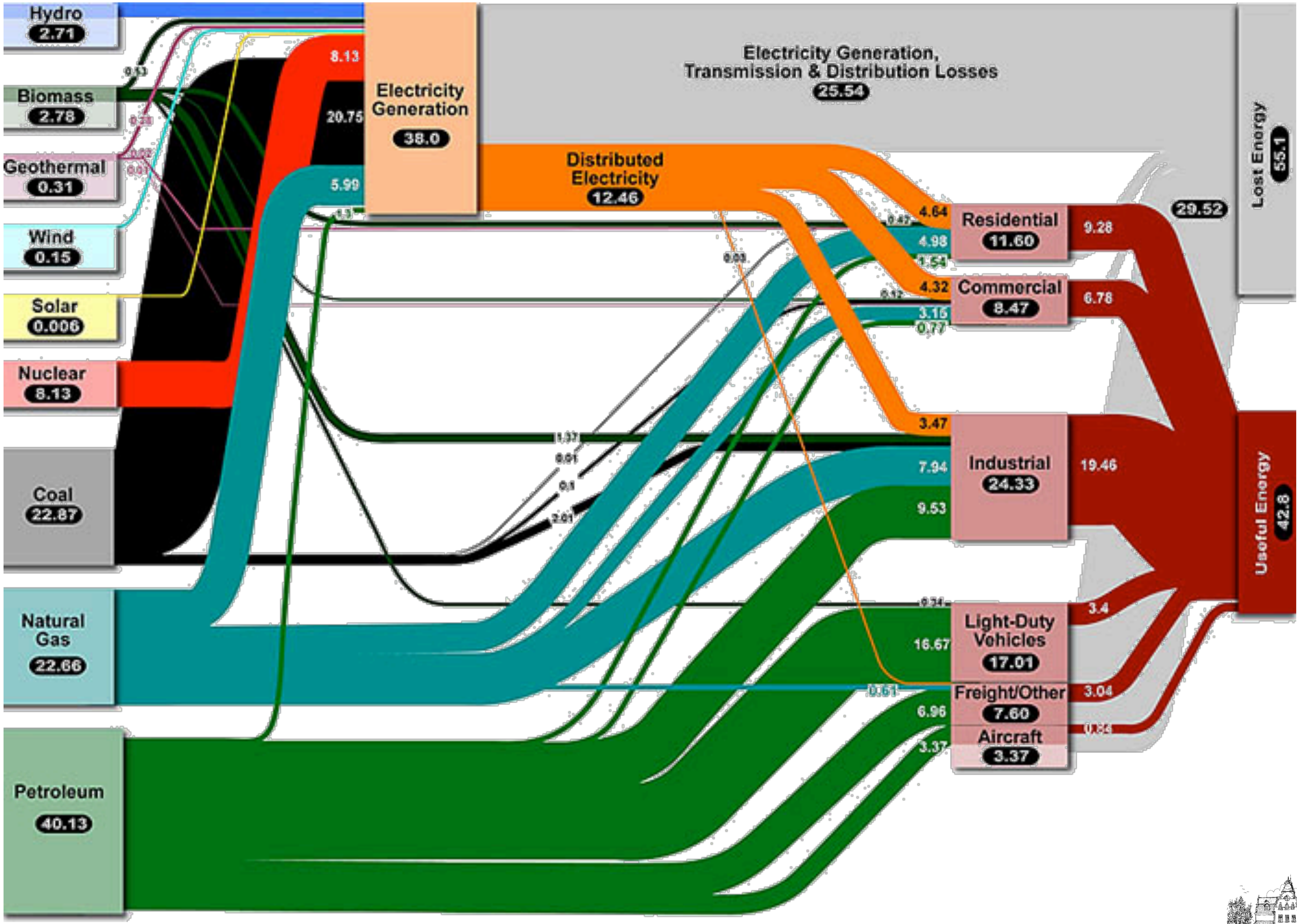


by Hippel and Quigg (1972)

Usage

$$T_l(s) = \frac{B_l^2(q)\Gamma}{m_r^2 - m^2 - \imath\rho B_l^2(q) m_0 \Gamma}$$

Input = Output



Outline of the Unitarity Approach

The most basic feature of an amplitude is UNITARITY

Everything which comes in has to get out again

no source and no drain of probability

Idea: Model a unitary amplitude

Realization: n-Rank Matrix of analytic functions, T_{ij}

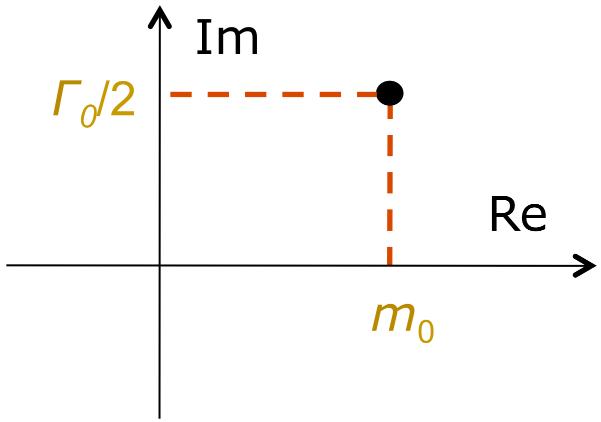
one row (column) for each decay channel

What is a resonance?

A pole in the complex energy plane $T_{ij}(m)$
with m being complex

Parameterizations: e.g. sum of poles

$$\frac{1}{m_0 - i\frac{\Gamma_0}{2}}$$



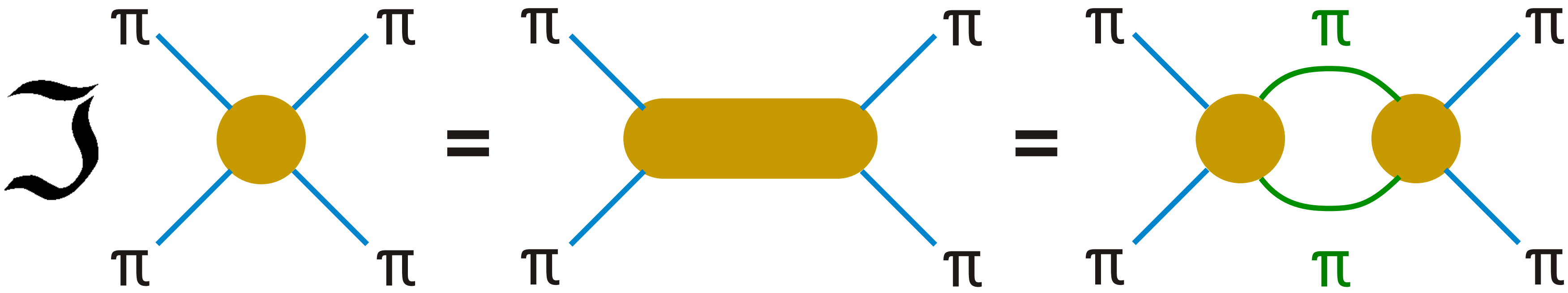
T-Matrix Unitarity Relations

Unitarity is a basic feature since probability has to be conserved

T is unitary if S is unitary

$$\sum_{j=0}^{n} S^*_{kj} S_{ij} = \delta_{ik} = \sum_{j=0}^{n} T^*_{kj} T_{ij}$$

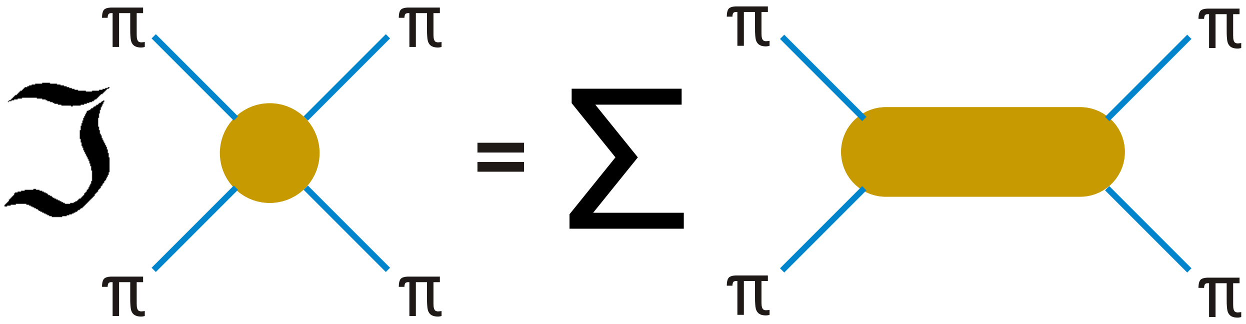
since $S = I + 2\imath\, T$ we get in addition $\Im\left[T_{ij}\right] = \sum_n T^*_{nj} T_{ni}$

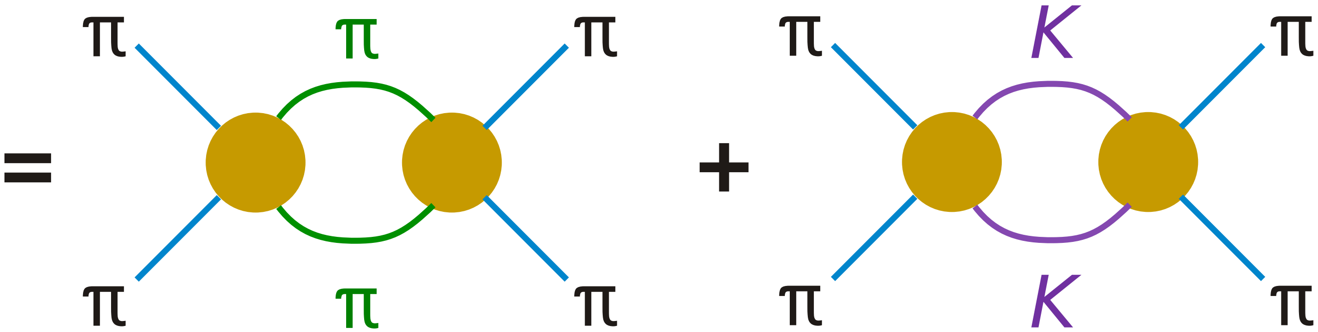


for a single channel $\Im\left[T_{11}\right] = T^*_{11} T_{11}$

but there a more than one channel involved….

$$\Im\left[T_{ij}\right] = T^*_{i1}T_{1j} + T^*_{i2}T_{2j} + \ldots$$





T-Matrix Dispersion Relations

Cauchy Integral on a closed contour

$$T_l(s) = \frac{1}{2\imath\pi}\int_C \frac{T_l(s')ds'}{s'-s}$$

By choosing proper contours and some limits one obtains the dispersion relation for $T_l(s)$

$$T_l(s) = \frac{1}{\pi}\int_{-\infty}^{s_L} \frac{\Im\left[T_l(s')\right]}{s'-s}ds' + \frac{1}{\pi}\int_{(m_1+m_2)^2}^{\infty} \frac{\Im\left[T_l(s')\right]}{s'-s}ds'$$

Satisfying this relation with an arbitrary parameterization is extremely difficult and is dropped in many approaches

much more elsewhere....

K-Matrix Definition

S (and *T*) is **n x n** matrix representing
n incoming and **n** outgoing channel

the Caley transformation generates a unitary matrix from a real and symmetric matrix *K*

$$S = (I + \imath K)(I - \imath K)^{-1} = (I - \imath K)^{-1}(I + \imath K)$$

then *T* commutes with K $[K, T] = 0$

and is defined like

$$T = K(I - \imath K)^{-1} = (I - \imath K)^{-1} K$$

then *T* is also unitary by design

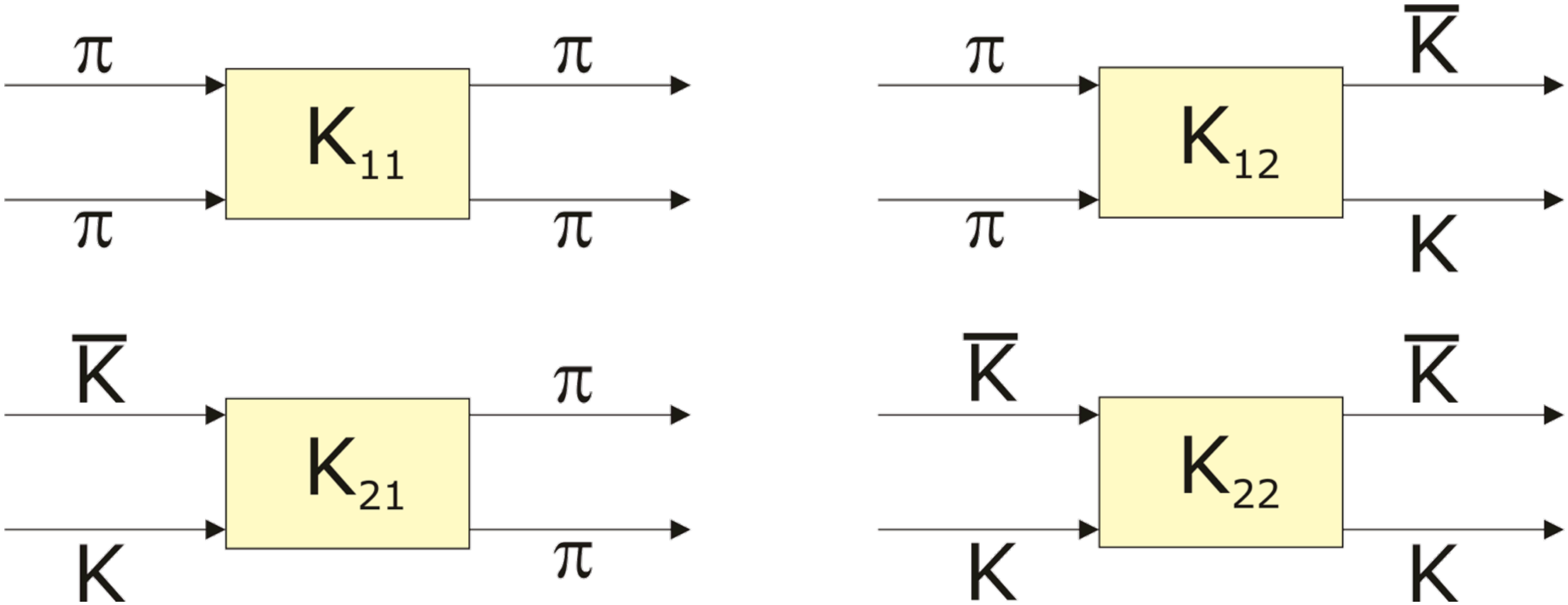
Some more properties

$$\begin{aligned} \Re[T] &= (I + K^2)^{-1} K = K(I + K^2)^{-1} \\ \Im[T] &= (I + K^2)^{-1} K^2 = K^2 (I + K^2)^{-1} \end{aligned}$$

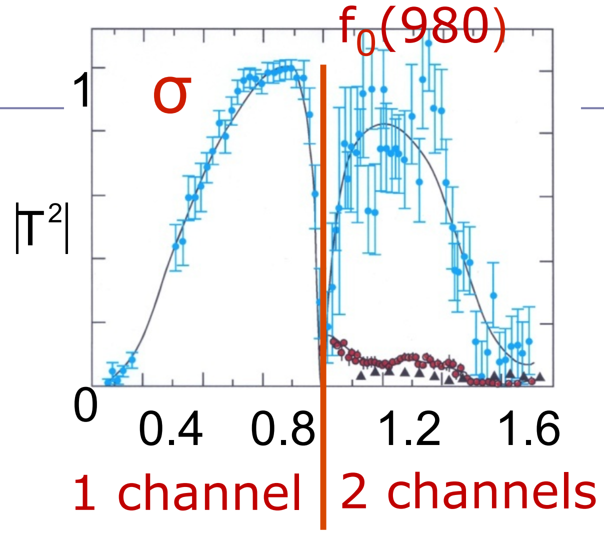
it can be shown, that this leads to $\Im[T] = T^* T = T T^*$

K-Matrix - Interpretation

Each element of the *K*-matrix describes
one particular propagation from initial to final states

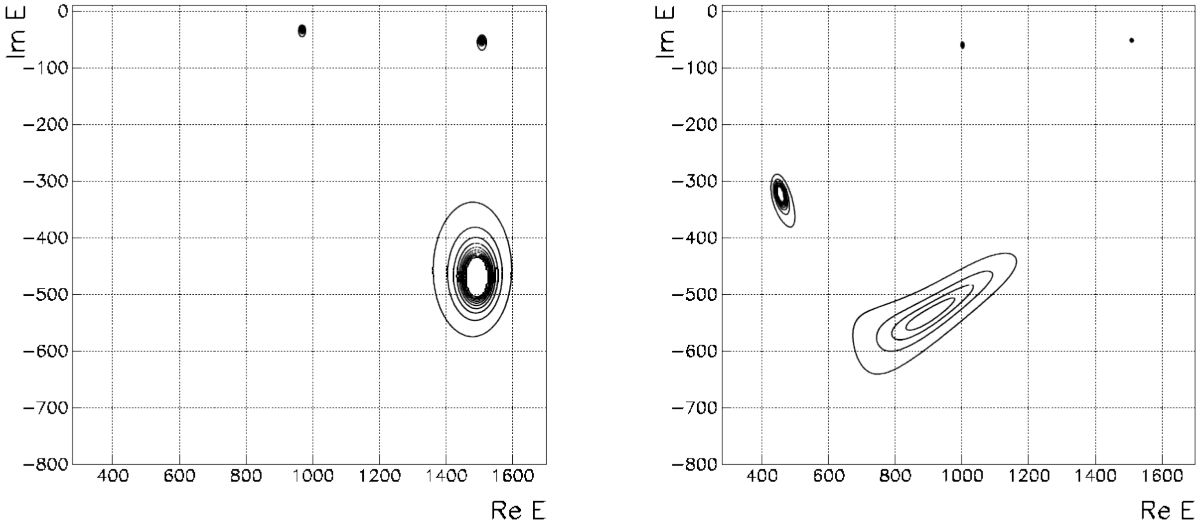


Example: ππ-Scattering



1 channel	2 channels
$\|S\| = 1$	$S_{ik} S^*_{jk} = \delta_{ij}$
$S = e^{2\iota\delta}$	$S_{11} = \eta\ e^{2i\delta_1}$ $S_{22} = \eta\ e^{2i\delta_2}$ $S_{12} = \iota\sqrt{1-\eta^2}\ e^{\iota\varphi_{12}}, \quad \varphi_{12} = \delta_1 + \delta_2$
$K = \tan\delta$	$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$
$T = e^{\iota\delta}\sin\delta$	$T = \frac{1}{1 - D - \iota(K_{11}+K_{22})} \begin{pmatrix} K_{11} - iD & K_{12} \\ K_{21} & K_{22} - iD \end{pmatrix}$
$\sigma = \left(\frac{4\pi}{q_i^2}\right)\sin^2\delta$	$D = K_{11}K_{22} - K_{12}^2$

Unitarity, cont‘d



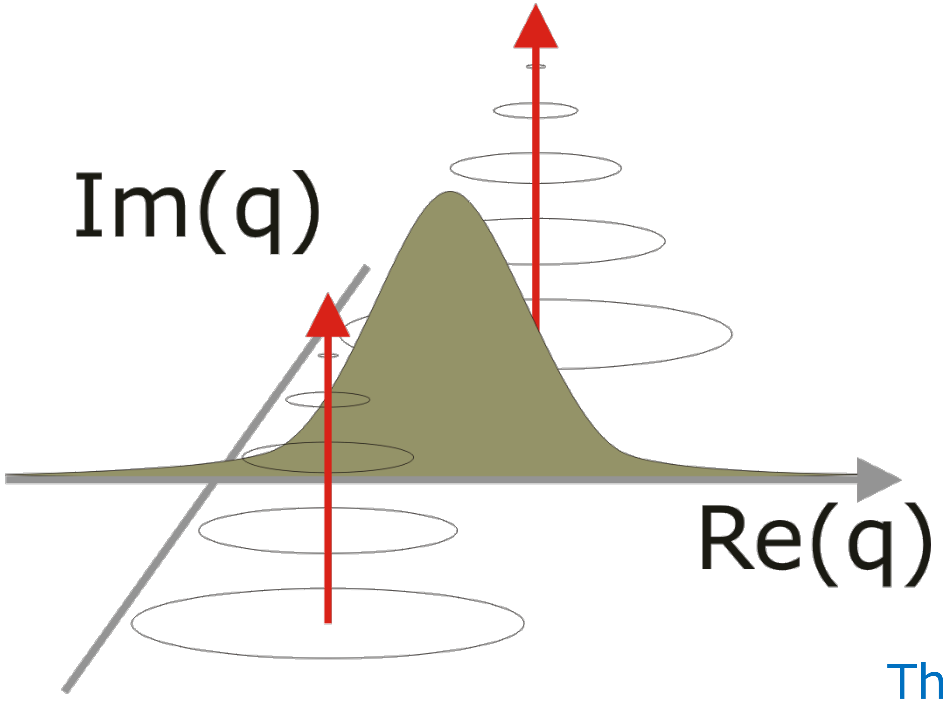
Goal: Find a reasonable parameterization

- The parameters are used to model the analytic function to follow the data
- Only a tool to identify the resonances in the complex energy plane
- Does not necessarily help to interpret the data!
- Poles and couplings have not always a direct physical meaning

Problem: Freedom and unitarity

- Find an approach where unitarity is preserved by construction
- And leave a lot of freedom for further extension

Nearest Pole Determines Real Axis



The pole nearest to the real axis

or more clearly to a point with mass *m* on the real axis

determines your physics results

Far away from thresholds this works nicely

Relativistic Treatment

So far we did not care about relativistic kinematics

covariant description $$T = \{\rho\}^{\frac{1}{2}}\ \widehat{T}\ \{\rho\}^{\frac{1}{2}}$$

or $$T_{ij} = \{\rho_i\}^{\frac{1}{2}}\ \widehat{T}_{ij}\ \{\rho_j\}^{\frac{1}{2}}$$

and $$S = I + 2\imath\ \{\rho\}^{\frac{1}{2}}\ \widehat{T}\ \{\rho\}^{\frac{1}{2}}$$

with $\rho = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix}$ $\quad \rho_1 = \dfrac{2q_1}{m}$ and $\rho_2 = \dfrac{2q_2}{m}$

therefore $\Im\left[\widehat{T}\right] = \widehat{T}^{*}\rho\widehat{T} = \widehat{T}\rho\widehat{T}^{*}$ $\quad \Im\left[\widehat{T}^{-1}\right] = -\rho$

and *K* is changed as well $K = \{\rho\}^{\frac{1}{2}}\ \widehat{K}\ \{\rho\}^{\frac{1}{2}}$ and

$$\widehat{K}^{-1} = \widehat{T}^{-1} + \imath\rho \qquad \boxed{\widehat{T} = \widehat{K}(I - \imath\rho\widehat{K})^{-1} = (I - \imath\widehat{K}\rho)^{-1}\widehat{K}}$$

Relativistic Treatment (cont'd)

So far we did not care about relativistic kinematics

covariant description $$T = \{\rho\}^{\frac{1}{2}}\ \widehat{T}\ \{\rho\}^{\frac{1}{2}}$$

with

$$\rho = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix} \qquad \rho_1 = \frac{2q_1}{m} \quad \text{and} \quad \rho_2 = \frac{2q_2}{m}$$

in detail

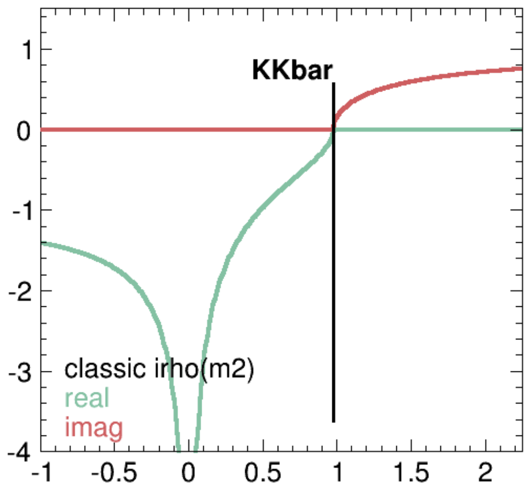
$$\rho_1 = \frac{2q_1}{m} = \sqrt{\left[1 - \left(\frac{m_a + m_b}{m}\right)^2\right]\left[1 - \left(\frac{m_a - m_b}{m}\right)^2\right]}$$

$$\rho_2 = \frac{2q_2}{m} = \sqrt{\left[1 - \left(\frac{m_c + m_d}{m}\right)^2\right]\left[1 - \left(\frac{m_c - m_d}{m}\right)^2\right]}$$

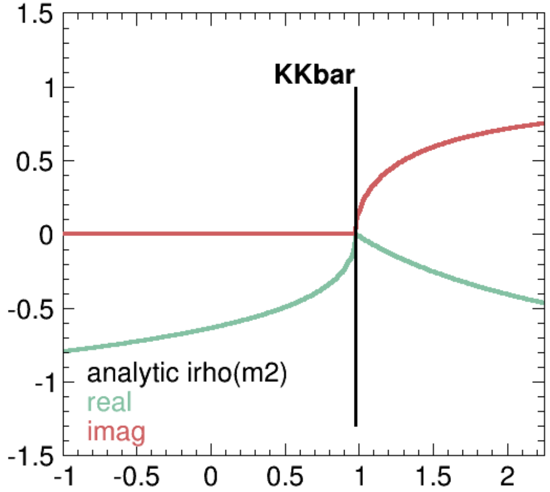
$$\rho_i \to 1 \quad as \quad m^2 \to \infty$$

Analytic extrapolation of ρ

iρ



$m^2(K\bar{K})$



$m^2(K\bar{K})$

Analytic extrapolation of ρ

$$\rho_i = \sqrt{|1-(m_a+m_b)^2/m^2|}$$

D. Asner (PDG)

iρ= $$-\frac{\rho_i}{\pi}\log\left|\frac{1+\rho_i}{1-\rho_i}\right|,\quad -\frac{2\rho_i}{\pi}\arctan\left(\frac{1}{\rho_i}\right),\quad -\frac{\rho_i}{\pi}\log\left|\frac{1+\rho_i}{1-\rho_i}\right|+i\rho_i$$

$$\text{for } m^2<0,\ 0<m^2<(m_a+m_b)^2, \text{ and } (m_a+m_b)^2<m^2,$$

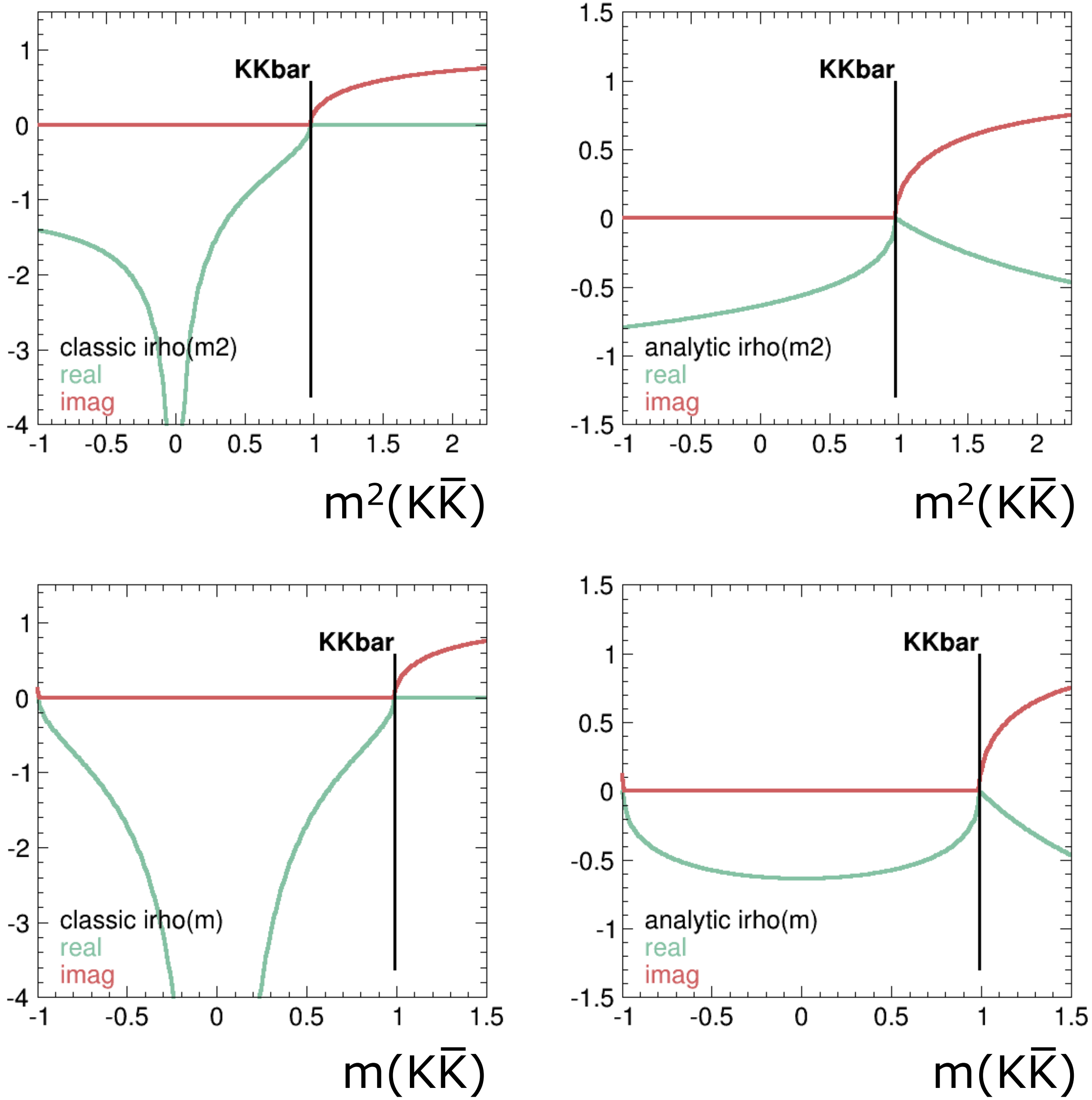
-iρ= $$CM(s) = \frac{\rho}{\pi}\log\left(\frac{\rho+1}{\rho-1}\right) = \frac{\rho}{\pi}\log\left(\frac{1+\rho}{1-\rho}\right) - i\rho$$

$$\mathbf{T}^{-1} = \overline{\mathbf{K}}^{-1} + \mathbf{CM}$$

M. Pennington (Lectures)

Analytic extrapolation of ρ

iρ



Relativistic Treatment – 2 channel

S-Matrix

$$
\begin{aligned}
S &= (I + \iota\{\rho\}^{\frac{1}{2}}\ \widehat{K}\{\rho\}^{\frac{1}{2}})(I - \iota\{\rho\}^{\frac{1}{2}}\ \widehat{K}\{\rho\}^{\frac{1}{2}})^{-1} \\
&= (I - \iota\{\rho\}^{\frac{1}{2}}\ \widehat{K}\{\rho\}^{\frac{1}{2}})^{-1}(I + \iota\{\rho\}^{\frac{1}{2}}\ \widehat{K}\{\rho\}^{\frac{1}{2}})
\end{aligned}
$$

2 channel *T*-Matrix

$$
\widehat{T} = \frac{1}{1 - \rho_1\rho_2\widehat{D} - \iota(\rho_1\widehat{K}_{11} + \rho_2\widehat{K}_{22})} \begin{pmatrix} \widehat{K}_{11} - \iota\rho_2\widehat{D} & \widehat{K}_{12} \\ \widehat{K}_{21} & \widehat{K}_{22} - \iota\rho_1\widehat{D} \end{pmatrix}
$$

$$
\widehat{D} = \widehat{K}_{11}\widehat{K}_{22} - \widehat{K}^2_{12}
$$

to be compared with the non-relativistic case

$$
T = \frac{1}{1 - D - \iota(K_{11} + K_{22})} \begin{pmatrix} K_{11} - iD & K_{12} \\ K_{21} & K_{22} - iD \end{pmatrix}
$$

$$
D = K_{11}K_{22} - K^2_{12}
$$

K-Matrix Poles

Now we introduce resonances
as poles (propagators)

$$K_{ij} = \sum_R \frac{g_{Ri}(m) g_{Rj}(m)}{m_R^2 - m^2} + c_{ij}$$

One may add c_{ij} a real polynomial
of m^2 to account for
slowly varying background
(not experimental background!!!)

$$\widehat{K}_{ij} = \sum_R \frac{g_{Ri}(m) g_{Rj}(m)}{(m_R^2 - m^2)\sqrt{\rho_i \rho_j}} + \hat{c}_{ij}$$

$$g_{Ri}^2(m) = m_R \Gamma_{Ri}(m)$$

Width/Lifetime

$$\Gamma_R(m) = \sum_i \Gamma_{Ri}(m)$$

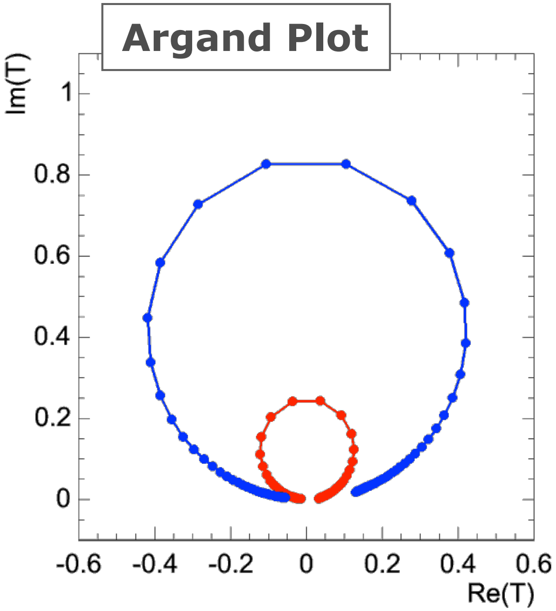
$$\Gamma_{Ri}(m) = \frac{g_{Ri}^2(m)}{m_R} = \gamma_{Ri}^2 \Gamma_R^0 \left[B_{Ri}^l(q, q_R) \right]^2 \rho_i$$

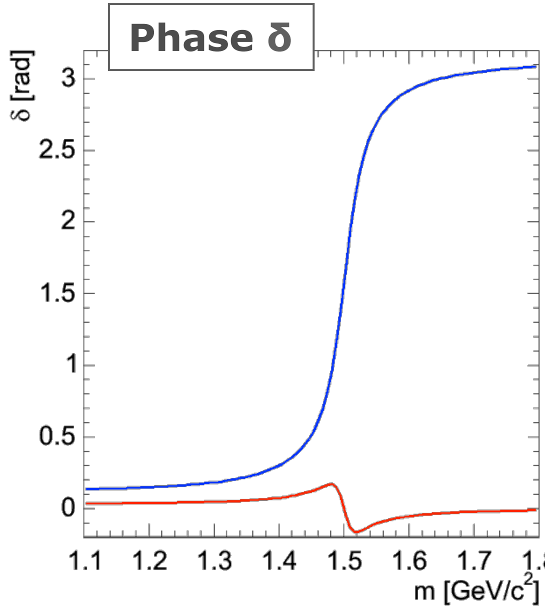
For a single channel and one pole we get

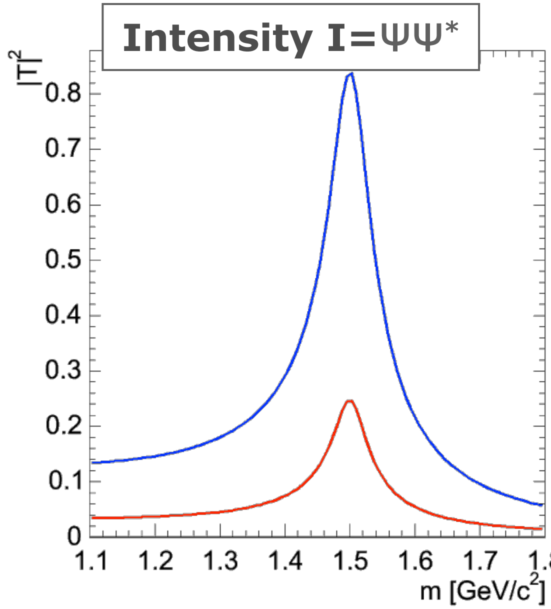
$$T = e^{i\delta} \,\mathrm{sn}\, \delta = \left[\frac{m_0 \Gamma_0}{m_0^2 - m^2 - i m_0 \Gamma(m)} \right] [B^l(q, q_0)]^2 \left(\frac{\rho}{\rho_0} \right)$$

T=+i
Pole
That=+i rho

Example: 1x2 K-Matrix







Strange effects in subdominant channels

Scalar resonance at 1500 MeV/c^2, Γ=100 MeV/c^2

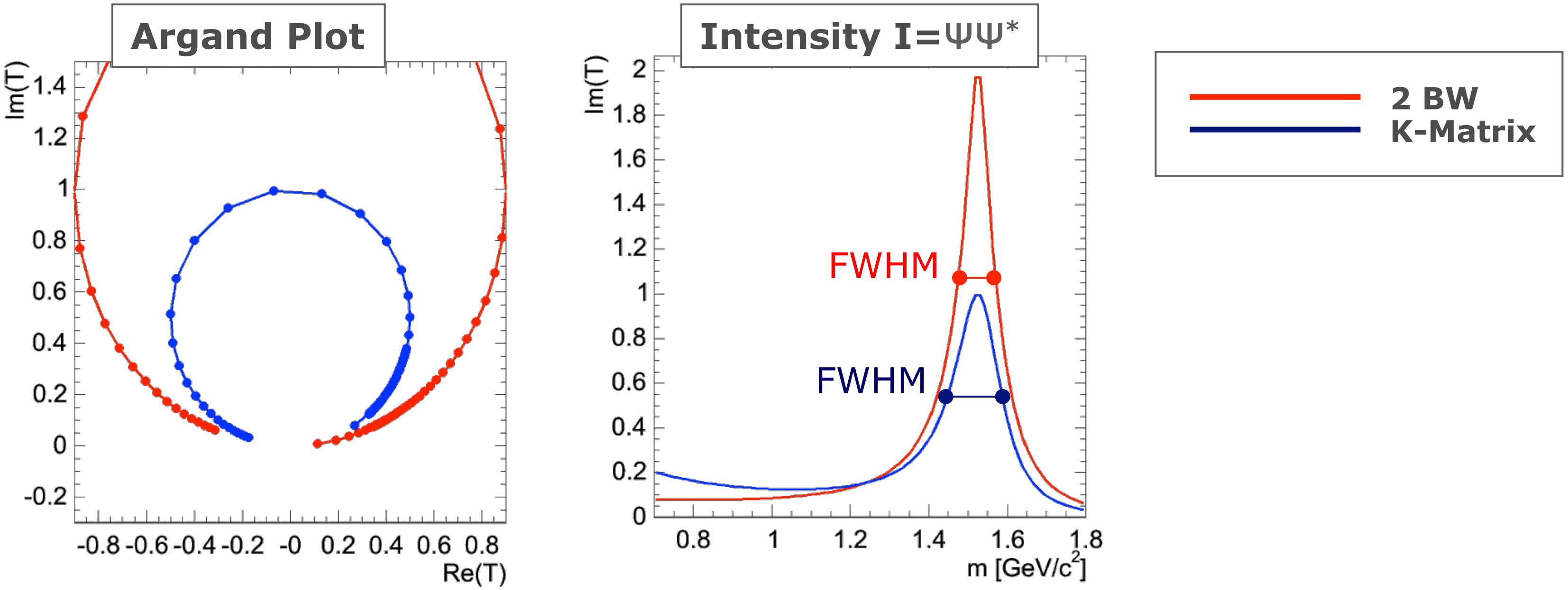
All plots show ππ channel

Blue: *ππ* dominated resonance ($\Gamma_{\pi\pi}$=80 MeV and $\Gamma_{K\bar{K}}$=20 MeV)

Red: $K\bar{K}$ dominated resonance ($\Gamma_{K\bar{K}}$=80 MeV and $\Gamma_{\pi\pi}$=20 MeV)

Look at the tiny phase motion in the subdominant channel

Example: 2x1 K-Matrix Overlapping Poles

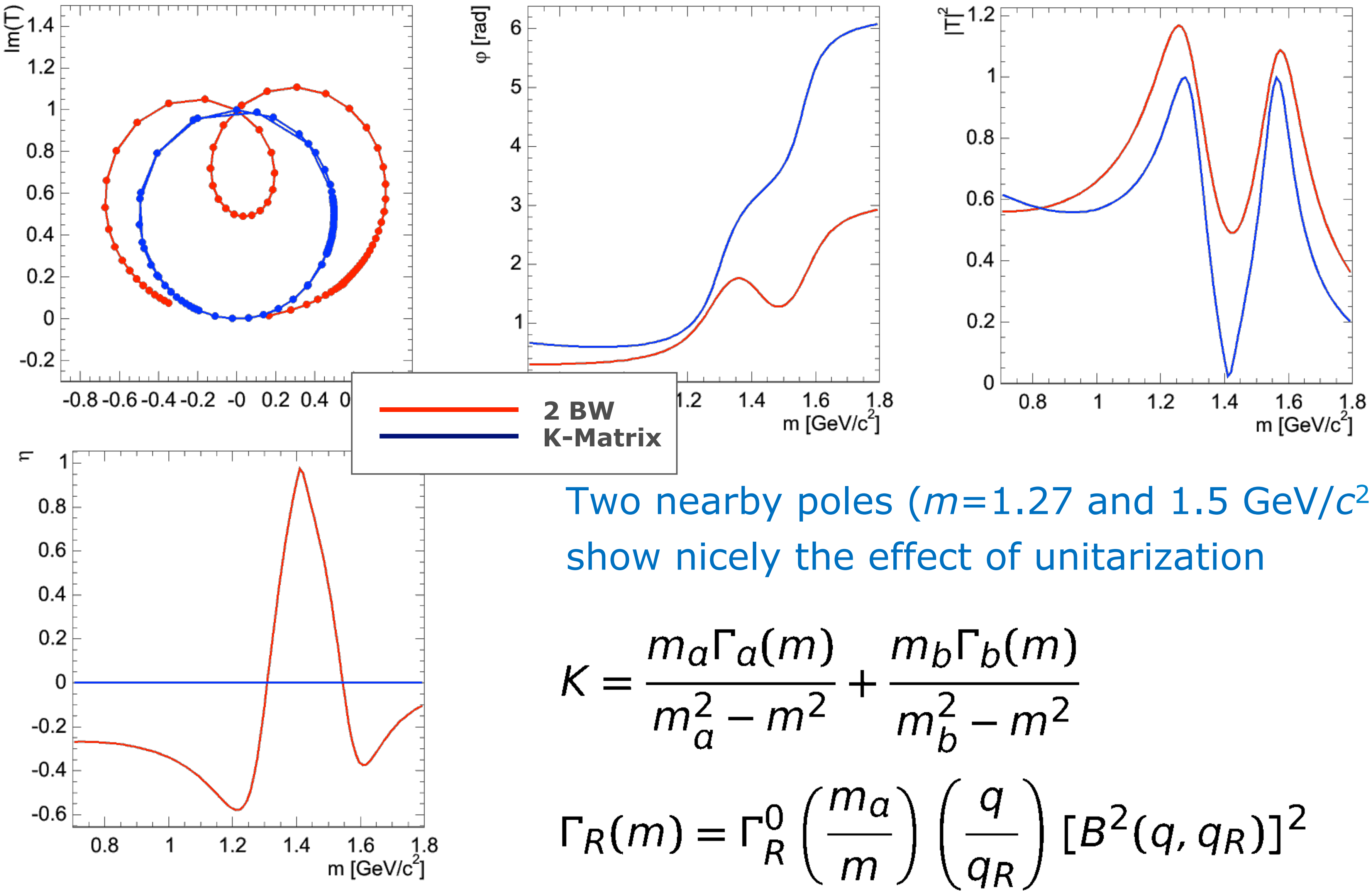


two resonances overlapping with different (100/50 MeV/c^2) widths are not so dramatic (except the strength)

The width is basically added

$$T = \frac{m_0[\Gamma_a(m) + \Gamma_b(m)]}{m_0^2 - m^2 - i m_0[\Gamma_a(m) + \Gamma_b(m)]}$$

Example: 1x2 K-Matrix Nearby Poles



Two nearby poles (m=1.27 and 1.5 GeV/c^2) show nicely the effect of unitarization

$$K = \frac{m_a \Gamma_a(m)}{m_a^2 - m^2} + \frac{m_b \Gamma_b(m)}{m_b^2 - m^2}$$

$$\Gamma_R(m) = \Gamma_R^0 \left(\frac{m_a}{m}\right)\left(\frac{q}{q_R}\right)[B^2(q, q_R)]^2$$

Example: Flatté 1x2 K-Matrix

2 channels for a single resonance at the threshold of one of the channels

with $\gamma_1^2 + \gamma_2^2 = 1$

$$\hat{K}_{11} = \frac{\gamma_1^2 m_0 \Gamma_0}{m_0^2 - m^2}$$

$$\hat{K}_{22} = \frac{\gamma_2^2 m_0 \Gamma_0}{m_0^2 - m^2}$$

$$\hat{K}_{12} = \hat{K}_{21} = \frac{\gamma_1 \gamma_2 m_0 \Gamma_0}{m_0^2 - m^2}$$

Leading to the T-Matrix

$$\hat{T} = \frac{m_0 \Gamma_0}{m_0^2 - m^2 - i m_0 \Gamma_0 (\rho_1 \gamma_1^2 + \rho_2 \gamma_2^2)} \begin{pmatrix} \gamma_1^2 & \gamma_1 \gamma_2 \\ \gamma_1 \gamma_2 & \gamma_2^2 \end{pmatrix}$$

and with

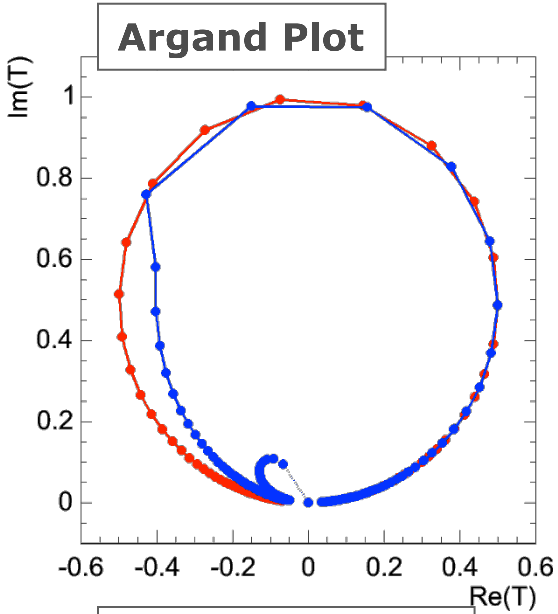
$$g_i = \gamma_i \sqrt{m_0 \Gamma_0}$$

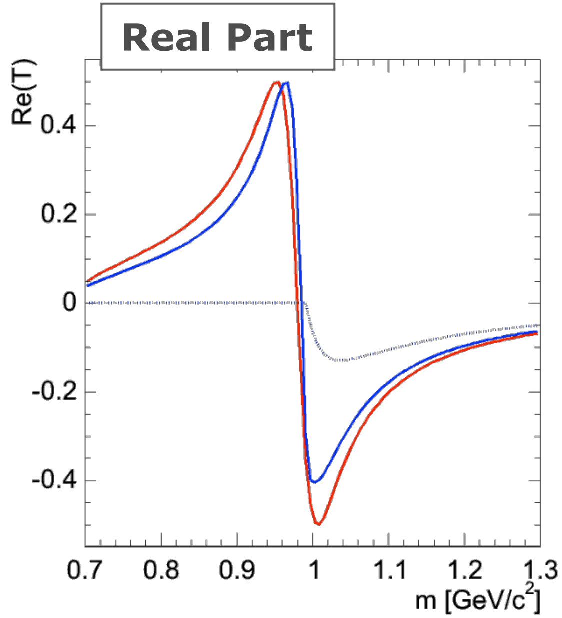
$$g_1^2 + g_2^2 = m_0 \Gamma_0$$

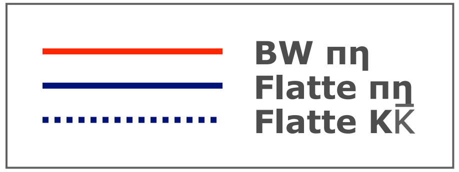
we get

$$\hat{T} = \frac{\begin{pmatrix} g_1^2 & g_1 g_2 \\ g_1 g_2 & g_2^2 \end{pmatrix}}{m_0^2 - m^2 - \imath(\rho_1 g_1^2 + \rho_2 g_2^2)}$$

Flatté

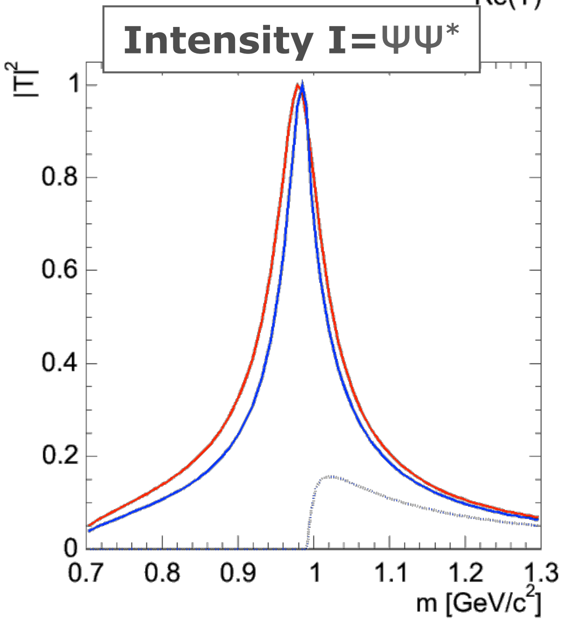


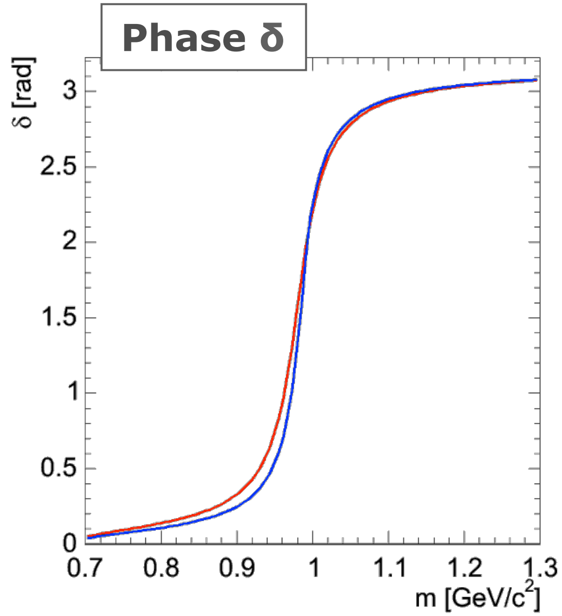




Example

$a_0(980)$ decaying into $\pi\eta$ and $K\bar{K}$



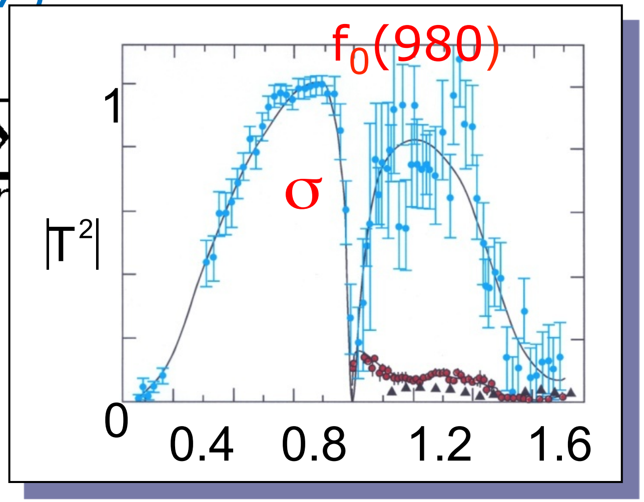


K-Matrix Parameterizations

Au, Morgan and Pennington (1987)

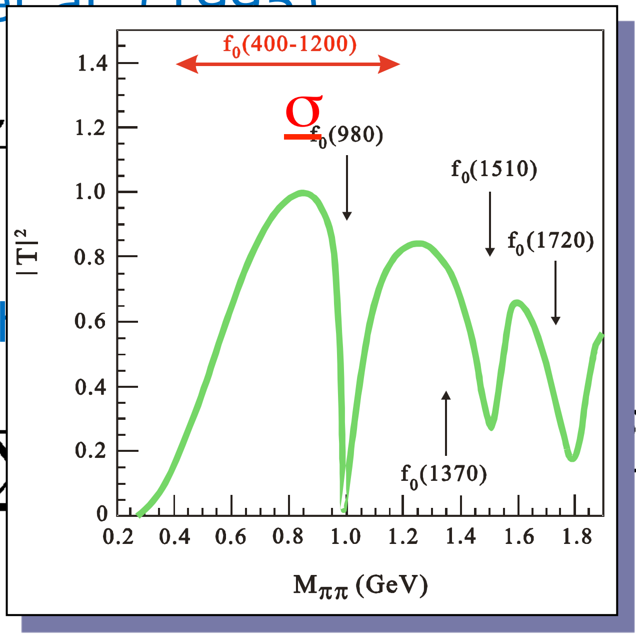
$$K_{ij} = \frac{s - s_0}{4m_K^2} \sum_r \frac{g_{r,i} g_{r,j}}{(s_r - s)(s_r - s_0)} + \sum_r \ldots$$

$$\equiv (s - s_0)\widehat{K}_{ij}$$



Amsler et al. (1995)

$$K_{ij} = \sum_r \frac{g_{r,} \ldots}{\ldots}$$

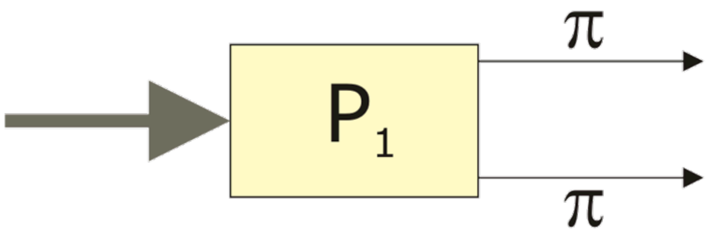


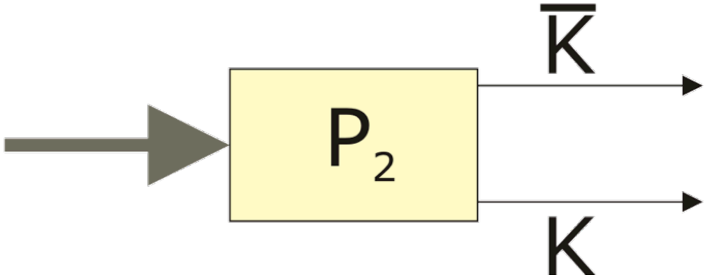
Anisovic…

$$K_{ij}(s) = \Bigl(\sum \ldots \frac{\ldots - s_A}{\ldots + s_{A0}}$$

P-Vector Definition

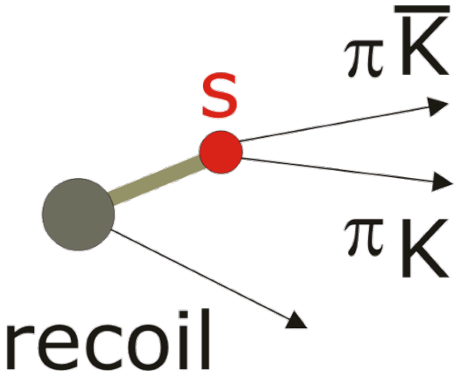
But in many reactions there is no scattering process but a production process, a resonance is produced with a certain strength and then decays

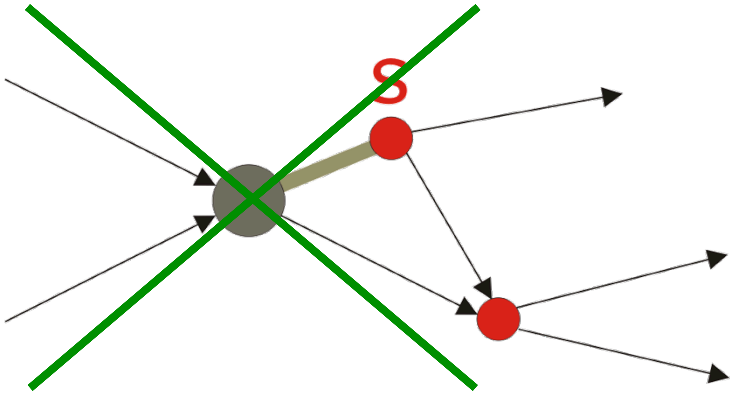




Aitchison (1972) $F = (I - iK)^{-1}P = TK^{-1}P$

$\hat{F} = (I - i\hat{K}\rho)^{-1}\hat{P} = \hat{T}\hat{K}^{-1}\hat{P}$ with $F = \{\rho\}^{\frac{1}{2}}\ \hat{F}$ and $P = \{\rho\}^{\frac{1}{2}}\ \hat{P}$





P-Vector Poles

The resonance poles are constructed as in the *K*-Matrix

$$P_i = \sum_R \frac{\beta_R^0 \; g_{Ri}(m)}{m_R^2 - m^2} \qquad \widehat{P}_i = \sum_R \frac{\beta_R^0 \; g_{Ri}(m)}{(m_R^2 - m^2)\sqrt{\rho_i}}$$

and one may add a polynomial ***d_i*** again $\quad P_i \rightarrow P_i + d_i$

For a single channel and a single pole

$$\widehat{F}(m) = \beta \frac{m_0 \Gamma_0}{m_0^2 - m^2 - i m_0 \Gamma(m)} B^l(q, q_0)$$

If the *K*-Matrix contains fake poles...

for non *s*-channel processes modeled in an s-channel model

...the corresponding poles in *P* are different

Q-Vector

A different Ansatz with a different picture: channel *n* is produced and undergoes final state interaction

$$Q = K^{-1}P \quad \text{and} \quad \{\rho\}^{\frac{1}{2}}Q = \widehat{Q} \quad \text{and} \quad \widehat{Q} = \widehat{K}^{-1}\widehat{P}$$

$$F = TQ \quad \text{and} \quad \widehat{F} = \widehat{T}\widehat{Q}$$

For channel 1 in 2 channels

$$F_1 = T_{11}Q_1 + T_{12}Q_2$$

N/D Method

To get the proper behavior for the left-hand cuts

Use $N_l(s)$ and $D_l(s)$ which are correlated by dispersion relations

$$T_l(s) = \frac{N_l(s)}{D_l(s)}$$

An example for this is the work of Bugg and Zhou (1993)

$$\begin{aligned}
K_{ij} &= \left(\frac{s - 2m_\pi^2}{s}\right)\left(\frac{\alpha_i\alpha_j}{s_A - s}\frac{\beta_i\beta_j}{s_B - s}\frac{\gamma_i\gamma_j}{s_C - s} + a_{ij} + b_{ij}s\right) \\
N_{\pi\pi}(s) &= N_{11}(s) = (c_1 + c_2 s)K_{11} + i\rho_2(c_3 + c_4 s) \\
&\quad (K_{11}K_{22} - K_{12}K_{21}) \\
N_{\eta\eta}(s) &= N_{22}(s) = c_1 K_{22} + i\rho_2 c_3(K_{11}K_{22} - K_{12}K_{21})
\end{aligned}$$

Complex Analysis Revisited

The Breit-Wigner example

$$T = e^{i\delta}\sin\delta = \left[\frac{m_0\Gamma_0}{m_0^2 - m^2 - i m_0\Gamma(m)}\right]\left[B^l(q,q_0)\right]^2\left(\frac{\rho}{\rho_0}\right)$$

shows, that *Γ(m)* implies *ρ(m)*

$$\Gamma_{Ri}(m) = \frac{g_{Ri}^2(m)}{m_R} = \gamma_{Ri}^2\Gamma_R^0\left[B_{Ri}^l(q,q_R)\right]^2\rho_i$$

Each *ρ(m)* which is a square root,

one obtains two solutions for $p>0$ or $p<0$ respectively

one obtains two solutions for $p>0$ or $p<0$ respectively

$$p > 0 \qquad\qquad p < 0$$

$$\begin{aligned} \rho_a &= \sqrt{\frac{2|q|}{m}} & \qquad \rho_a &= \iota\sqrt{\frac{2|q|}{m}} \\ \rho_b &= -\sqrt{\frac{2|q|}{m}} & \qquad \rho_b &= -\iota\sqrt{\frac{2|q|}{m}} \end{aligned}$$

But the two values ($w=2q/m$) have some phase in between and are not identical

$$\sqrt{w} - \sqrt{w^*} = \pm\sqrt{|w|}\left(e^{\iota\frac{\varphi}{2}} + e^{-\iota\frac{\varphi}{2}}\right) = \cosh\frac{\varphi}{2}\Big|_{\varphi=0} \neq 0$$

So you define a new complex plane for each solution, which are 2^n complex planes, called Riemann sheets they are continuously connected. The borderlines are called CUTS.

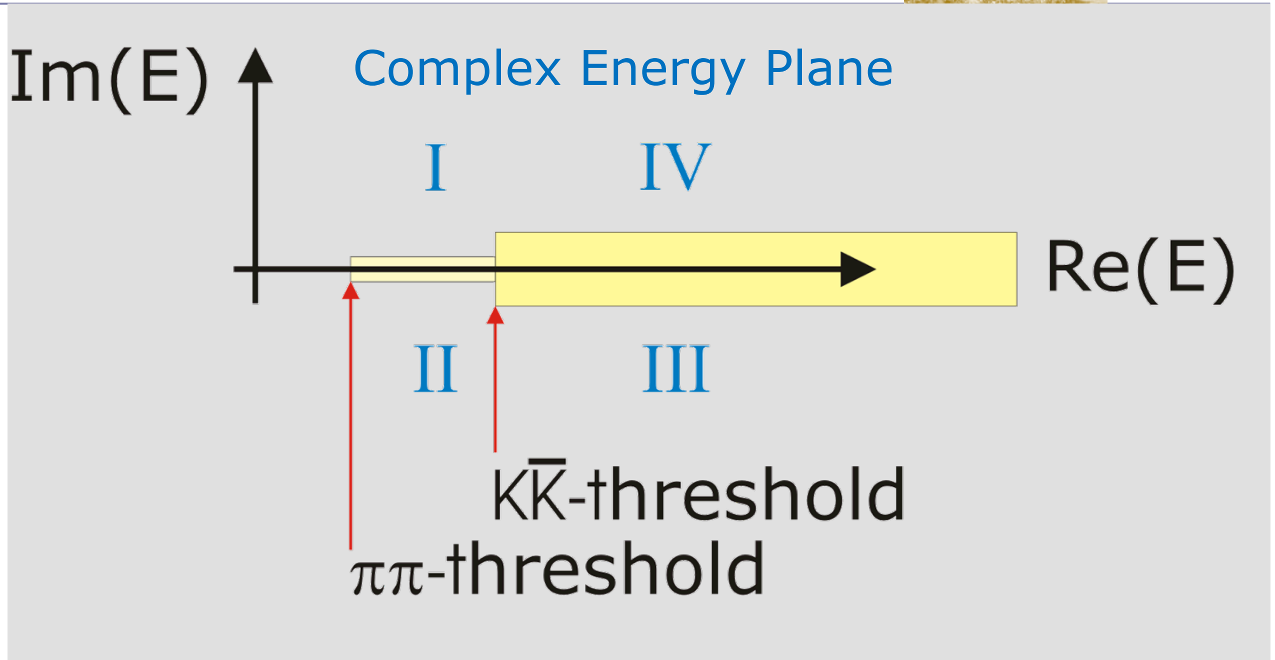
Riemann Sheets in a 2 Channel Problem

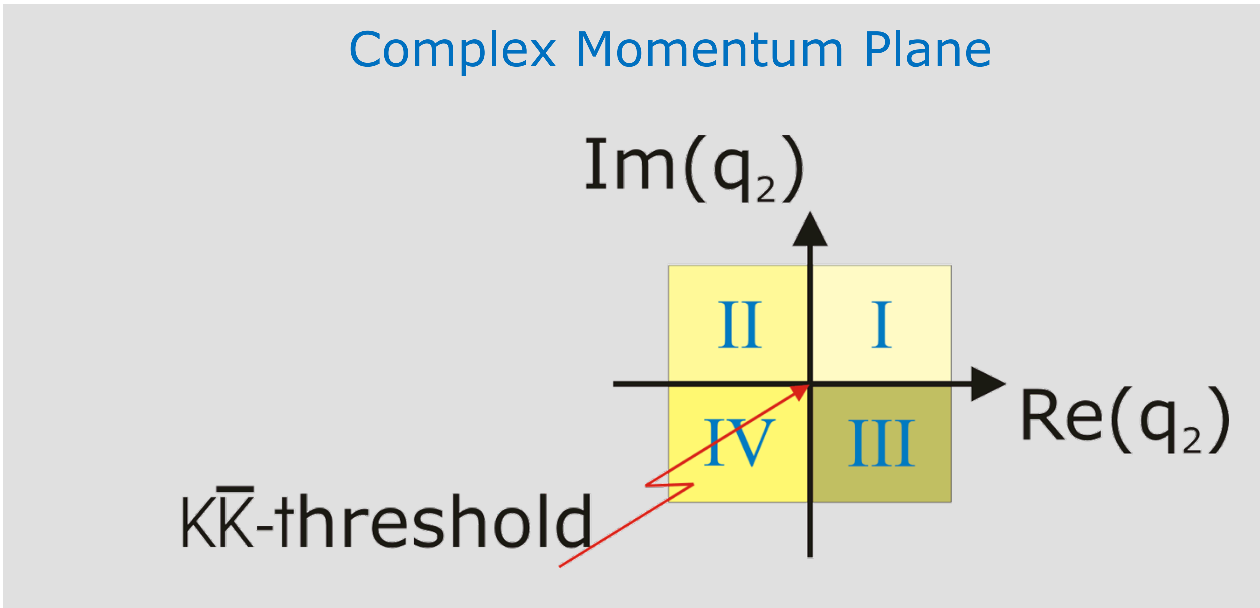
Usual definition

sheet	$\mathrm{sgn}(q_1)$	$\mathrm{sgn}(q_2)$
I	+	+
II	−	+
III	−	−
IV	+	+

Implication

$$\left(\hat{T}^{III}\right)^{-1} = \left(\hat{T}^{II}\right)^{-1} + i\rho_2$$

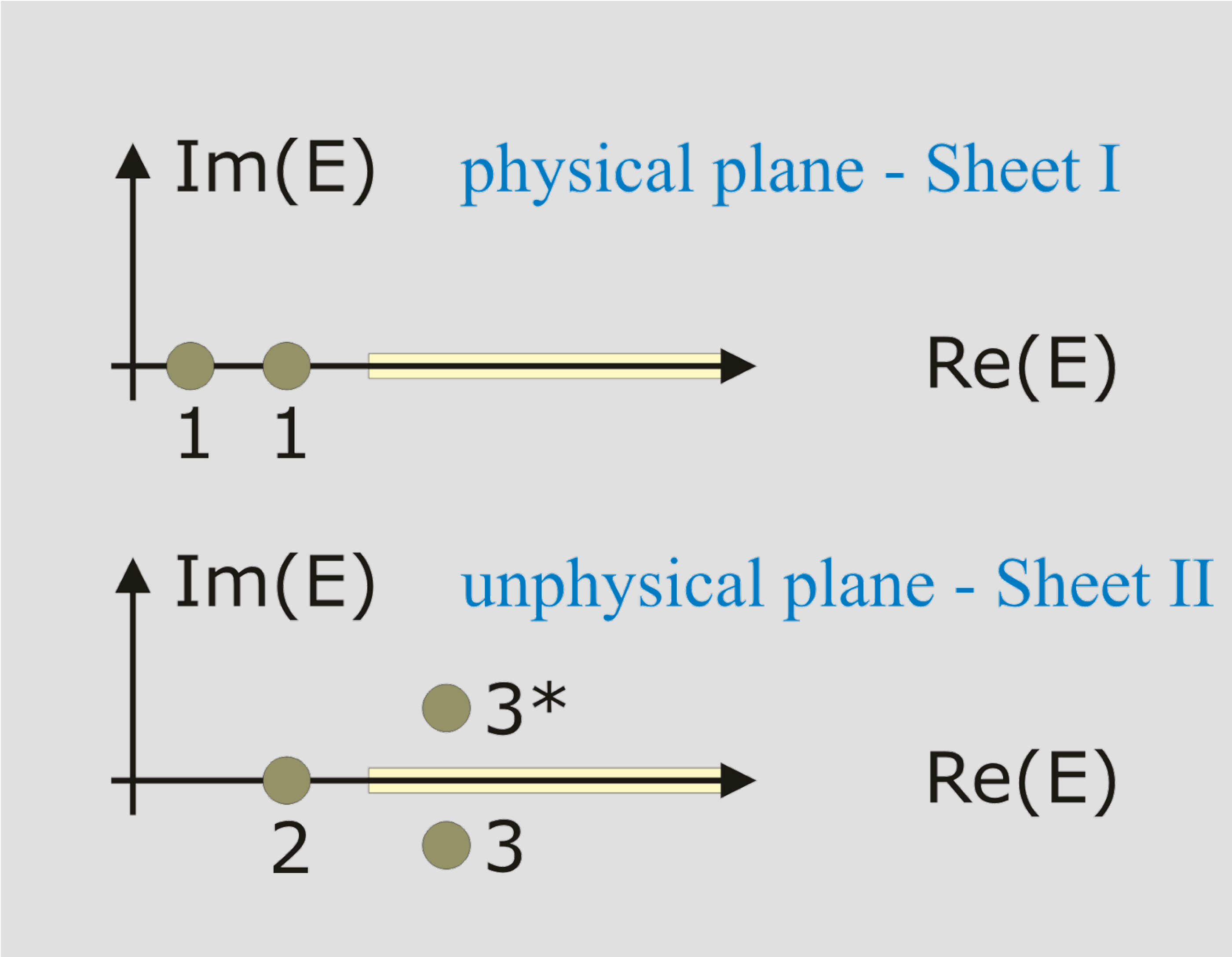




States on Sheets of the Energy Plane

Singularities appear naturally where

$$T(E + \iota\frac{\Gamma}{2}) = 0$$



Singularities might be

1 – bound states

2 – anti-bound states

3 – resonances

or

artifacts due to wrong treatment of the model

States on Momentum Sheets

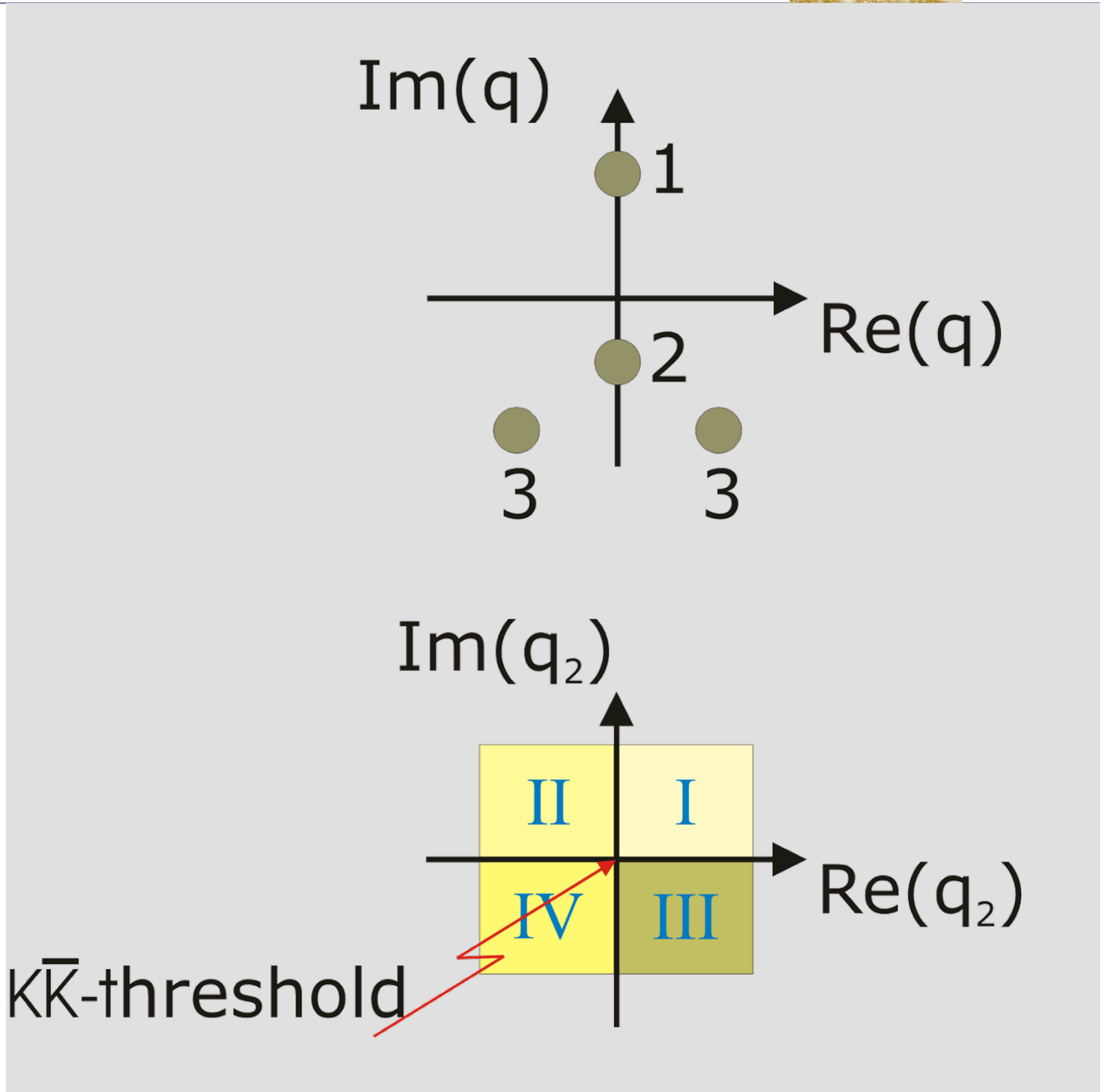
→ complex momentum plane

Singularities might be

1 – bound states

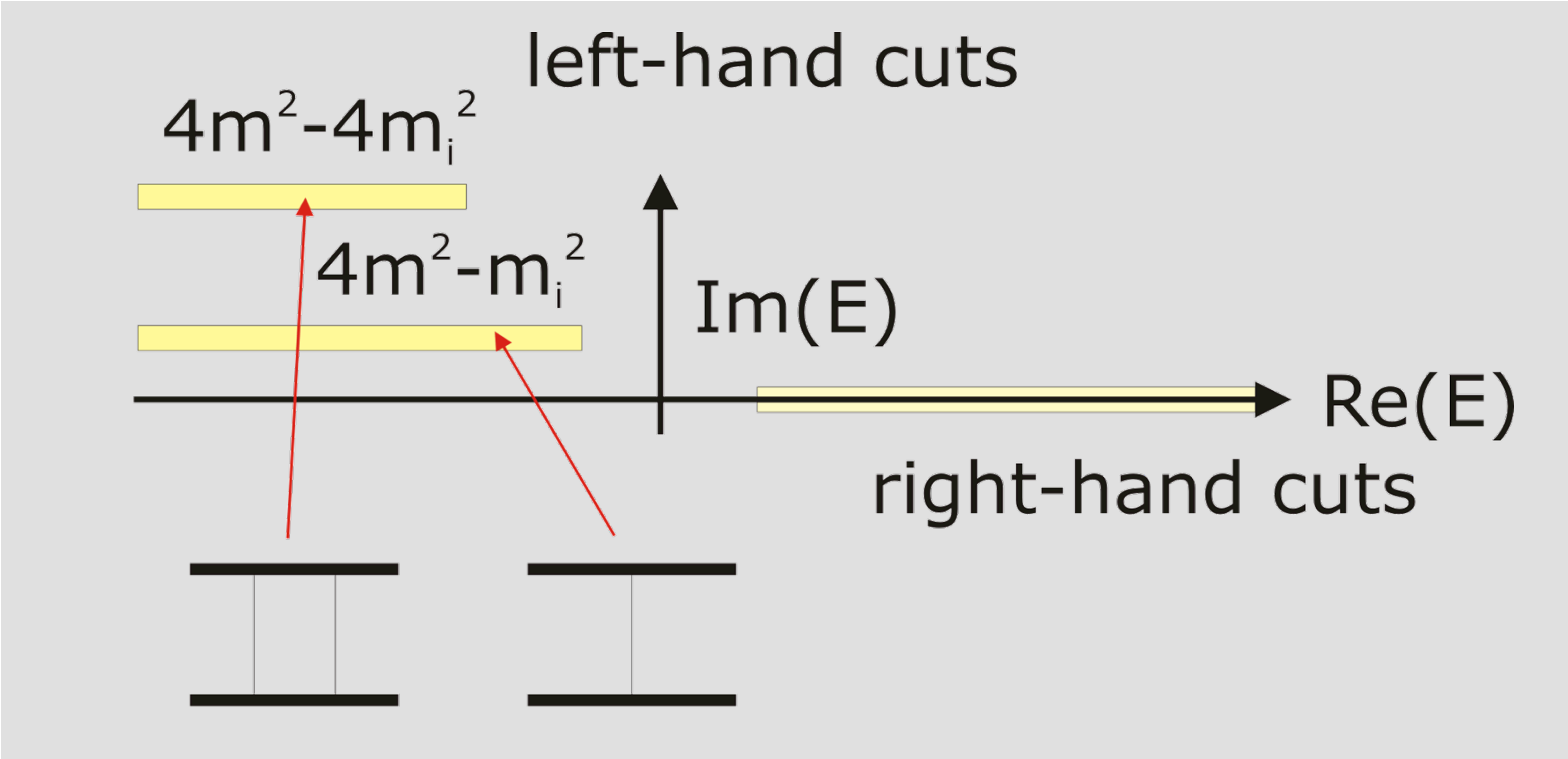
2 – anti-bound states

3 – resonances



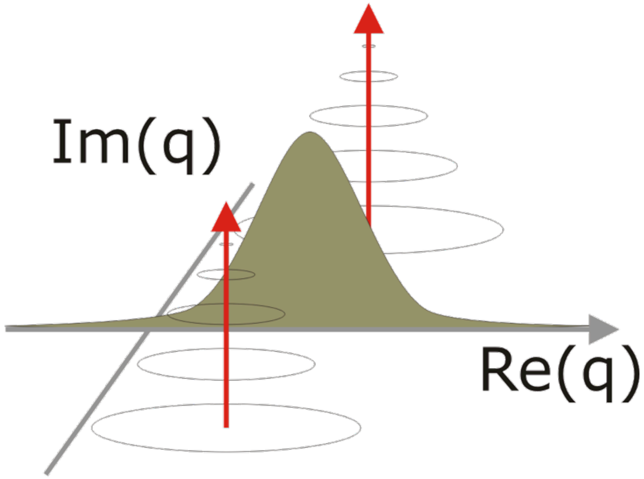
Left-hand and Right-hand Cuts

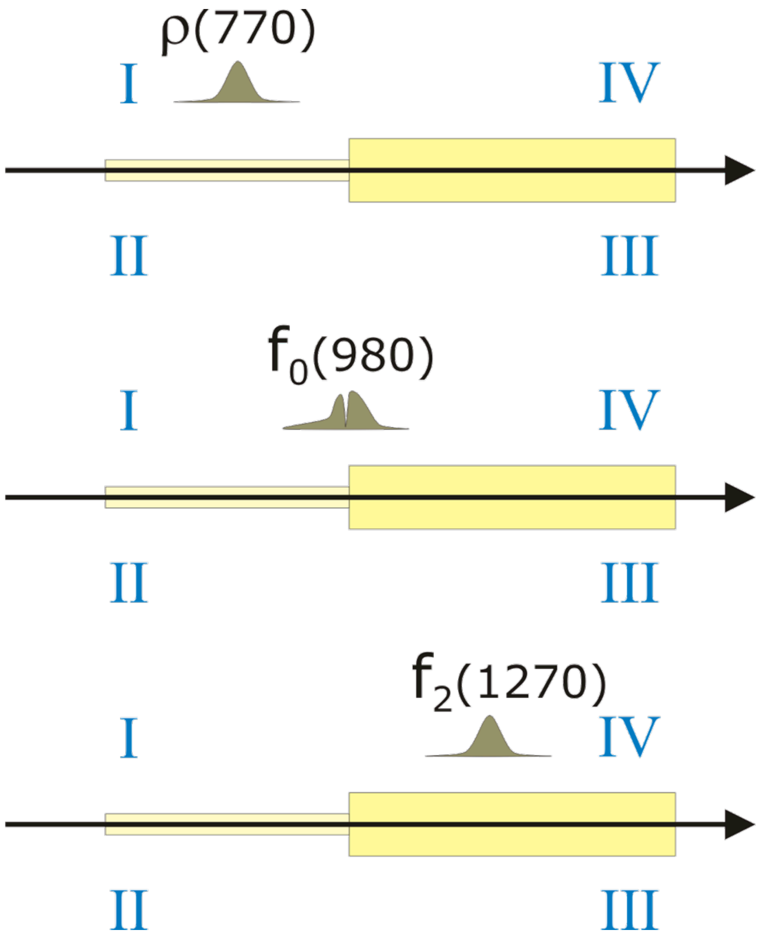
The right hand CUTS (RHC) come from the open channels in an n channel problem



But also exchange processes and other effects introduce CUTS on the left-hand side (LHC)

Nearest Pole Determines Real Axis





The pole nearest to the real axis

> or more clearly to a point with mass *m* on the real axis

determines your physics results

Far away from thresholds this works nicely

At thresholds, the world is more complicated

While $\rho(770)$ in between two thresholds has a beautiful shape

the $f_0(980)$ or $a_0(980)$ have not

Pole and Shadows near Threshold (2 Channels)

For a real resonance one always obtains poles on sheet II and III

due to symmetries in T_l

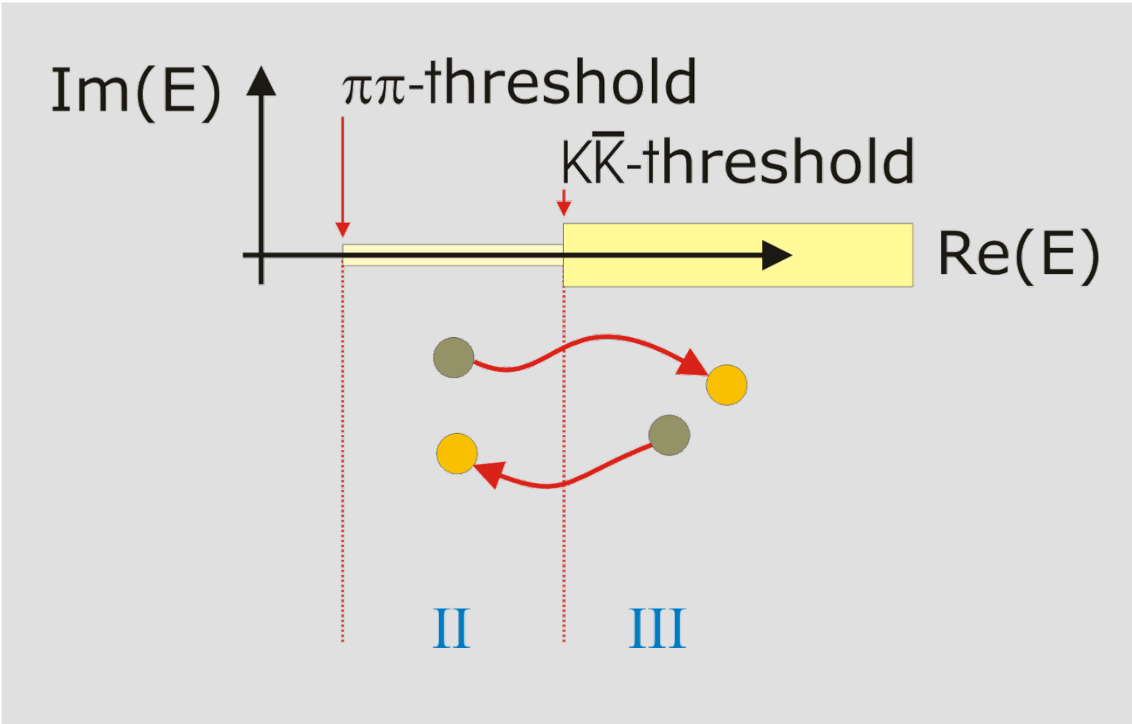
$$\hat{T}_l(q) = \hat{T}_l^*(-q^*) \qquad \text{and} \qquad \hat{T}_l(s) = \hat{T}_l^*(s^*)$$

Usually

$$\Gamma_r^{\mathrm{BW}} \approx \frac{1}{2}\left(\Gamma_r^{II} + \Gamma_r^{III}\right)$$

To make sure that pole and shadow match and form an s-channel resonance, it is mandatory to check if the pole on sheets II and III match

Done by artificially changing ρ_2 smoothly from q_2 to $-q_2$



t-channel Effects (also u-channel)

They may appear resonant and non-resonant

Formally they cannot be used with Isobars

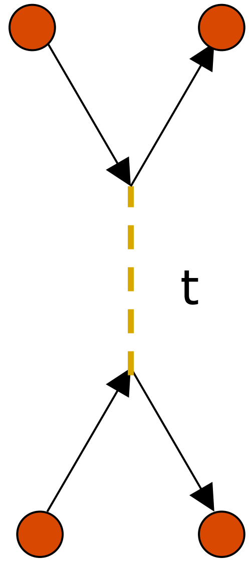
But the interaction is among two particles

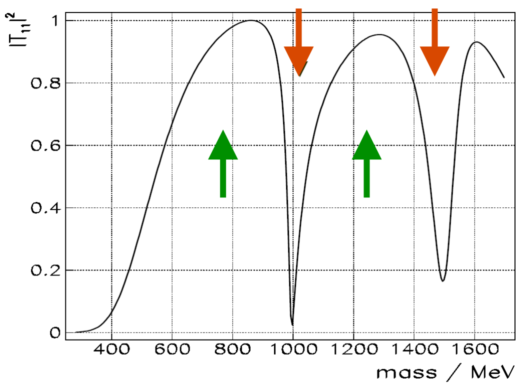
To save the Isobar Ansatz (workaround)

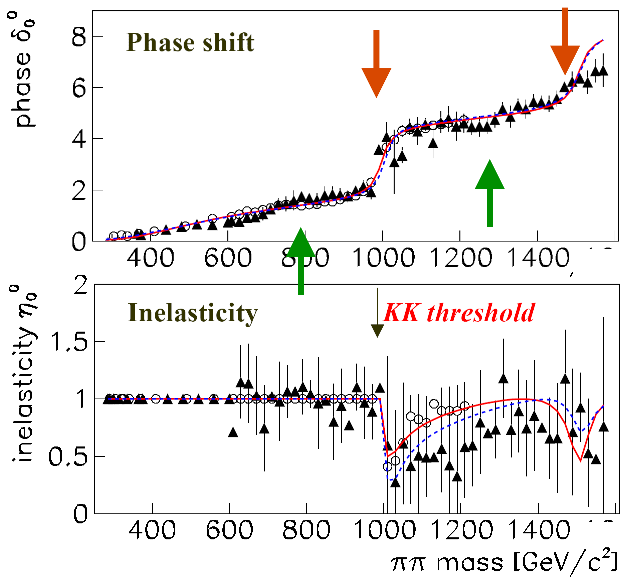
they may appear as unphysical poles in *K*-Matrices

or as polynomial of s in *K*-Matrices

background terms in unitary form



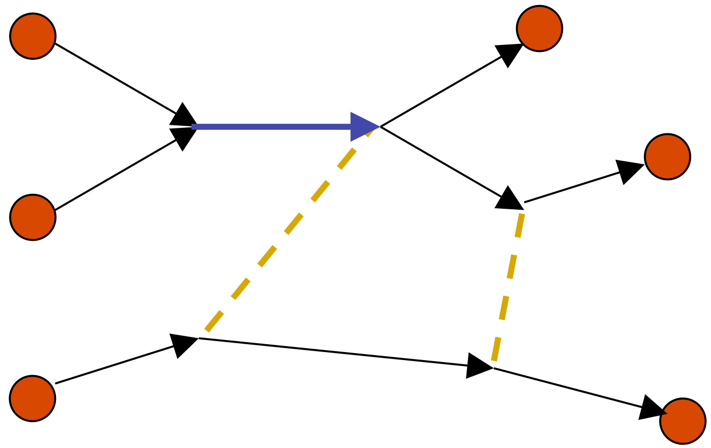


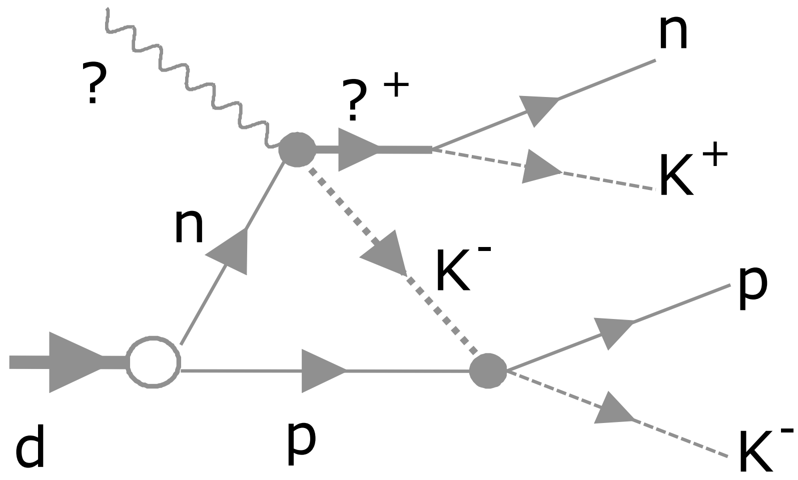


Rescattering

No general solution

Specific models needed





Problems of the method are

- performance (complex matrix-inversions!)

- numerical instabilities
  - singularities

- unitarity constraints
  - for *P*-Vectors

- cut structure
  - behavior at left- and right-hand cuts

Problems of the method are

unmeasured channels

yield huge problems if numerous or dominant

systematic errors of the experiment

relative efficiency, shift in mass, different resolutions

damping factors (sizes) for respective objects

Problems in terms of interpretation are

mapping *K*-Matrix to *T*-Matrix poles

number might be different

branching ratios

K-matrix strength is unequal T-matrix coupling

Problems in terms of interpretation are

validity of *P*-vectors

all channels need to have identical production processes

FSI has to be dominant

singularities

not all are resonances limit of the isobar model

Summary

K-Matrix is a good tool

if one obeys a few rules

ideally one would like to use an unbiased parameterization which fulfills everything

use the best you can for your case and document well, what you have done