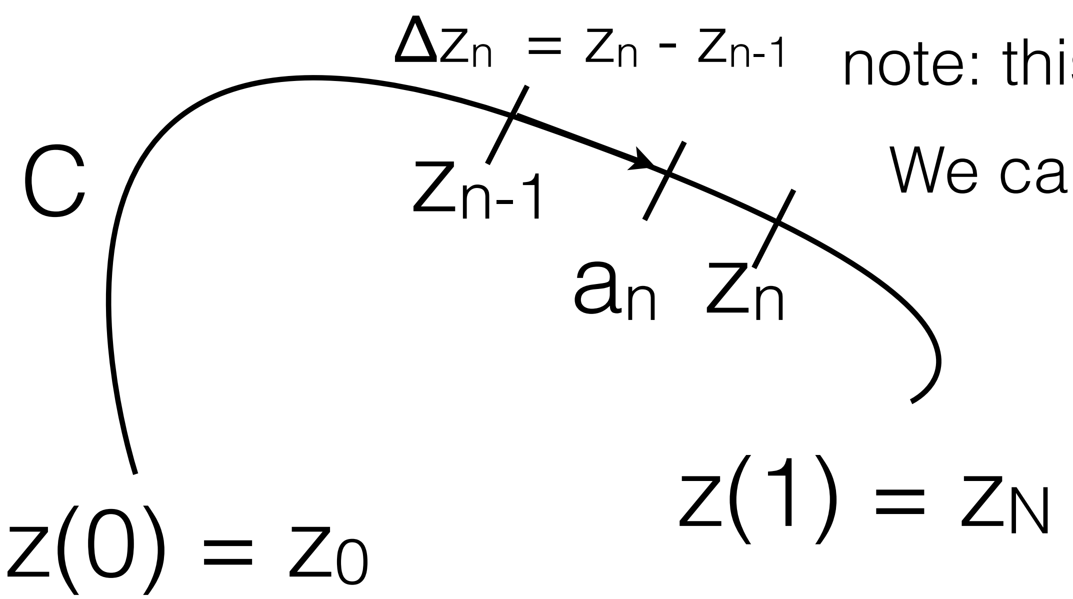
Complex calculus:
Complex integrals

Real calculus $\int_a^b dx\, f(x)$

Complex calculus $\int_C dz\, f(z)$ C = curve in z-plane

Line integrals: given a curve C in the complex plane parametrized by a real number $0 \leq t \leq 1$, $t \rightarrow z(t) = x(t) + iy(t)$ the integral of f over C is defined by

$$\int_C f(z)dz = \int_{t=0}^{1} f(z(t))\frac{dz}{dt}dt = \lim_{|\Delta z_n|\to 0, N\to\infty} \sum_{n=1}^{N} f(a_n)\Delta z_n$$



note: this is an ordered path

We can estimate the integral: if $|f(z)| \leq M > 0$ along C then

$$|\int_C f(z)dz| \leq Ms$$

where s it the length of the path

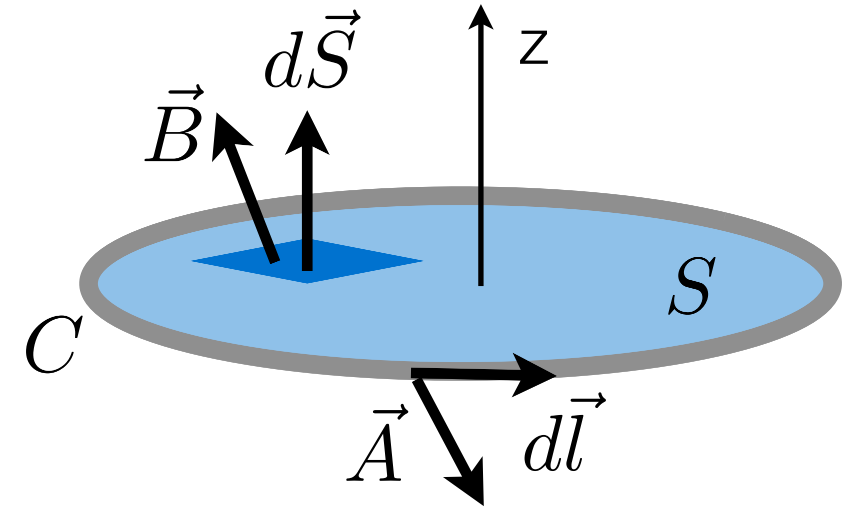
Cauchy-Goursat theorem: If f(z) is holomorphic in some region G and C is a closed contour (consisting of continuous or discontinuous cycles, double cycles, etc.) then

$$\oint f(z)dz = 0$$

(converse is also true)

Proof: according to Stoke's theorem

$$\int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{S} = \oint_C \vec{A} \cdot d\vec{l}$$



(e.g. Magnetic flux $\vec{B} \equiv \vec{\nabla} \times \vec{A}$ over open surfaces = circulation of vector potential over its boundary)

$$\int_S \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dxdy = \oint (A_x dx + A_y dy)$$

(Cauchy relation for u,v)

use: $A_y = u(x,y)$, $A_x = v(x,y)$ then $\frac{\partial A_y}{\partial x} = \frac{\partial A_x}{\partial y}$ and l.h.s=0

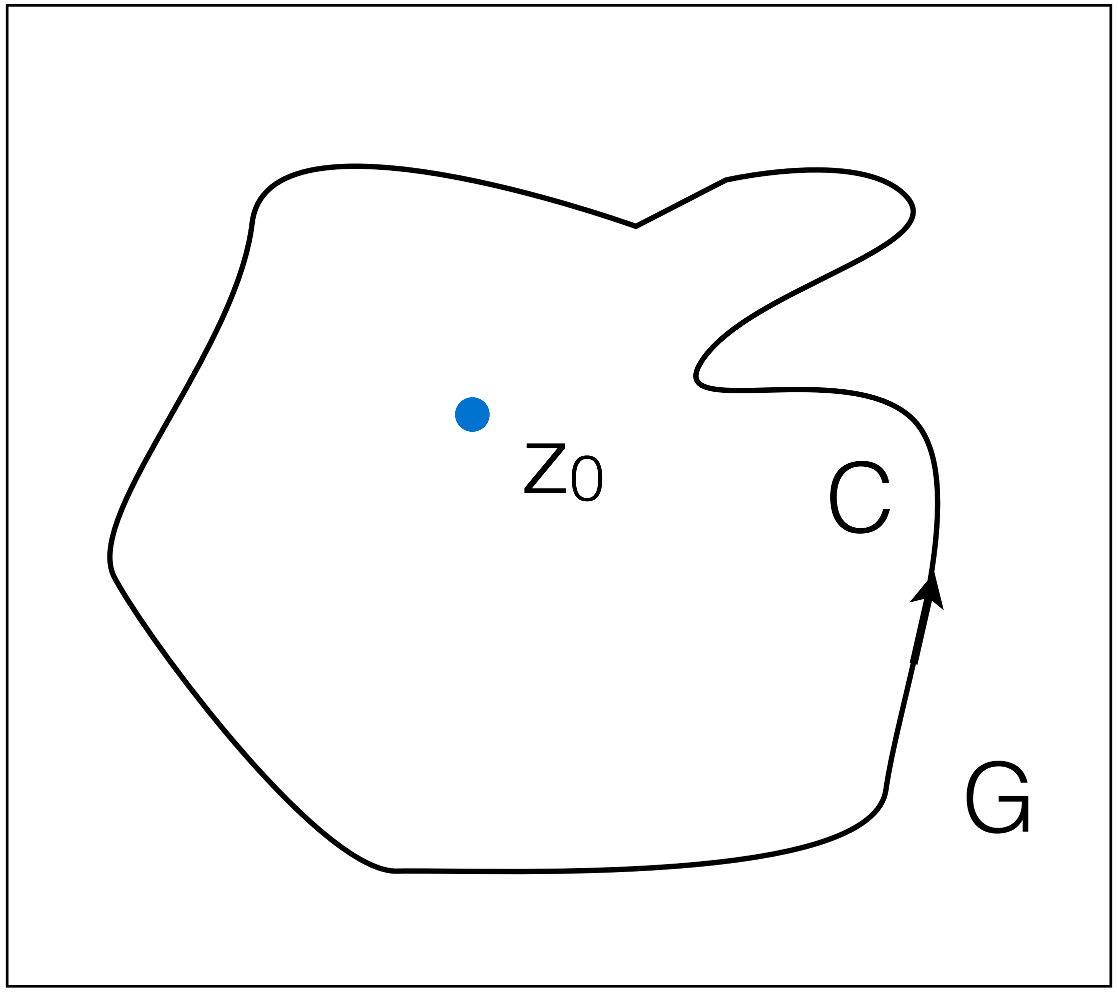
$$\oint (vdx + udy) = 0$$

use: $A_y = v(x,y)$, $A_x = -u(x,y)$ then $\frac{\partial A_y}{\partial x} = \frac{\partial A_x}{\partial y}$ and l.h.s=0

$$\oint (-udx + vdy) = 0$$

$$\oint f(z)dz = \oint [u + iv][dx + idy] = \oint [udx - vdy] + i \oint [vdx + udy] = 0$$

The Cauchy integral formula: if f(z) holomorphic in G, $z_0 \in G$, and C a closed curve (cycle), which goes around z_0 once in positive (counterclockwise) direction, then

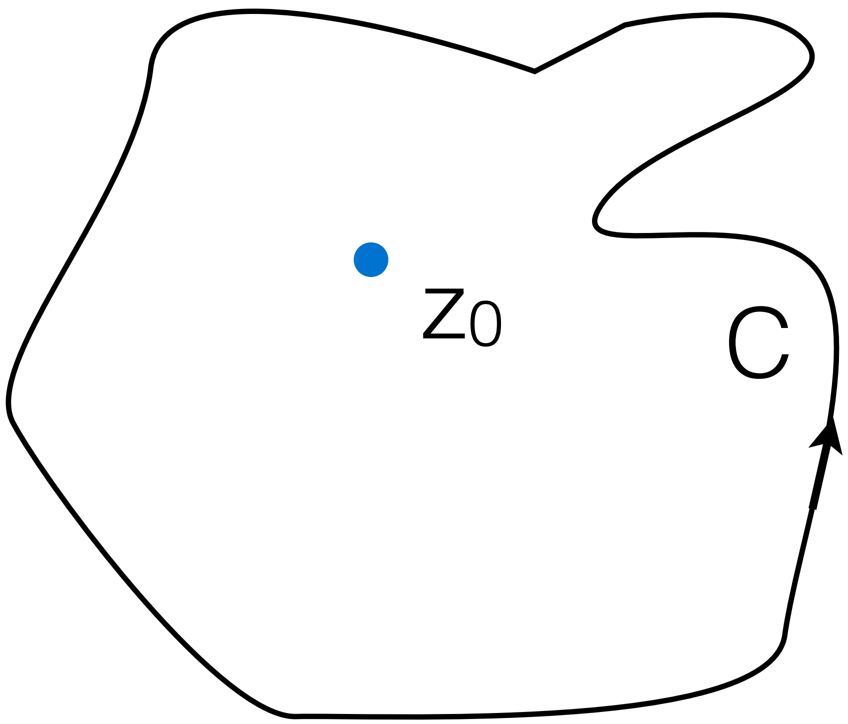


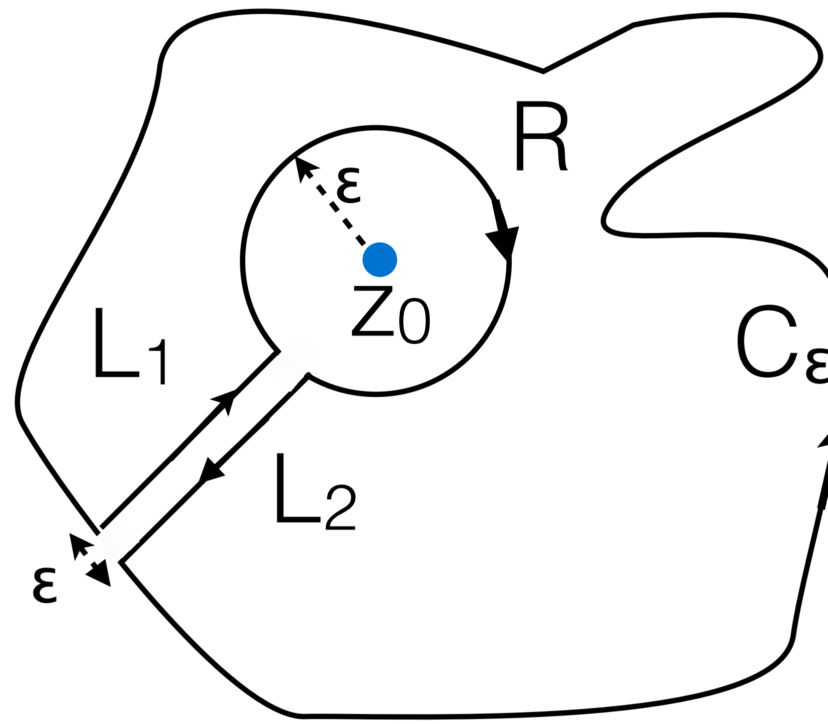
$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{z - z_0}$$

The Cauchy formula solves a boundary-value problem. The values of the function on C determine its value in the interior. There is no analogy in the theory of real functions. It is related though to the uniqueness of the Dirichlet boundary-value problem for harmonic functions (in 2dim)

Proof:

$\lim_{\varepsilon \to 0} C_\varepsilon = C \quad \lim_{\varepsilon \to 0} L_1 = -L_2$





$$\oint_{C'} \frac{f(z)dz}{z-z_0} = 0$$

$C' = C_\varepsilon + L_1 + L_2 + R$

$$0 = \oint_{C'} = \lim_{\epsilon \to 0} \left[\int_{L_1} + \int_{L_2} + \int_R + \int_{C_\epsilon} \right] = \lim_{\epsilon \to 0} \int_R + \int_{C_\epsilon}$$

$$\int_R \frac{f(z)dz}{z-z_0} = f(z_0) \int_R \frac{dz}{z-z_0} + \int_R \frac{f(z)-f(z_0)}{z-z_0} dz$$

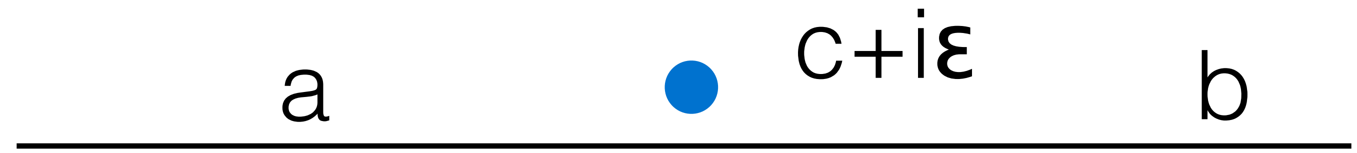
$\varepsilon \to 0$: $\quad -2\pi i \qquad O(\epsilon) \to 0$

$$z - z_0 = \epsilon e^{i\phi}$$

$$-2\pi i f(z_0) + \int_C = 0$$

(very) useful formula

$$I = \int_a^b dx \frac{f(x)}{x - c - i\epsilon}$$



$$\frac{1}{x - c - i\epsilon} = \frac{x - c + i\epsilon}{(x-c)^2 + \epsilon^2}$$

$$\frac{1}{x - c - i\epsilon} = \frac{x - c + i\epsilon}{(x-c)^2 + \epsilon^2} = P.V. \frac{1}{x - c} + \frac{i\epsilon}{(x-c)^2 + \epsilon^2}$$

$$I = P.V. I + i\pi f(c)$$

Examples

Derivatives: f(z)g(z)

of elementary functions (may) have singularities

Integrals:

$$\int_\gamma dz \qquad \int_\gamma z^n dz$$

$$\int_\gamma \frac{dz}{z} \qquad \int_{\gamma'} \frac{dz}{z}$$

γ = unit circle

γ' = unit square

$$\int_\gamma \frac{dz}{z^2}$$

Series Expansion:

Series expansion approximates the function near a point.

Complex functions are determined by their singularities and series expansion will also “probe” their singularity structure.

Holomorphic functions are “very smooth”, e.g. existence of 1st derivative implies existence of infinite number of derivatives. This is not true for real functions, e.g.

$$f(x) = \begin{cases} x^2 \text{ for } x \geq 0 \\ -x^2 \text{ for } x < 0 \end{cases} \qquad f'(x) = 2|x|$$

so f’(0) = 0 but f’’(0) does not exist

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} \text{ for } x \neq 0 \\ 0 \text{ for } x = 0 \end{cases}$$

all derivative vanish at x=0, $f^{(k)}(0) = 0$, and the resulting (trivial) Taylor series does not reproduce the function

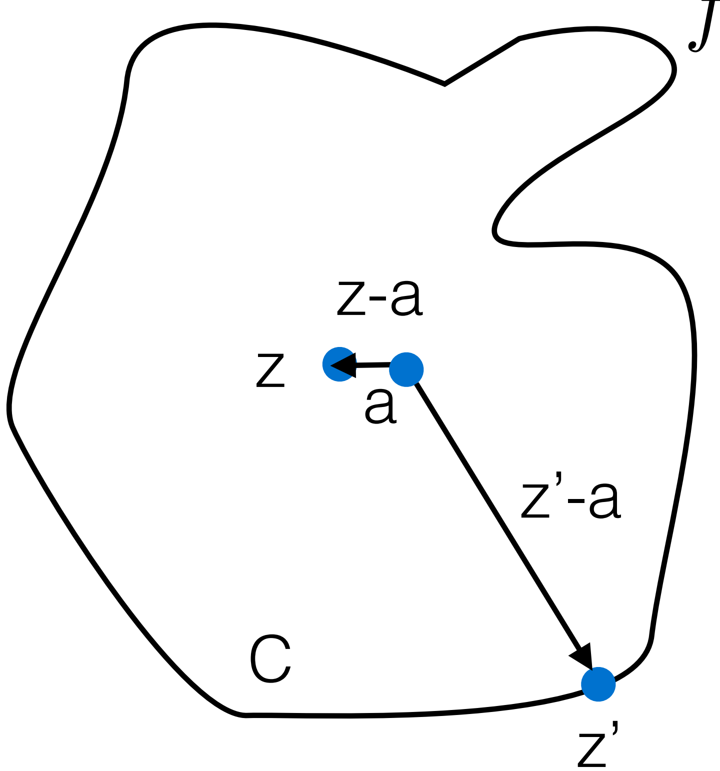
Hadamard’s formula: The sum of powers $\sum a_n z^n$ defines a holomorphic function inside the circle of convergence R given by

$$\frac{1}{R} = \overline{\lim}_{n\to\infty}|a_n|^{1/n}$$

If f(z) is holomorphic in G, a ∊ G and C is a cycle:

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(z')dz'}{z'-z} = \frac{1}{2\pi i}\oint \frac{f(z')}{z'-a}\frac{1}{1-\frac{z-a}{z'-a}}dz'$$

for |z'-a| > |z-a| we have:



$$f(z) = \frac{1}{2\pi i}\oint \frac{f(z')}{z'-a}\left[1+\frac{z-a}{z'-a}+\frac{(z-a)^2}{(z'-a)^2}+\cdots\right]$$

or integrating each term :

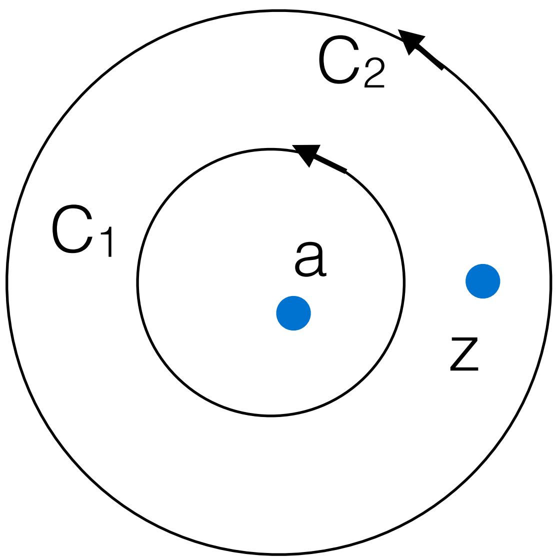
$$f(z) = f(a) + f'(a)(z-a) + \frac{1}{2!}f''(a)(z-a)^2 + \cdots$$

this is the Taylor series

If f(z) is holomorphic between two circles C_1 and C_2 and z is a point inside the ring, and a is a point inside the small circle C_1 then

$$f(z) = \frac{1}{2\pi i}\left(\oint_{C_1} \frac{f(z')dz'}{z'-z} - \oint_{C_2} \frac{f(z')dz'}{z'-z}\right)$$

the expansions are convergent on C_2 and C_1 respectively



$$\frac{1}{z'-z} = \frac{1}{z'-a}\left[1 + \frac{z-a}{z'-a} + \cdots\right] \quad \text{on } C_2$$

$$= -\frac{1}{z-a}\left[1 + \frac{z'-a}{z-a} + \cdots\right] \quad \text{on } C_1$$

we have:

$$f(z) = \sum_{\nu=-\infty}^{\infty} A_\nu (z-a)^\nu = \cdots \frac{A_{-2}}{(z-a)^2} + \frac{A_{-1}}{z-a} + A_0 + A_1(z-a) + A_2(z-a)^2 + \cdots$$

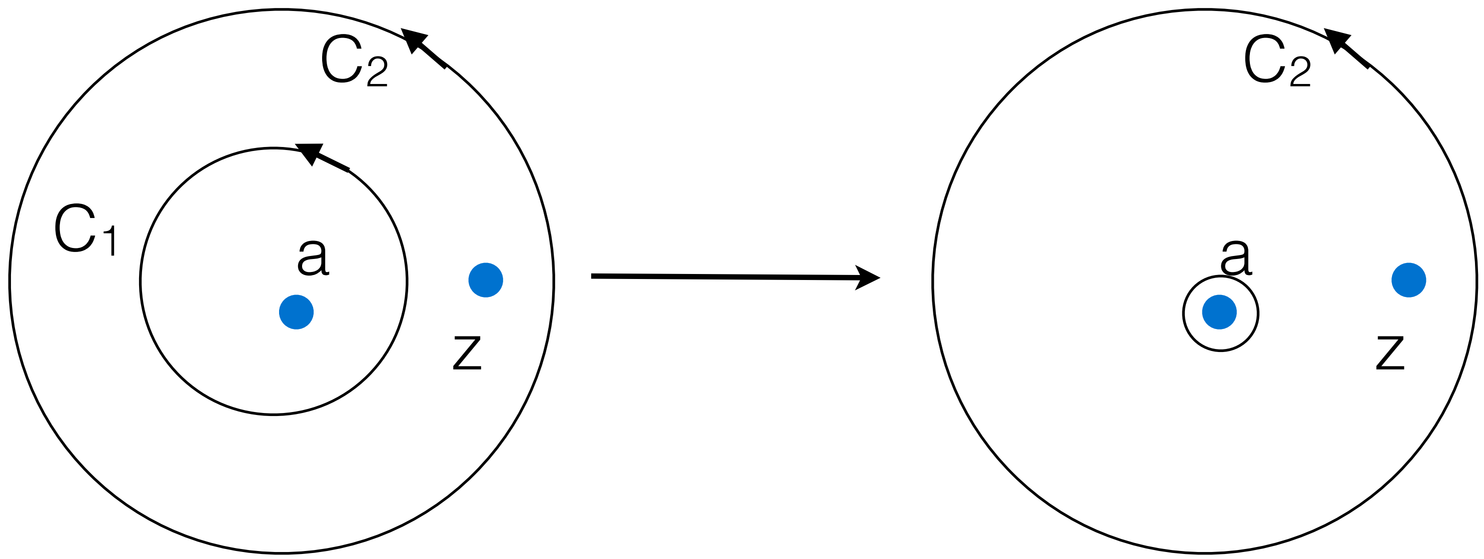
$$A_\nu = \frac{1}{2\pi i}\oint_{C_2} \frac{f(z')dz'}{(z'-a)^{\nu+1}} \qquad \nu \geq 0 \qquad A_n = \frac{f^{(n)}(a)}{n!}, \ (n = \nu \geq 0)$$

$$A_\nu = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')dz'}{(z'-a)^{\nu+1}} \qquad \nu < 0$$

This is Laurent series

Classification of singularities

Assume radius of C_1 is 0, i.e. f(z) is holomorphic in C_2 -{a} called "deleted neighborhood" of a



$$f(z) = \sum_{\nu=-\infty}^{\infty} A_\nu (z-a)^\nu = \frac{A_{-m}}{(z-a)^m} + \frac{A_{-m+1}}{(z-a)^{m-1}} + \cdots \sum_{n=0} A_n (z-a)^n$$

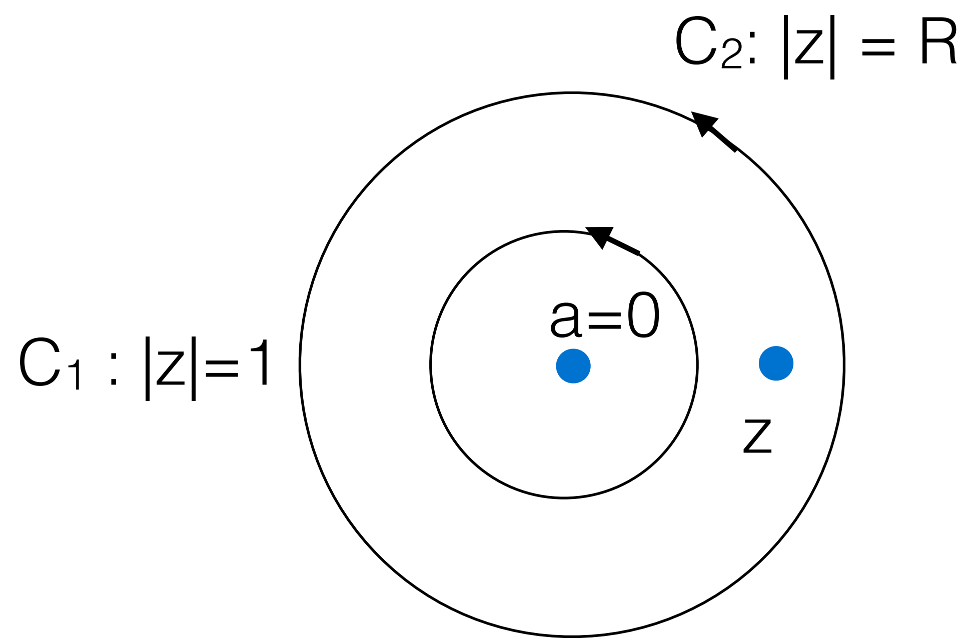
point a is called a pole of order m, if m=∞ it is called an essential singularity, if m=1 it is called a simple pole (or just a pole). A_{-1} plays a special role since

$$2\pi i A_{-1} = \oint dz\, f(z)$$

A_{-1} is called the residue.

Examples:

$$f(z) = \frac{1}{z(z-1)}$$



since f(z) is holomorphic for |z| > 1,R can be chosen as large as one pleases. This implies A_n must be 0 for all n > 0 (otherwise $\sum A_n z^n$ would diverge for large |z| = R, contrary to being holomorphic)

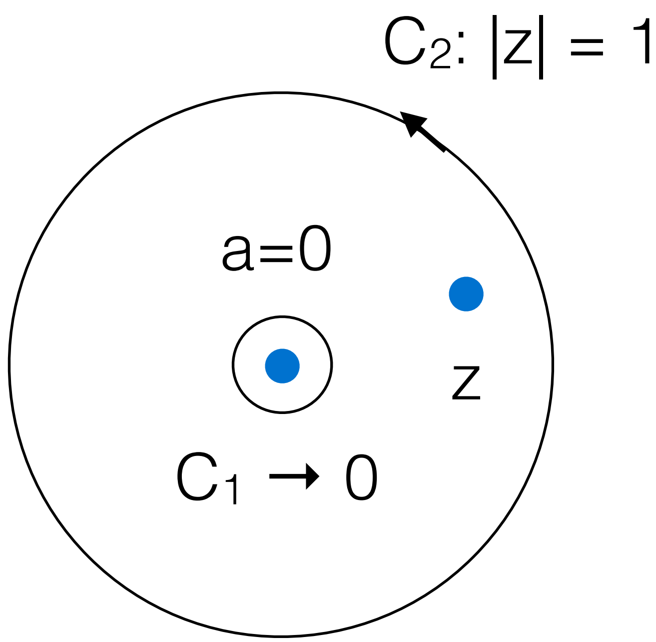
For |z| > 1 Laurent series is

$$f(z) = \frac{1}{z(z-1)} = \frac{1}{z}\left[\frac{1}{z}\frac{1}{1-\frac{1}{z}}\right] = \frac{1}{z^2} + \frac{1}{z^3}\cdots$$

a=0 is NOT essential singularity because G is not a "deleted neighborhood" (radius of C_1 is finite)

Example:

$$f(z) = \frac{1}{z(z-1)}$$



For 0< |z| < 1 G = "deleted neighborhood" of a=0 and the Laurent series is

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z}\left(1 + z + z^2 + \cdots\right) = -\frac{1}{z} - 1 - z - z^2 \cdots$$

this shows (as expected) that a=0 is a simple pole with residue A_{-1} = -1

Application of

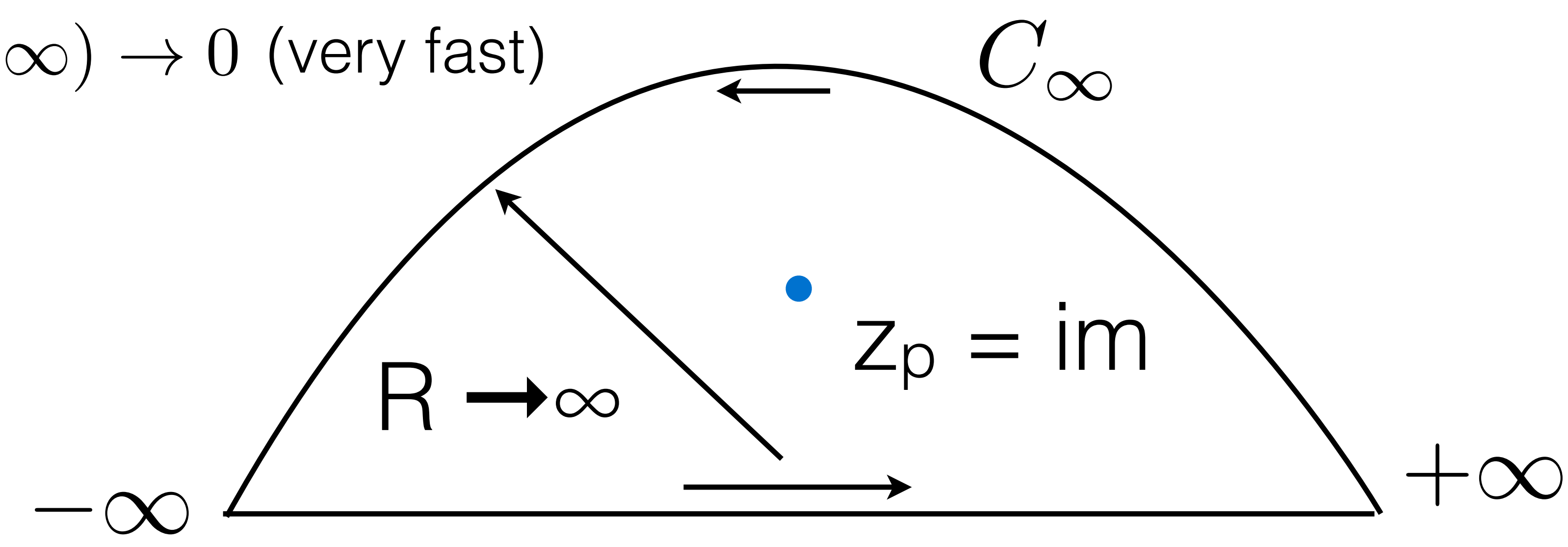
$$2 \pi i A_{-1} = \oint dz\, f(z)$$

This is likely the most common used consequence of complex calculus, since it can be also applied to compute real integrals

Suppose you want to compute $\int_{-\infty}^{\infty} dp \frac{e^{irp}}{p^2+m^2}$ with m,r > 0

consider an integral of f(z) over a contour C $f(z) = \frac{e^{irz}}{z^2+m^2}$

$f(z = Re^{i\phi} \text{ with } R \to \infty) \to 0$ (very fast)



$$\oint dz f(z) = \int_{-\infty}^{\infty} dx f(x) + \int_{C_\infty} f(z) dz \to \int_{-\infty}^{\infty} dx f(x)$$

$$f(z \sim z_p) = \frac{e^{irz}}{z^2+m^2} = \frac{e^{irz}}{(z+im)(z-im)} \sim \frac{e^{-rm}}{2im} \frac{1}{z-z_p}$$

$= A_{-1}$

$$\int_{-\infty}^{\infty} dx \frac{e^{irx}}{x^2+m^2} = \frac{\pi}{m} e^{-rm}$$

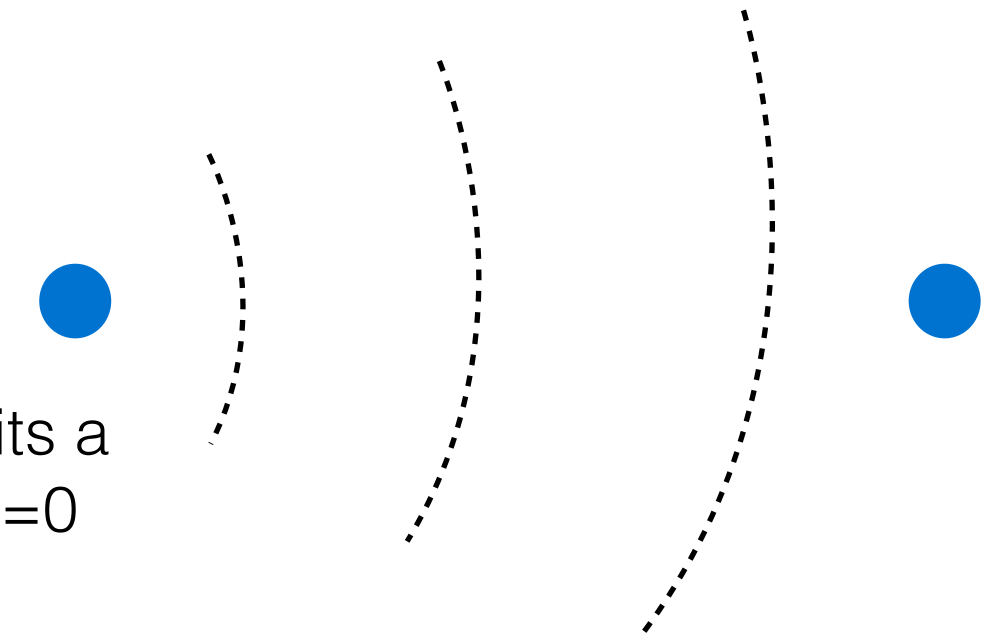
Which branch cut to use

Examples to consider

$$\int_{-1}^{1} dx \frac{1}{\sqrt{1-x^2}}$$

$$\int_{1}^{\infty} dx \frac{1}{x\sqrt{x^2-1}}$$

Dispersion relations



source emits a signal at t=0

causality: receiver receives at t>0 and not at t<0

amplitude of the signal

$$f(t) \propto \theta(t)$$

consider the Fourier transform (E → energy)

$$f(E) \equiv \int dt e^{iEt} f(t)$$

and extend definition to complex plane E → z, then f(z) is holomorphic for Im E > 0

The idea is to determine all singularities of f(E). Once this is done one can reconstruct f(E) outside the region of singularities.

Suppose f(E) was also analytical for Im E ≤ 0 and f(∞) → 0

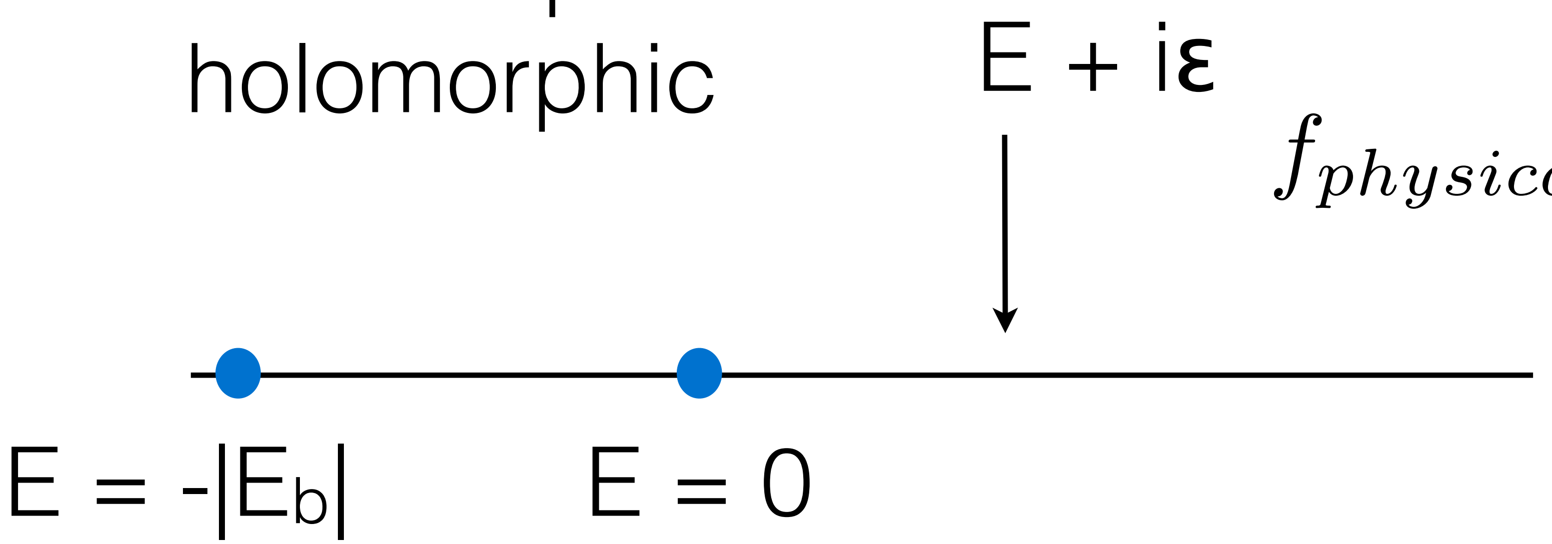
Then f(E) = constant !

($f(E) = \Sigma_n f_n E^n$ and infinite radius of convergence implies $f_1, f_2, .. = 0$)

Singularities of f(E) in the complex E-plane

Im E > 0 amplitudes holomorphic

$$f_{physical}(E) = \lim_{\epsilon \to 0} f(E + i\epsilon)$$



Im E = 0, Re E < 0 amplitudes have singularities (bound states = poles)

no scattering for Re E < 0, at E=0 change in physics → branch point

Reconstruction of amplitudes from its singularities : dispersion relations

Example (1)

$$f(E) = \frac{a_1}{E - E_1} + \frac{a_2}{E - E_2} + \sum_{n=0} b_n E^n$$

Need to specify behavior at ∞

1. $f(\infty) \rightarrow$ const $b_n = 0$, $n>0$

2. $f(\infty) \rightarrow 1/s$ $b_n = 0$

3. $f(\infty) \rightarrow 1/s^2$ $b_n = 0$, $a_1 = -a_2$

Reconstruction of amplitudes from its singularities : dispersion relations

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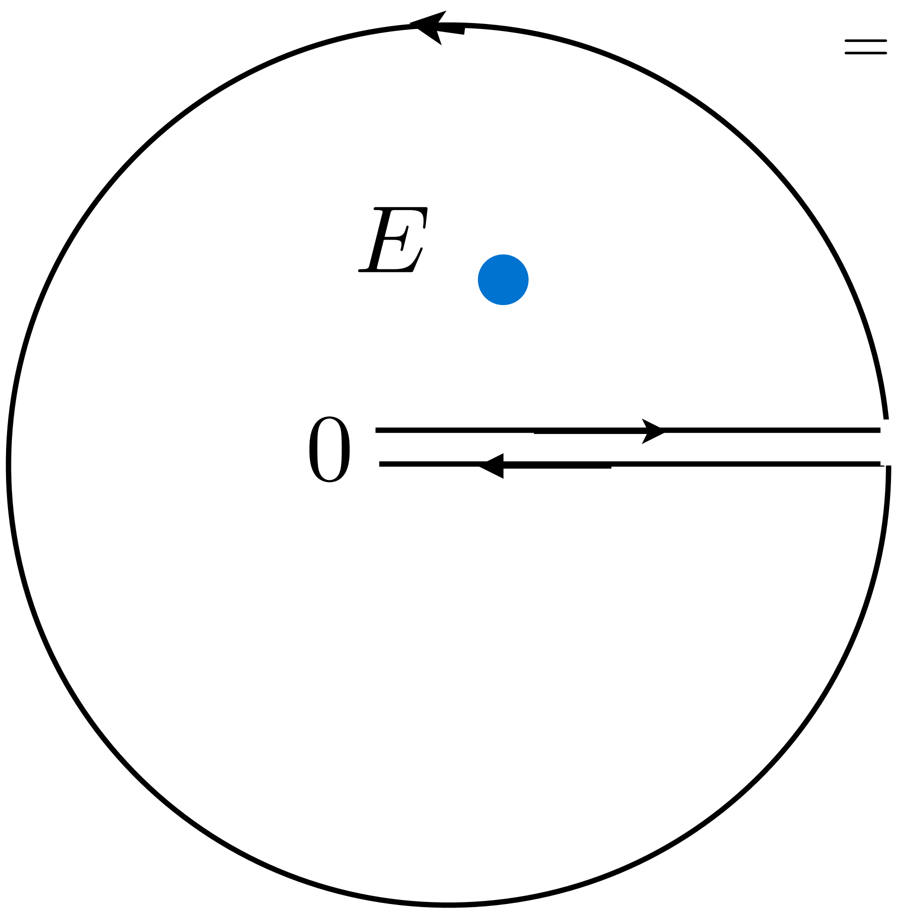
Example (2)

$$\text{Dis. } f(E) = f(E+i\epsilon) - f(E-i\epsilon) = 2i\sqrt{E} \text{ for } E > 0$$

in addition f(0)=1, and is analytical everywhere else what is f(E) ? Can f(∞) be a constant ?

$$f(E) = \frac{1}{2\pi i}\oint dz \frac{f(z)}{z-E}$$



$$= \frac{1}{2\pi i}\int_{-\infty}^{0} dE' \frac{f(E'-i\epsilon)}{E'-i\epsilon-E} + \int_{0}^{\infty} dE' \frac{f(E'+i\epsilon)}{E'+i\epsilon-E} + \int_R \cdots$$

$$= \frac{1}{2\pi i}\int_{0}^{\infty} dE' \frac{2i\sqrt{E'}}{E'-E} + \frac{1}{2\pi}\int_R d\phi f(Re^{i\phi})$$

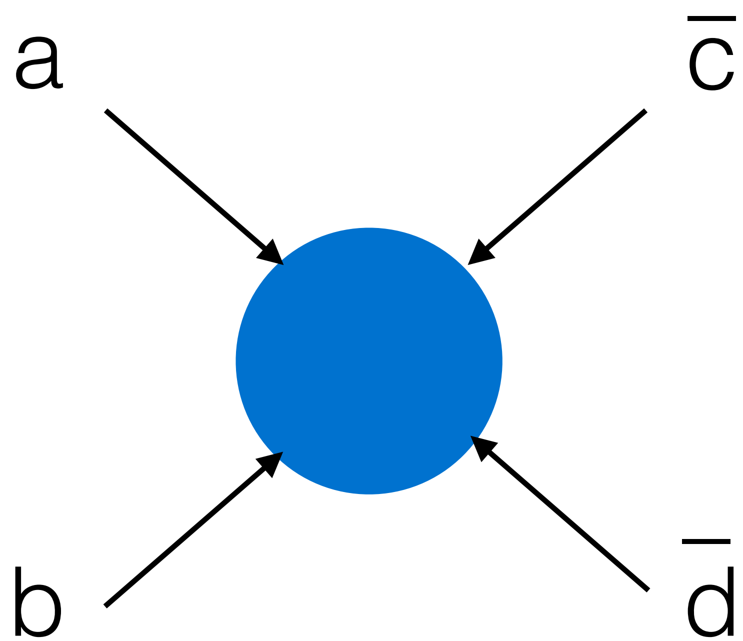
const.

$$f(E) = -\sqrt{-E} + 1$$

in scattering, Dis f(E) is related to observables (unitarity)

f(0) is “subtraction constant”: one trades the large-s’ behavior for small-s one

Relativistic scattering



$$s = (p_a + p_b)^2$$
$$t = (p_a + p_{\bar{c}})^2 = (p_a - p_c)^2$$
$$u = (p_a + p_{\bar{d}})^2 = (p_a - p_d)^2$$
$$s + t + u = \sum_i m_i^2 = 4m^2$$

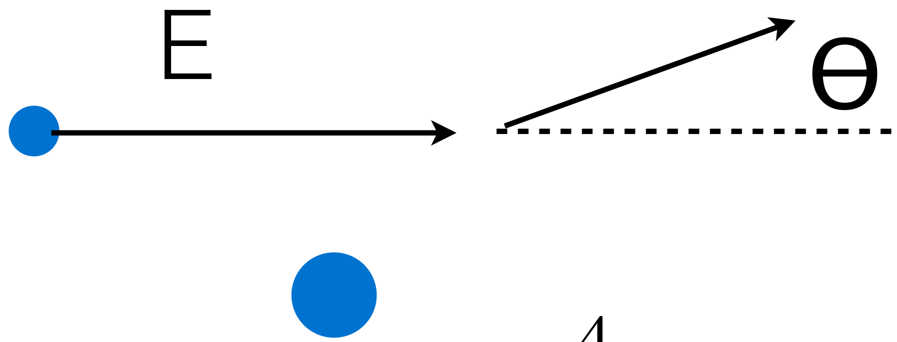
A(s,t) for $s > 4m^2$ $t < 0$ describes $a + b \rightarrow c + d$

A(s,t) for $s > 0$ $t < 4m^2$ describes $a + \bar{c} \rightarrow \bar{b} + d$

A(s,t) for $u > 0$ $s < 4m^2$ describes $a + \bar{d} \rightarrow \bar{b} + c$

In relativistic scattering

$$a + b \to c + d$$



$$s = M^2 + m^2 + 2EM$$

$$A(s, \cos\theta) = A(s, t)$$

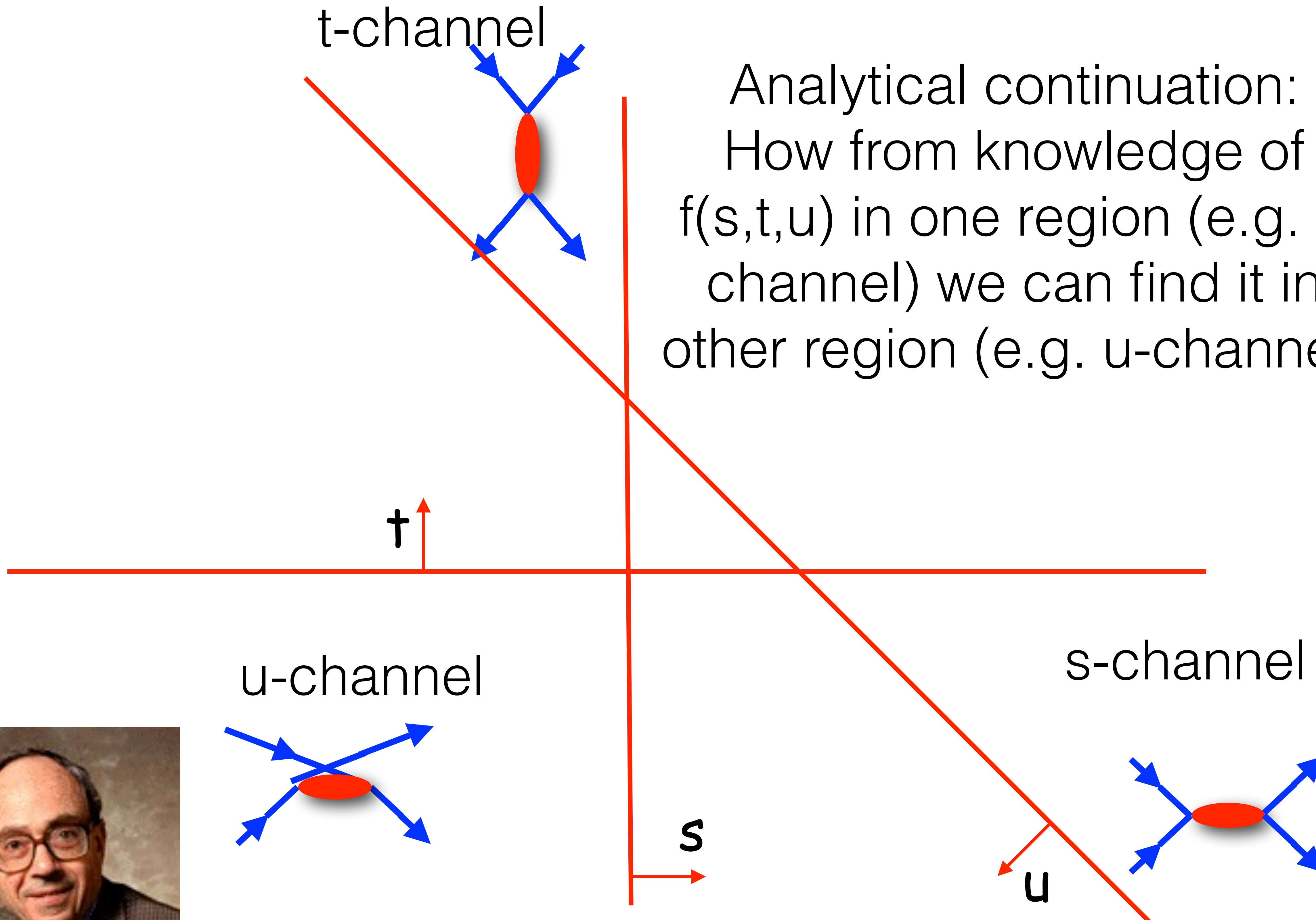
$$s + t + u = \sum_{i=1}^{4} m_i^2$$

In S-matrix theory it is assumed that a single complex function A(s,t,u) describes all reactions related by crossing

similarly to s for $a + b \to c + d$, the variable u is related to energy for the reaction $a + \bar{d} \to c + \bar{b}$

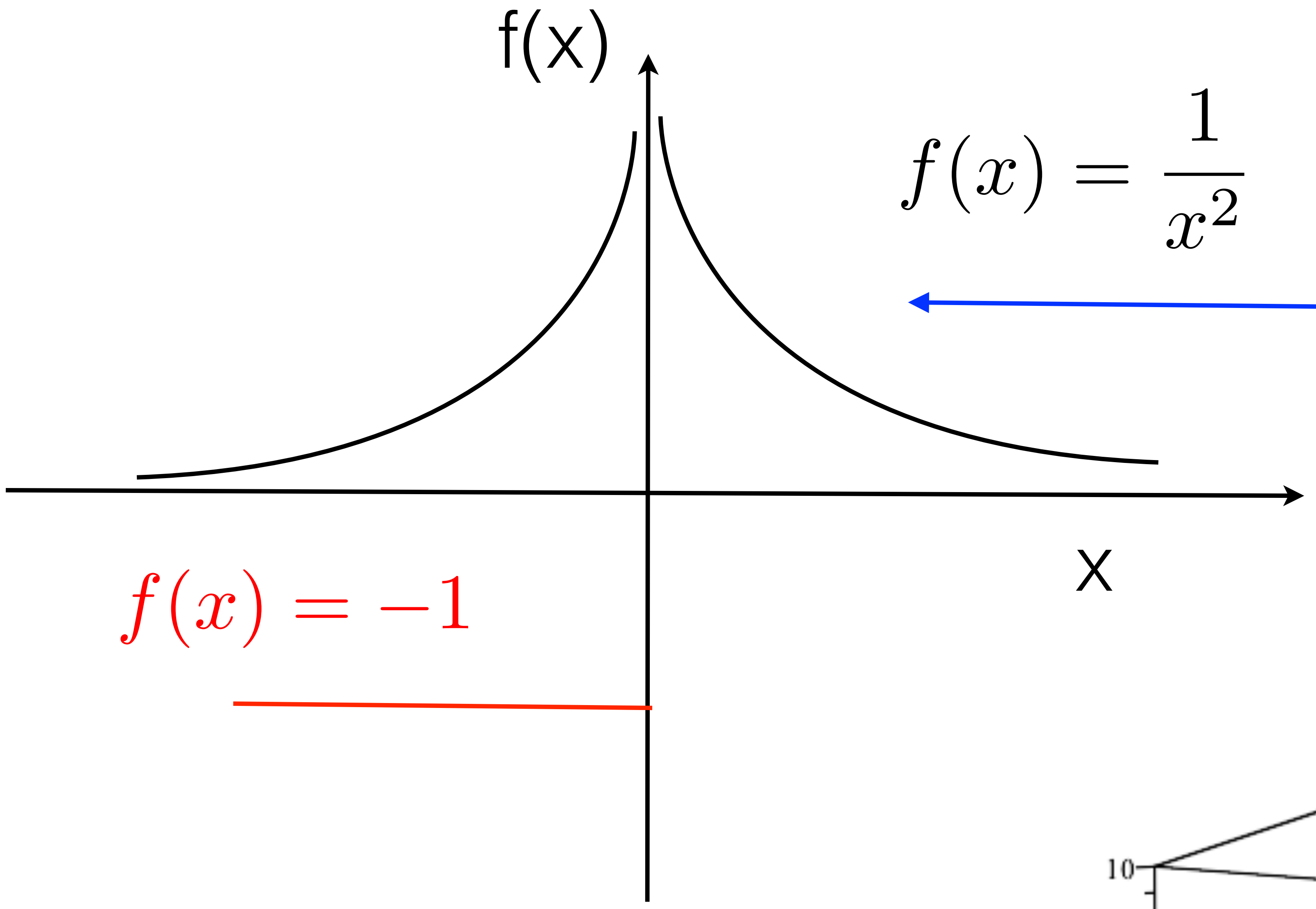
since u = $\sum m^2$ - s - t if A(u) is holomorphic for Im u > 0 it is also holomorphic in s for Im s < 0

Analytical continuation: How from knowledge of f(s,t,u) in one region (e.g. t-channel) we can find it in other region (e.g. u-channel)

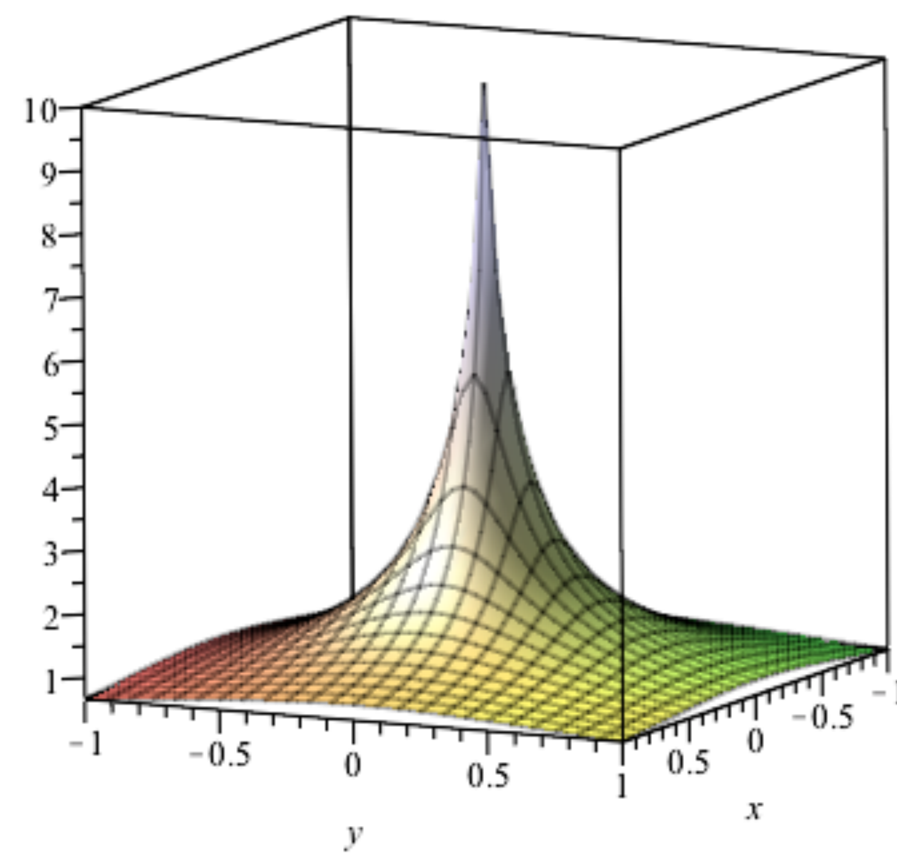


Analytical continuation

For real functions it does not work



but for complex functions you can go continuously around the z=0 singularity and *analytically continue* from one region to another



Theorem: If f(z) is holomorphic on G and f(z)=0 on an arc A in G, then f(z)=0 everywhere in G

Proof: f(z)=0 on A implies f'(z)=0 on A, because we can take the limit $\Delta z \to 0$ along the arc. Thus all derivatives vanish along A. Then by Taylor expansion around some point z_0 of A, $f(z) = \sum f^{(n)}(z_0)(z-z_0)^n/n! = 0$, for z inside some circle C. Now we take another arc A' along f(z)=0, etc. Continuing this process everywhere in G we prove the theorem.

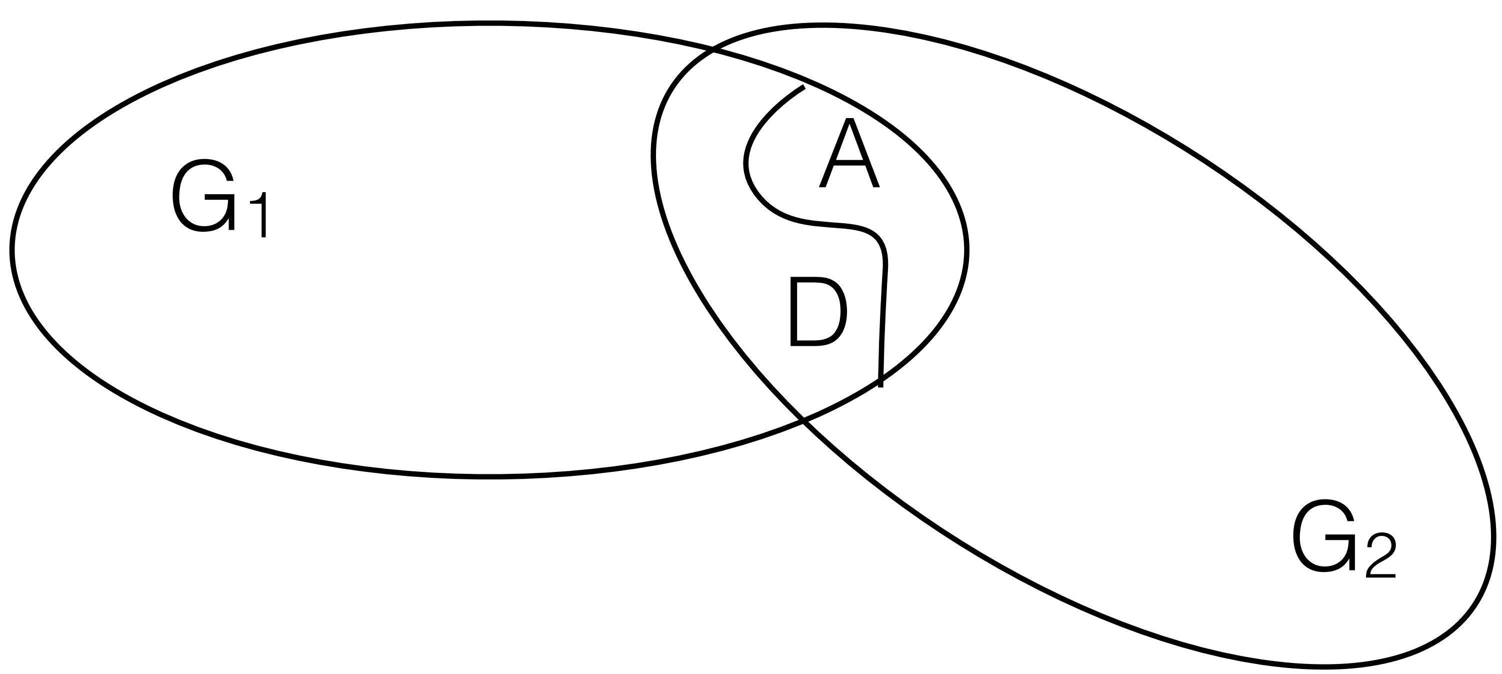
If f(z) is holomorphic on G then f(z) is uniquely defined by its values on an arc A in G.

Analytical continuation

Let $f_1(z)$ be holomorphic in G_1 and $f_2(z)$ in G_2, G_1 and G_2 intersect on an arch A (or domain D), and $f_1 = f_2$ on A (or D) then f_1 and f_2 are analytical continuation of each other and

$$f(z) = \begin{cases} f_1(z), z \in G_1 \\ f_2(z), z \in G_2 \end{cases}$$

is holomorphic in the union of G_1 and G_2



Examples:

$1 + z + z^2 + \cdots$ is holomorphic in |z|<1

$\int_0^\infty e^{-(1-z)t} dt$ is holomorphic in Re z < 0

$-(1 + 1/z + 1/z^2 + \cdots)$ is holomorphic in |z|>1

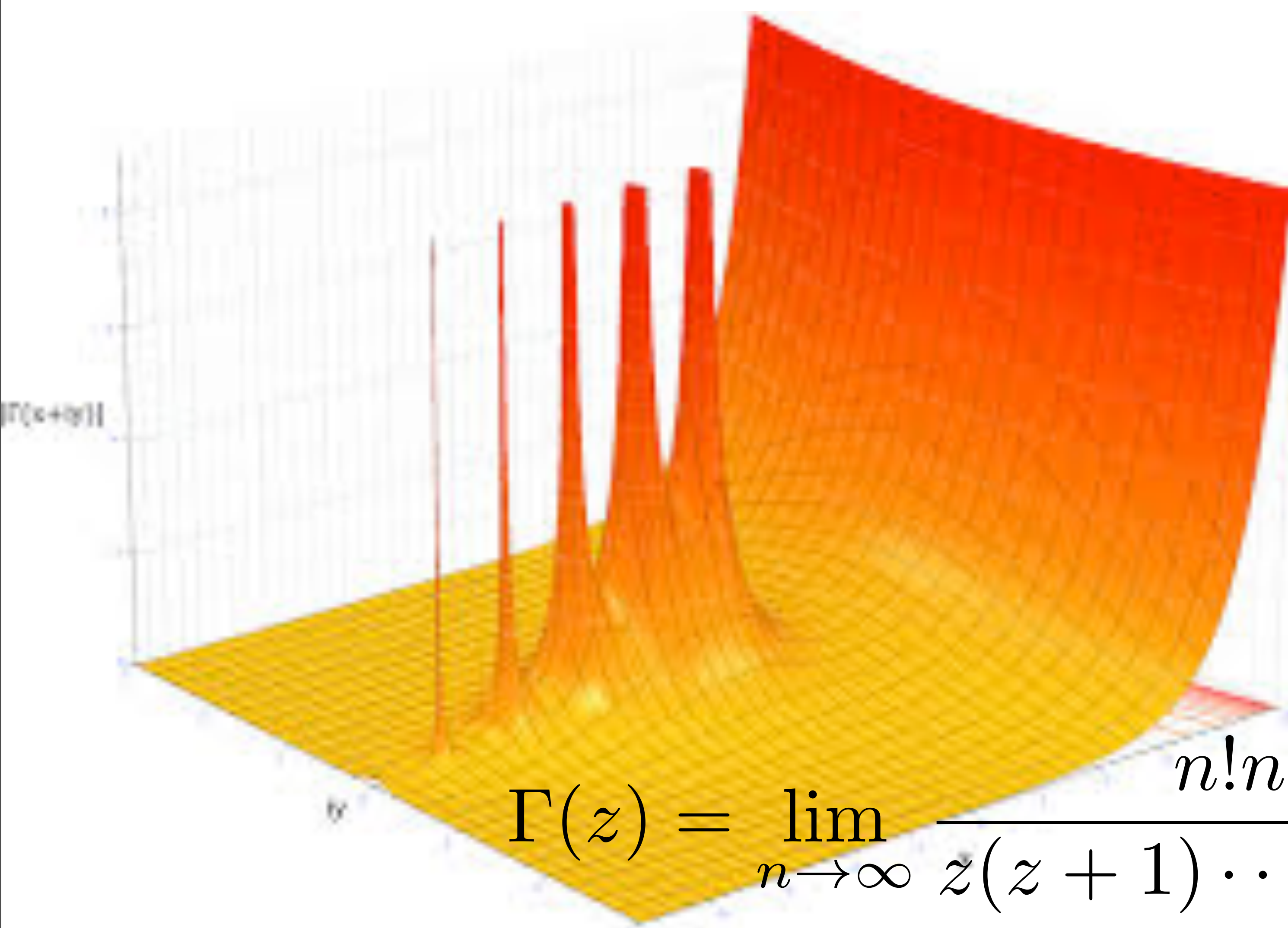
all these functions represent f(z) = 1/(1-z) in different domains, which is holomorphic everywhere except at the point z=1

Γ(z) function:

$\Gamma(z+1) = z\Gamma(z)$: generalization of factorial

$n! = n\,(n-1)!$ so $\Gamma(n) = (n-1)!$

$$\Gamma(z) = \int_0^\infty \frac{dt}{t} t^z e^{-t} \qquad \Gamma(0) \sim \log 0 \qquad \Gamma(-1) \sim \frac{1}{0} \qquad \Gamma(-n) \sim \frac{1}{0^n}$$



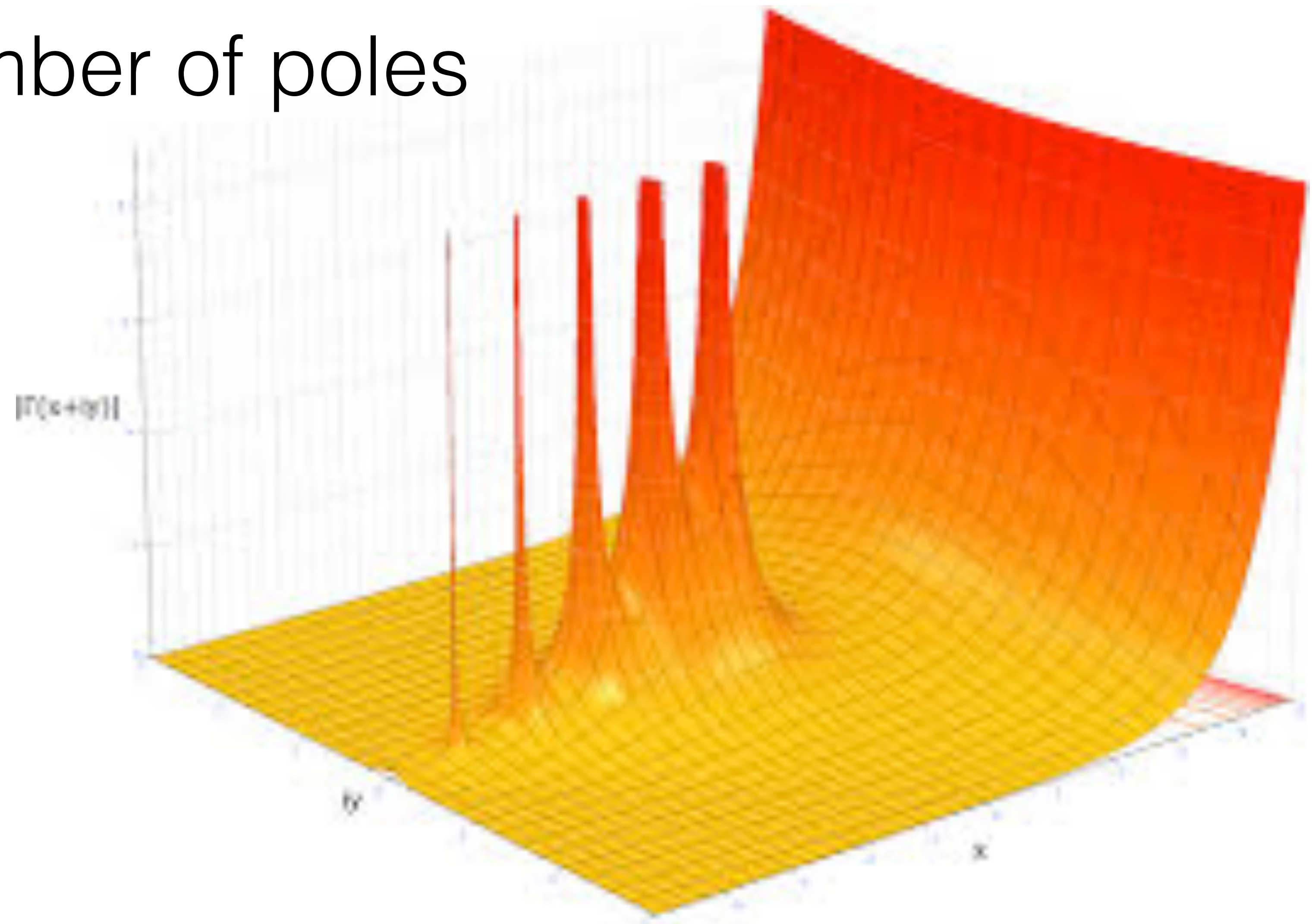
$$\Gamma(z) \sim \frac{(-1)^n}{n!} \frac{1}{z+n}$$

for z~-n

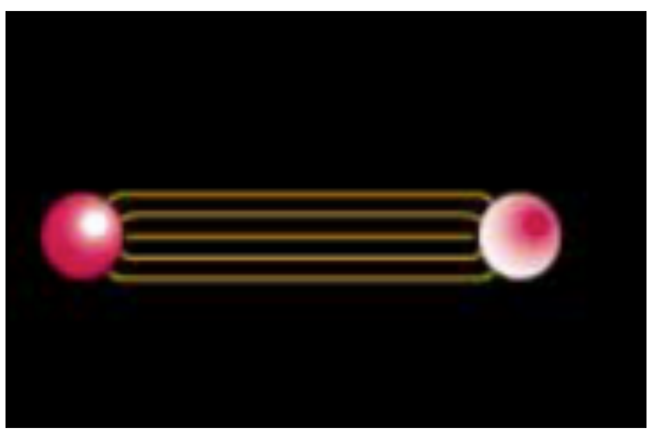
$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\cdots(z+n)}$$

Why would you ever care about the Γ function (?)

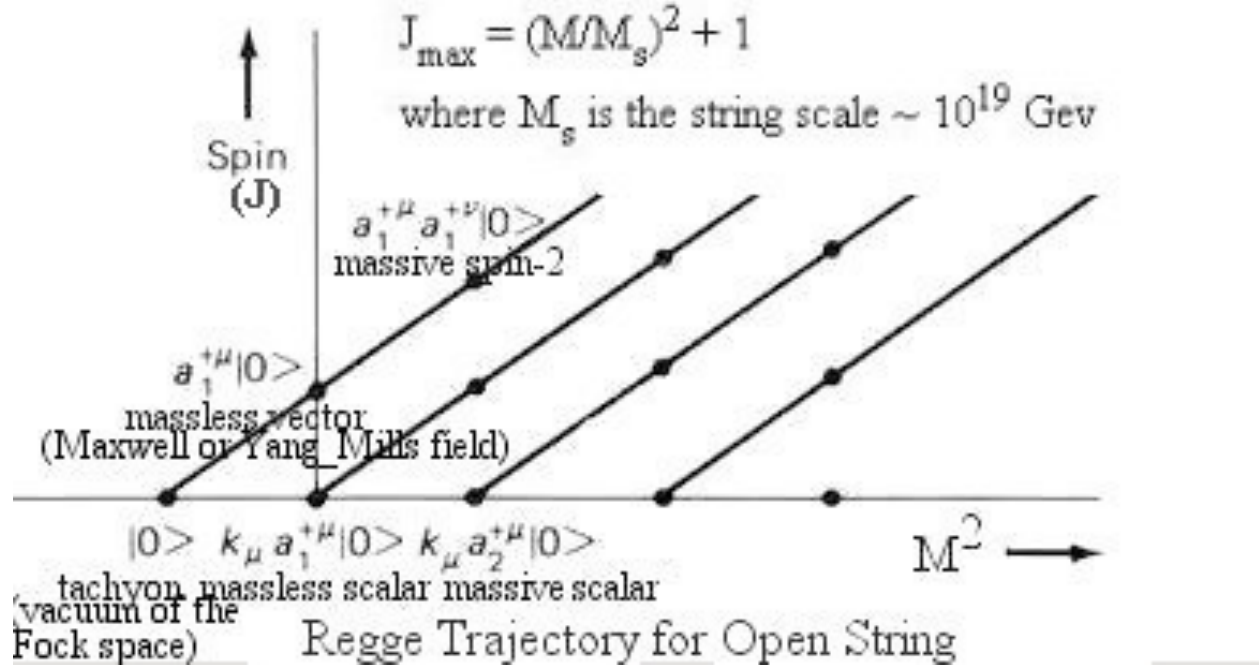
Infinite number of poles



If QCD were confined it would have ∞ of poles !



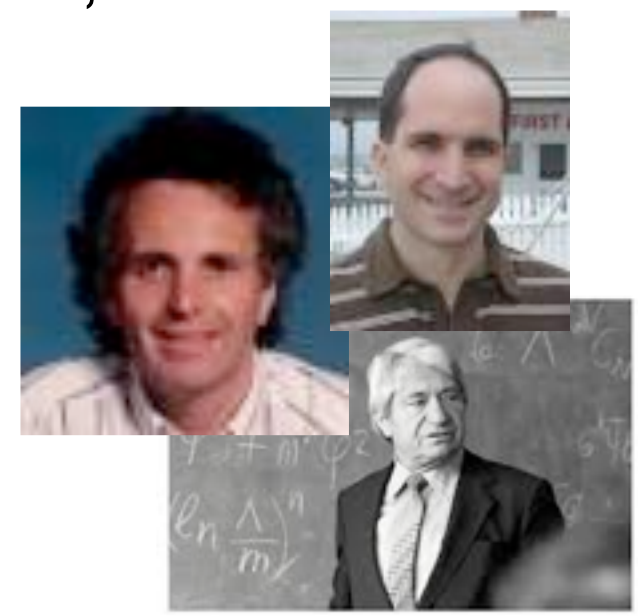
$$J(M^2) = \frac{1}{2\pi\sigma} M^2 = \alpha' M^2$$



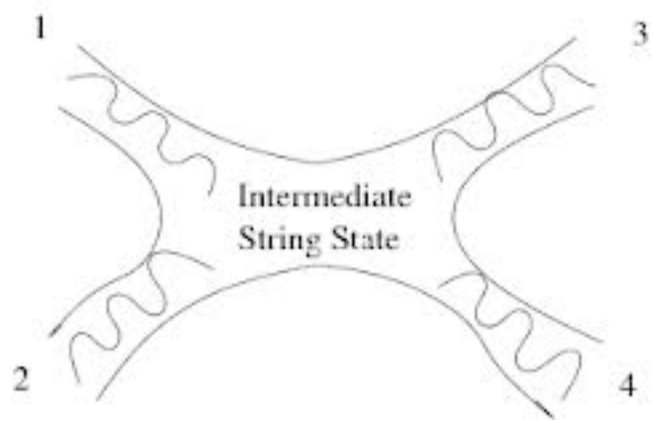
relativistic h.o.

string of relativistic oscillators

QCD, loop representation, large-N_c, AdS/CFT, ...



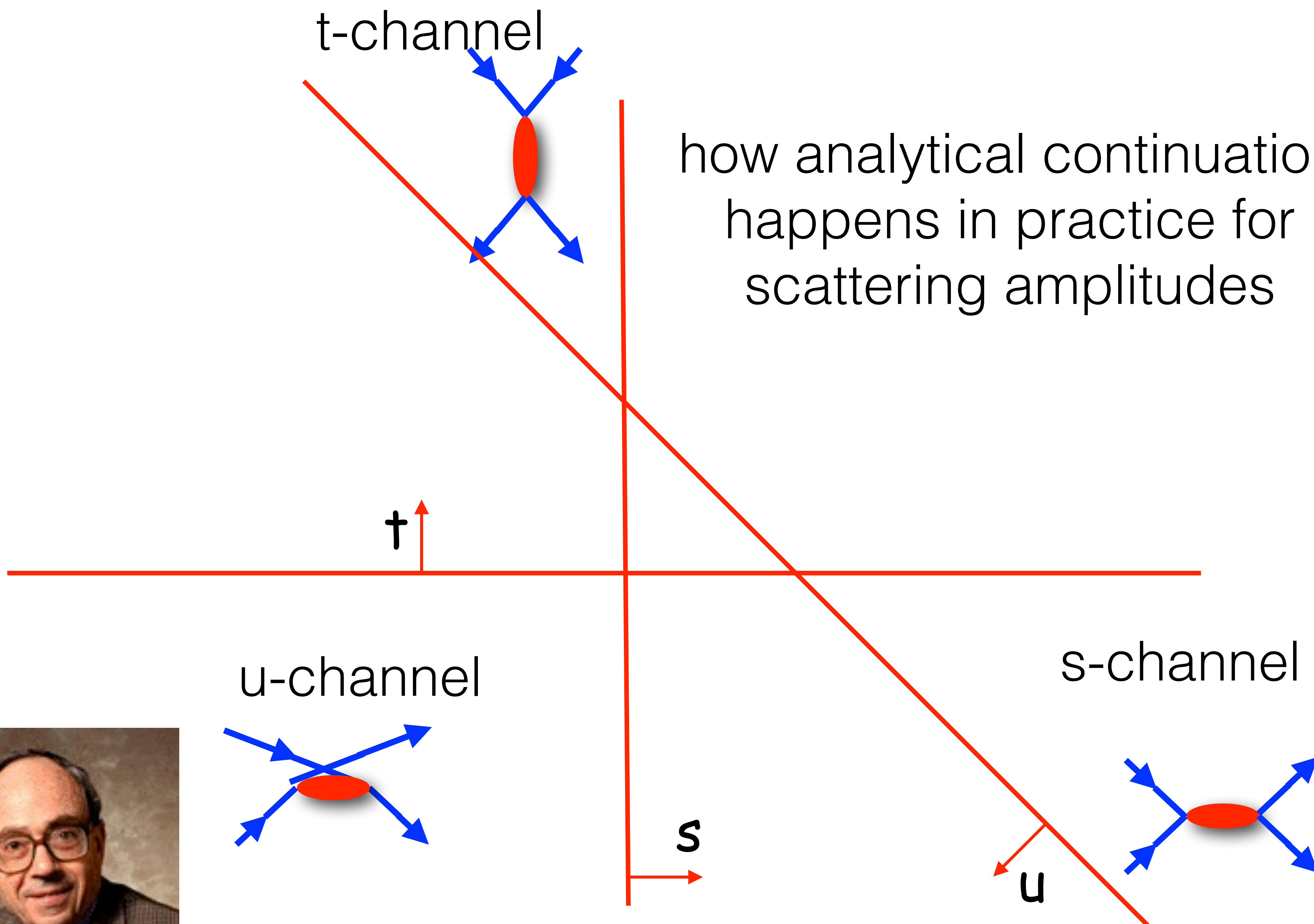
$\omega \to 3\pi$

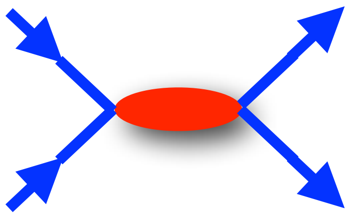


string revolution

$$A(s,t) = \frac{\Gamma(-J(s))\Gamma(-J(t))}{\Gamma(-J(s)-J(t))}$$

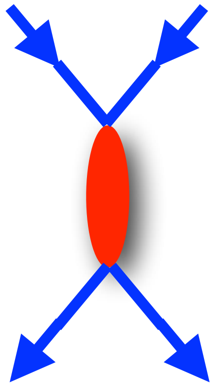
how analytical continuation happens in practice for scattering amplitudes





$$f(s,t) = \sum_n f_n(s) t^n$$

unitarity in s-channel $\text{Disc.}\ f_n(s) \neq 0$



$$f(s,t) = \sum_n f'_n(t) s^n$$

unitarity in t-channel $\text{Disc.}\ f'_n(t) \neq 0$

s-channel sum over t must diverge to reproduce a t-channel singularity in t (and vice versa)

sum over n in s-channel p.w. is replaced by an integral (Mandelstam)

$$A(s,t) = \int dt_1 dt_2 K(s,t_1,t_2,t) A(s,t_1) A^*(s,t_2)$$

Continuation of integral representation $g(w) = \int_C f(z,w)dz$

what are the possibilities for g(w) to be singular?

Let D be a neighborhood of the arc C and G be a domain in the w-plane, f(z,w) be regular in both variables, except for a finite number of isolated singularities or branch points.

g(w) can be singular at $w_0 \in G$ only if

1. $f(z,w_0)$ in z-plane has a singularity coinciding with the end points of the arc C (end-point singularity)

2. two singularities of f, $z_1(w)$ and $z_2(w)$, approach the arc C from opposite sides and pinch the arc precisely at $w=w_0$. (pinch singularity)

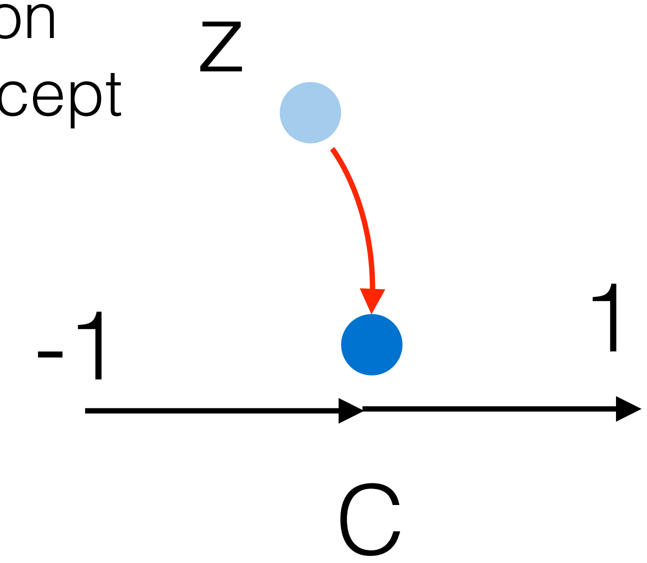
3. a singularity z(w) tents to infinity as $w \to w_0$ deforming the contour with itself to infinity; one has to change variables to bring the point ∞ to the finite plane to see what happens.

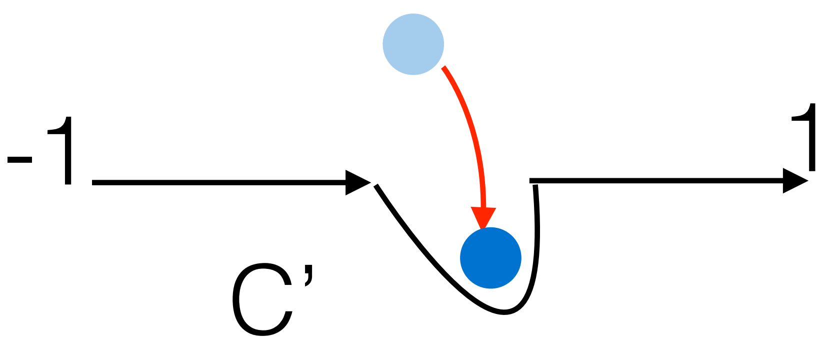
Examples

Apparent singularities need not be one!

$$f(z) = \int_{-1}^{1} \frac{dx}{x - z} \quad C = [-1,1]$$

looks like a regular function of z in the entire plane except for the interval z ∈ [-1,1]

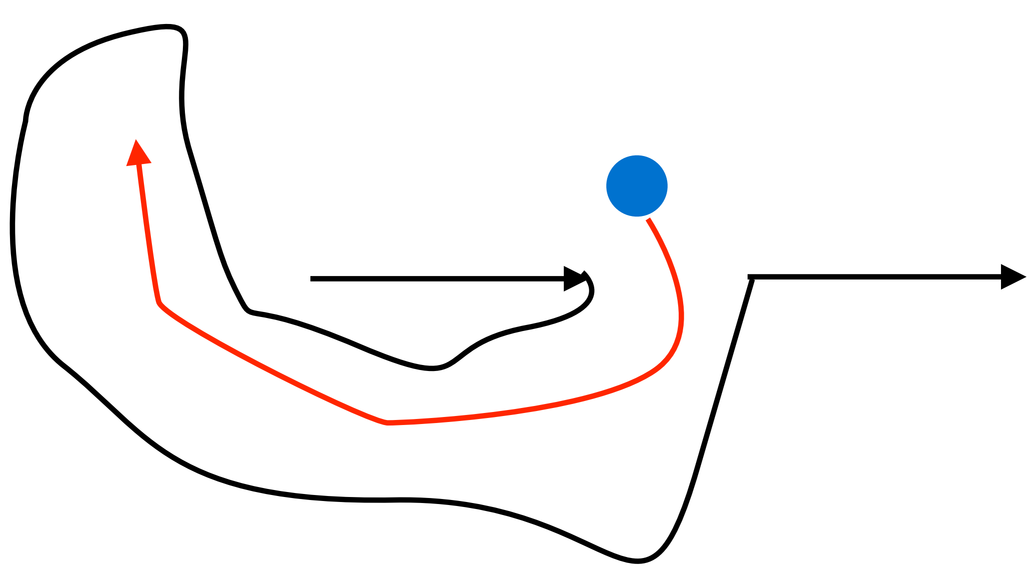


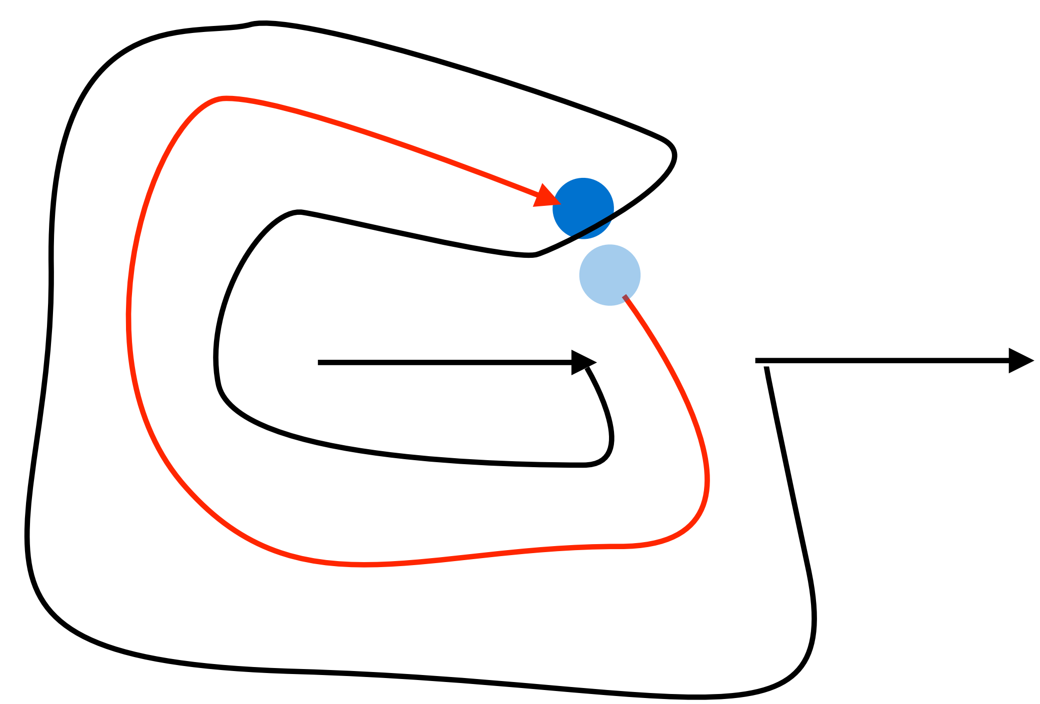


as long a z does not hit the C f(z) changes smoothly

when z approaches deforming C allows to define a function f(z) which continues changing

... however when z returns to the original we end up with a different function value. f(z) is multivalued and -1 is a branch point.



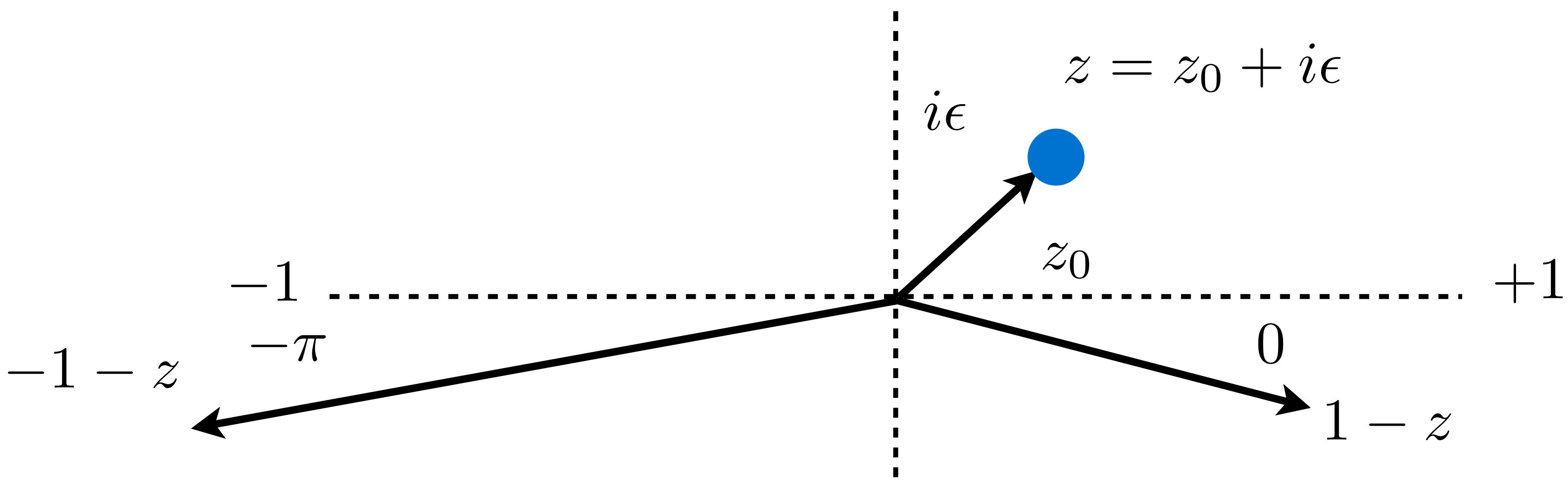


$$f(z) = \int_{-1}^{1} \frac{dx}{x - z}$$

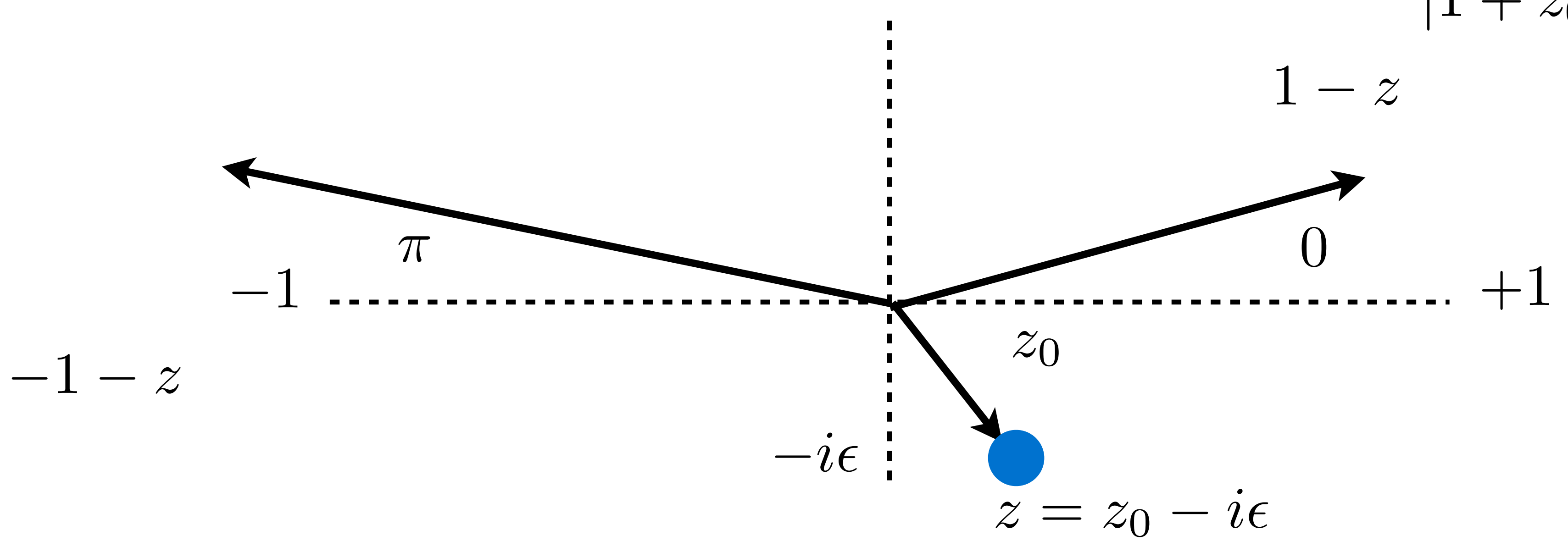
if we don't deform the contour, then f(z) is analytical everywhere except on the real axis between [-1,1]

$$f(z) = \mathrm{Log}(1 - z) - \mathrm{Log}(-1 - z)$$

$$f(z_0 + i\epsilon) = (\log|1 - z_0| + 0i) - (\log|1 + z_0| - i\pi) = \log\frac{|1 - z_0|}{|1 + z_0|} + i\pi$$



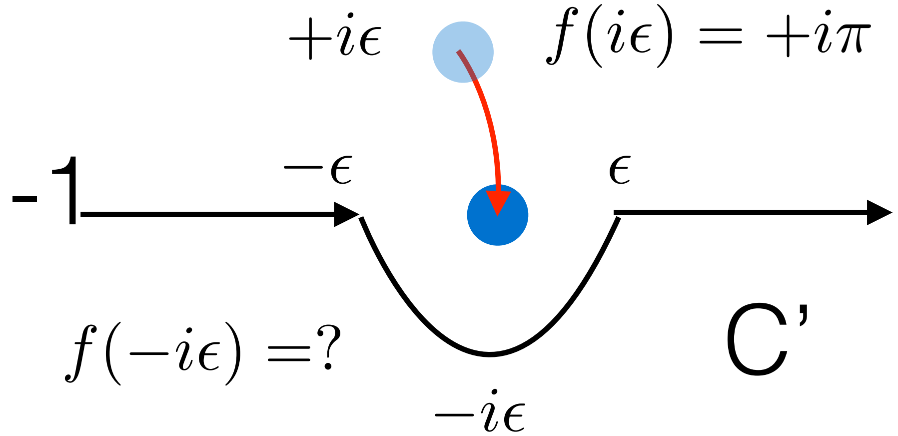
$$f(z_0 - i\epsilon) = (\log|1 - z_0| + 0i) - (\log|1 + z_0| + i\pi) = \log\frac{|1 - z_0|}{|1 + z_0|} - i\pi$$



f(z) jumps as z crosses the real axis, $f(z_0+i\epsilon)-f(z_0-i\epsilon) = 2\pi$. We say f(z) has a cut [-1:1] in 2π is the value of the discontinuity across the cut (happens to be constant) i.e. f(z) is analytical everywhere except [-1:1]

how distorting contour makes f(z) continuos
e.g. take z = 0 + i**ε** and move towards 0 - i**ε**

$$f(z) = \int_{-1}^{1} \frac{dx}{x - z}$$



$$f(0) = \int_{C'} \frac{dx}{x - 0} = \int_{-1}^{-\epsilon} \frac{dx}{x} + \int_{\epsilon}^{1} \frac{dx}{x} + \int_{-\pi}^{0} \frac{id\phi \epsilon e^{i\phi}}{\epsilon e^{i\phi}}$$

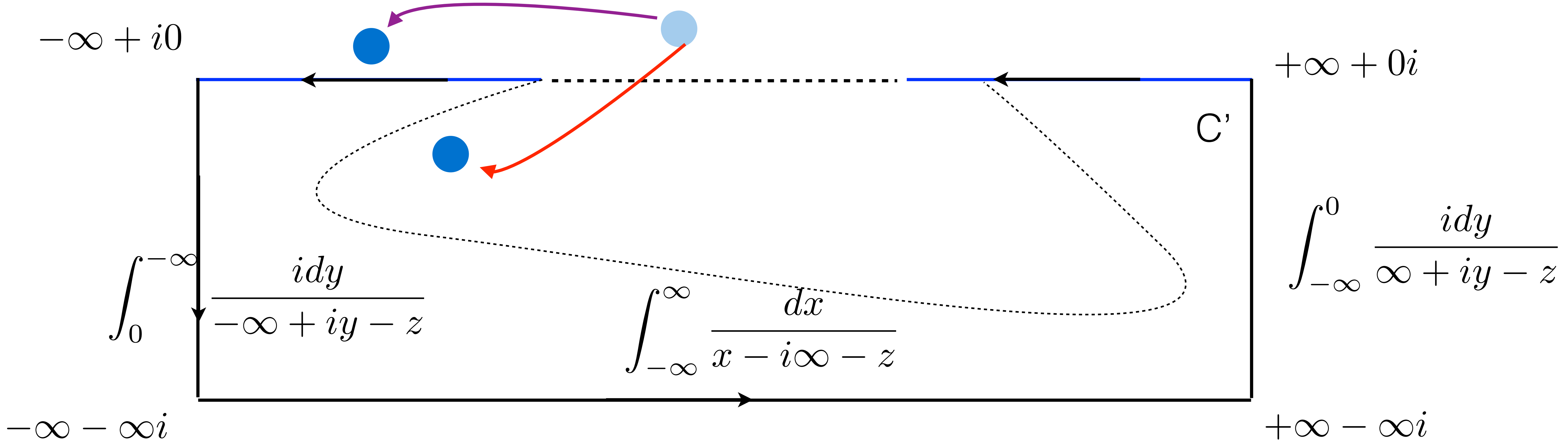
$$= \log \epsilon - \log \epsilon + i\pi = i\pi$$

as promised, f(z) varies smoothly as z crosses the real axis (provided the contour is distorted) It is no longer discontinuous across [-1:1] we can define (single-valued) f(z) in a different domain, e,g with a cut [-∞,-1]

$$f(z) = \int_{C'} \frac{dx}{x-z}$$

$$\int_{-1}^{-\infty} \frac{dx}{x-z} = \int_{1}^{\infty} \frac{dx}{x+z}$$

$$\int_{\infty}^{1} \frac{dx}{x-z} = -\int_{1}^{\infty} \frac{dx}{x-z}$$



$$f(z) = -2z \int_{1}^{\infty} \frac{dx}{x^2 - z^2}$$

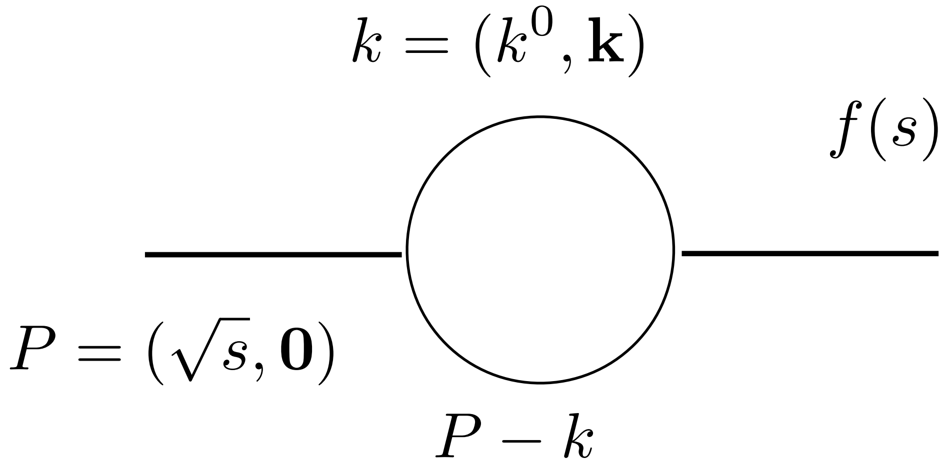
$f(z+i\epsilon) - f(z-i\epsilon) = -2\pi i$ for z<-1 or z>1

$f(z) = -\log(1+z) - \log(-1-z)$ everywhere else

and it is the same function as the one which had the [-1:1] cut outside the real axis

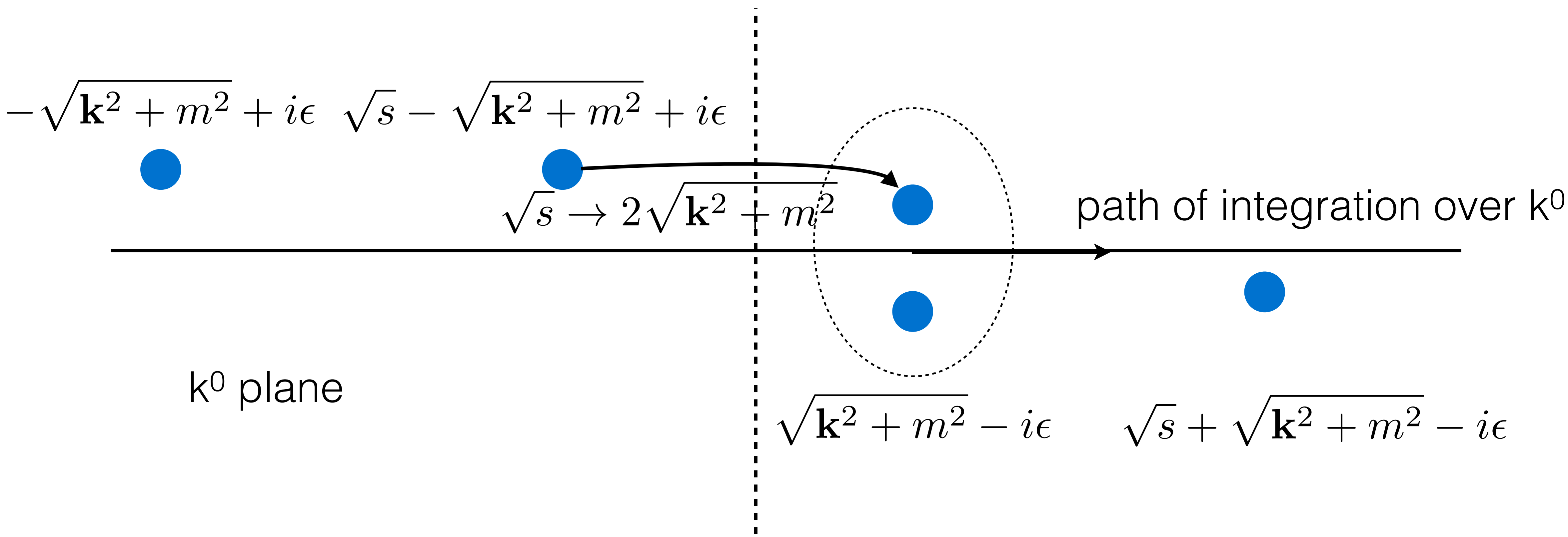
Note that z=±1 are singular (branch) points (end-point singularities)

Example: pinch singularity



$$f(s) = i \int \frac{d^4k}{(2\pi)^3} \frac{1}{(k^0)^2 - \mathbf{k}^2 - m^2 + i\epsilon} \frac{1}{(\sqrt{s} - k^0)^2 - \mathbf{k}^2 - m^2 + i\epsilon}$$

as **ε➞**0 these two poles “pinch” the contour i.e. it cannot be deformed without crossing one of them



the happens for any **k**, so we expect f(s) to be singular for all $s>4m^2$

$$f(s + i\epsilon) - f(s - i\epsilon) \propto \sqrt{1 - \frac{4m^2}{s}}\theta(s - 4m^2)$$

Hunting for a resonance $Im f_l(s) = \rho(s) f_l(s) f_l^*(s)$

f(s) is a real-analytic function : $f(s^*) = f^*(s)$

$$f_l(s+i\epsilon) - f_l(s-i\epsilon) = 2i\rho(s) f_l(s+i\epsilon) f_l(s-i\epsilon)$$

1^{st} sheet $s_1 = 3 + 0.01\,i$: $f(s_1)$

1^{st} sheet $s_2 = 3 - 0.01\,i$: $f(s_2)$

$f(s_1) - f(s_2) =$ "large"

$$f_l(s+i\epsilon) = \frac{f_l(s-i\epsilon)}{1 - 2i\rho(s) f_l(s-i\epsilon)}$$

$$f^{2nd}(s) = \frac{f_l(s)}{1 - 2i\rho(s) f_l(s)} \quad (*)$$

use (*) to define analytical continuation of f to the second sheet

1^{st} sheet $s_1 = 3 + 0.01\,i$: $f(s_1)$

2^{st} sheet $s_2 = 3 - 0.01\,i$: $f^{2nd}(s_2)$

$$f(s_1) - f^{2nd}(s_2) = O(0.01)$$

f(s) has not singularities but $f^{2nd}(s)$ may have when

$$f_l(s) = \frac{1}{2i\rho(s)}$$

Enjoy the rest of the School