Complex analysis in a nut shell

Lecture 1: Introduction

Complex algebra and geometric interpretation

Elementary functions, domains, maps

Differentiation, Cauchy relations, harmonic functions

Lecture 2: Complex integrals

Cauchy theorem implications and applications

Lecture 3: Analytic continuation

Multivalued functions

Branch points, Riemann sheets

Conditions for singularities of integral transforms

History and motivation

Positive integers: 5=3+2 ok but 3-5 is unaccounted for

Were sufficient for about 2000y, Geeks did not use negatives and even after 0 was introduced by Brahmagupta ~628 they were not used until development of axiomatic algebra

Fractional numbers: 3/2 ok but $x^2=2$ unaccounted for

Positive integers and fractions were the pillars of Greek's natural number system, who assumed they are continuously distributed. In1872 Richard Dedekind showed that they "leave holes" for irrational numbers.

Imaginary numbers: $x^2 = -1$

Introduced by Girolano Cardano in 1545, Leonhard Euler introduced "i" in eighteen century, in 1799 Friedrich Gauss introduced 2dim geometric interpretation, which was abandoned till reintroduced in 1806 by Robert Argand. Complex calculus was pioneered by Augustin Cauchy in nineteen century.

Physical quantities are Real.

However, they often come in pairs, e.g. amplitude and phase that have simple representation in terms of complex numbers. In such cases complex numbers simplify how physical laws are expressed and manipulated.

Complex algebra

$z_1 = a + b\,i,\ z_2 = c + d\,i$

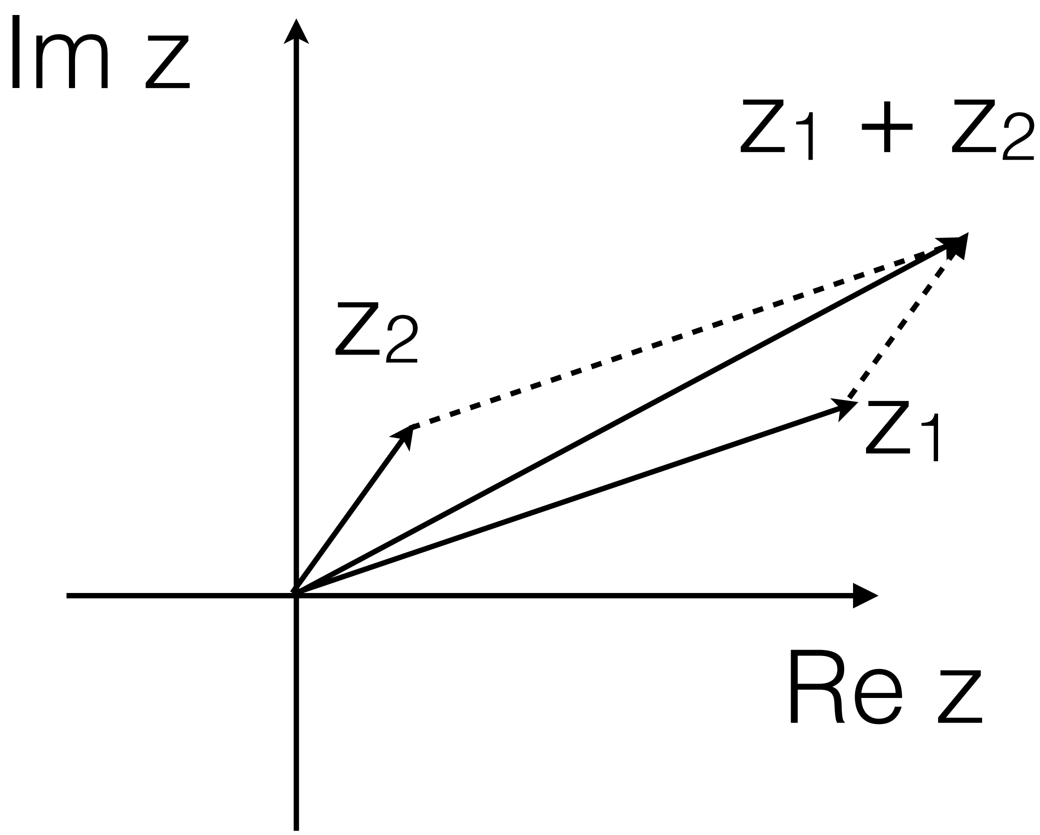
$z_1 + z_2 = (a + b) + (b + d)\,i$

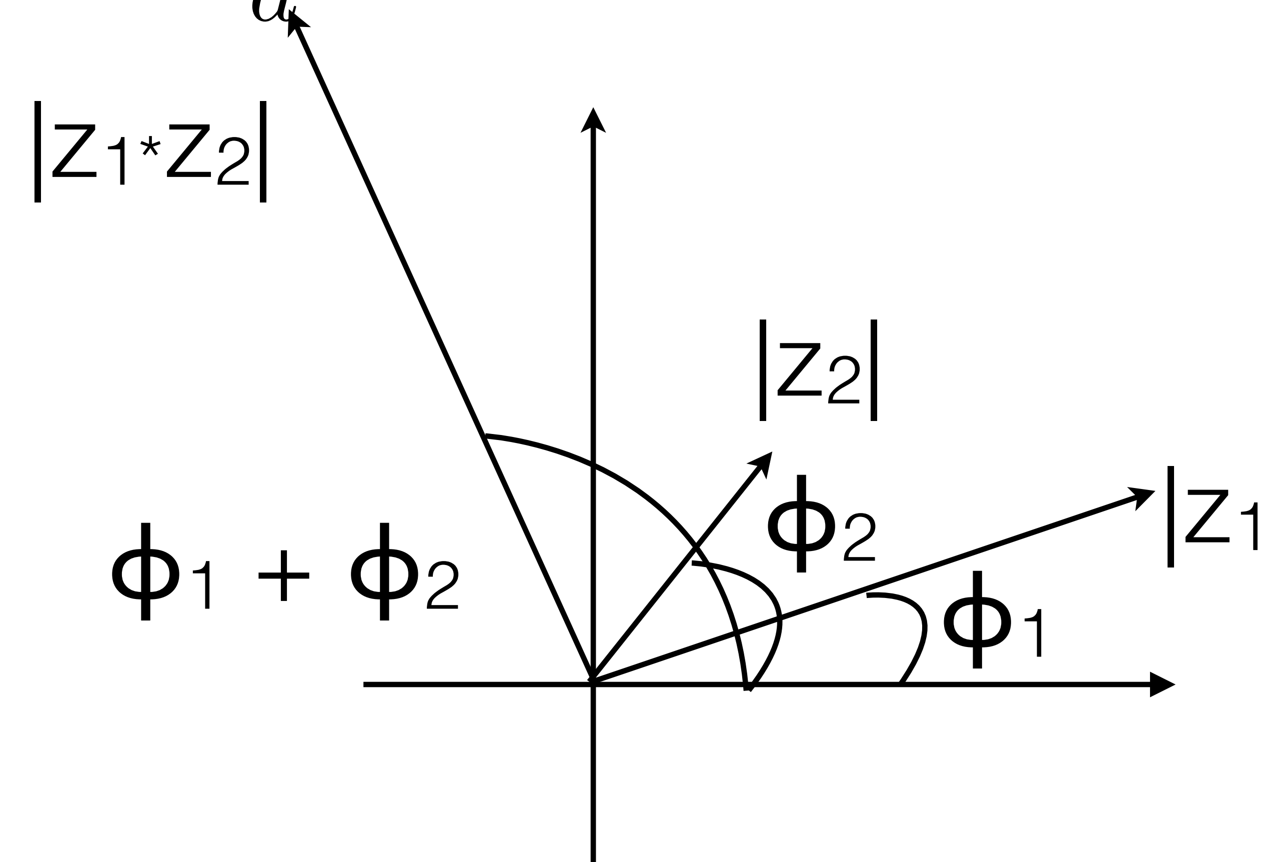
$z_1 * z_2 = (a\,c - b\,d) + (a\,d + b\,c)\,i$

$z_1 = |z_1| \cos\phi_1 + |z_1|\, i \sin\phi_1$

$$|z_1| = \sqrt{a^2 + b^2} \qquad \phi_1 = \arctan \frac{b}{a}$$

complex numbers = field (+, . and distribution law)
z(u +w) = zu + zw





De Moivre's formula

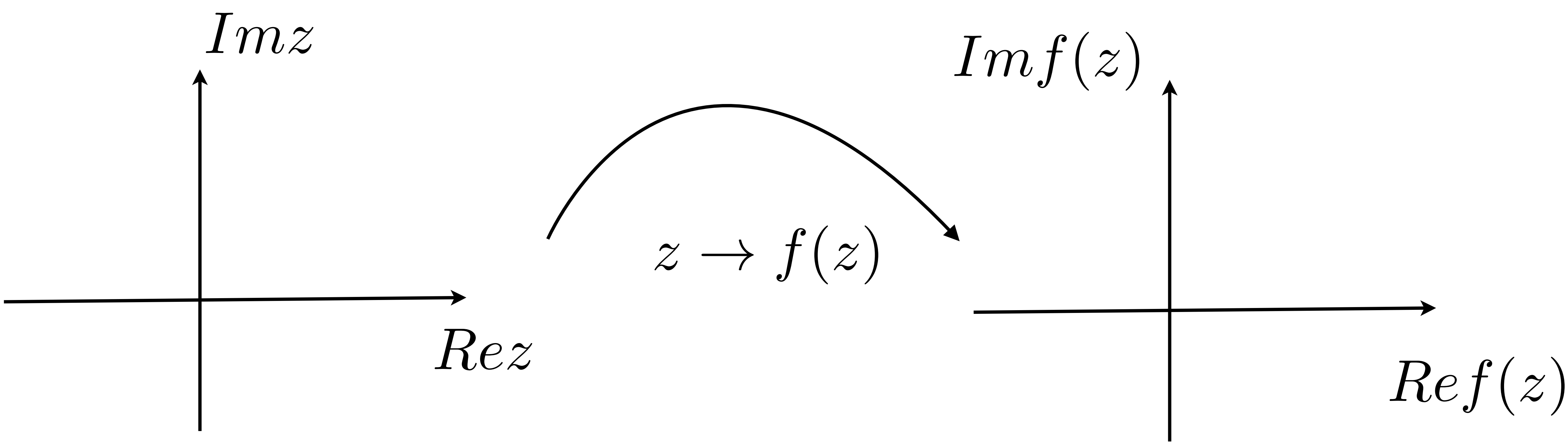
$$z^n = |z|^n (\cos n\phi + i \sin n\phi)$$

Complex functions: definitions

Complex functions

$$z = a + bi \to f(z) = Ref(z) + iImf(z)$$

Elementary functions: you can also think of them as maps of one complex plane (z) to another (f(z)): z→ f(z)



To define a function we can use the algebraic relations e.g

$f(z) = \sqrt{z}$ is such that $z = f(z) \times f(z)$

Complex functions (and complex analysis)

1. Continuity imposes very strong conditions of functions (much stronger than in the case of real variables)

2. "Smooth" (holomorphic, analytic) functions are "boring" all "action" is in the singularities.

3. Singularities also determine functions "far away" from location of the singularity (e.g. charge determines potentials)

4. Physical observables are functions of real parameters, however physics law can be generalized to complex domains and are "smooth", however "constraints" result in singularities.

Example:

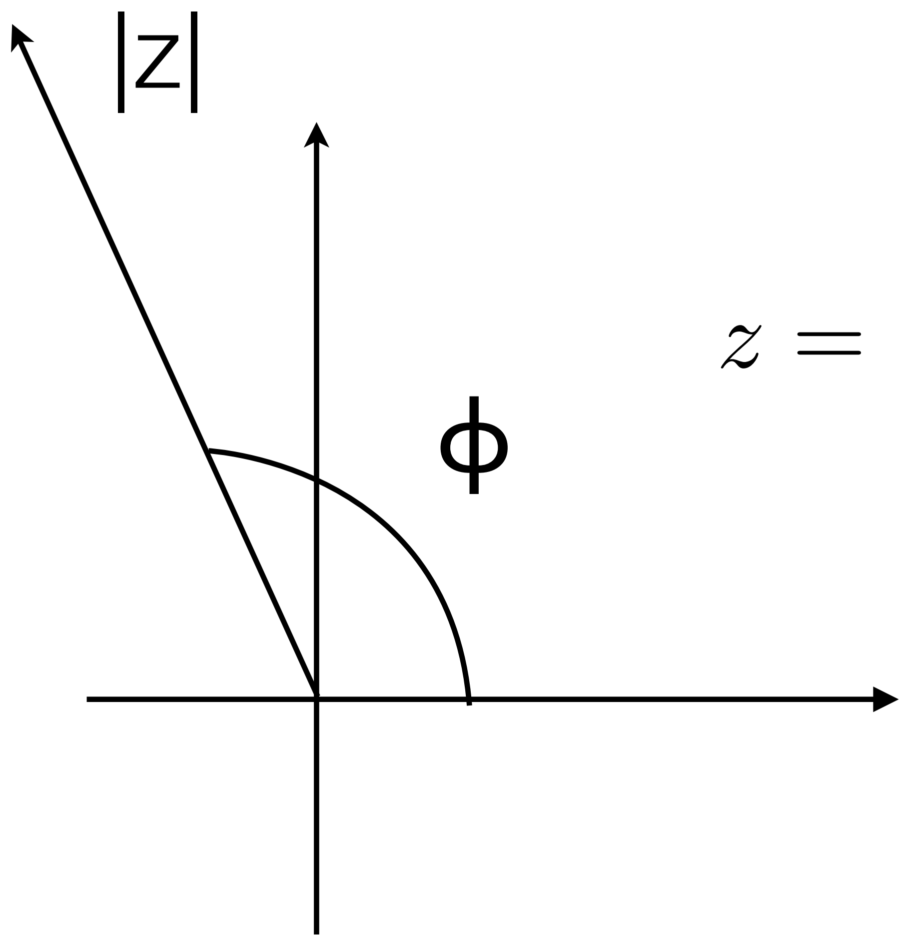
(exp of complex argument has the same algebraic properties as exp of real arg., e.g. $\exp(z_1 z_2) = \exp(z_1)\exp(z_2)$)

$$e^{i\phi} = [1 - \frac{\phi^2}{2} + \cdots] + i[\phi - \frac{\phi^3}{3!} + \cdots] = \cos\phi + i\sin\phi$$

Example (De Moivre's formula)

$$e^{3i\phi} = cos 3\phi + i\sin 3\phi = (\cos\phi + i\sin\phi)^3$$

$$= (cos^3\phi - 3\cos\phi\sin\phi^2) + i(3\cos\phi^2\sin\phi - \sin^3\phi^3)$$



$$z = |z|e^{i\phi} = |z|(\cos\phi + i\sin\phi)$$

Examples

find solutions of $z^8 = 1$

simplify $\dfrac{1+i}{2-i}, \quad \sqrt{1+\sqrt{i}}$

show that maximum absolute value of z^2+1 on a unit disk $|z| \leq 1$ is 2

show that

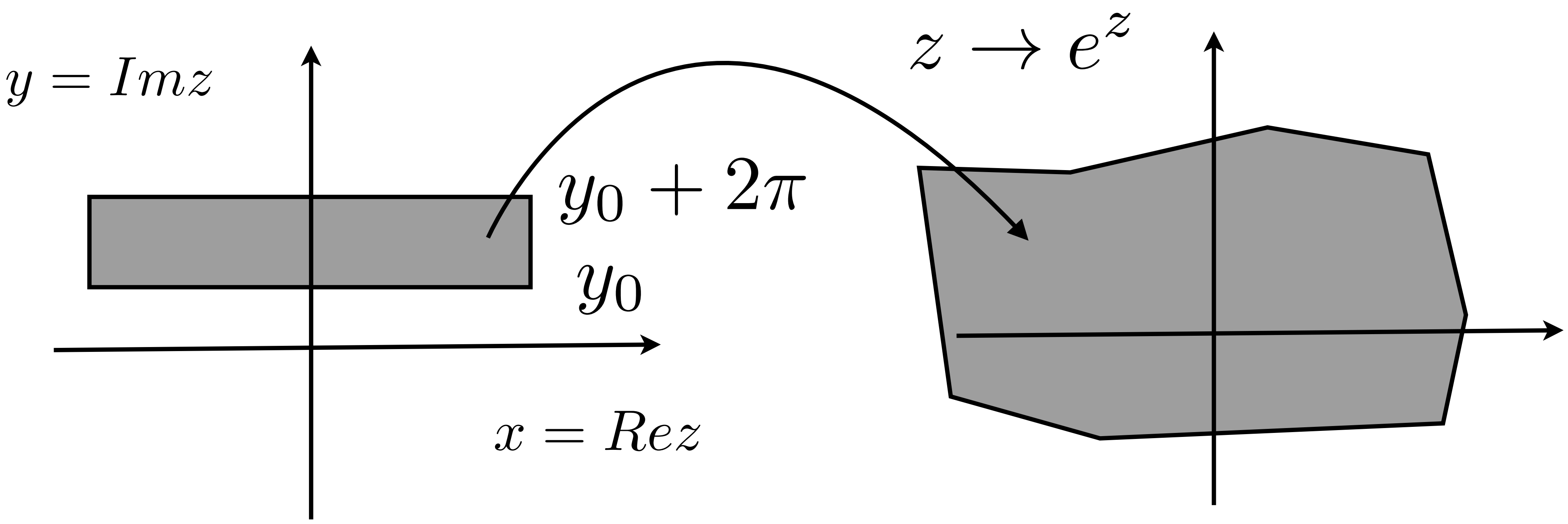
$$1 + \cos\phi + \cos 2\phi + \cdots \cos n\phi = \frac{1}{2} + \frac{\sin(n+\frac{1}{2})\phi}{2\sin\frac{\phi}{2}}$$

solve $\dfrac{d^2x(t)}{dt^2} + \omega^2 x^2(t) = 0$

Complex functions: branches

exp(z) is periodic!

$$z \to e^z = e^{Rez + iImz} = e^{Rez}(\cos Imz + i \sin Imz)$$
$$e^{z+2\pi i} = e^z$$



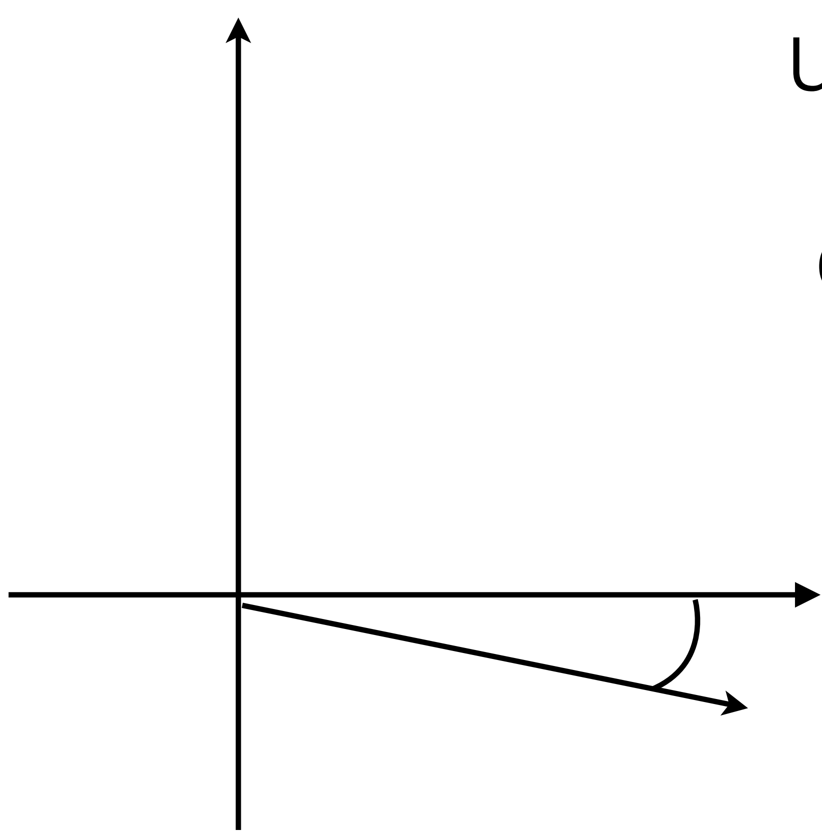
one needs to be careful when defining its inverse i.e. logarithm*:*
the z-plane can be mapped back in many different ways

similar issue with the √z

$$z = |z|e^{i\phi}$$

$$\sqrt{z} \equiv \sqrt{|z|}e^{i\frac{\phi}{2}}$$

$$\sqrt{z}\sqrt{z} = \sqrt{|z|}e^{i\frac{\phi}{2}}\sqrt{|z|}e^{i\frac{\phi}{2}} = |z|e^{i\phi}$$



using $\phi = [-\pi, \pi)$

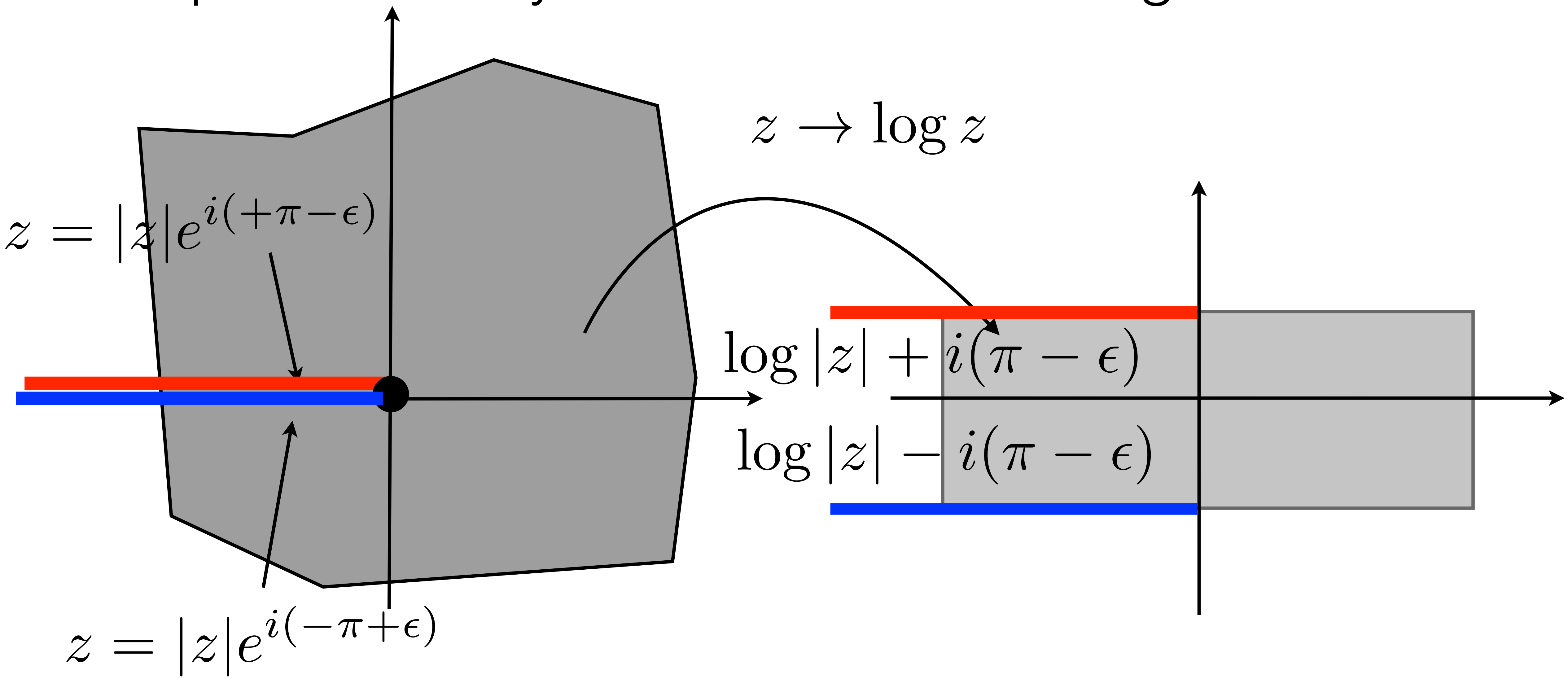
or $\phi = [0, 2\pi)$

gives different results for $\sqrt{z}$

ϕ = ~0 or ~ 2π

log: maps C-{0} → C with range defined $y_0 \leq \text{Im} \log z < y_0 + 2\pi$: $\log z = \log|z| + i \arg z$ (many "log"-functions depending on the choice of y_0)

Principal branch: $y_0 = -\pi$ so that $-\pi \leq \arg z < \pi$



log is discontinuous on its *branch line* (e.g. Im z = 0, Re z < 0).and z=0 ● is the *branch point*

Evaluation of the log: (make sure you stay with the chosen branch)

A: $-\pi \leq \text{Im} \log < \pi$

$$\log(-1+i) = \log(\sqrt{2}) + i\frac{3\pi}{4}$$

$$\log(1-i) = \log(\sqrt{2}) - i\frac{\pi}{4}$$

B: $0 \leq \text{Im} \log < 2\pi$

$$\log(-1+i) = \log(\sqrt{2}) + i\frac{3\pi}{4}$$

$$\log(1-i) = \log(\sqrt{2}) + i\frac{7\pi}{4}$$

$$\log[(-1+i)(1-i)] = \log(2I) = \log(2e^{i\frac{\pi}{2}}) = \log(2) + i\frac{\pi}{2}$$

or use $\log(z_1 z_2) = \log z_1 + \log z_2$

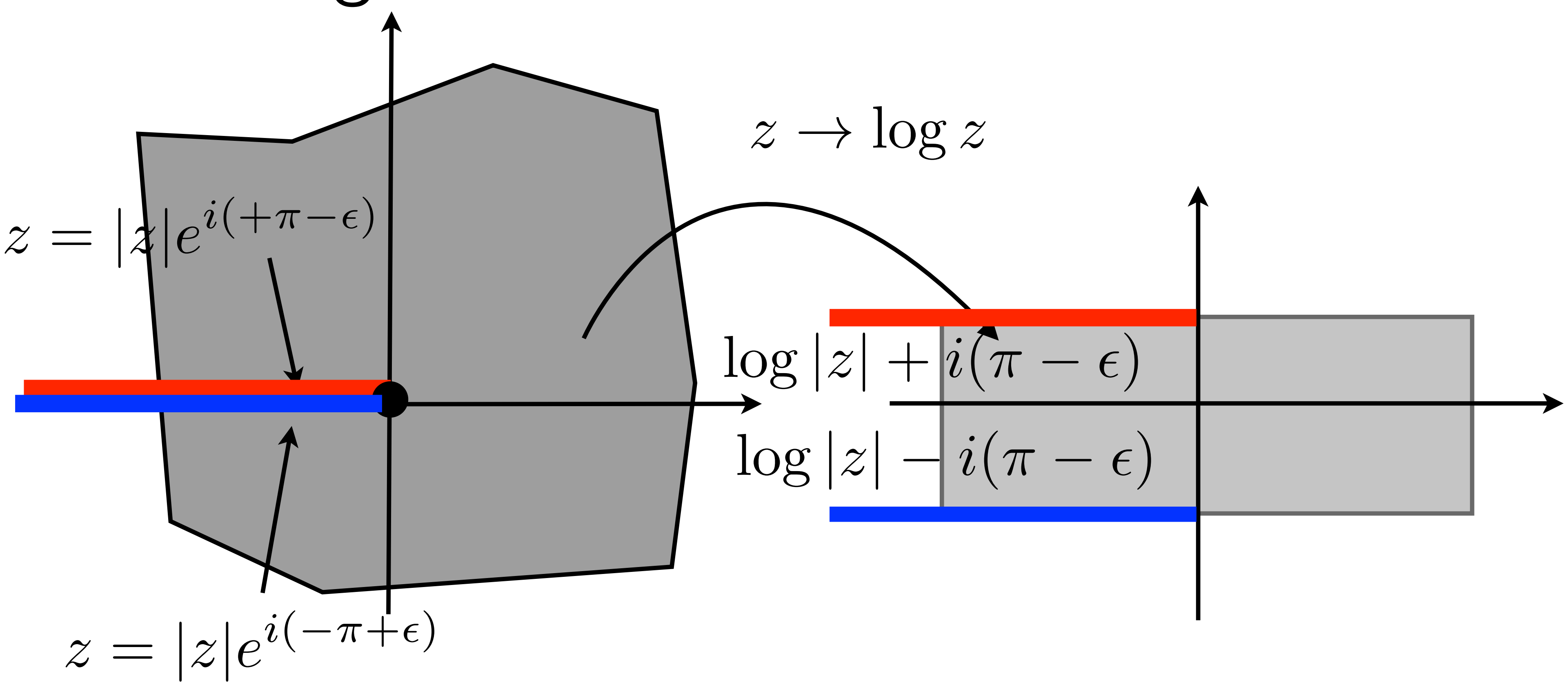
A: $\dfrac{3\pi}{4} - \dfrac{\pi}{4} = \dfrac{\pi}{2}$

B: $\dfrac{3\pi}{4} - \dfrac{7\pi}{4} = \dfrac{\pi}{2} + 2\pi$

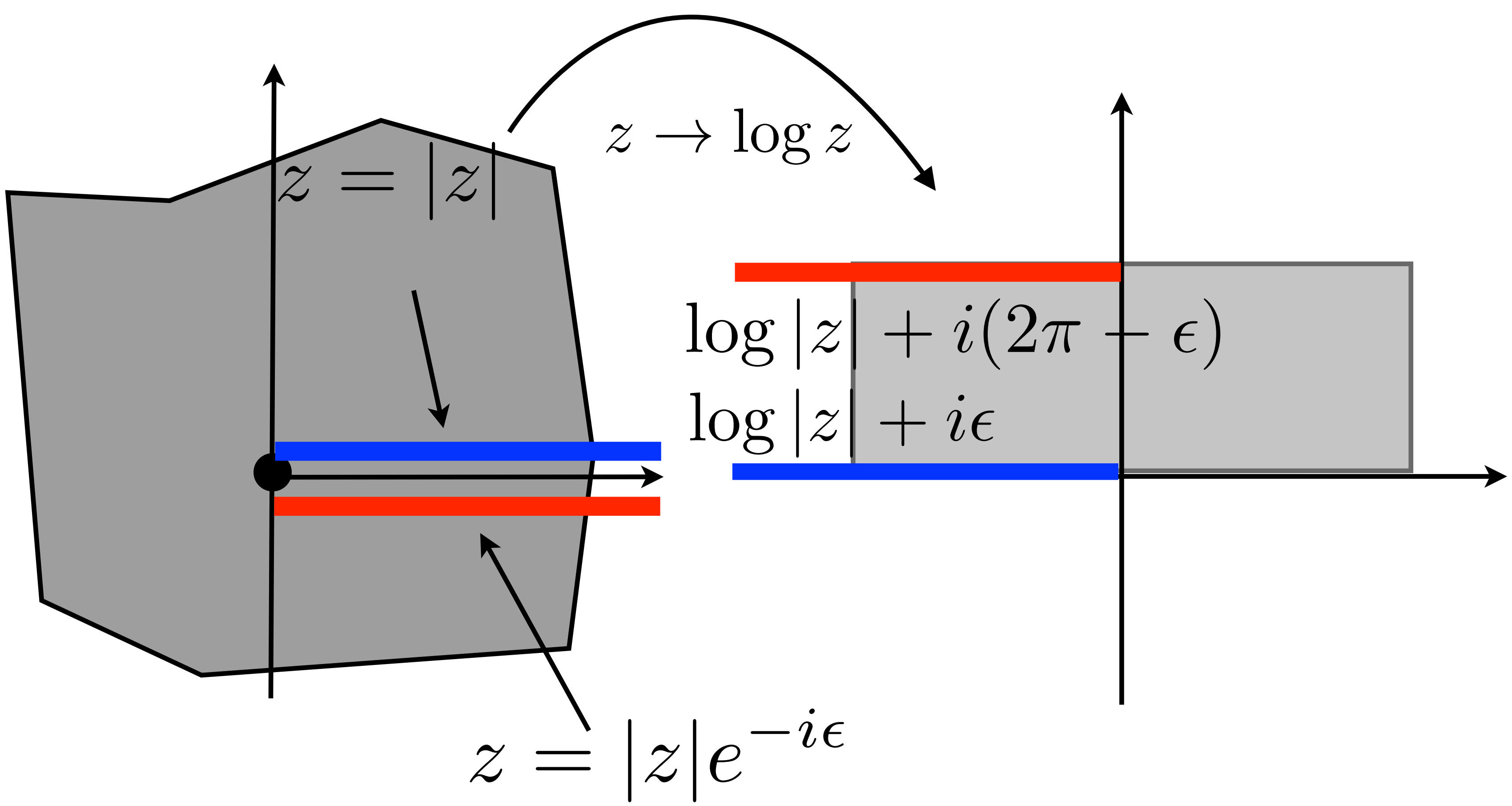
OK as is

subtract $2\pi i$ to keep inside the defining region of $[0,2\pi)$

Case A: -π ≤Im log < π



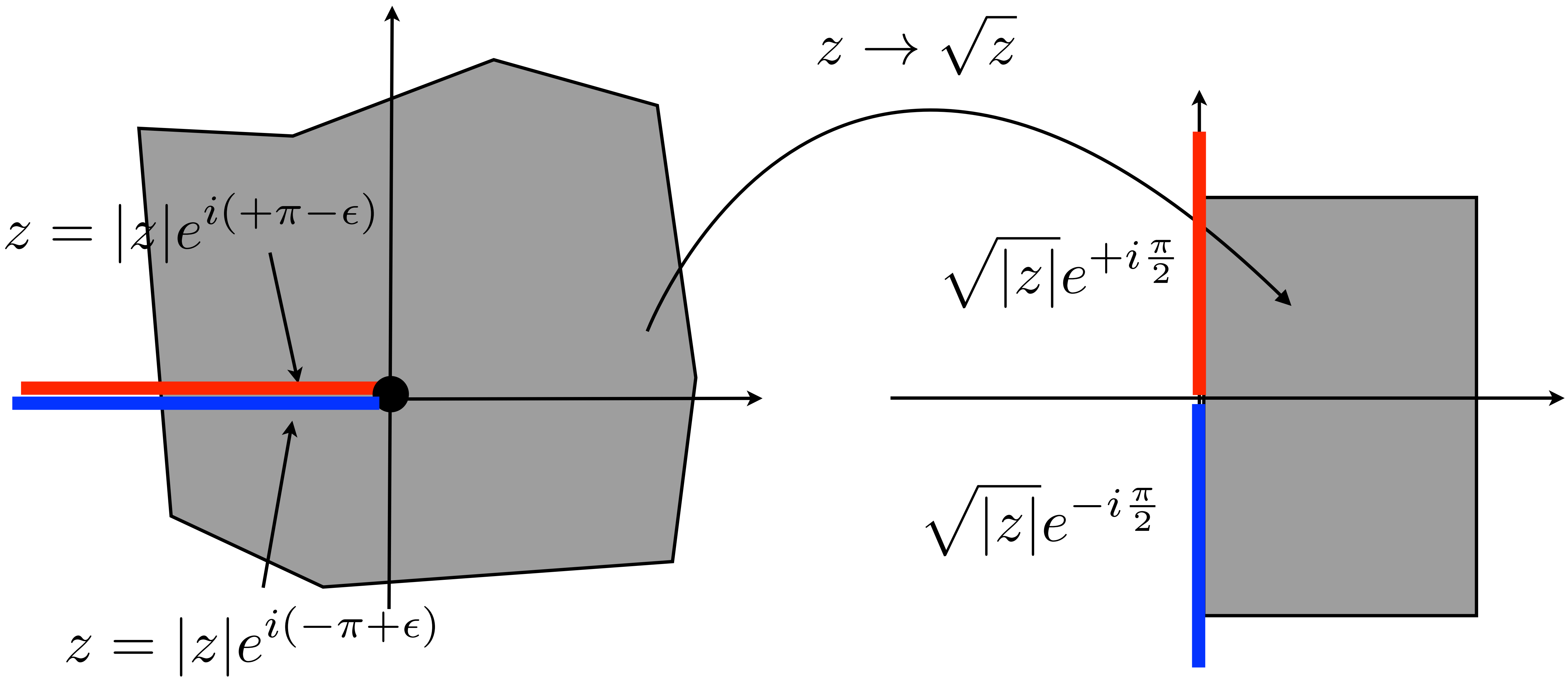
Case B: 0 ≤Im log 2< π



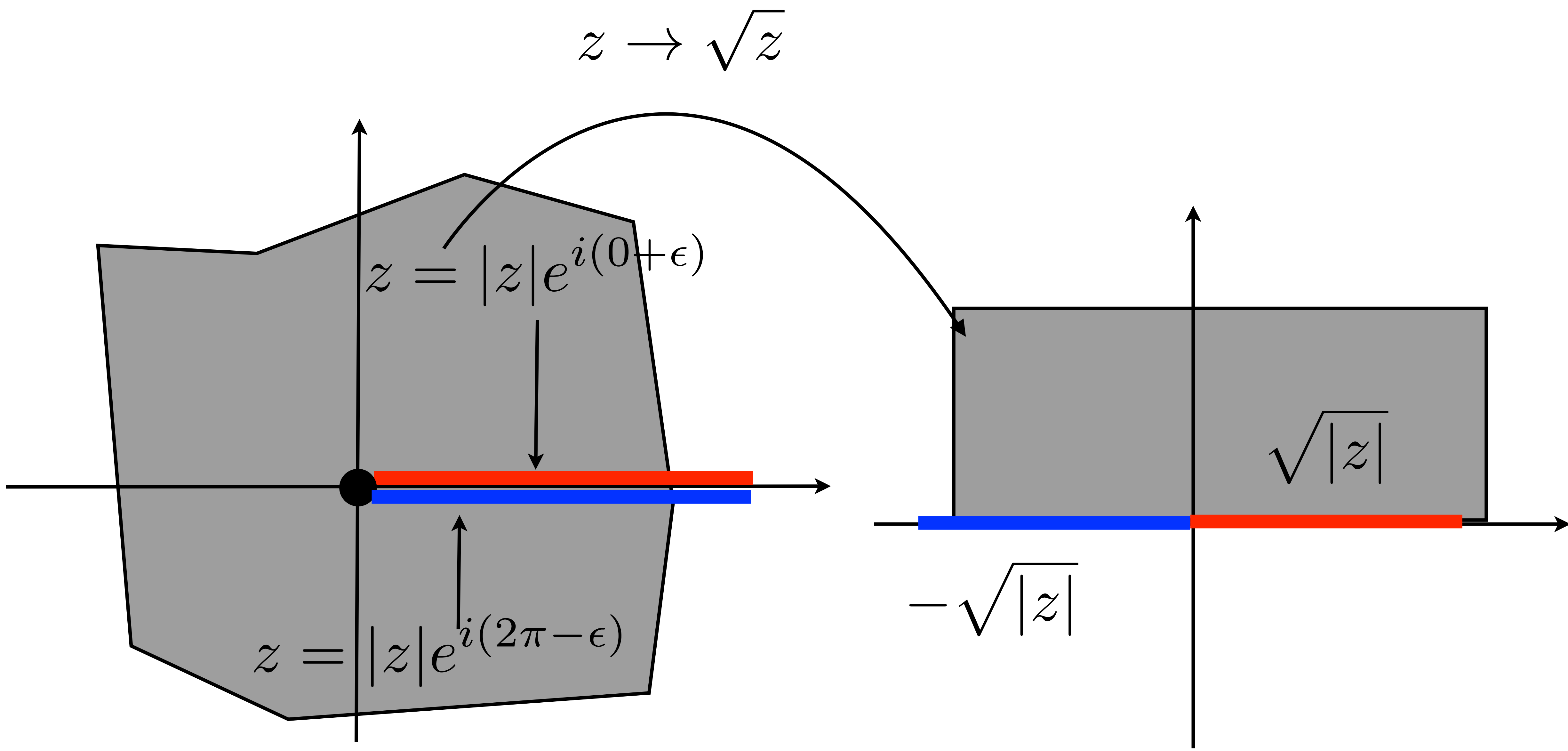
Powers: $a^b = e^{b\log(a)}$ (for chosen branch of log)

$$\sqrt{z} = e^{\frac{1}{2}\log(z)} = \sqrt{|z|}e^{\left[i\frac{arg z}{2}+(mod\, i\pi)\right]}$$

for example: using the principal branch ($-\pi \leq \arg z < \pi$)

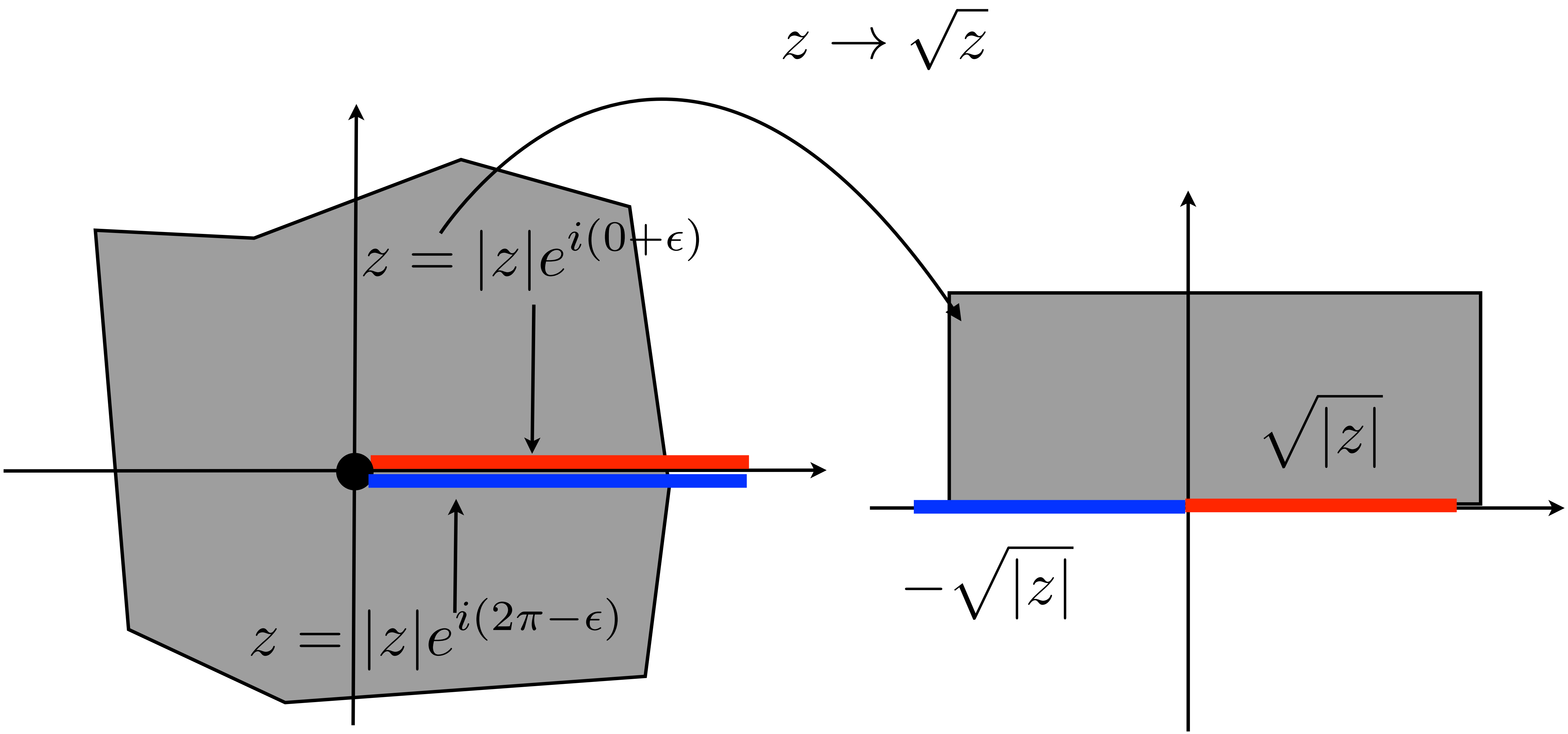


... or using the [0,2π) branch



function has different value when evaluated above vs below a branch line:

$$\lim_{\delta z \to 0} [f(z+\delta z) - f(z-\delta z)] \equiv \text{ Dis. } f(z) \neq 0$$

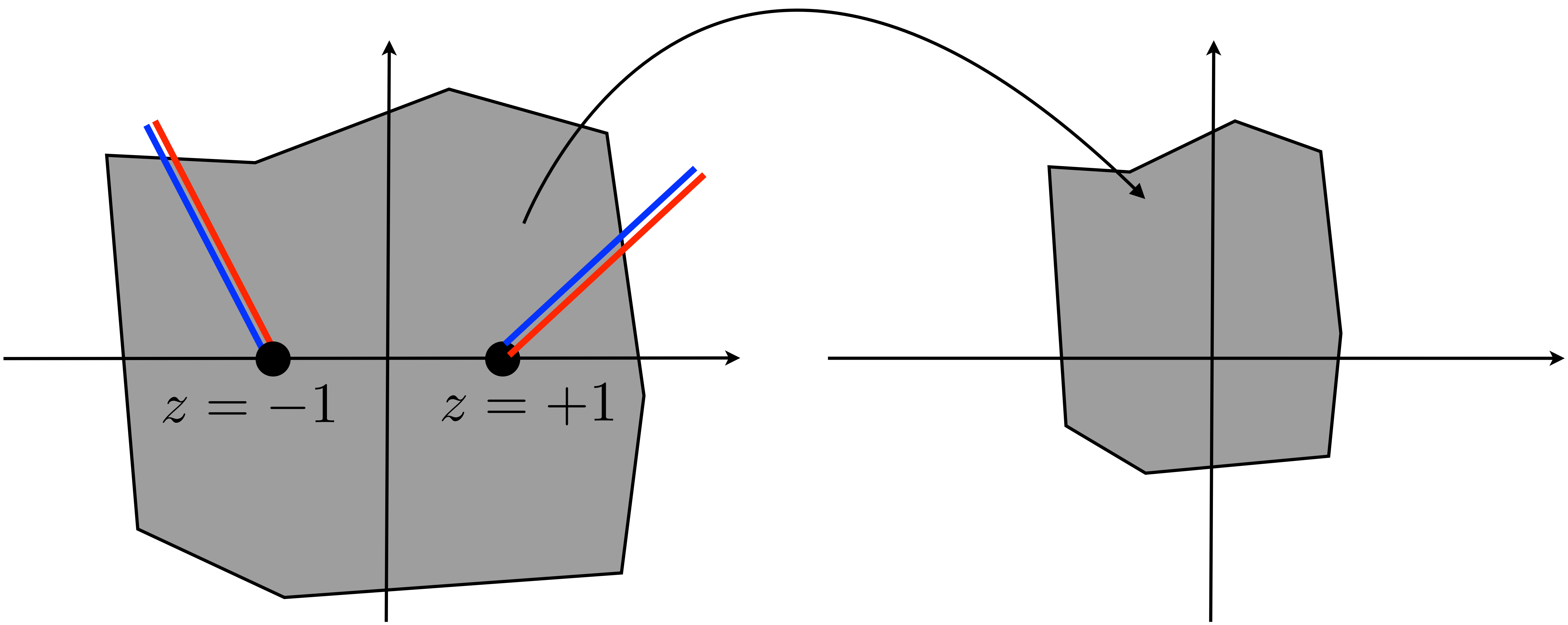


$$\text{Dis.}\sqrt{z} = 2\sqrt{z} \text{ for } z \text{ real and positive}$$

Composite functions

the key is to define one-to-one mapping which requires specification of branch lines

for example $z \to \sqrt{z^2 - 1}$ has two branch points and one needs to define orientation of two branch lines



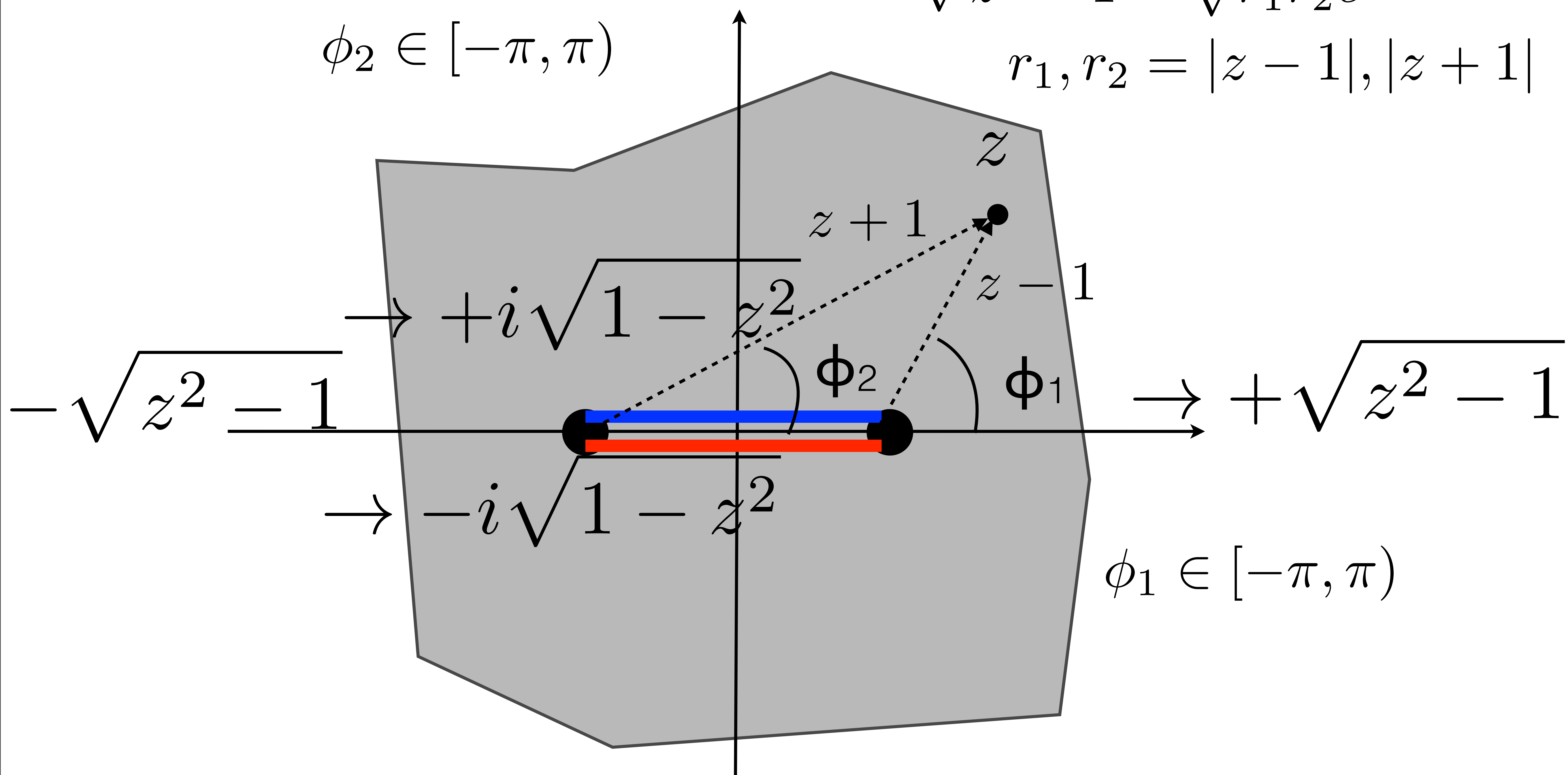
A. $z \to \sqrt{z^2-1}$

$= \sqrt{z-1}\sqrt{z+1}$

and use principal branches

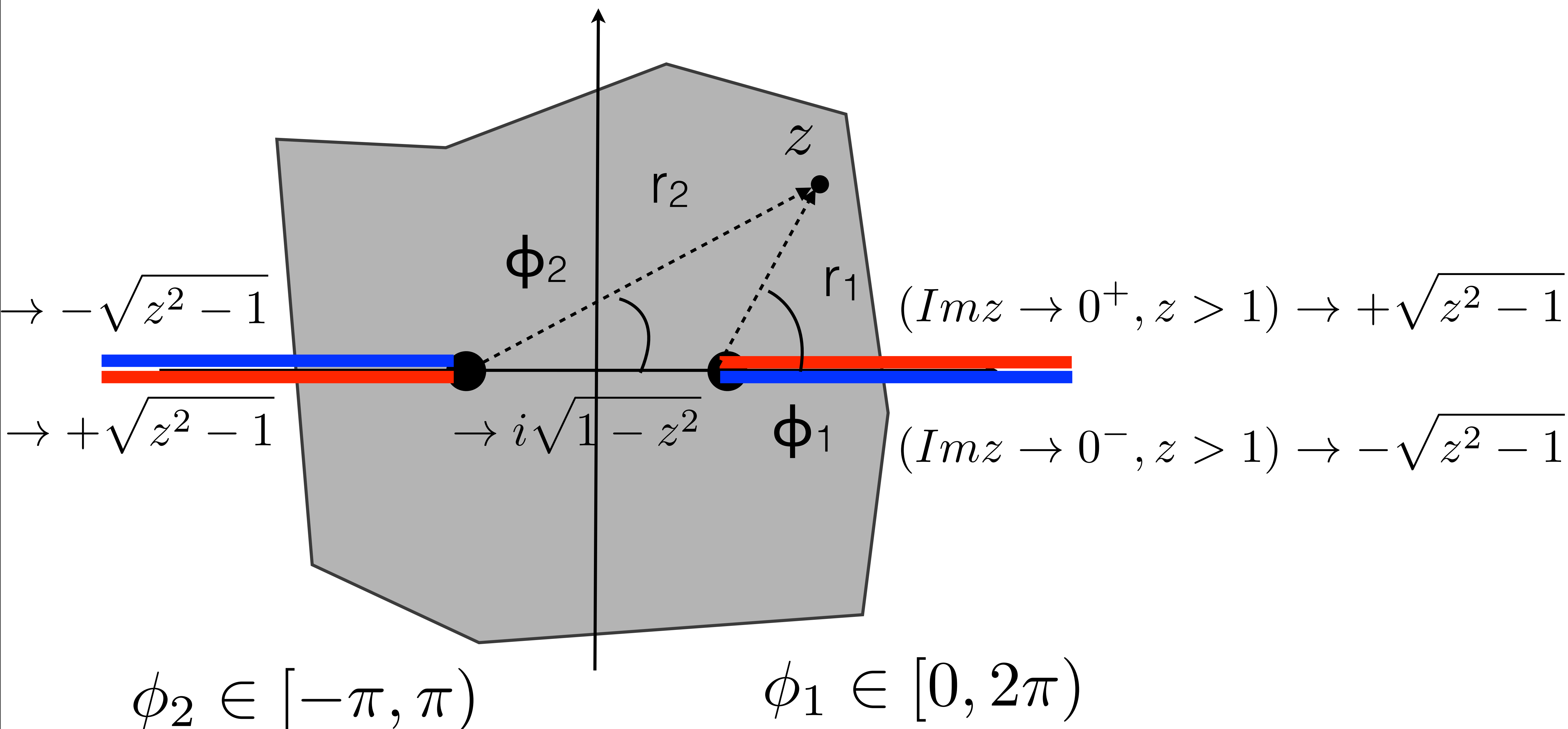
$$\sqrt{z^2-1} = \sqrt{r_1 r_2}\, e^{i\frac{\phi_1+\phi_2}{2}}$$

$$r_1, r_2 = |z-1|, |z+1|$$



B. $z \to \sqrt{z^2-1}$

$$\sqrt{z^2-1} = \sqrt{r_1 r_2}\, e^{i\frac{\phi_1+\phi_2}{2}}$$



All these definite different complex functions which on the real axis relate to the real function

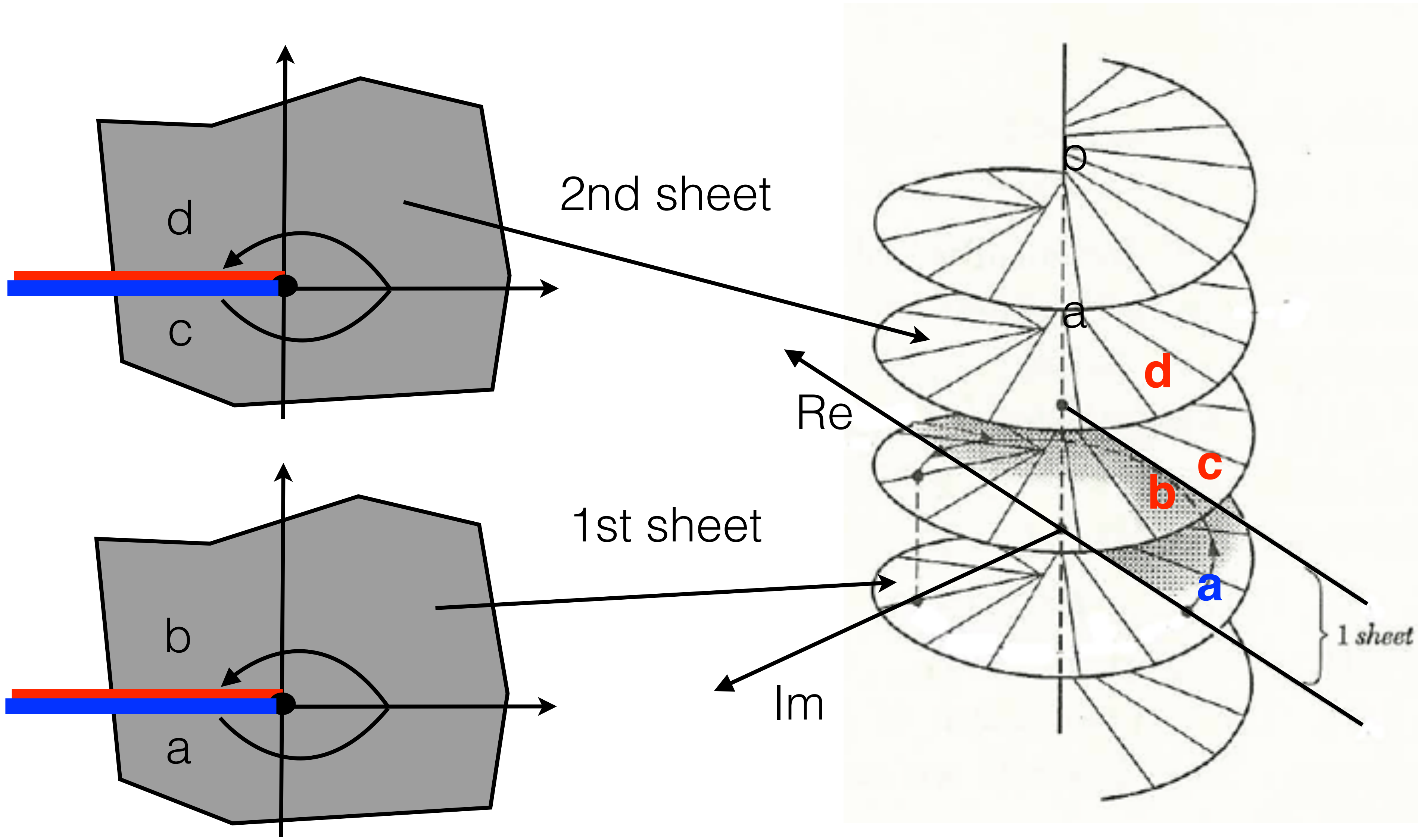
$$\sqrt{x^2 - 1}$$

Which one to use depends on a specific application (more later)

Complex functions: Riemann sheets

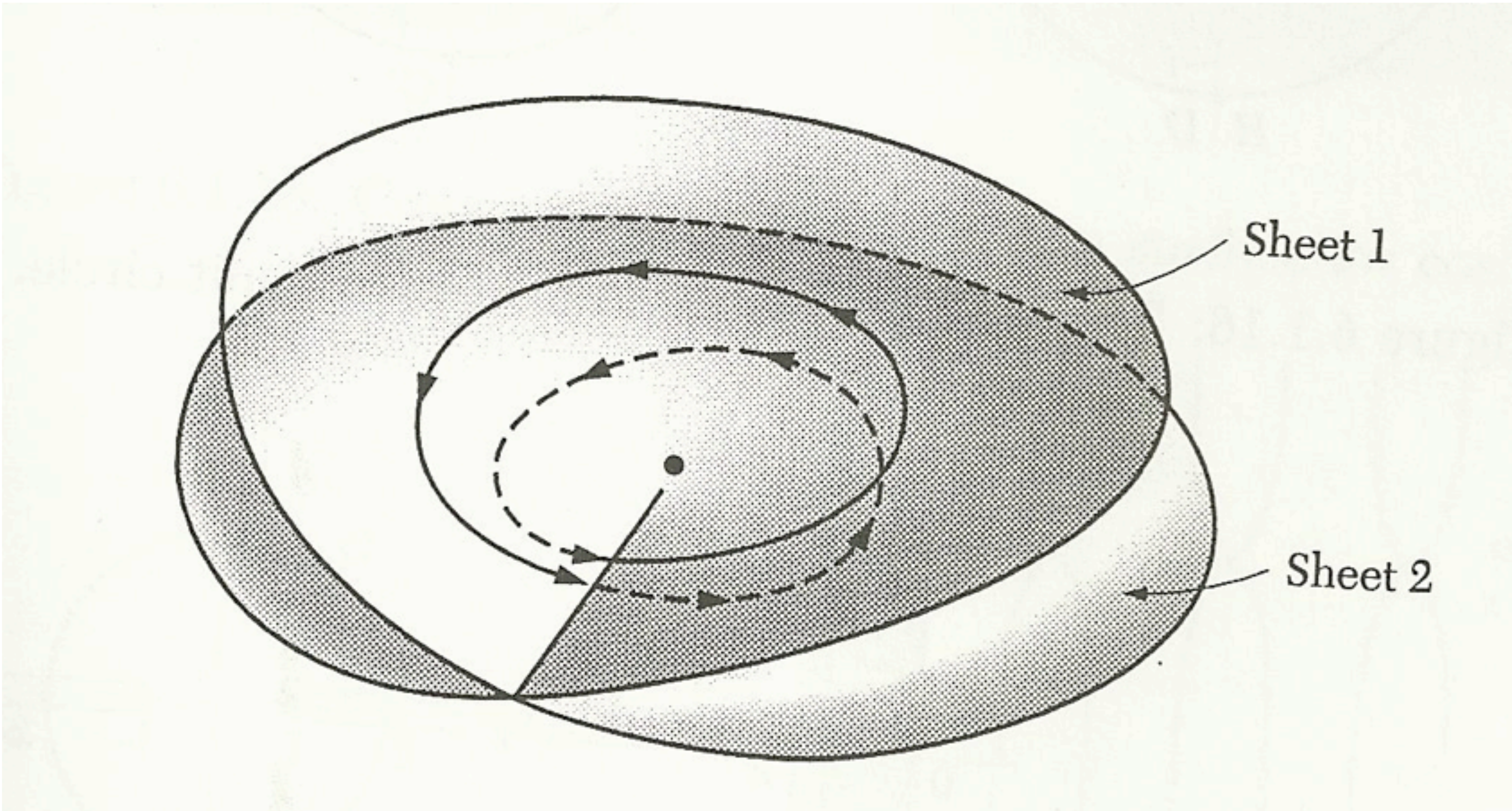
Is there a definition of a multivalued function which does not require branch cuts. (Georg Riemann, PhD. 1851)

Example: $z \rightarrow \log z$



When z moves from a to b arg (Im log) changes from 0 to 2π.
The 2nd Riemann sheet is a copy of the z-plane attached ("glued") at the branch line, such that c (on the 2nd sheet, infinitesimally below real axis) is close to b (on the 1st sheet, just above the real axis).

Riemann sheet for z → √z



Riemann construction: change the “shape” (Riemann sheet) of the “input” complex plane z, so that f(z) is single-valued when defined on this modified “shape”

Examples:

show that $\cos z = \frac{1}{2}$ has only real solutions

Find all values of i^i

show that $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \sin z_2 \cos z_1$

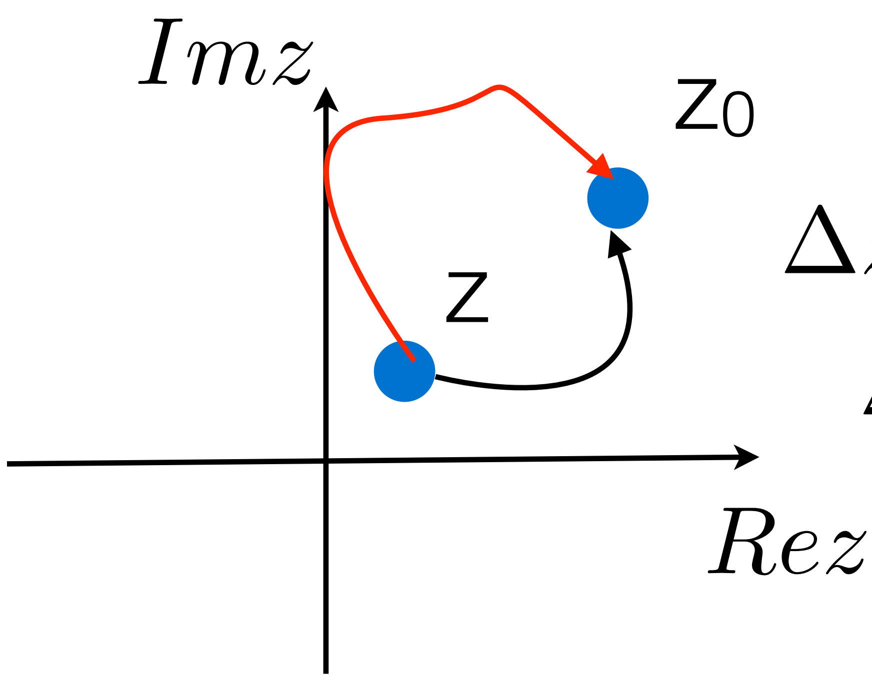
Show that under $z \rightarrow \sin(z)$ lines parallel to the real axis are mapped to ellipses and that lines parallel the the imaginary axis are mapped to hyperbolas

Complex Calculus:

Preliminaries:

Definitions (continuity, limits) similar to functions of real variables, except that variations "**Δ**" can be taken anywhere along paths in the complex plane

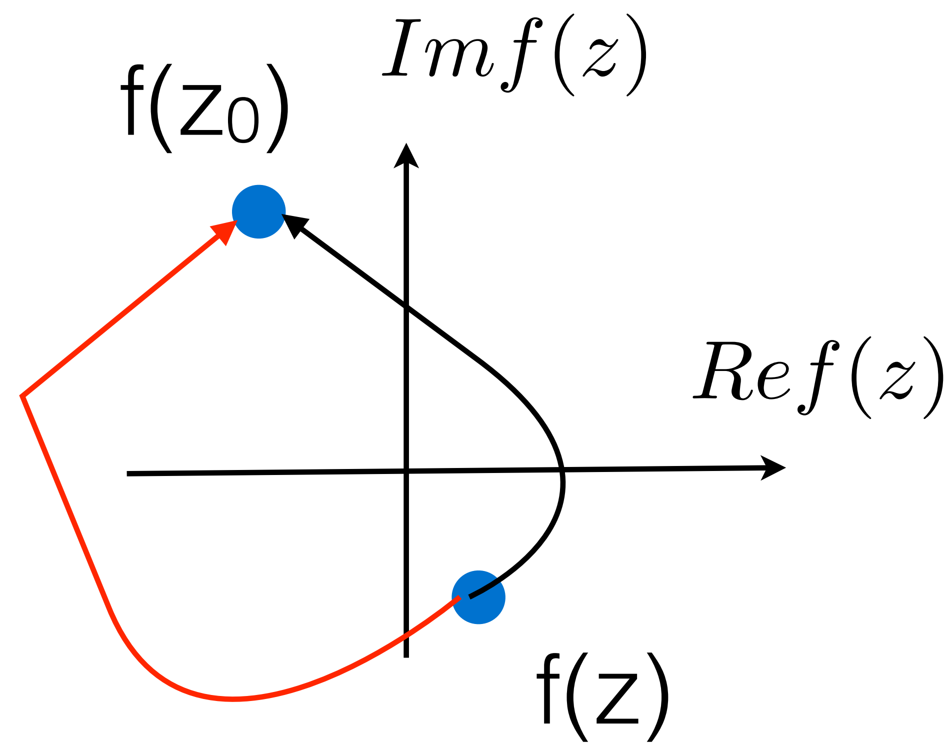
e.g. continuity: f(z) is continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$



$$z \to f(z)$$

$$\Delta z = z - z_0$$

$$\Delta f = f(z) - f(z_0)$$

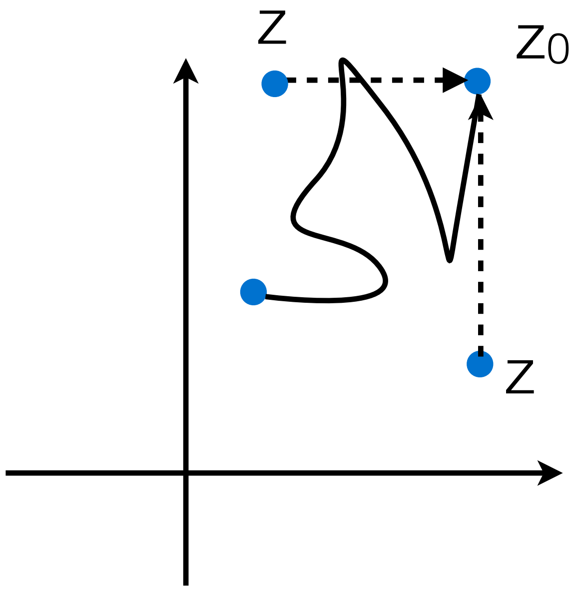


f(z) is a function of two real variables since z=x + iy. However, f(z) refers to a function of z and not of independent variables. The whole point is to explore the consequences of this "unique" combination of x and y "coupled" by i

Differentiation:

f(z) is differentiable (holomorphic) if $\lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0} \equiv f'(z_0)$ exists

write z = x + iy and f(z) as f(z) = u(x,y) + i v(x,y). Since limit in definition of f'(z_0) is independent of the path taken in z→z_0, you can take two independent paths e.g. x = x_0 + **ε**, y = y_0 and x = x_0, y = y + **ε**: Cauchy relations:



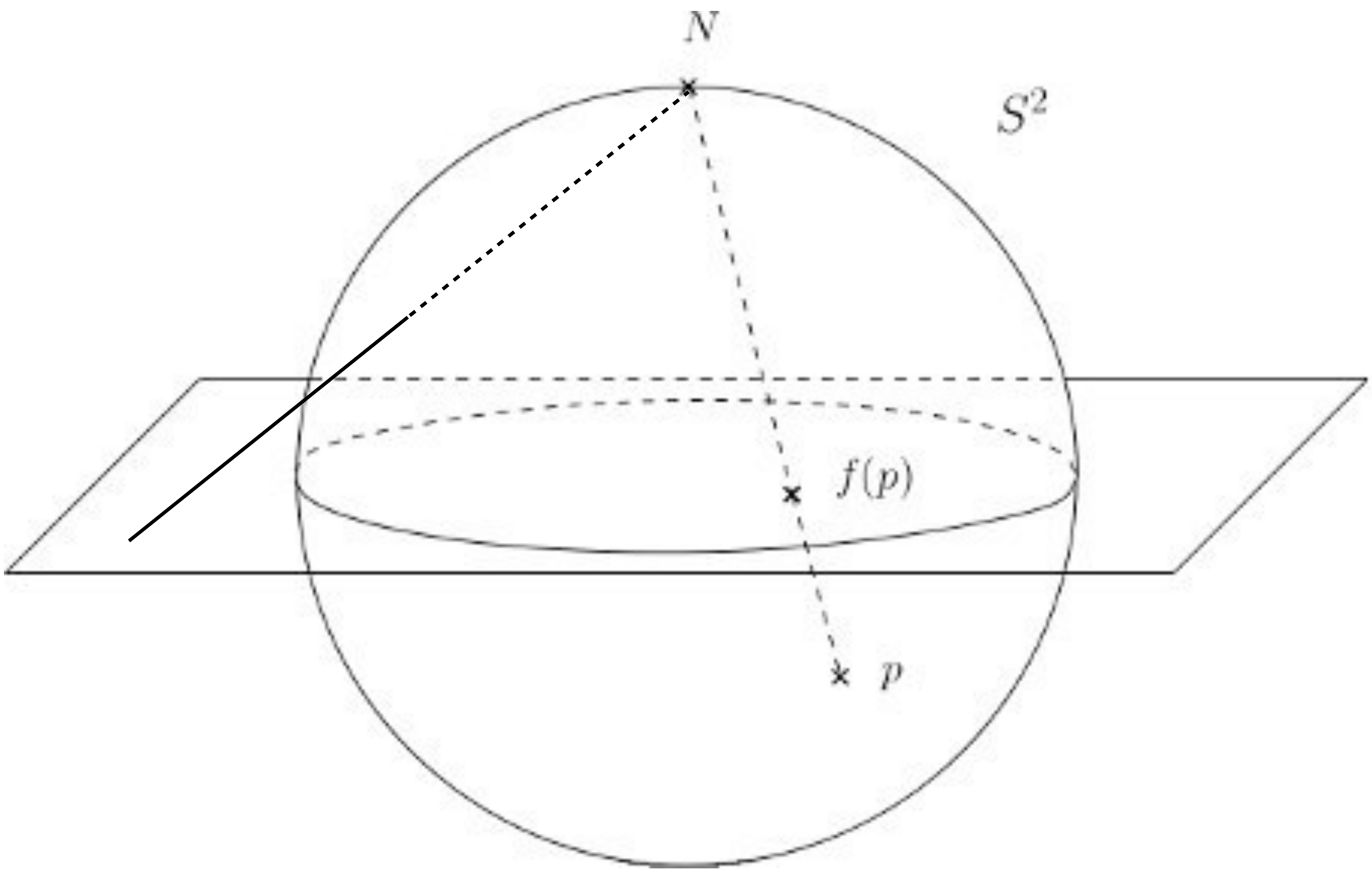
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

This implies **Δ**u = **Δ**v = 0
where **Δ** is 2-dim Laplacian
u,v : harmonic functions

Infinity: on the real axis there are two (axis is oriented) but on the complex plane (calculus) there is no preferred direction: one infinity (somewhat counter intuitive)

$$\frac{df}{dz}(\infty) = -\frac{df}{dw}\left(\frac{1}{w}\right)_{w=0}$$

Stereogrphic projection $S^2 \rightarrow (x,y) = z$



N pole is mapped at the point at infinity

Summary of Lecture 1

1. Complex algebra

2. Elementary functions : not so simple
some functions cannot be defined on the entire plane:
branch lines where function is undefined

$z = |z|e^{i\phi}$ then $\sqrt{z} = \sqrt{|z|}e^{i\frac{\phi}{2}}$ is undefined for
Im z =0, Re z > 0

since $\sqrt{x + i\epsilon} \to \sqrt{x}$

$0 < \phi < 2\pi$ but $\sqrt{x - i\epsilon} \to -\sqrt{x}$

so the z-plane (domain plane) needs to be "cut" along the positive real axis

3. It is (often) possible to eliminate the cut by "glueing" another sheet(s) and effectively replacing the z-plane (domain) by a more complicated surface and continue defining f(z).

f(z) is differentiable (holomorphic) if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \equiv f'(z_0)$ exists

Key feature: existence $f'(z_0)$ means finite and independent how the limit $z \to z_0$ is taken

writing $z = x + iy$ and $f(z) = u(x,y) + iv(x,y)$ this implies

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

This implies $\Delta u = \Delta v = 0$
where Δ is 2-dim Laplacian
u,v : harmonic functions