Summary of Lecture 1

1. Complex algebra

2. Elementary functions : not so simple

some functions cannot be defined on the entire plane: branch lines where function is undefined

$$z = |z|e^{i\phi} \quad \text{then} \quad \sqrt{z} = \sqrt{|z|}e^{i\frac{\phi}{2}}$$

is undefined for Im z =0, Re z > 0

since $\sqrt{x + i\epsilon} \to \sqrt{x}$

$$0 < \phi < 2\pi$$

but $\sqrt{x - i\epsilon} \to -\sqrt{x}$

so the z-plane (domain plane) needs to be “cut” along the positive real axis

3. It is (often) possible to eliminate the cut by “glueing” another sheet(s) and effectively replacing the z-plane (domain) by a more complicated surface and continue defining f(z).

4 f(z) is differentiable (holomorphic) if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \equiv f'(z_0)$ exists

Key feature: existence f'(z_0) means finite and independent how the limit $z \rightarrow z_0$ is taken

writing z = x + iy and f(z) = u(x,y) + iv(x,y) this implies

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

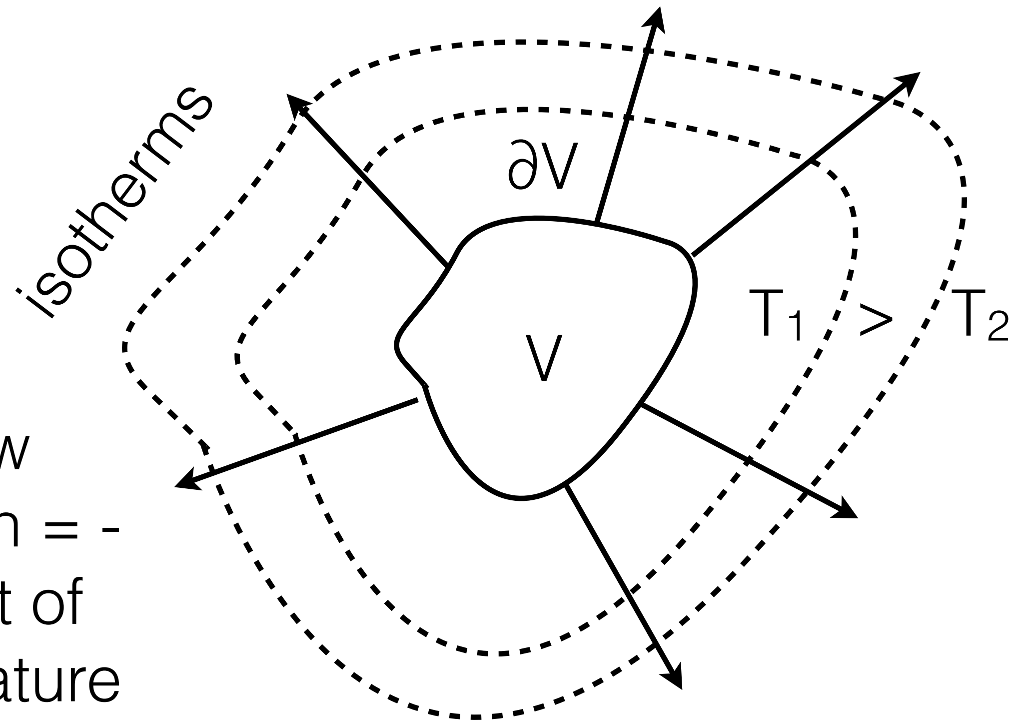
This implies Δu = Δv = 0
where Δ is 2-dim Laplacian
u,v : harmonic functions

Applications:
Explore the connection between harmonic functions and holomorphic functions

Harmonic functions represent solutions to physical problems relating “flows” to “sources”

e.g. mass density vs. velocity flux,
temperature vs heat flow,
electric charge vs electric field
magnetic charge (monopole) vs magnetic field
etc.

Heat flow due to Temperature gradient



$$\text{Temperature gradient} = \vec{\nabla} T$$

$$\text{Energy density} = \rho c T$$

$$-\frac{d}{dt}\int_{Volume} \text{Energy density } dV = \text{Conductivity} \int_{\partial Volume} -\text{Temperature gradient} \cdot d\vec{S}$$

Gauss's law:

$$\int_V \vec{\nabla} \cdot \vec{f} dV = \int_{\partial V} \vec{f} \cdot d\vec{S}$$

$$\rho c \frac{\partial T}{\partial t} = \kappa \Delta T$$

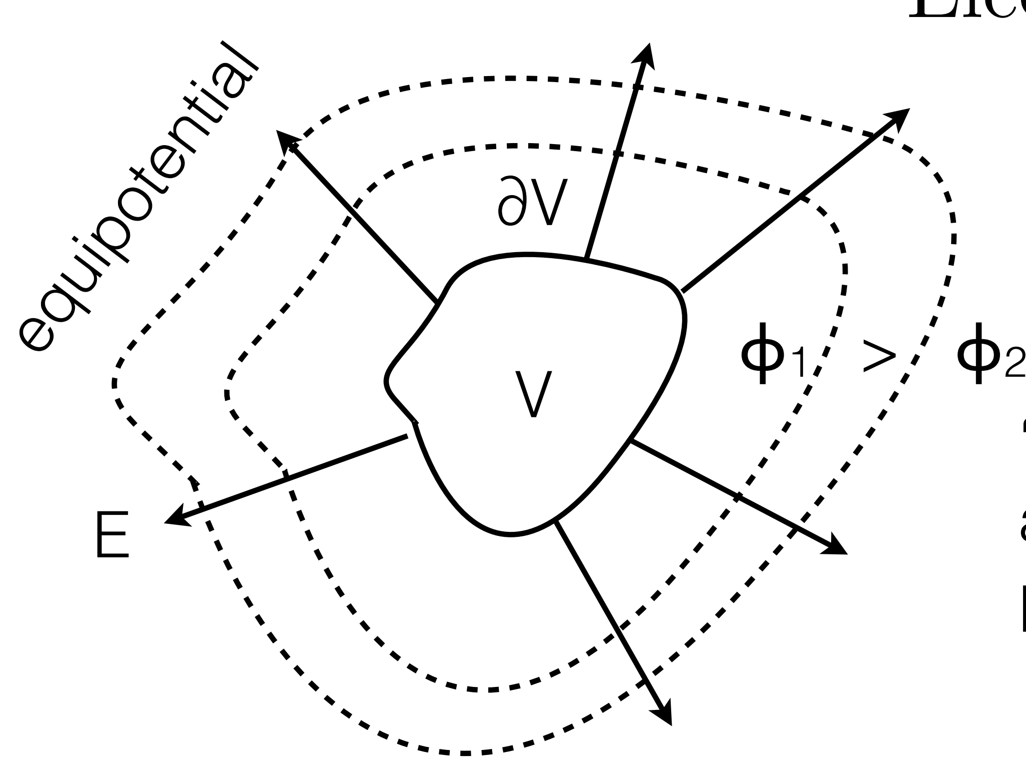
if T is kept constant, then spacial distribution is a harmonic function $\Delta T=0$

For a given isotherm,spacial distribution of temperature can be found by “guessing” a complex function whose real (or imaginary) part has the prescribed value on a line segment (isotherm)

Electric charge vs Electric field (or potential)

$$\int_{Volume} \text{Charge density } dV = \epsilon_0 \int_{\partial Volume} \text{Electric field} \cdot d\vec{S}$$

$$\text{Electric field} = -\text{ Potential gradient} = -\vec{\nabla}\phi$$



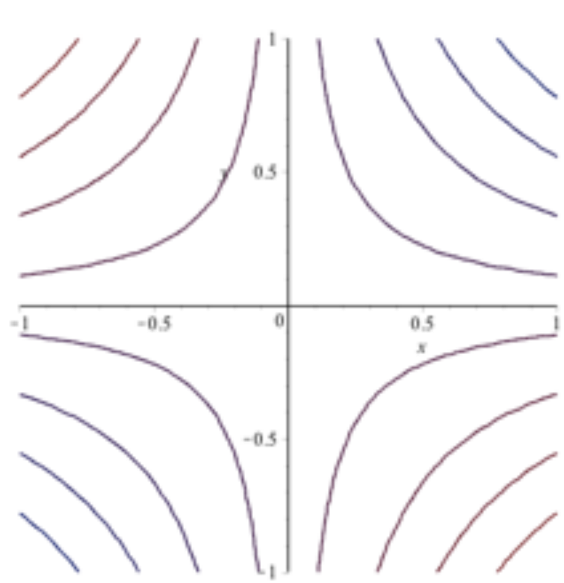
Gauss's law: $\int_V \vec{\nabla} \cdot \vec{f} dV = \int_{\partial V} \vec{f} \cdot d\vec{S}$

$$\Delta\phi = -\frac{\rho}{\epsilon_0}$$

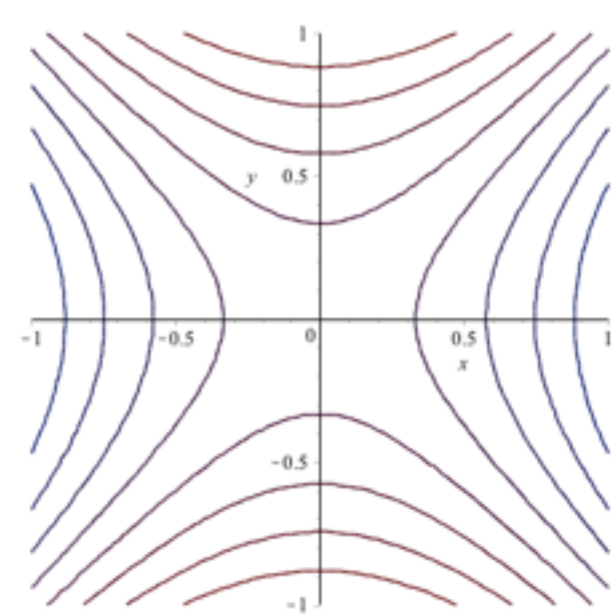
"guess" complex function to represent ϕ
any holomorphic function solves some problem in electrostatics e.g.

$$f(z) = z^2 = (x^2 - y^2) - 2ixy$$

contourplot$(x \cdot y, x = -1..1, y = -1..1)$;

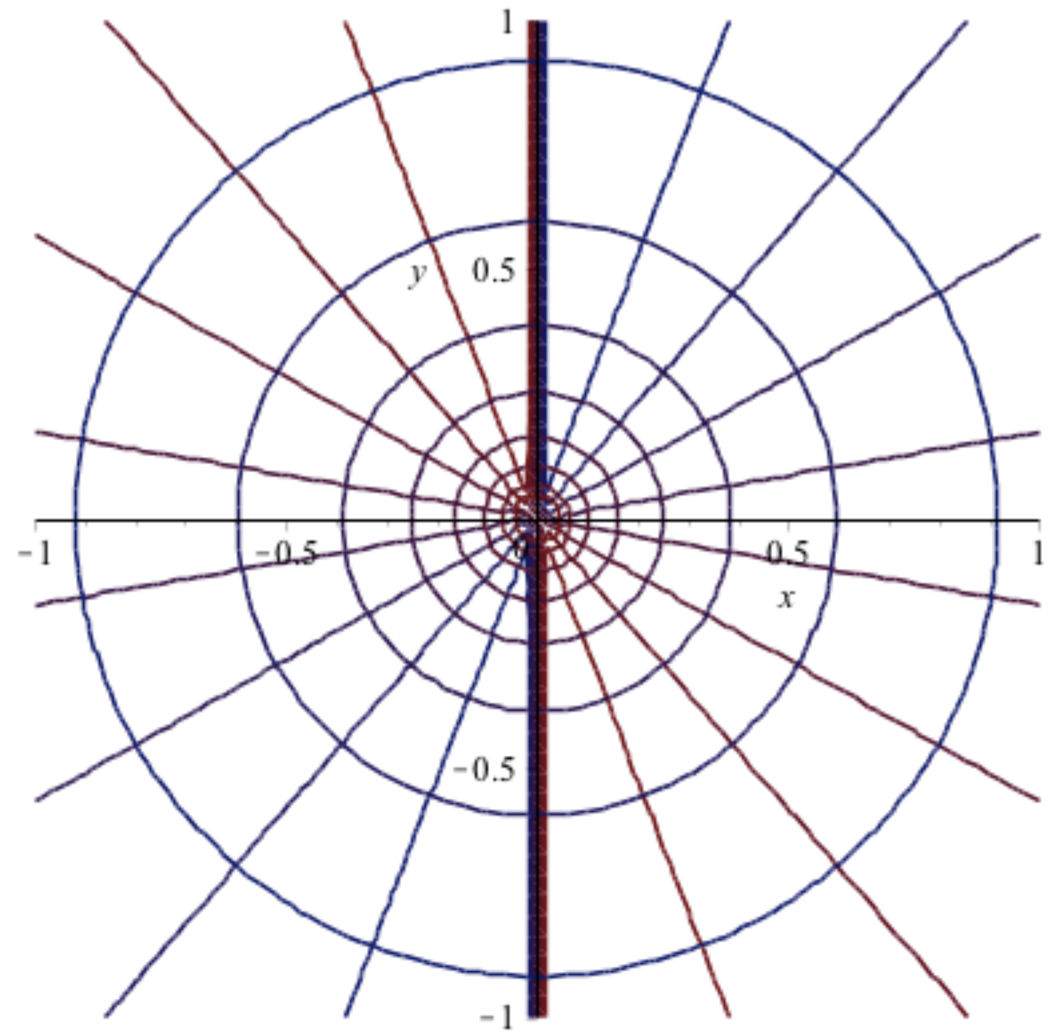


contourplot$(x^2 - y^2, x = -1..1, y = -1..1)$;



Potential of a single charge in 2 dim

$$contourplot\left(\left\{\log\left(x^2+y^2\right), \arctan\left(\frac{y}{x}\right)\right\}, x=-1..1, y=-1..1\right);$$



f(z) = log z: holomorphic except at z=0

$$\Delta \log r = 2\pi\delta^2(\vec{r})$$

$$\int_{S_1} \vec{\nabla} \log r \cdot d\vec{S} = \int_0^{2\pi} d\phi \frac{\vec{r}\cdot\vec{n}}{r} = 2\pi = \int dV \Delta \log r = \int_{S_2} dxdy \Delta \log r$$

$$\vec{\nabla} \log r = \frac{r^i}{r^2}$$

Intermezzo: Magnetic monopoles

EM Fields in a tensor from

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix} \qquad \overline{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} = \begin{pmatrix} 0 & -H_x & -H_y & -H_z \\ H_x & 0 & E_z & -E_y \\ H_y & -E_z & 0 & E_x \\ H_z & E_y & -E_x & 0 \end{pmatrix}$$

Maxwell equations

$$\partial_\nu F^{\nu\mu} = j^\mu \qquad \partial_\nu \overline{F}^{\nu\mu} = j^\mu_{mag}$$

$$\partial_\nu F^{\nu\mu} = j^\mu \;\rightarrow\; \vec{\nabla}\cdot\vec{E} = \rho \qquad -\partial_t\vec{E} + \vec{\nabla}\times\vec{H} = \vec{j}$$

$$\partial_\nu \overline{F}^{\nu\mu} = j^\mu_{mag} \;\rightarrow\; \vec{\nabla}\cdot\vec{H} = 0 \qquad -\partial_t\vec{H} - \vec{\nabla}\times\vec{E} = 0$$

EM Fields in a tensor from

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix} \qquad \overline{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} = \begin{pmatrix} 0 & -H_x & -H_y & -H_z \\ H_x & 0 & E_z & -E_y \\ H_y & -E_z & 0 & E_x \\ H_z & E_y & -E_x & 0 \end{pmatrix}$$

Maxwell equations

$$\partial_\nu F^{\nu\mu} = j^\mu \qquad \partial_\nu \overline{F}^{\nu\mu} = j^\mu_{mag}$$

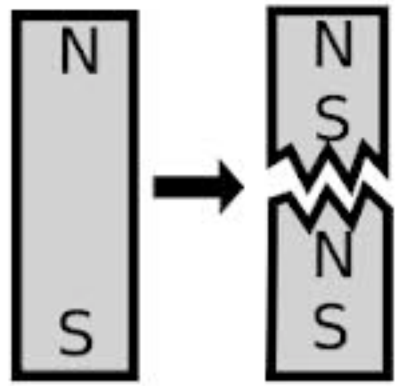
$$\partial_\nu F^{\nu\mu} = j^\mu \;\rightarrow\; \vec{\nabla}\cdot\vec{E} = \rho \qquad -\partial_t\vec{E} + \vec{\nabla}\times\vec{H} = \vec{j}$$

$$\partial_\nu \overline{F}^{\nu\mu} = j^\mu_{mag} \;\rightarrow\; \vec{\nabla}\cdot\vec{H} = \rho_{mag} \qquad -\partial_t\vec{H} - \vec{\nabla}\times\vec{E} = \vec{j}_{mag}$$

adding magnetic charges and currents makes equations more symmetric !!

... but cannot introduce EM **potentials** in a standard way
(divergence of H no longer vanishes)

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \qquad \vec{E} = -\partial_t\vec{A} - \vec{\nabla}\phi \qquad \vec{H} = \vec{\nabla}\times\vec{A}$$

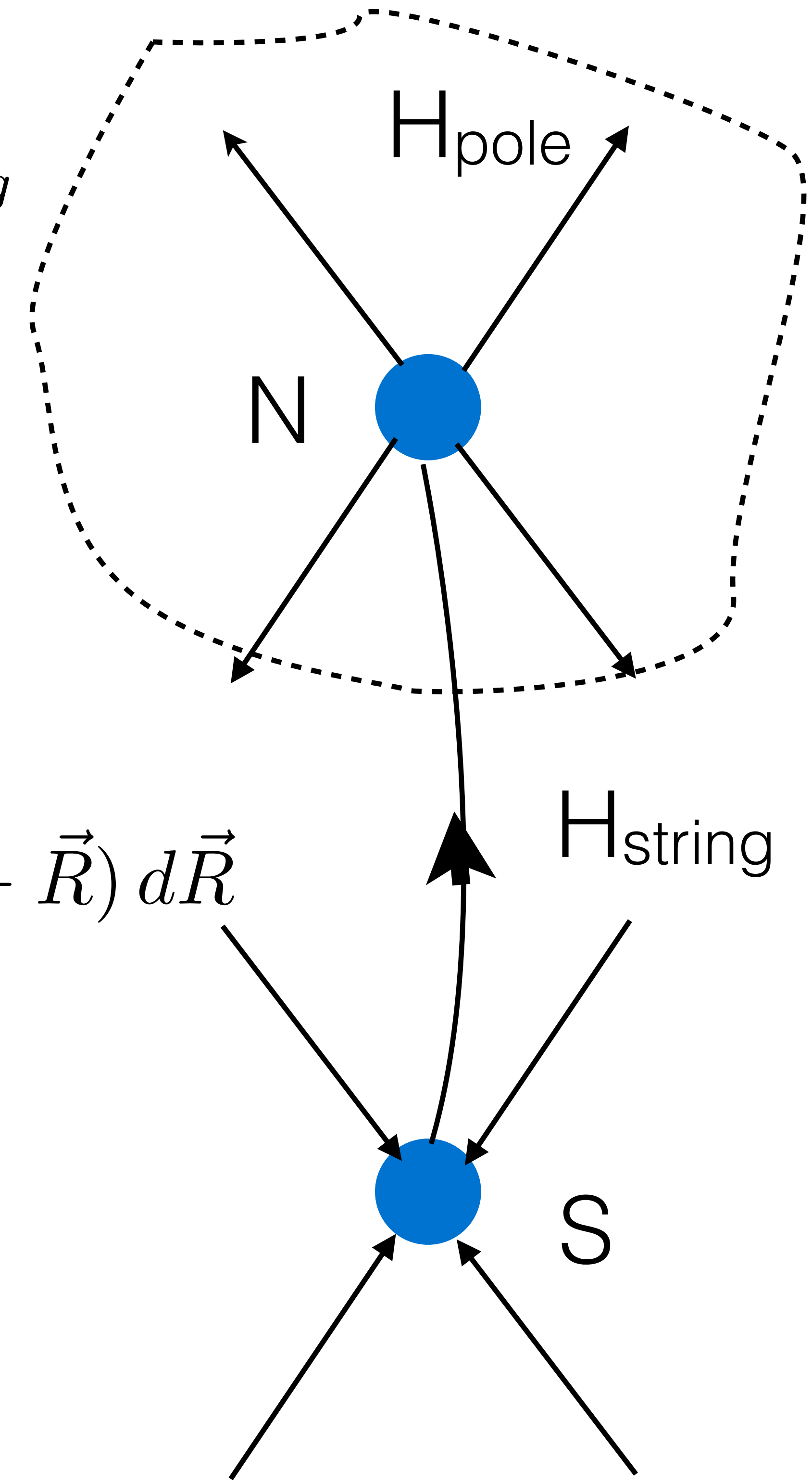


Instead of isolated charge, think of a very long magnet/solenoid

$$\vec{H} \equiv \vec{H}_{pole} - \vec{H}_{string}$$

$$\vec{H}_{pole} = \vec{\nabla} \times \vec{A} + \vec{H}_{string}$$

$$\vec{A} = -\frac{g}{4\pi}\vec{\nabla} \times \int_L \frac{d\vec{R}}{|\vec{r} - \vec{R}|}$$

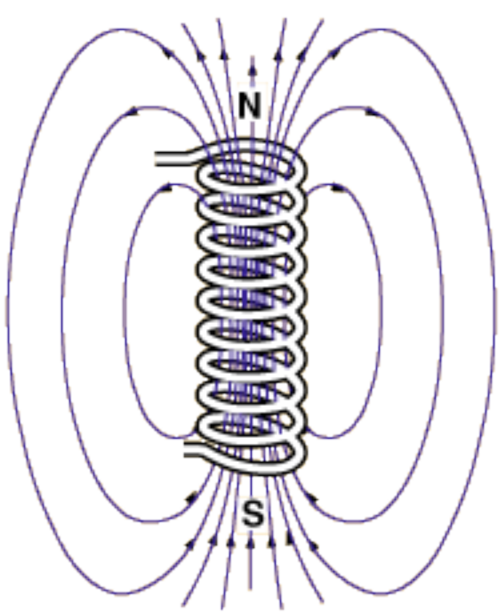


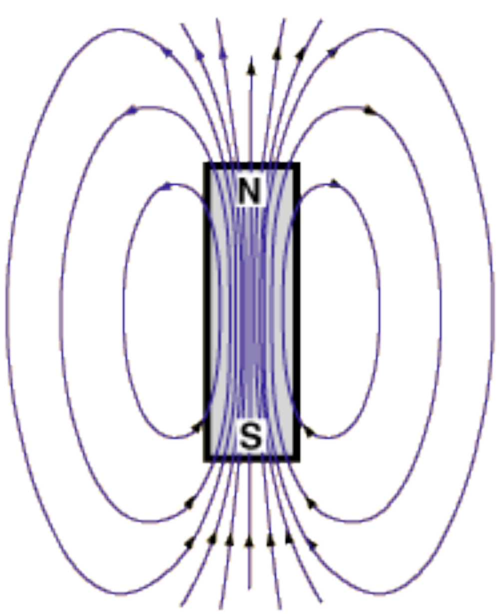
Since $\vec{\nabla}\vec{H} = 0$ it is possible

to introduce A associated with H

For infinitesimally thin solenoid ("string") it magnetic field is along its direction

$$\vec{H}_{string} = g\int_L \delta(\vec{r} - \vec{R})\, d\vec{R}$$





$$\vec{H}_{pole} = \frac{g}{4\pi}\frac{\vec{r}}{r^3}$$

Monopoles in QCD $\vec{B} \to \vec{B}^a, \ a = 1, \cdots N_c^2 - 1 \quad \text{QCD} \ : N_c = 3$

(simplify using N_c=2)

$$B_i = \partial_j A_k - \partial_k A_j \to B_i^a = \partial_j A_k^a - \partial_k A_j^a + \epsilon_{abc}\epsilon_{ijk} A_j^b A_k^c$$

Maxwell (YM) equations are nonlinear

$$\partial^\nu F_{\nu\mu}^a + \epsilon_{abc} A_\nu^b F_{\mu\nu}^c = 0 \qquad B_i^a \sim \frac{x_i x^a}{r^4}$$

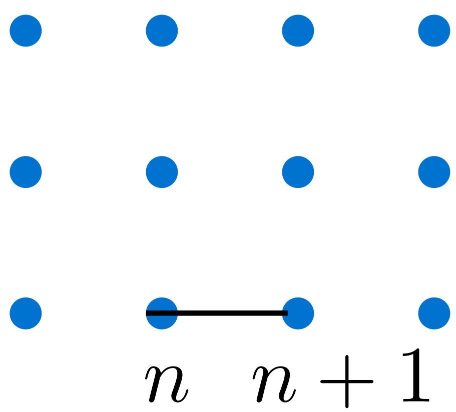
and even in absence of external source have monopole-like solutions (Wu-Yang monopoles)

Unfortunately they are singular (infinite energy (YM equations have no non-trivial classical solutions with finite energy (eg. solitons) or classical glueballs do not exists (Coleman)

But lattice “regularizes” short distances: and monopoles can be fund in lattice simulations

QCD on the lattice : unbound vector potential becomes replaced by an angular variable:

$$\text{" Link Variable "} = e^{i\int_n^{n+1} d\vec{l}\cdot\vec{A}} \to e^{iaA} \in SU(N_c)$$

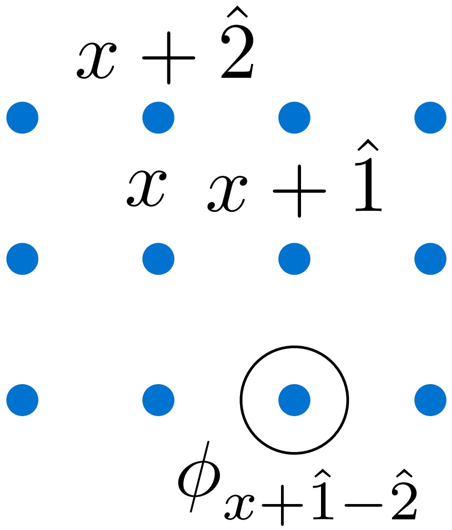


Here A = $A^a T^a$ with T generators of SU(N_c) but consider a simpler theory: QED in 2 dim (N_c =0 and A^a → A = vector potential). Then at each lattice link one defines exp(i a A (along the link)) complex number of unit length. Consider even simpler model, by replacing a vector A by a scalar exp(i a A) → exp(i a ϕ).The simplest interaction which a) couples next-neighbor (eg. local in continuum limit) and b) preserves the angular nature of a ϕ is of the type

$$1 - \cos(a\phi_x - a\phi_{x+\delta})$$

$$H = \frac{1}{a^2}\sum_{x,\delta}[1 - \cos(a\phi_x - a\phi_{x+\delta})] \to \frac{1}{2}a^2\sum_{x,\delta}\frac{(\phi_x - \phi_{x+\delta})^2}{a^2} \to \frac{1}{2}\int dxdy(\partial_i\phi)^2$$

Partition function is then given by: $Z = \int_{-\pi}^{\pi} \Pi_x \frac{d\phi_x}{2\pi} \exp(-\beta\sum_{x,\delta}[1 - \cos(\phi_x - \phi_{x,\delta})]$



configurations are as important

$$\phi_{x+\delta} \sim +\pi - \epsilon \qquad \phi_{x+\delta} \sim -\epsilon$$

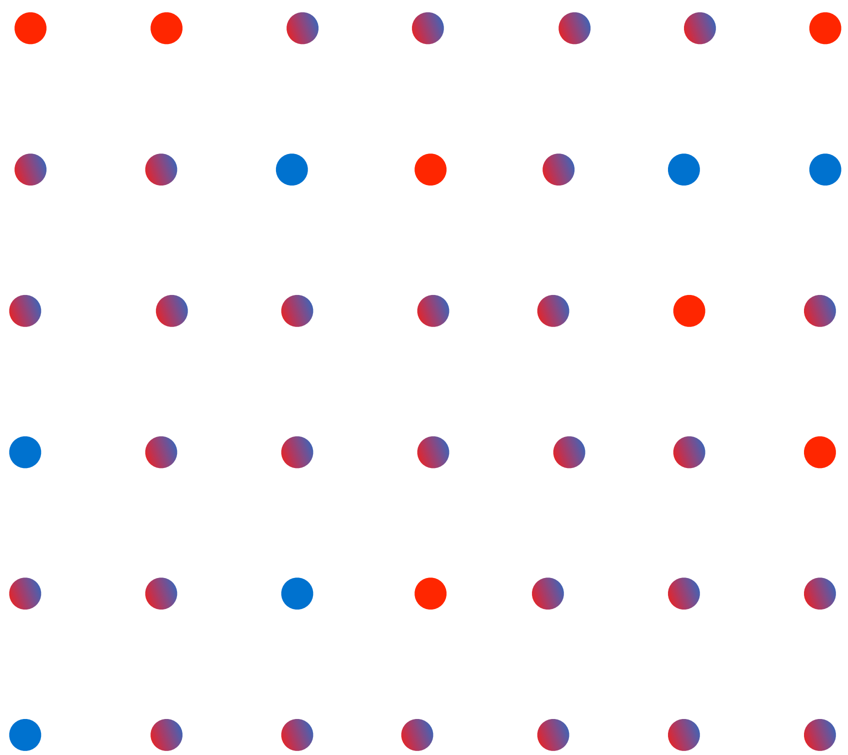
$$\phi_x \sim -\pi + \epsilon \qquad \phi_x \sim +\epsilon$$

There could give “fracture lines” between lattice sites across which ϕ changes by 2π

● $\phi \sim +\pi$ ● $\phi \sim -\pi$

● $-\pi < \phi < +\pi$

high temperature (small-β)

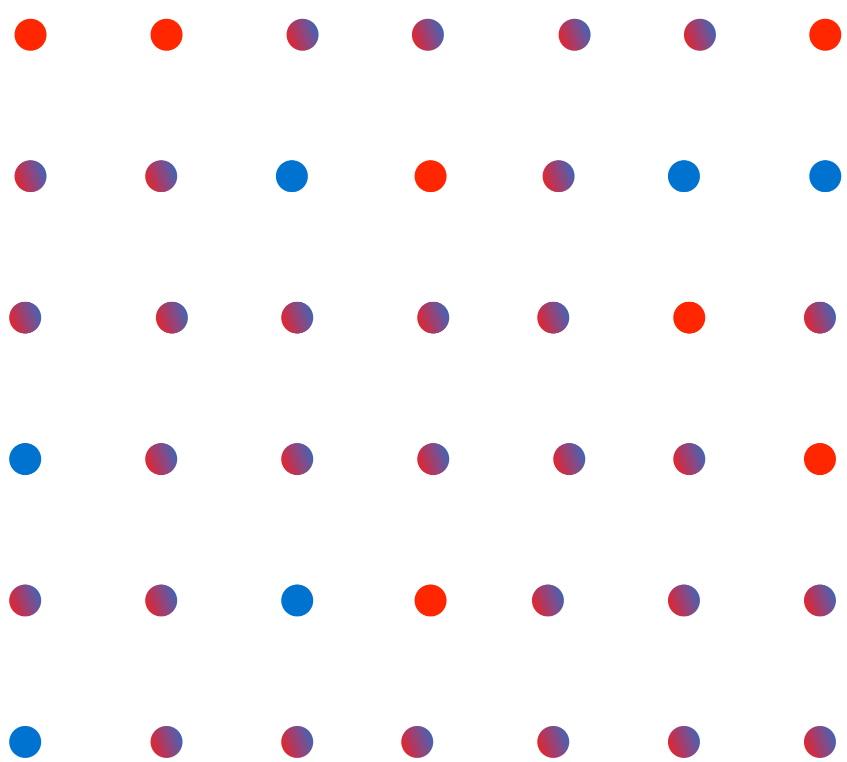


system is disordered

$\phi \sim +\pi$ $\phi \sim -\pi$

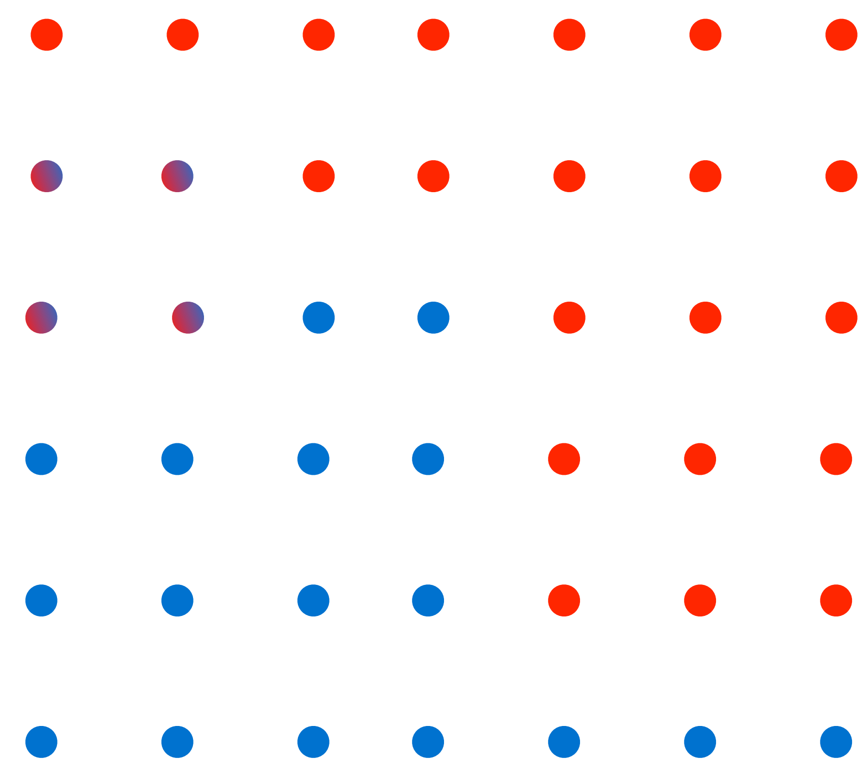
$-\pi < \phi < +\pi$

high temperature (small-β)



system is disordered

low temperature (large -β)

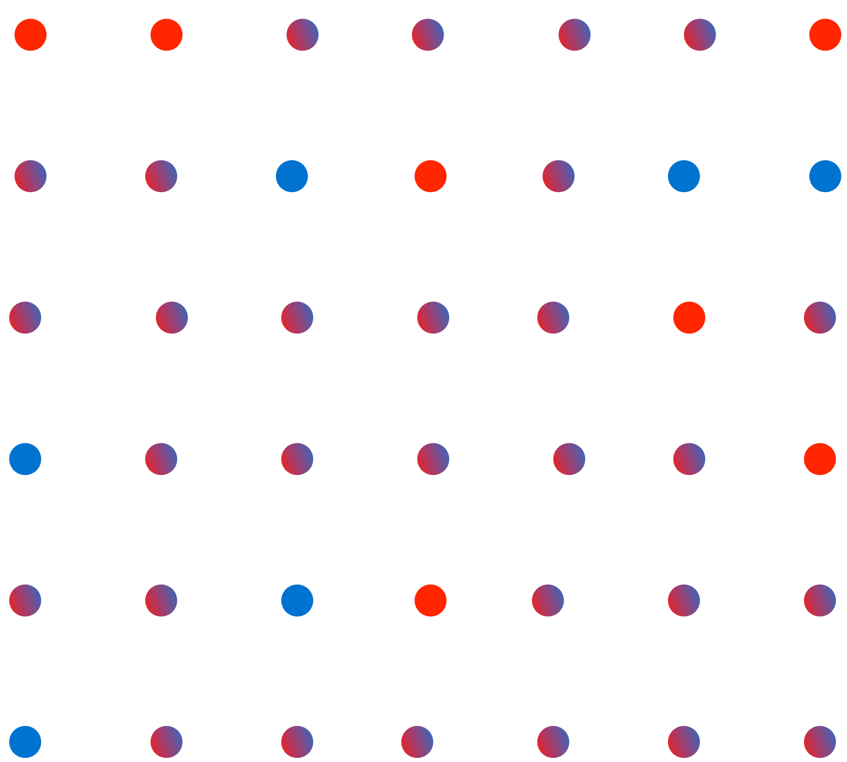


system is ordered

● $\phi \sim +\pi$ ● $\phi \sim -\pi$

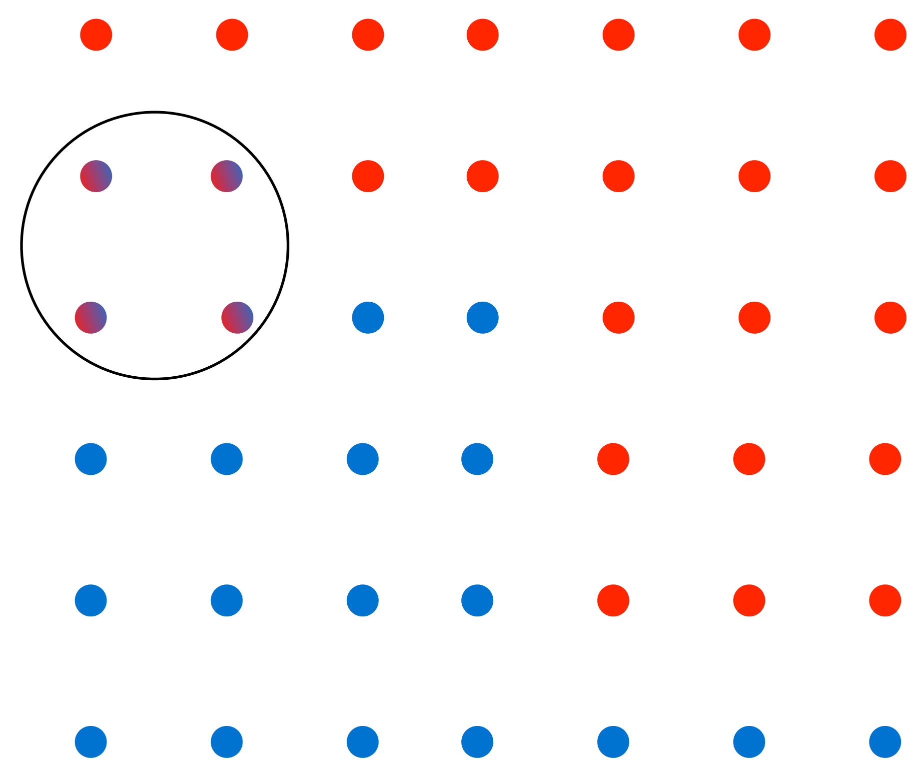
● $-\pi < \phi < +\pi$

high temperature (small-β)



system is disordered

low temperature (large -β)

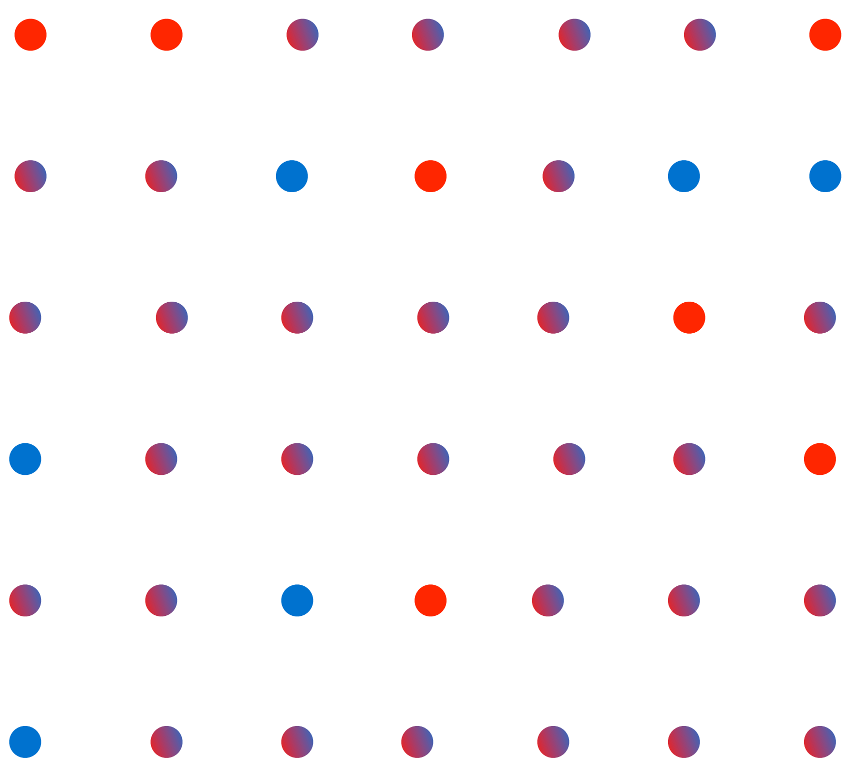


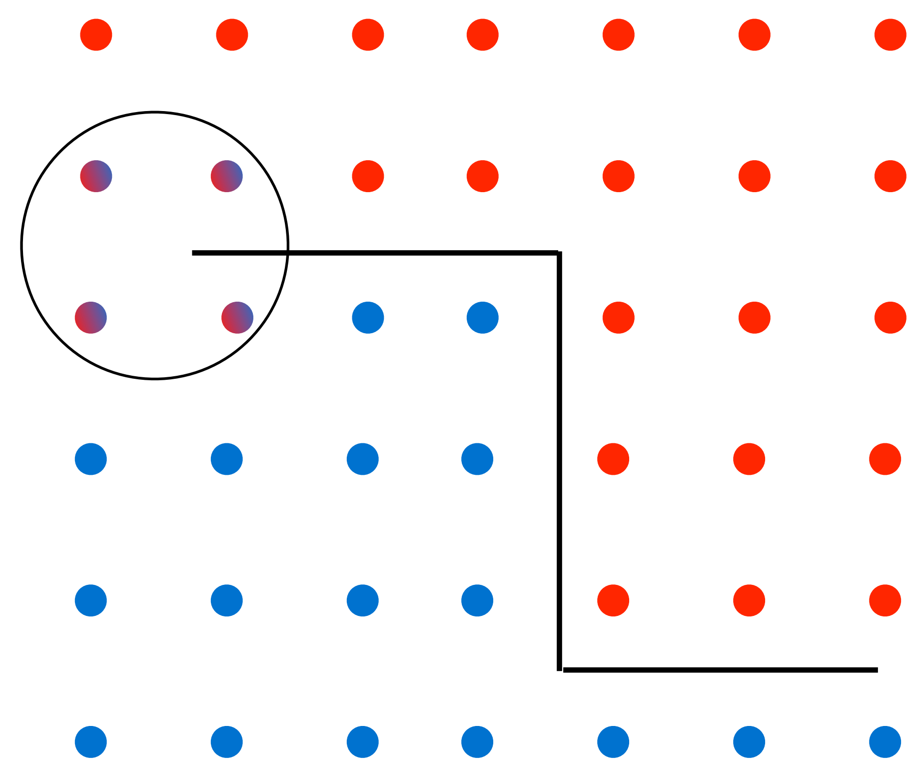
system is ordered

$\phi \sim +\pi$ $\phi \sim -\pi$

$-\pi < \phi < +\pi$

high temperature (small-β) low temperature (large -β)

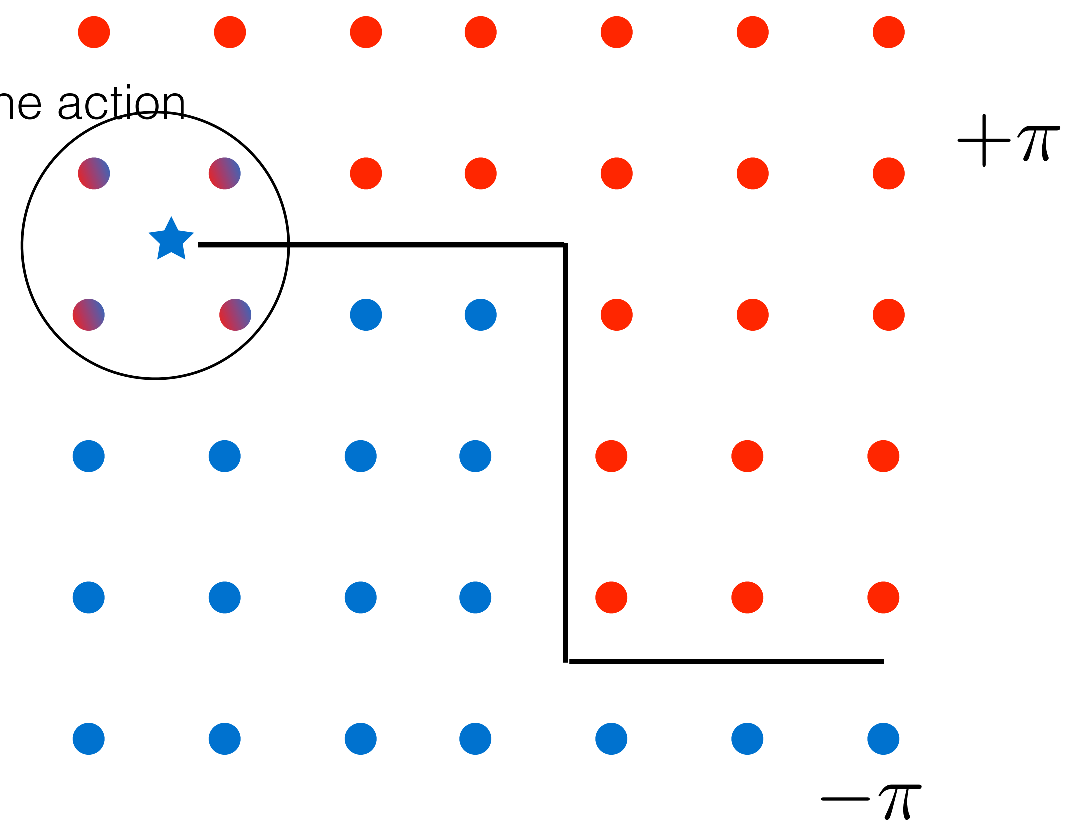




system is disordered system is ordered

looks like a monopole with a string !

region of contribution to the action
due to “fraction”



In the continuum limit $H = \frac{1}{2}\int dxdy(\partial_i \phi)^2$ and minimum of the energy satisfies **Δϕ** = 0. Once we have introduced a set of monopoles (in 2dim called vortices) placed at points x_a with strength q_a it means that we have introduced a multivalued **ϕ** which changes by $2\pi q_a$ every time we go around a vortex. In this case the harmonic function solution to the 2dim Laplace equation has the form

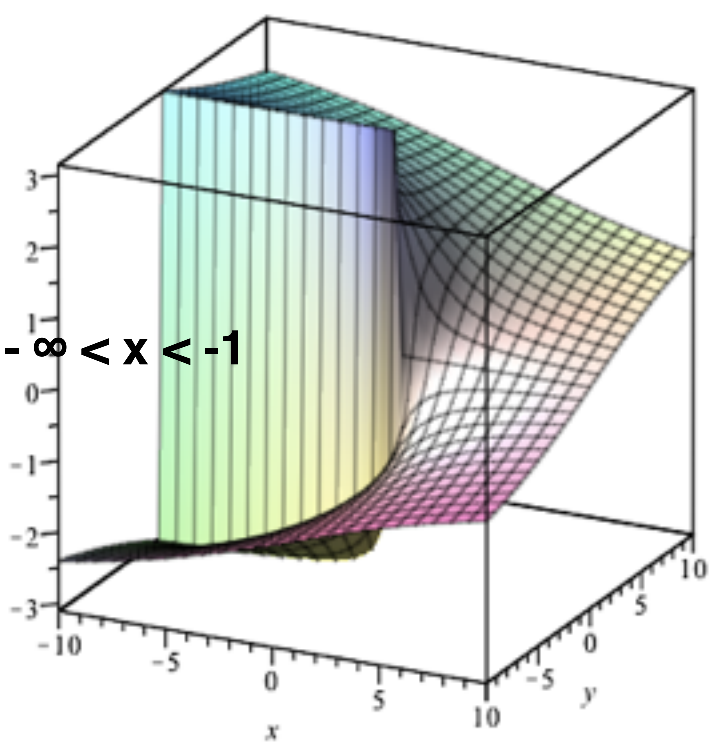
$$\phi = \sum_{a=1}^{N} q_a \,\mathrm{Im}\log(z - z_a)$$

with $z_a = x_a + iy_a$ being the location of the vortex

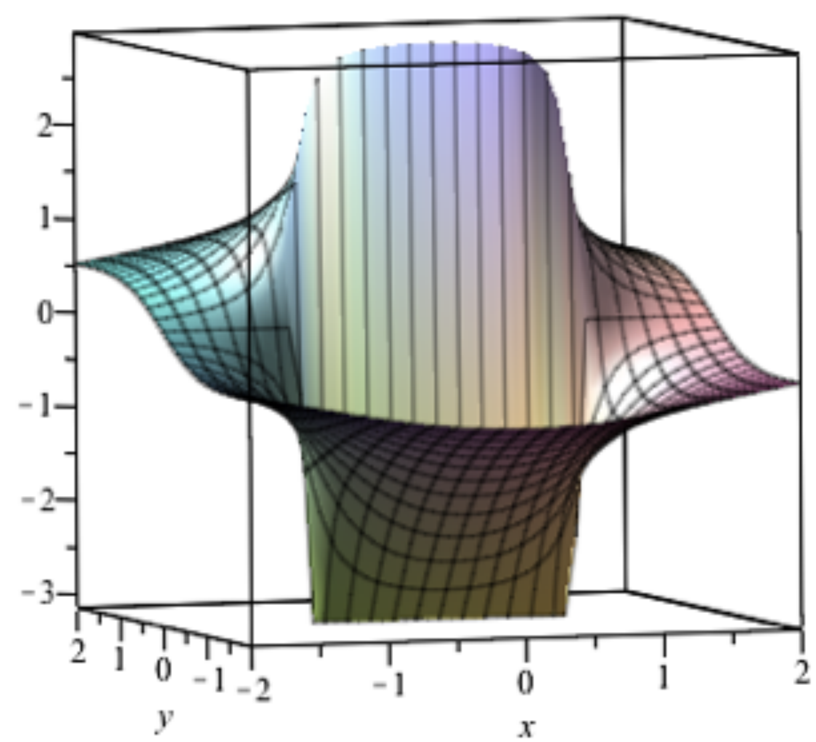
$$N = 2\; q_1 = 1\; z_1 = -1 + 0i$$
$$q_2 = -1\; z_2 = +1 + 0i$$

$$N = 1\; q = 1\; z = -1 + 0i$$

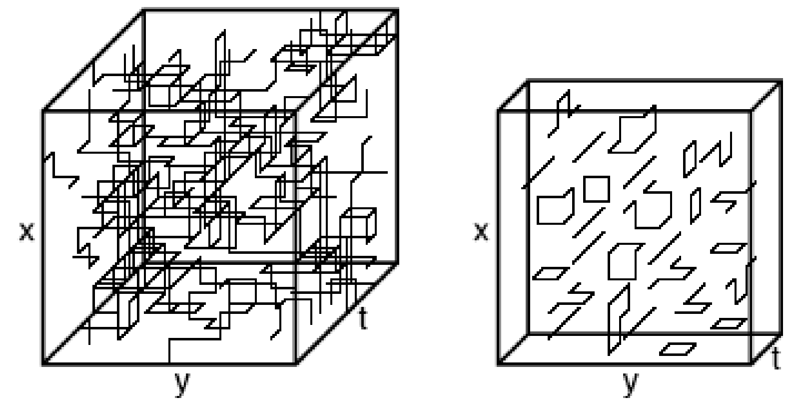
String: - ∞ < x < -1



String: - 1< x < +1



QCD: Confinement due to percolating (center) vortices



confined phase   deconfined phase