Chiral Perturbation Theory for Baryons

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0. Motivation

1. QCD and chiral symmetry

2. Spontaneous symmetry breaking

3. Chiral perturbation theory for mesons

4. Chiral perturbation theory for baryons

0. Motivation of keywords

- Chiral perturbation theory (ChPT) is the effective field theory (EFT) of the Standard Model/strong interactions at low energies.

- EFTs are low-energy approximations to (more) fundamental theories.

- Instead of solving the underlying theory, low-energy physics is described with a set of variables (effective degrees of freedom) that is suited for the particular energy region you are interested in.

- In our case: Pions and nucleons instead of the more fundamental quarks and gluons of QCD.

- Calculate physical quantities in terms of an expansion in q/Λ, where q stands for momenta or masses that are smaller than a certain momentum scale Λ.

- There exists a regime where both fundamental and effective theories yield the same results.

- EFTs are based on the most general Lagrangian, which includes all terms that are compatible with the symmetries of the underlying theory. $\Rightarrow$ Infinite number of terms. Each term is accompanied by a low-energy coupling constant (LEC).

- One needs a method that allows one to decide which terms contribute in a calculation up to a certain accuracy: Weinberg's power counting.

- In actual calculations only a finite number of terms in the expansion in q/Λ has to be considered. $\Rightarrow$ Predictive power.

- Effective field theories are non-renormalizable in the traditional sense. However, as long as one considers all terms that are allowed by the symmetries, divergences that occur in calculations up to any given order of q/Λ can be renormalized by redefining fields and parameters of the Lagrangian of the effective field theory. The so-called non-renormalizable theories are actually just as renormalizable as renormalizable theories.

	Fundamental theory	Effective field theory
	QCD	ChPT
dof	quarks & gluons	Goldstone bosons (+ other hadrons)
parameters	g_3 + quark masses	(∞ # of) LECs + quark masses

Simplified analogies between multipole expansion and EFT

Multipole expansion	EFT
$\lvert\vec{x}\,\rvert \gg R$	$q \ll \Lambda_\chi$
$\phi(\vec{x}) = \sum_{lm} \boxed{q_{lm}} \boxed{\frac{1}{2l+1}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}}$	$\mathcal{L}_{\text{eff}} = \sum_{lm} \boxed{c_{lm}} \boxed{\mathcal{L}_{lm}}$
multipole moment q_{lm}	LEC c_{lm}
$\frac{1}{2l+1}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$	Structures $\mathcal{L}_{lm}$

- In principle, infinite number of terms.
  Actual calculation: Truncation at finite order.

- Systematic improvement possible.

- **Aim of these lectures:**

  Most general description of the strong interactions at low energies: $\pi\pi$, πN.

- **Challenge:**

  We need the

  1. **the most general Lagrangian;**

  2. **a consistent power counting scheme to perform perturbative calculations.**

1. QCD and chiral symmetry

Apply the gauge principle with respect to (color) SU(3)

Matter fields: Quarks

flavor	u	d	s
charge [e]	$2/3$	$-1/3$	$-1/3$
mass [MeV]	$1.5 - 3.0$	$3 - 7$	95 ± 25
flavor	c	b	t
charge [e]	$2/3$	$-1/3$	$2/3$
mass [GeV]	1.25 ± 0.09	4.20 ± 0.07	174.2 ± 3.3

(Masses from PDG: Review of Particle Physics, 2006)

Quark field components $q_{f A \alpha}$

$f = 1, 2, 3, 4, 5, 6$: flavor index (u, d, s, c, b, t)
$A = 1, 2, 3$: color index (red, green, blue)
$\alpha = 1, 2, 3, 4$: Dirac spinor index

Gauge potentials (gluons) $\mathcal{A}_{a\mu}$

$a = 1, \cdots, 8$: color index
$\mu = 0, \cdots, 3$: Lorentz index

and field strengths $\mathcal{G}_{a\mu\nu}$

$$\mathcal{G}_{a\mu\nu} = \partial_\mu \mathcal{A}_{a\nu} - \partial_\nu \mathcal{A}_{a\mu} - g_3 f_{abc} \mathcal{A}_{b\mu} \mathcal{A}_{c\nu}.$$

Recall Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \cdots, \quad \left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2} \right] = i f_{abc} \frac{\lambda_c}{2}.$$

$$\mathcal{L}_{\rm QCD} = \sum_{f,f'=1}^{6}\sum_{A,A'=1}^{3}\sum_{\alpha,\alpha'=1}^{4} \bar{q}_{fA\alpha}\Big[(\gamma^{\mu}_{\alpha\alpha'} i\partial_{\mu} - m_f \delta_{\alpha\alpha'})\delta_{AA'}$$

$$\underbrace{-g_3 \sum_{a=1}^{8} \mathcal{A}_{a\mu} \frac{\lambda^{c}_{a,AA'}}{2}\gamma^{\mu}_{\alpha\alpha'}}_{\text{from gauge principle}}\Big]\delta_{ff'} q_{f'A'\alpha'} - \frac{1}{4}\sum_{a=1}^{8} \mathcal{G}_{a\mu\nu}\mathcal{G}^{\mu\nu}_{a}.$$

Short version

$$\mathcal{L}_{\rm QCD} = \sum_{f=\substack{u,d,s,\\ c,b,t}} \bar{q}_f (i\not{D} - m_f) q_f - \frac{1}{4}\mathcal{G}_{a\mu\nu}\mathcal{G}^{\mu\nu}_{a}.$$

Extremely short version ($\mathcal{G}_{\mu\nu} = \mathcal{G}_{a\mu\nu}\lambda_a/2$)

$$\mathcal{L}_{\rm QCD} = \bar{q}(i\not{D} - \mathcal{M})q - \frac{1}{2}\mathrm{Tr}_c(\mathcal{G}_{\mu\nu}\mathcal{G}^{\mu\nu}).$$

In principle (quark masses originate from electroweak symmetry breaking)

$$\mathcal{L}_\theta = -\frac{g_3^2\bar{\theta}}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}\sum_{a=1}^{8}\mathcal{G}_a^{\mu\nu}\mathcal{G}_a^{\rho\sigma} = -\frac{g_3^2\bar{\theta}}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\text{Tr}_c(\mathcal{G}^{\mu\nu}\mathcal{G}^{\rho\sigma}),\ \epsilon_{0123} = 1.$$

So-called θ term implies explicit P and CP violation in the strong interactions:

$$\left.\begin{array}{c} P : \mathcal{G}_a^{\mu\nu}(t,\vec{x}) \mapsto \mathcal{G}_{a\mu\nu}(t,-\vec{x}), \\ \epsilon_{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma}, \end{array}\right\} \Rightarrow \mathcal{L}_\theta(t,\vec{x}) \mapsto -\mathcal{L}_\theta(t,-\vec{x}).$$

1. Electric dipole moment of the neutron; empirical information: very small.

2. $\eta \to 2\pi$, $\eta \to 4\pi^0$ $(\to 8\gamma)$ (not observed).

3. Strong CP problem: Why is $\bar{\theta}$ so small?

Accidental, global symmetries of $\mathcal{L}_{\mathrm{QCD}}$

The pion is special!

quark content	mesons
$u\bar{d}$	π^+, ρ^+
$(d\bar{d} - u\bar{u})/\sqrt{2}$	π^0, ρ^0
$d\bar{u}$	π^-, ρ^-

$$M_{\pi^+} = 140\,\text{MeV} \;\ll\; M_\rho = 776\,\text{MeV},$$
$$M_\pi \;\ll\; m_p = 938\,\text{MeV}.$$

$$M_{\pi^+} \;<\; M_{K^+} = 494\,\text{MeV} \ll M_{\underbrace{D^+}_{c\bar{d}}} = 1869\,\text{MeV}.$$

$$\begin{pmatrix} m_u = (1.5 - 3.0)\,\text{MeV} \\ m_d = (3 - 7)\,\text{MeV} \\ m_s = (95 \pm 25)\,\text{MeV} \end{pmatrix} \ll \Lambda_\chi \approx 1\,\text{GeV} \leq \begin{pmatrix} m_c = (1.25 \pm 0.09)\,\text{GeV} \\ m_b = (4.20 \pm 0.07)\,\text{GeV} \\ m_t = (174.2 \pm 3.3)\,\text{GeV} \end{pmatrix}$$

Motivation

$$m_p \gg 2m_u + m_d$$

Consider light-flavor quarks in so-called chiral limit $m_u, m_d, m_s \to 0$ as starting point in discussion of low-energy QCD:

$$\mathcal{L}^0_{\text{QCD}} = \sum_{l=u,d,s} \bar{q}_l i \not{D}\, q_l - \frac{1}{4} \sum_{a=1}^{8} \mathcal{G}_{a\mu\nu} \mathcal{G}_a^{\mu\nu} + \text{heavy quarks}$$

Projection operators: $q = \left[\frac{1}{2}(1 - \gamma_5) + \frac{1}{2}(1 + \gamma_5)\right] q \equiv q_L + q_R$

$$= \sum_{l=u,d,s} (\bar{q}_{L,l} i \not{D}\, q_{L,l} + \bar{q}_{R,l} i \not{D}\, q_{R,l}) - \frac{1}{4} \sum_{a=1}^{8} \mathcal{G}_{a\mu\nu} \mathcal{G}_a^{\mu\nu} + \cdots$$

What are the global symmetries of $\mathcal{L}^0_{\mathrm{QCD}}$?

$\mathcal{L}^0_{\mathrm{QCD}}$ is invariant under (covariant derivative flavor independent!)

$$\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \mapsto U_L \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} = \exp\left(-i \sum_{a=1}^{8} \Theta_a^L \frac{\lambda_a}{2}\right) e^{-i\Theta^L} \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix}$$

plus analogous expression for right-handed fields

classical global $\mathrm{U}(3)_L \times \mathrm{U}(3)_R$ symmetry

After quantization: $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ symmetry $\Rightarrow 2\times 8+1 = 17$ conserved currents

But: Spectrum does not follow (approximate) $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetry

Solution: Spontaneous symmetry breaking (Goldstone theorem)

Explicit chiral symmetry breaking due to quark masses

$$\mathcal{M} = \text{diag}(m_u, m_d, m_s)$$

Quark-mass term mixes left- and right-handed fields

$$\mathcal{L}_{\mathcal{M}} = -\bar{q}\mathcal{M}q = -(\bar{q}_R\mathcal{M}q_L + \bar{q}_L\mathcal{M}q_R)$$

$$\begin{aligned}
\partial_\mu V_a^\mu &= i\bar{q}[\mathcal{M}, \frac{\lambda_a}{2}]q \\
\partial_\mu A_a^\mu &= i\bar{q}\{\frac{\lambda_a}{2}, \mathcal{M}\}\gamma_5 q \\
\partial_\mu V^\mu &= 0 \\
\partial_\mu A^\mu &= 2i\bar{q}\mathcal{M}\gamma_5 q + \frac{3g^2}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathcal{G}_a^{\mu\nu}\mathcal{G}_a^{\rho\sigma}, \qquad \epsilon_{0123} = 1
\end{aligned}$$

Summary

- Massless quarks: 16 conserved currents $L^{\mu,a}$ and $R^{\mu,a}$ ($V^{\mu,a}$ and $A^{\mu,a}$) + 1 conserved singlet vector current V^{μ}. Singlet axial-vector current A^{μ} has an anomaly.

- For any value of quark masses: flavor currents $\bar{u}\gamma^{\mu}u$, $\bar{d}\gamma^{\mu}d$, and $\bar{s}\gamma^{\mu}s$ are always conserved.

- Equal quark masses $m_u = m_d = m_s$:

  8 conserved vector currents $V^{\mu,a}$ ($[\lambda_a, 1] = 0$).
  SU(3) flavor symmetry.

  8 axial-vector currents $A^{\mu,a}$ are not conserved.
  Microscopic origin of the PCAC relation (partially conserved axial-vector current).

- $m_u = m_d$: isospin symmetry.

2. Spontaneous symmetry breaking

Example: O(3) sigma model

$$\begin{aligned}\mathcal{L}(\vec{\Phi}, \partial_\mu \vec{\Phi}) &= \mathcal{L}(\Phi_1, \Phi_2, \Phi_3, \partial_\mu \Phi_1, \partial_\mu \Phi_2, \partial_\mu \Phi_3) \\ &= \frac{1}{2}\partial_\mu \Phi_i \partial^\mu \Phi_i - \frac{m^2}{2}\Phi_i \Phi_i - \frac{\lambda}{4}(\Phi_i \Phi_i)^2\end{aligned}$$

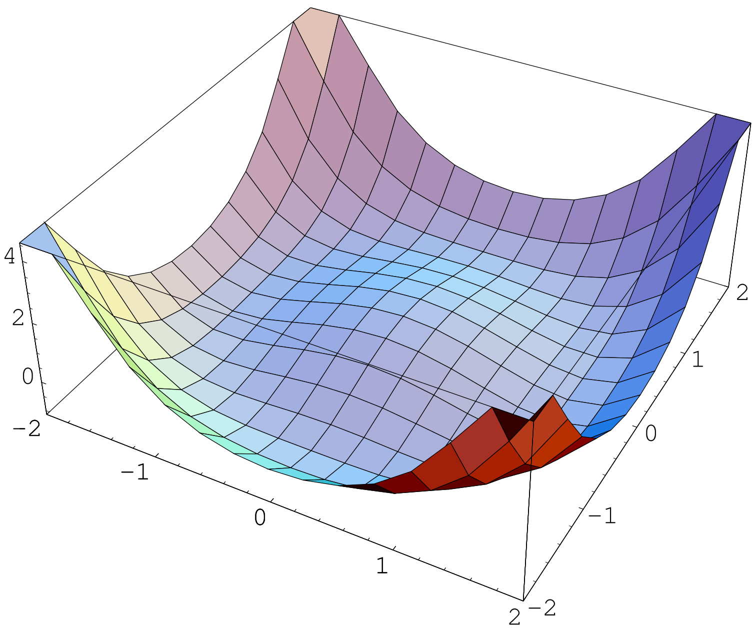
$m^2 < 0$, $\lambda > 0$, Hermitian fields Φ_i

$\mathcal{L}$ invariant under a global "isospin" rotation

$$g \in \mathbf{SO(3)}: \; \Phi_i \to \Phi_i' = D_{ij}(g)\Phi_j = (e^{-i\alpha_k T_k})_{ij}\Phi_j$$

$$[T_i, T_j] = i\epsilon_{ijk}T_k$$

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



Two-dimensional rotationally invariant potential: $\mathcal{V}(x, y) = -(x^2 + y^2) + \frac{(x^2+y^2)^2}{4}$

Exercise: Determine the minimum of the potential

$$\mathcal{V}(\Phi_1, \Phi_2, \Phi_3) = \frac{m^2}{2}\Phi_i\Phi_i + \frac{\lambda}{4}(\Phi_i\Phi_i)^2$$

We find

$$|\vec{\Phi}_{\mathrm{min}}| = \sqrt{\frac{-m^2}{\lambda}} \equiv v, \quad |\vec{\Phi}| = \sqrt{\Phi_1^2 + \Phi_2^2 + \Phi_3^2}$$

$\vec{\Phi}_{\rm min}$ can point in any direction in isospin space
$\Rightarrow$ non-countably infinite number of degenerate vacua

Spontaneous symmetry breaking (hidden symmetry)

Any infinitesimal external perturbation which is not invariant under SO(3) will select a particular direction.
Appropriate orientation of the internal coordinate frame $\Rightarrow$

$$\vec{\Phi}_{\rm min} = v\hat{e}_3 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$$

$\vec{\Phi}_{\rm min}$ not invariant under full group $G =$ SO(3)

Rotations about the 1 and 2 axis change $\vec{\Phi}_{\rm min}$

$$T_1\vec{\Phi}_{\rm min} = v\begin{pmatrix} 0 \\ -i \\ 0 \end{pmatrix}, \qquad T_2\vec{\Phi}_{\rm min} = v\begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}$$

$\vec{\Phi}_{\min}$ invariant under subgroup H of G: rotations about the 3 axis

$$h \in H: \quad \vec{\Phi}' = D(h)\vec{\Phi} = e^{-i\alpha_3 T_3}\vec{\Phi}, \quad D(h)\vec{\Phi}_{\min} = \vec{\Phi}_{\min}, \quad T_3\vec{\Phi}_{\min} = 0$$

Exercise: Expand Φ_3 with respect to v: $\Phi_3(x) = v + \eta(x)$

New expression for the potential

$$\tilde{\mathcal{V}} = \frac{1}{2}(-2m^2)\eta^2 + \lambda v\eta(\Phi_1^2 + \Phi_2^2 + \eta^2) + \frac{\lambda}{4}(\Phi_1^2 + \Phi_2^2 + \eta^2)^2 - \frac{\lambda}{4}v^4$$

$$m_{\Phi_1}^2 = m_{\Phi_2}^2 = 0, \quad m_\eta^2 = -2m^2$$

Model-independent feature of the above example:

1. For each of the two generators T_1 and T_2 which do not annihilate the ground state one obtains a massless Goldstone boson

2. Number of Goldstone bosons is determined by the structure of the symmetry groups:

   - G symmetry group of the Lagrangian, n_G generators

   - H subgroup with n_H generators which leaves the ground state after spontaneous symmetry breaking invariant

   - # of Goldstone bosons: $n_G - n_H$

3. Criterion for ssb: Non-vanishing vacuum expectation value of some Hermitian operator, here $\langle 0|\Phi_3(0)|0\rangle = v$.

Explicit symmetry breaking: A first look

Modify potential by adding $a\Phi_3$,

$$\mathcal{V}(\Phi_1, \Phi_2, \Phi_3) = \frac{m^2}{2}\Phi_i\Phi_i + \frac{\lambda}{4}(\Phi_i\Phi_i)^2 + a\Phi_3,$$

$m^2 < 0$, $\lambda > 0$, $a > 0$ and real fields Φ_i.

New potential has lower symmetry: O(2) symmetry (rotations about the 3 axis)

Conditions for the new minimum (from $\vec{\nabla}_\Phi \mathcal{V} = 0$) read

$$\Phi_1 = \Phi_2 = 0, \quad \lambda\Phi_3^3 + m^2\Phi_3 + a = 0$$

Exercise: Solve using a perturbative ansatz

$$\langle\Phi_3\rangle = \Phi_3^{(0)} + a\Phi_3^{(1)} + \mathcal{O}(a^2).$$

Result

$$\Phi_3^{(0)} = \pm\sqrt{-\frac{m^2}{\lambda}}, \quad \Phi_3^{(1)} = \frac{1}{2m^2}.$$

$\Phi_3^{(0)}$: Result without explicit breaking.

Expand potential with $\Phi_3 = \langle \Phi_3 \rangle + \chi \Rightarrow$

$$m^2_{\Phi_1} = m^2_{\Phi_2} \; = \; a\sqrt{\frac{\lambda}{-m^2}}, \qquad \left(m^2_\chi = -2m^2 + 3a\sqrt{\frac{\lambda}{-m^2}} \right).$$

Remarks:

- The Goldstone bosons have acquired a mass.
- Squared masses proportional to a.
- Quantum corrections lead to observables which are nonanalytic in the symmetry breaking parameter a, e.g. $a\ln(a)$ (so-called chiral logarithms).
- Analogue of a in QCD: Quark masses.

Spontaneous Symmetry Breaking in QCD

Indications from the Hadron Spectrum

Example: $H_{\rm str}$ is isospin invariant

$$[H_{\rm str}, T_i] = 0, \qquad [T_i, T_j] = i\epsilon_{ijk}T_k$$

Hadrons can be classified as irreducible multiplets of isospin SU(2)

$$T = 0: \quad d$$

$$T = \frac{1}{2}: \quad \begin{pmatrix} p \\ n \end{pmatrix}, \quad \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}, \quad \begin{pmatrix} \bar{K}^0 \\ K^- \end{pmatrix}$$

$$T = 1: \quad \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}$$

$$T = \frac{3}{2}: \quad \begin{pmatrix} \Delta^{++} \\ \Delta^+ \\ \Delta^0 \\ \Delta^- \end{pmatrix}$$

- Q: Where does this symmetry come from?

- A: Accidental global symmetry of QCD

Consider linear combinations

$$\begin{aligned} Q_V^a &= Q_R^a + Q_L^a \overset{P}{\mapsto} Q_V^a \\ Q_A^a &= Q_R^a - Q_L^a \overset{P}{\mapsto} -Q_A^a \end{aligned}$$

$$[H^0_{\mathrm{QCD}}, Q_V^a] = [H^0_{\mathrm{QCD}}, Q_A^a] = 0$$

Naive expectation: Parity doubling

Let

$$H^0_{\mathrm{QCD}}|\Psi\rangle = E|\Psi\rangle, \qquad P|\Psi\rangle = |\Psi\rangle$$

Construct new state $|\Phi\rangle = Q_A|\Psi\rangle$ **(superscript** a **suppressed)**

$$\begin{aligned} H^0_{\mathrm{QCD}}|\Phi\rangle &= H^0_{\mathrm{QCD}}Q_A|\Psi\rangle = Q_A \underbrace{H^0_{\mathrm{QCD}}|\Psi\rangle}_{E|\Psi\rangle} = E|\Phi\rangle \\ P|\Phi\rangle &= PQ_A|\Psi\rangle = \underbrace{PQ_AP^{-1}}_{-Q_A}\underbrace{P|\Psi\rangle}_{|\Psi\rangle} = -|\Phi\rangle \end{aligned}$$

Not observed in hadronic spectrum

- **Q: What's wrong?**

- **A: We have tacitly assumed that the ground state of QCD is annihilated by Q_A.**

Solution: Spontaneous symmetry breaking

Symmetry of $|0\rangle \neq$ symmetry of $H^0_{\rm QCD}$

- **Coleman theorem:[1] The symmetry of the ground state determines the symmetry of the spectrum (reverse argument: infer symmetry of the ground state from the symmetry of the spectrum)**

- **Goldstone theorem:[2] To each generator that does not annihilate the ground state exists a massless Goldstone boson**

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[1] S. Coleman, J. Math. Phys. 7, 787 (1966)

[2] J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962)

- **Here**
  - $H^0_{\rm QCD}$ **invariant under** $G = \mathbf{SU(3)}_L \times \mathbf{SU(3)}_R$
  - $|0\rangle$ **invariant under**

$$H = \{(V,V)\} \cong \mathbf{SU}(3)_V \quad \textbf{flavor SU(3)}$$

  - **idealized: 8 massless Goldstone bosons** π, K, η

Another (sufficient but not necessary) criterion: [3] **Nonvanishing scalar quark condensate**

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[3] G. Colangelo, J. Gasser, and H. Leutwyler, Phys. Rev. Lett. 86, 5008 (2001)

3. Chiral perturbation theory for mesons [4]

- Starting point: Chiral SU(3)$\times$SU(3) symmetry of the QCD Lagrangian

- Spontaneous symmetry breaking: Ground state (vacuum) has a lower symmetry, namely flavor SU(3)$_V$, than the Lagrangian

- Goldstone theorem: $16 - 8 = 8$ massless Goldstone bosons

- Goldstone bosons interact "weakly" at low energies

- Include explicit chiral symmetry breaking through quark masses as a perturbation

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[4] J. Gasser and H. Leutwyler, Nucl. Phys. B250, 465 (1985)

- Map symmetries onto the most general (effective) Lagrangian for the interaction of Goldstone bosons (π, K, η)

- Organization of the Lagrangian in the number of (covariant) derivatives and number of quark mass terms [5] (π, K, η)

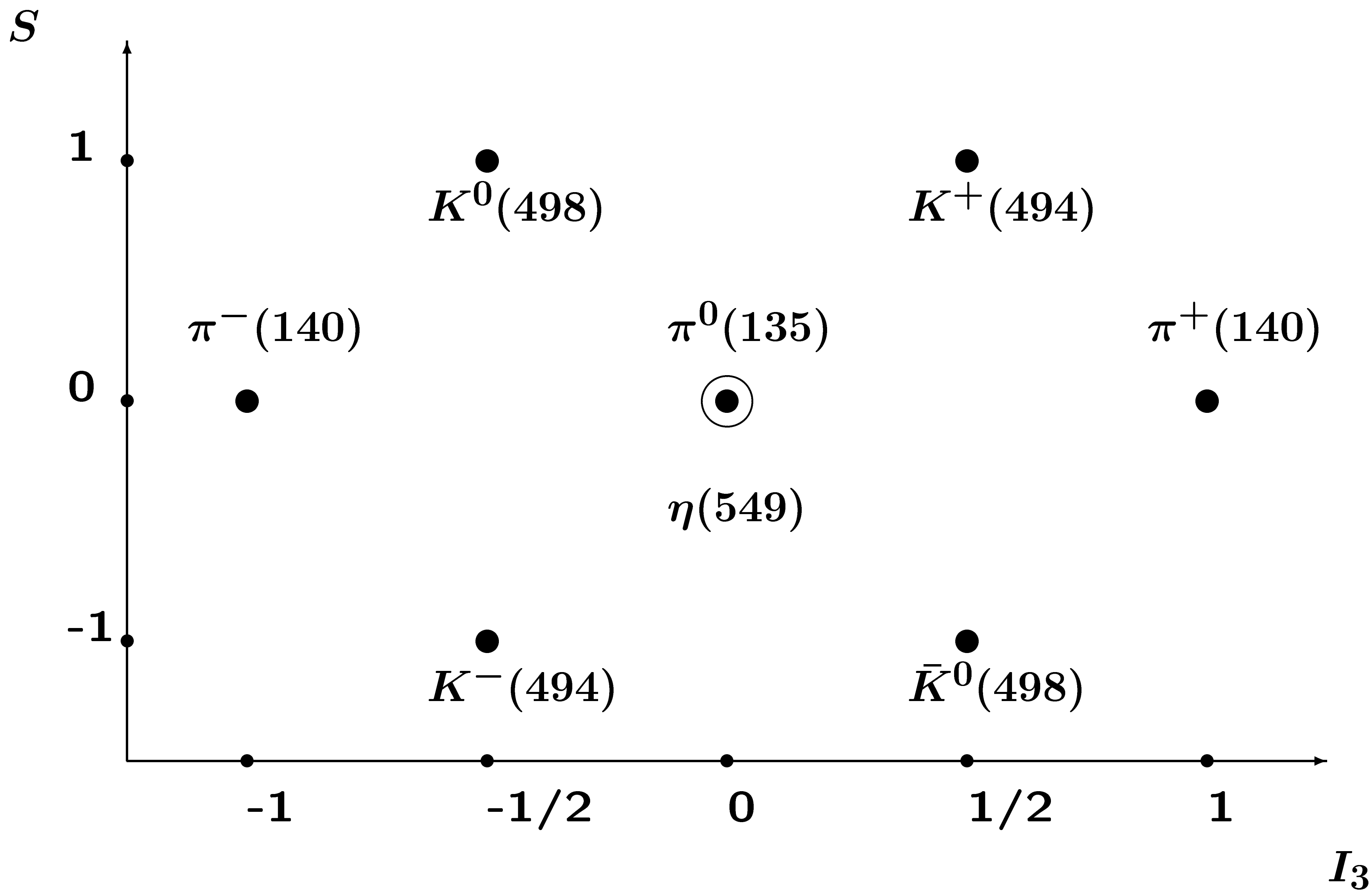
$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \cdots$$

$$\underbrace{2}_{\mathcal{O}(q^2)} \quad + \quad \underbrace{10+2}_{\mathcal{O}(q^4)} \quad + \quad \underbrace{90+4+23}_{\mathcal{O}(q^6)} \quad + \quad \cdots$$

  - q: Small quantity such as a pion mass
  - Even powers

[5] J. Gasser and H. Leutwyler, Nucl. Phys. B250, 465 (1985);
H. W. Fearing and S. S. , Phys. Rev. D 53, 315 (1996);
J. Bijnens, G. Colangelo, G. Ecker, JHEP 02, 020 (1999);
T. Ebertshäuser, H. W. Fearing, S. S., Phys. Rev. D 65, 054033 (2002);
J. Bijnens, L. Girlanda, P. Talavera, Eur. Phys. J. C 23, 539 (2002)

- **Pseudoscalar meson octet (masses in MeV)**



$$\phi(x) = \sum_{a=1}^{8} \lambda_a \phi_a(x) \equiv \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix},$$

$$U(x) = \exp\left(i\frac{\phi}{F_0}\right)$$

- **Lowest-order Lagrangian (chiral limit, no external fields)**

$$\mathcal{L}_{\text{eff}} = \frac{F_0^2}{4}\text{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right).$$

- $F_0 \approx 93$ **MeV: Pion-decay constant in the chiral limit**

- **Expand exponential** $U = 1 + i\phi/F_0 + \cdots$, $\partial_\mu U = i\partial_\mu \phi/F_0 + \cdots$, $\Rightarrow$

$$\mathcal{L}_{\text{eff}} = \frac{F_0^2}{4}\text{Tr}\left[\frac{i\partial_\mu \phi}{F_0}\left(-\frac{i\partial^\mu \phi}{F_0}\right)\right] + \cdots = \frac{1}{4}\text{Tr}(\lambda_a \partial_\mu \phi_a \lambda_b \partial^\mu \phi_b) + \cdots$$

$$= \frac{1}{4}\partial_\mu\phi_a\partial^\mu\phi_b \underbrace{\mathrm{Tr}(\lambda_a\lambda_b)}_{2\delta_{ab}} + \cdots = \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a + \mathcal{L}_{\rm int}.$$

No other terms containing only two fields $\Rightarrow$ eight fields ϕ_a describe eight independent massless particles.

- Finite quark masses (isospin symmetry: $m_u = m_d = \hat{m}$)

$$\mathcal{L}_{\rm s.b.} = \frac{F_0^2 B_0}{2}\mathrm{Tr}(\mathcal{M}U^\dagger + U\mathcal{M}^\dagger).$$

Expand in ϕ

$$\mathcal{L}_{\rm s.b.} = F_0^2 B_0(2\hat{m} + m_s) \underbrace{- \frac{B_0}{2}\mathrm{Tr}(\phi^2\mathcal{M})}_{\text{generate mass terms}} + \mathcal{L}_{\rm int}.$$

Lowest-order results (Exercise)

$$\begin{aligned} M_\pi^2 &= 2B_0\hat{m}, \\ M_K^2 &= B_0(\hat{m} + m_s), \\ M_\eta^2 &= \frac{2}{3}B_0\left(\hat{m} + 2m_s\right). \end{aligned}$$

- Using $M_\pi = 135$ MeV, $M_K = 496$ MeV, and $M_\eta = 547$ MeV $\Rightarrow$ quark mass ratios

$$\frac{M_K^2}{M_\pi^2} = \frac{\hat{m} + m_s}{2\hat{m}} \;\Rightarrow\; \frac{m_s}{\hat{m}} = 25.9,$$
$$\frac{M_\eta^2}{M_\pi^2} = \frac{2m_s + \hat{m}}{3\hat{m}} \;\Rightarrow\; \frac{m_s}{\hat{m}} = 24.3.$$

- B_0: $3F_0^2 B_0 = -\langle \bar{q}q \rangle_0$

- Analogy with a ferromagnet

$$-\langle \vec{M} \rangle \cdot \vec{H} \leftrightarrow \langle \bar{u}u \rangle m_u + \langle \bar{d}d \rangle m_d + \langle \bar{s}s \rangle m_s = -F_0^2 B_0 (m_u + m_d + m_s)$$

- **Weinberg's power counting for the mesonic sector** [6]

  Q: How do different diagrams compare?

  Analyze given diagram under

  1. **linear rescaling of all external momenta, $p_i \mapsto tp_i$,**

  2. **quadratic rescaling of light quark masses, $m_q \mapsto t^2 m_q$ (corresponds to $M^2 \mapsto t^2 M^2$).**

  Chiral dimension D:

$$\mathcal{M}(tp_i, t^2 m_q) = t^D \mathcal{M}(p_i, m_q) = \mathcal{O}(q^D).$$

  For small enough momenta (and masses) contributions with increasing D become less important.

[6] S. Weinberg, Physica A 96, 327 (1979)

$$
\begin{aligned}
D &= nN_L - 2N_I + \sum_{k=1}^{\infty} 2kN_{2k} \\
&= 2 + (n-2)N_L + \sum_{k=1}^{\infty} 2(k-1)N_{2k} \\
&\geq 2 \text{ in 4 dimensions.}
\end{aligned}
$$

– n: Number of space-time dimensions.

– N_L: Number of independent loops.

– N_I: Number of internal Goldstone boson lines.

– N_{2k}: Number of vertices from $\mathcal{L}_{2k}$.

– Loops suppressed by $(n-2)N_L$.

– Relation between the momentum and loop expansion:

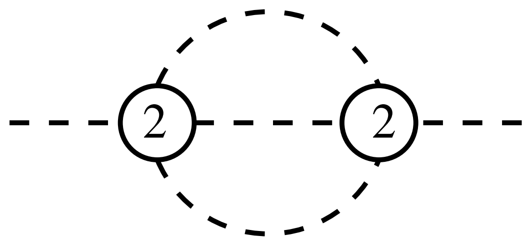
1. $\mathcal{O}(q^2)$: No loops.

2. $\mathcal{O}(q^4)$: No loops and 1 loop.

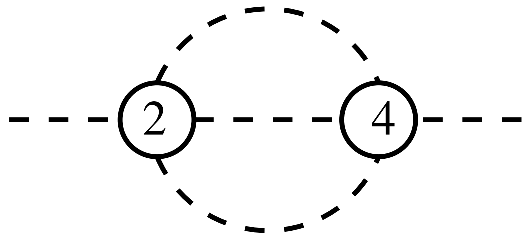
3. $\mathcal{O}(q^6)$: No loops, 1 loop, and 2 loops, etc.

– Perturbative scheme in terms of external momenta and quark masses ($\rightarrow$ meson masses^2) which are small compared to some scale [here: $4\pi F_0 = \mathcal{O}$ (1 GeV)].

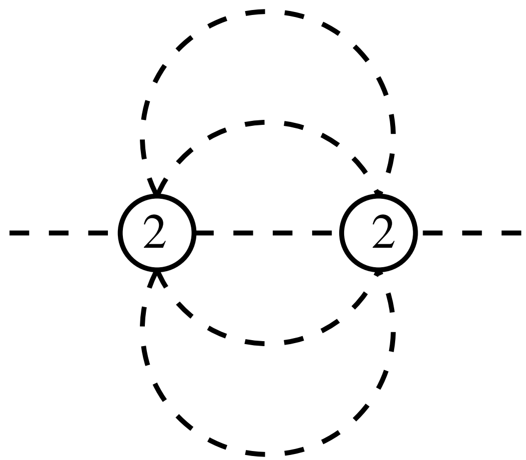
Examples ($n = 4$ dimensions):



$$\begin{aligned} D &= 4 \cdot 2 - 2 \cdot 3 + 2 \cdot 2 = 6 \\ &= 2 + 2 \cdot 2 + (2 - 2) \cdot 2 \end{aligned}$$



$$D = 4 \cdot 2 - 2 \cdot 3 + 1 \cdot 2 + 1 \cdot 4 = 8$$

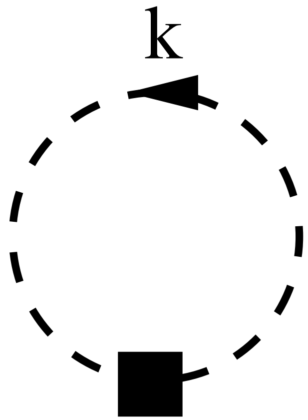


$$D = 4 \cdot 4 - 2 \cdot 5 + 2 \cdot 2 = 10$$

$D \geq 4$: We need to discuss loops!

Dimensional regularization: Basics

Simple example



$$I = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - M^2 + i0^+}, \quad k^2 = k_0^2 - \vec{k}^{\,2}.$$

Introduce

$$a \equiv \sqrt{\vec{k}^{\,2} + M^2} > 0$$

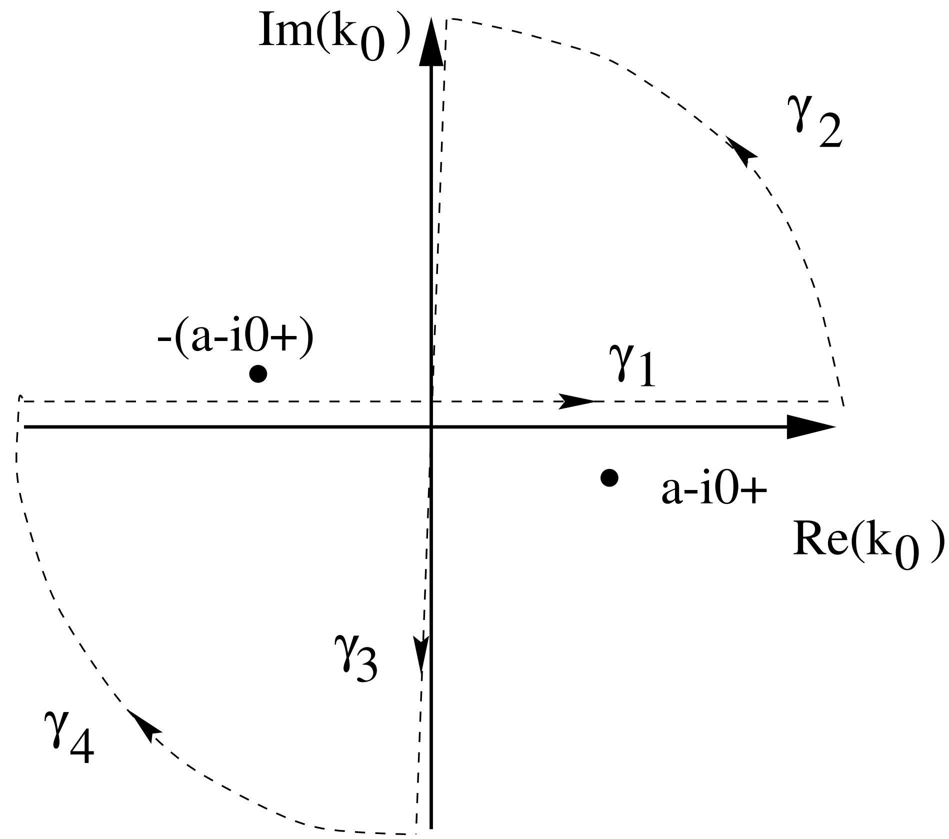
and define

$$f(k_0) = \frac{1}{[k_0 + (a - i0^+)][k_0 - (a - i0^+)]}$$

Consider f in the complex k_0 plane and make use of Cauchy's theorem

$$\oint_C \mathrm{d}z f(z) = 0$$

for functions which are differentiable in every point inside the closed contour C.



$$\mathbf{0 = \sum_{i=1}^{4} \int_{\gamma_i} \mathrm{d}z f(z).}$$

Recall $\int_\gamma f(z)\mathrm{d}z = \int_a^b f[\gamma(t)]\gamma'(t)\mathrm{d}t \quad \Rightarrow$

$$
\begin{aligned}
\int_{\gamma_1} f(z)\mathrm{d}z &= \int_{-\infty}^{\infty} f(t)\mathrm{d}t, \\
\int_{\gamma_2} f(z)\mathrm{d}z &= \lim_{R\to\infty}\int_0^{\frac{\pi}{2}} f(Re^{it})iRe^{it}\mathrm{d}t = 0, \\
&\quad \text{because } \lim_{R\to\infty} \underbrace{Rf(Re^{it})}_{\sim \frac{1}{R}} = 0, \\
\int_{\gamma_3} f(z)\mathrm{d}z &= \int_{\infty}^{-\infty} f(it)i\mathrm{d}t, \\
\int_{\gamma_4} f(z)\mathrm{d}z &= 0.
\end{aligned}
$$

$\Rightarrow$ So-called Wick rotation

$$\int_{-\infty}^{\infty} f(t)\mathrm{d}t = -i\int_{\infty}^{-\infty}\mathrm{d}t f(it) = i\int_{-\infty}^{\infty}\mathrm{d}t f(it).$$

Intermediate result

$$
\begin{aligned}
I &= \frac{1}{(2\pi)^4} i \int \mathrm{d}^3k \int_{-\infty}^{\infty} \mathrm{d}k_0 \frac{i}{(ik_0)^2 - \vec{k}^{\,2} - M^2 + i0^+} \\
&= \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{1}{l^2 + M^2 - i0^+}
\end{aligned}
$$

$l^2 = l_1^2 + l_2^2 + l_3^2 + l_4^2$ denotes a Euclidian scalar product

I diverges (quadratically) for large values of l (ultraviolet divergence)

Dimensional regularization: Generalize from 4 to n dimensions and introduce polar coordinates

$$
\begin{aligned}
l_1 &= l\cos(\theta_1), \\
l_2 &= l\sin(\theta_1)\cos(\theta_2), \\
l_3 &= l\sin(\theta_1)\sin(\theta_2)\cos(\theta_3), \\
&\vdots \\
l_{n-1} &= l\sin(\theta_1)\sin(\theta_2)\cdots\cos(\theta_{n-1}), \\
l_n &= l\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{n-1}),
\end{aligned}
$$

$$
0 \le l, \quad \theta_i \in [0,\pi], i = 1,\cdots,n-2, \quad \theta_{n-1} \in [0, 2\pi].
$$

A general integral is then symbolically of the form

$$
\begin{aligned}
&\int \mathrm{d}^n l \cdots = \int_0^\infty l^{n-1}\mathrm{d}l \\
&\times \int_0^{2\pi} \mathrm{d}\theta_{n-1} \int_0^{\pi} \mathrm{d}\theta_{n-2}\sin(\theta_{n-2}) \cdots \int_0^{\pi} \mathrm{d}\theta_1 \sin^{n-2}(\theta_1) \cdots
\end{aligned}
$$

If the integrand does not depend on the angles, the angular integration can explicitly be carried out:

$$\int \mathrm{d}\Omega_n = 2\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

Example

$$n = 3: \qquad 4\pi = 2\frac{\pi}{1/2} = 2\frac{\pi^{3/2}}{\sqrt{\pi}/2} = 2\frac{\pi^{3/2}}{\Gamma(3/2)}$$

We define the integral for n dimensions (n integer) as

$$I_n(M^2, \mu^2) = \mu^{4-n} \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{i}{k^2 - M^2 + i0^+}$$

Scale μ: Unit of mass, ’t Hooft parameter, renormalization scale (integral has the same dimension for arbitrary n)

Integral formally reads

$$I_n(M^2,\mu^2) = \mu^{4-n} \underbrace{2\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}}_{\text{angular integration}} \frac{1}{(2\pi)^n} \underbrace{\int_0^\infty \mathrm{d}l \frac{l^{n-1}}{l^2+M^2}}_{\text{elementary}}$$

$$= \mu^{4-n}\, 2\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\, \frac{1}{(2\pi)^n}\, \frac{1}{2}(M^2)^{\frac{n}{2}-1}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(1-\frac{n}{2}\right)}{\underbrace{\Gamma(1)}_{1}}$$

$$= \frac{\mu^{4-n}}{(4\pi)^{\frac{n}{2}}}(M^2)^{\frac{n}{2}-1}\Gamma\left(1-\frac{n}{2}\right)$$

$\Gamma(z)$ is single valued and analytic over the entire complex plane, save for the points $z=-n$, $n=0,1,2,\cdots$, where it possesses simple poles with residue $(-1)^n/n!$

$a^z = \exp[\ln(a)z]$, $a \in R^+$ is an analytic function in C

Define (as a function of a complex variable n)

$$I(M^2,\mu^2,n) = \frac{M^2}{(4\pi)^2}\left(\frac{4\pi\mu^2}{M^2}\right)^{2-\frac{n}{2}}\Gamma\left(1-\frac{n}{2}\right)$$

As $n \to 4$ Gamma function has a pole $\Rightarrow I(M^2,\mu^2,n)$ has a pole

How is this pole is approached?

Important property: $\Gamma(z+1) = z\Gamma(z)$

$$\Gamma\left(1-\frac{n}{2}\right) \;=\; \frac{\Gamma\left(1-\frac{n}{2}+1\right)}{1-\frac{n}{2}} = \frac{\Gamma\left(2-\frac{n}{2}+1\right)}{\left(1-\frac{n}{2}\right)\left(2-\frac{n}{2}\right)} = \frac{\Gamma\left(1+\frac{\epsilon}{2}\right)}{(-1)\left(1-\frac{\epsilon}{2}\right)\frac{\epsilon}{2}}$$

where $\epsilon \equiv 4-n$.

$$a^x = \exp[\ln(a)x] = 1+\ln(a)x + O(x^2)$$

$$I \;=\; \frac{M^2}{16\pi^2}\left[-\frac{2}{\epsilon} \underbrace{-\Gamma'(1)}_{\gamma_E\,=\,0.5772\cdots} -1-\ln(4\pi)+\ln\left(\frac{M^2}{\mu^2}\right)+O(\epsilon)\right]$$

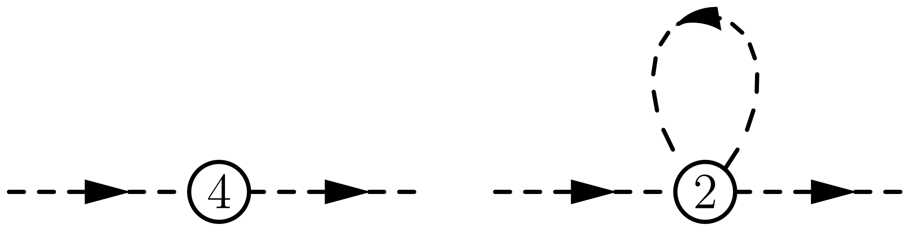
Summary

$$I(M^2, \mu^2, n) = \frac{M^2}{16\pi^2}\left[R + \ln\left(\frac{M^2}{\mu^2}\right)\right] + O(n-4)$$

where

$$\underbrace{R}_{\widetilde{\mathrm{MS}}} = \underbrace{\underbrace{\frac{2}{n-4}}_{\textbf{MS}} - [\ln(4\pi) + \Gamma'(1)]}_{\overline{\mathrm{MS}}} - 1$$

- **Example at $\mathcal{O}(q^4)$:**



$$-i\Sigma = \quad \sim L_i \qquad \sim I(M^2, \mu^2, n)$$

... the cancellation of ultraviolet divergences does not really depend on renormalizability; as long as we include every one of the infinite number of interactions allowed by symmetries, the so-called non-renormalizable theories are actually just as renormalizable as renormalizable theories.[7]

Conclusion: Adjust (renormalize) parameters of $\mathcal{L}_4$ to cancel one-loop infinities!

---

[7] S. Weinberg, The Quantum Theory of Fields, Vol. I, Chap. 12

$$
\begin{aligned}
M^2_{\pi,4} &= M^2_{\pi,2}\left\{1+\frac{M^2_{\pi,2}}{32\pi^2F_0^2}\ln\left(\frac{M^2_{\pi,2}}{\mu^2}\right)-\frac{M^2_{\eta,2}}{96\pi^2F_0^2}\ln\left(\frac{M^2_{\eta,2}}{\mu^2}\right)\right. \\
&\left.+\frac{16}{F_0^2}\left[(2\hat{m}+m_s)B_0(2L_6^r-L_4^r)+\hat{m}B_0(2L_8^r-L_5^r)\right]\right\}
\end{aligned}
$$

– $\sim M^2_{\pi,2}=2B_0\hat{m}$, vanishes in the chiral limit

– contains nonanalytical terms $\sim m_q^2\ln(m_q)$

– contains analytical terms $\sim m_q^2$ with new low-energy constants

- Applications at the two-loop level: See J. Bijnens, "Chiral perturbation theory beyond one loop," Prog. Part. Nucl. Phys. 58, 521 (2007) [arXiv:hep-ph/0604043]

4. Chiral perturbation theory for baryons

- Most general Lagrangian

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\pi} + \mathcal{L}_{\pi N} = \mathcal{L}_{\pi}^{(2)} + \mathcal{L}_{\pi}^{(4)} + \cdots + \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(2)} + \cdots$$

πN Lagrangian [SU(2)×SU(2)×U(1)] [8]

$$\underbrace{2}_{\mathcal{O}(q)} + \underbrace{7}_{\mathcal{O}(q^2)} + \underbrace{23}_{\mathcal{O}(q^3)} + \underbrace{118}_{\mathcal{O}(q^4)} + \quad \cdots$$

- Odd and even powers (spin)
- One-loop level

---

[8] J. Gasser, M. E. Sainio, A. Švarc, Nucl. Phys. B307, 779 (1988);
V. Bernard, N. Kaiser, U.-G. Meißner, Int. J. Mod. Phys. E 4, 193 (1995);
G. Ecker and M. Mojžiš, Phys. Lett. B 365, 312 (1996);
N. Fettes, U.-G. Meißner, M. Mojžiš, S. Steininger, Ann. Phys. (N.Y.) 283, 273 (2000)

Define

$$\Psi = \begin{pmatrix} p \\ n \end{pmatrix}, \qquad u = \sqrt{U} = \exp\left(\frac{i\vec{\tau}\cdot\vec{\phi}}{2F}\right).$$

- **Lowest-order Lagrangian**

$$\mathcal{L}_{\pi N}^{(1)} = \bar{\Psi}\left(i\not{D} - \boxed{m} + \frac{\boxed{\mathrm{g}_A}}{2}\gamma^\mu\gamma_5 u_\mu\right)\Psi,$$

where (without external fields)

$$\begin{aligned}
D_\mu\Psi &= (\partial_\mu + \Gamma_\mu)\Psi, \\
\Gamma_\mu &= \frac{1}{2}\left(u^\dagger\partial_\mu u + u\partial_\mu u^\dagger\right) = \frac{i}{4F^2}\vec{\tau}\cdot\vec{\phi}\times\partial_\mu\vec{\phi} + \mathcal{O}(\phi^4), \\
u_\mu &= i\left(u^\dagger\partial_\mu u - u\partial_\mu u^\dagger\right) = -\frac{\vec{\tau}\cdot\partial_\mu\vec{\phi}}{F} + \mathcal{O}(\phi^3).
\end{aligned}$$

m, g_A, and F denote the chiral limit of the physical nucleon mass, the axial-vector coupling constant, and the pion-decay constant, respectively.

- **Power counting:** Associate chiral order D with a diagram
  - Square of the lowest-order pion mass:
    $M^2 = B(m_u + m_d) \sim \mathcal{O}(q^2)$
  - Nucleon mass in the chiral limit $m \sim \mathcal{O}(q^0)$
  - Loop integration in n dimensions $\sim \mathcal{O}(q^n)$
  - Vertex from $\mathcal{L}_\pi^{(2k)} \sim \mathcal{O}(q^{2k})$
  - Vertex from $\mathcal{L}_{\pi N}^{(k)} \sim \mathcal{O}(q^k)$
  - Nucleon propagator $\sim \mathcal{O}(q^{-1})$
  - Pion propagator $\sim \mathcal{O}(q^{-2})$

- Renormalization
  - Regularize (typically dimensional regularization)

$$\begin{aligned} I(M^2,\mu^2,n) &= \mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\frac{i}{k^2-M^2+i0^+} \\ &= \frac{M^2}{16\pi^2}\left[R+\ln\left(\frac{M^2}{\mu^2}\right)\right]+O(n-4), \end{aligned}$$

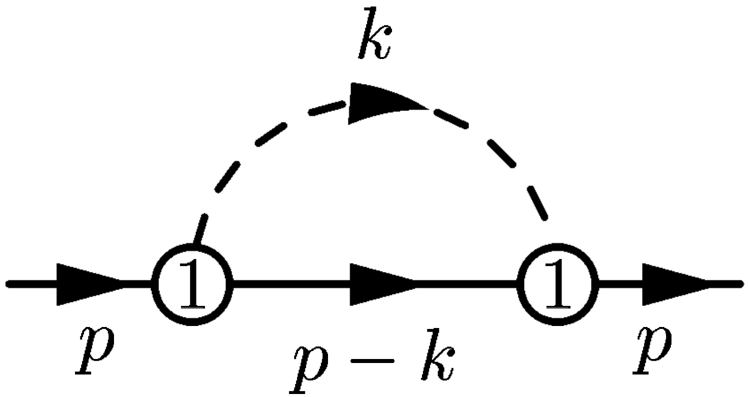
where

$$\boxed{R} = \frac{2}{n-4} - [\ln(4\pi)+\Gamma'(1)] - 1 \to \boxed{\infty}$$

Scale μ: 't Hooft parameter (integral has the same dimension for arbitrary n)

  - Adjust counterterms such that they absorb all the divergences occurring in the calculation of loop diagrams
  - Renormalization prescription: Adjust finite pieces such that renormalized diagrams satisfy a given power counting

- **Example: Contribution to nucleon mass**



Goal: $D = n \cdot 1 - 2 \cdot 1 - 1 \cdot 1 + 2 \cdot 1 = n - 1$

$$\Sigma = -\frac{3g_{A0}^2}{4F_0^2}\left[(\not{p}+m)I_N + M^2(\not{p}+m)I_{N\pi}(-p,0)+\cdots\right]$$

Apply $\widetilde{\mathrm{MS}}$ renormalization scheme

$$\begin{aligned}\Sigma_r &= -\frac{3g_{Ar}^2}{4F_r^2}\left[M^2(\not{p}+m)\underbrace{I_{N\pi}^r(-p,0)}_{-\frac{1}{16\pi^2}+\cdots}+\cdots\right]\\ &= \mathcal{O}(q^2)\end{aligned}$$

GSS [9]: It turns out that loops have a much more complicated low-energy structure if baryons are included. Because the nucleon mass m_N does not vanish in the chiral limit, the mass scale m (nucleon mass in the chiral limit) occurs in the effective Lagrangian $\mathcal{L}_{\pi N}^{(1)}$ ...
This complicates life a lot.

[9] J. Gasser, M. E. Sainio, A. Švarc, Nucl. Phys. B307, 779 (1988)

Solutions

- **Heavy-baryon chiral perturbation theory** [10]

- **Infrared regularization (IR)** [11]

  Special treatment of (the Feynman parameterization of) one-loop integrals

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1-x)]^2}$$

$$a = (k-p)^2 - m^2 + i0^+, \qquad b = k^2 - M^2 + i0^+$$

$$H = \int_0^1 dx \cdots = \int_0^\infty dx \cdots - \int_1^\infty dx \cdots \equiv I + R$$

---

[10] E. Jenkins and A. V. Manohar, Phys. Lett. B 255, 558 (1991);
V. Bernard, N. Kaiser, J. Kambor, U.-G. Meißner, Nucl. Phys. B388, 315 (1992)

[11] T. Becher and H. Leutwyler, Eur. Phys. J. C 9, 643 (1999)

- I: power counting o.k.
- R: violates power counting; regular, i.e., can be absorbed in counterterms

- Extended on-mass-shell (EOMS) scheme [12]

  Main idea: Perform additional subtractions such that renormalized diagrams satisfy the power counting

  Motivation for this approach [13]

  Terms violating the power counting are analytic in small quantities (and can thus be absorbed in a renormalization of counterterms)

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[12] T. Fuchs, J. Gegelia, G. Japaridze, S. S., Phys. Rev. D 68, 056005 (2003)
[13] J. Gegelia and G. Japaridze, Phys. Rev. D 60, 114038 (1999)

– Example (chiral limit)

$$H(p^2, m^2; n) = \int \frac{d^n k}{(2\pi)^n} \frac{i}{[(k-p)^2 - m^2 + i0^+][k^2 + i0^+]}$$

Small quantity

$$\Delta = \frac{p^2 - m^2}{m^2} = \mathcal{O}(q)$$

We want the renormalized integral to be of order

$$D = n - 1 - 2 = n - 3$$

Result of integration [14]

$$H \sim F(n, \Delta) + \Delta^{n-3} G(n, \Delta)$$

F and G are hypergeometric functions; analytic in Δ for arbitrary n

[14] J. Gegelia, G. Japaridze, K. S. Turashvili, Theor. Math. Phys. 101, 1313 (1994)

Observation

F corresponds to first expanding the integrand in small quantities and then performing the integration

$\Rightarrow$ Algorithm: Expand integrand in small quantities and subtract those (integrated) terms whose order is smaller than suggested by the power counting

Here:

$$
\begin{aligned}
H^{\text{subtr}} &= \int \frac{d^n k}{(2\pi)^n} \frac{i}{(k^2 - 2k\cdot p + i0^+)(k^2 + i0^+)}\bigg|_{p^2=m^2} \\
&= -2\bar{\lambda} + \frac{1}{16\pi^2} + O(n-4)
\end{aligned}
$$

where

$$
\bar{\lambda} = \frac{m^{n-4}}{(4\pi)^2}\left\{\frac{1}{n-4} - \frac{1}{2}\left[\ln(4\pi) + \Gamma'(1) + 1\right]\right\}
$$

$$
H^R = H - H^{\text{subtr}} = \mathcal{O}(q^{n-3})
$$

- **General case including pion mass**

$$
\begin{aligned}
&\left.\frac{1}{\left(k^2-2k\cdot p+i0^+\right)\left(k^2+i0^+\right)}\right|_{p^2=m^2}\\
&+(p^2-m^2)\left[\frac{1}{2m^2}\frac{1}{\left(k^2-2k\cdot p+i0^+\right)^2}+\cdots\right]_{p^2=m^2}\\
&+M^2\left.\frac{1}{\left(k^2-2k\cdot p+i0^+\right)\left(k^2+i0^+\right)^2}\right|_{p^2=m^2}\\
&+\cdots
\end{aligned}
$$

- **Reformulation of IR in terms of EOMS** [15]
  - **Formal equivalence shown at one-loop level**
  - **Heavy degrees of freedom** [16]
  - **Higher-order loops** [17]

---

[15] M. R. Schindler, J. Gegelia, S. S., Phys. Lett. B 586, 258 (2004)

[16] T. Fuchs, M. R. Schindler, J. Gegelia, S. S., Phys. Lett. B 575, 11 (2003)

[17] M. R. Schindler, J. Gegelia, S. S., Nucl. Phys. B 682, 367 (2004)

Exercise: The EOMS approach in more detail

Calculation of the nucleon mass up to and including order $\mathcal{O}(q^3)$

Full propagator

$$S_0(p) = \frac{1}{\not p - m_0 - \Sigma_0(\not p)} \equiv \frac{1}{\not p - m - \Sigma(\not p)}$$

- m_0 **bare mass**
- m **nucleon mass in the chiral limit**
- $\Sigma_0(\not p)$ **self energy**

Definition of the nucleon mass

$$m_N - m_0 - \Sigma_0(m_N) = m_N - m - \Sigma(m_N) = 0$$

- **Tree-level contribution**

Recall πN Lagrangian at order $\mathcal{O}(q^2)$

$$
\begin{aligned}
\mathcal{L}_{\pi N}^{(2)} &= c_1\,\mathrm{Tr}(\chi_+)\bar{\Psi}\Psi - \frac{c_2}{4m^2}\left[\bar{\Psi}\,\mathrm{Tr}(u_\mu u_\nu)D^\mu D^\nu \Psi + \mathrm{H.c.}\right] \\
&+ \bar{\Psi}\left[\frac{c_3}{2}\mathrm{Tr}(u_\mu u^\mu) + i\frac{c_4}{4}[u_\mu, u_\nu] + c_5\left[\chi_+ - \frac{1}{2}\mathrm{Tr}(\chi_+)\right]\right. \\
&\qquad \left. + \frac{c_6}{2} f_{\mu\nu}^{+} + \frac{c_7}{2} v_{\mu\nu}^{(s)}\right]\sigma^{\mu\nu}\Psi
\end{aligned}
$$

Only c_1 term contributes to the self energy

$$-4c_1 M^2$$

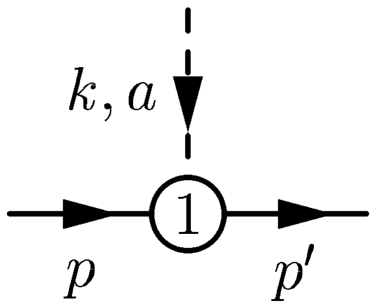
No contact contributions from the Lagrangian $\mathcal{L}_{\pi N}^{(3)}$

- **Loop contributions**

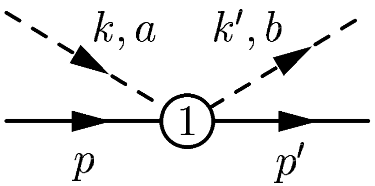
Expand $\mathcal{L}_{\pi N}^{(1)}$ up to and including two pion fields:

$$\mathcal{L}_{\rm int}^{(1)} = -\frac{1}{2}\frac{g_{A0}}{F_0}\bar{\Psi}\gamma^\mu\gamma_5\tau^b\partial_\mu\phi^b\Psi - \frac{1}{4F_0^2}\bar{\Psi}\gamma^\mu\vec{\tau}\cdot\vec{\phi}\times\partial_\mu\vec{\phi}\Psi$$

Feynman rules

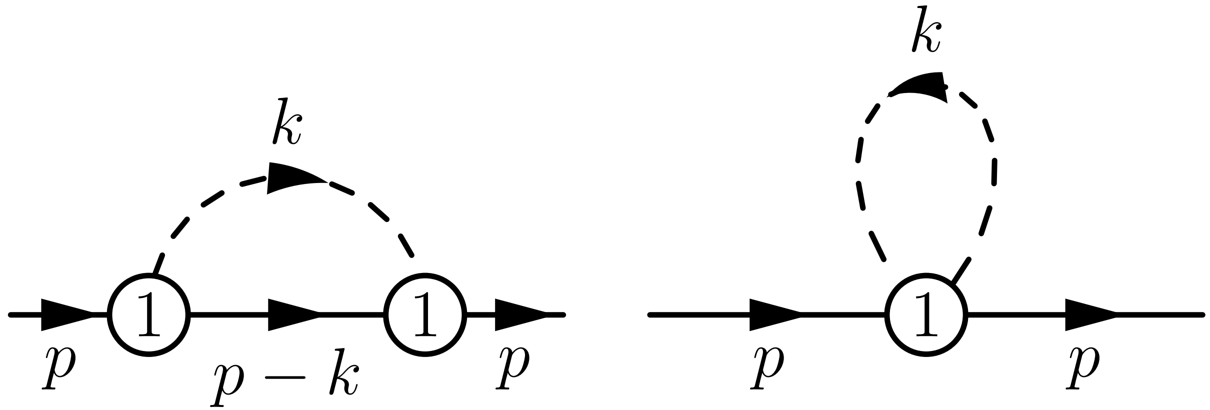


$$-\frac{g_{A0}}{2F_0}\not{k}\gamma_5\tau_a$$



$$\frac{1}{4F_0^2}(\not{k}+\not{k}')\epsilon_{abc}\tau_c$$

Two types of loop contributions at order $\mathcal{O}(q^3)$



Second diagram does not contribute: $\epsilon_{aac} = 0$

Feynman rules + propagators + $\tau_a \tau_a = 3$

$$
\begin{aligned}
i\Delta_\pi(p) &= \frac{i}{p^2 - M^2 + i0^+} \\
iS_N(p) &= i\,\frac{\not{p} + m - i0^+}{p^2 - m^2 + i0^+}
\end{aligned}
$$

⇒ **contribution of the first diagram in dim. reg.**

$$-i\Sigma^{\text{loop}}(\not p) =$$

$$-i\frac{3g_{A0}^2}{4F_0^2}\, i\mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\frac{\not k(\not p - m - \not k)\not k}{[(p-k)^2 - m^2 + i0^+][k^2 - M^2 + i0^+]}$$

Simplify numerator using $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$

$$-(\not p + m)\underbrace{k^2}_{k^2 - M^2 + M^2} + (p^2 - m^2)\not k - \left[(p-k)^2 - m^2\right]\not k$$

Intermediate result

$$\Sigma^{\text{loop}}(\not p) = \frac{3\text{g}_{A0}^2}{4F_0^2}\left\{-(\not p + m)\mu^{4-n} i \int \frac{d^n k}{(2\pi)^n} \frac{1}{[(p-k)^2 - m^2 + i0^+]}\right.$$

$$-(\not p + m)M^2 \mu^{4-n} i \int \frac{d^n k}{(2\pi)^n} \frac{1}{[(p-k)^2 - m^2 + i0^+][k^2 - M^2 + i0^+]}$$

$$+(p^2 - m^2)\mu^{4-n} i \int \frac{d^n k}{(2\pi)^n} \frac{\not k}{[(p-k)^2 - m^2 + i0^+][k^2 - M^2 + i0^+]}$$

$$\left. -\mu^{4-n} i \int \frac{d^n k}{(2\pi)^n} \frac{\not k}{[k^2 - M^2 + i0^+]}\right\}$$

Last term vanishes (integrand odd)

Convention

$$I_{N\cdots\pi\cdots}(p_1, \cdots, q_1, \cdots)$$

$$= \mu^{4-n} i \int \frac{d^n k}{(2\pi)^n} \frac{1}{[(k+p_1)^2 - m^2 + i0^+]\cdots[(k+q_1)^2 - M^2 + i0^+]\cdots}$$

To determine the vector integral use the ansatz

$$\mu^{4-n} i \int \frac{d^n k}{(2\pi)^n} \frac{k_\mu}{[(p-k)^2 - m^2 + i0^+][k^2 - M^2 + i0^+]} = p_\mu C$$

Multiply by $p^\mu \Rightarrow$

$$C = \frac{1}{2p^2}\left[I_N - I_\pi + (p^2 - m^2 + M^2) I_{N\pi}(-p,0)\right]$$

Using the above convention the loop contribution to the nucleon self energy reads

$$\begin{aligned}\Sigma^{\text{loop}}(\not p) &= -\frac{3\mathrm{g}_{A0}^2}{4F_0^2}\{(\not p + m)I_N + (\not p + m)M^2 I_{N\pi}(-p,0) \\ &\quad -(p^2 - m^2)\frac{\not p}{2p^2}\left[I_N - I_\pi + (p^2 - m^2 + M^2)I_{N\pi}(-p,0)\right]\}\end{aligned}$$

The explicit expressions for the integrals are given by

$$I_\pi = \frac{M^2}{16\pi^2}\left[R + \ln\left(\frac{M^2}{\mu^2}\right)\right]$$

$$
\begin{aligned}
I_N &= \frac{m^2}{16\pi^2}\left[R + \ln\left(\frac{m^2}{\mu^2}\right)\right] \\
I_{N\pi}(p,0) &= \frac{1}{16\pi^2}\left[R + \ln\left(\frac{m^2}{\mu^2}\right) - 1\right. \\
&\left. + \frac{p^2 - m^2 - M^2}{p^2}\ln\left(\frac{M}{m}\right) + \frac{2mM}{p^2}F(\Omega)\right]
\end{aligned}
$$

where

$$
\begin{aligned}
R &= \frac{2}{n-4} - [\ln(4\pi) + \Gamma'(1) + 1] \\
\Omega &= \frac{p^2 - m^2 - M^2}{2mM}
\end{aligned}
$$

and

$$
F(\Omega) = \begin{cases}
\sqrt{\Omega^2 - 1}\ln\left(-\Omega - \sqrt{\Omega^2 - 1}\right), & \Omega \leq -1, \\
\sqrt{1 - \Omega^2}\ \arccos(-\Omega), & -1 \leq \Omega \leq 1, \\
\sqrt{\Omega^2 - 1}\ln\left(\Omega + \sqrt{\Omega^2 - 1}\right) - i\pi\sqrt{\Omega^2 - 1}, & 1 \leq \Omega\,.
\end{cases}
$$

- Σ^{loop} contains divergences as $n \to 4$ (the terms proportional to R) $\Rightarrow$ needs to be renormalized

- For convenience: $\mu = m$

- $\widetilde{\text{MS}}$ renormalization:
  - drop terms proportional to R
  - replace all bare coupling constants ($c_{10}, \text{g}_{A0}, F_0$) with the renormalized ones, now indicated by a subscript r

$\Rightarrow$ $\widetilde{\text{MS}}$ renormalized self energy contribution

$$\begin{aligned}\Sigma_r^{\text{loop}}(\not{p}) &= -\frac{3\text{g}_{Ar}^2}{4F_r^2}\{(\not{p}+m)M^2 I_{N\pi}^r(-p,0) \\ &\quad -(p^2-m^2)\frac{\not{p}}{2p^2}\left[(p^2-m^2+M^2)I_{N\pi}^r(-p) - I_\pi^r\right]\}\end{aligned}$$

Using

$$I_{N\pi}^{r}(-p,0) = -\frac{1}{16\pi^2} + \cdots$$

$\Rightarrow$ contribution of $\mathcal{O}(q^2)$

Solve for the nucleon mass

$$\begin{aligned} m_N &= m + \Sigma_r^{\text{contact}}(m_N) + \Sigma_r^{\text{loop}}(m_N) \\ &= m - 4c_{1r}M^2 + \Sigma_r^{\text{loop}}(m_N) \end{aligned}$$

- $m_N - m = \mathcal{O}(q^2)$

- We need $\Sigma_r^{\text{loop}}(m_N)$ to $\mathcal{O}(q^3)$

- Expansion of $I_{N\pi}^{r}$

$$\arccos(-\Omega) = \frac{\pi}{2} + \cdots$$

$$I^r_{N\pi} = \frac{1}{16\pi^2}\left(-1 + \frac{\pi M}{m} + \cdots\right)$$

- This yields

$$m_N = m - 4c_{1r}M^2 + \boxed{\frac{3g^2_{Ar}M^2}{32\pi^2 F_r^2}m} - \frac{3g^2_{Ar}M^3}{32\pi^2 F_r^2}$$

- Power counting problem

Solution

Term violating the power counting is analytic in small quantities and can thus be absorbed in counter terms

Rewrite

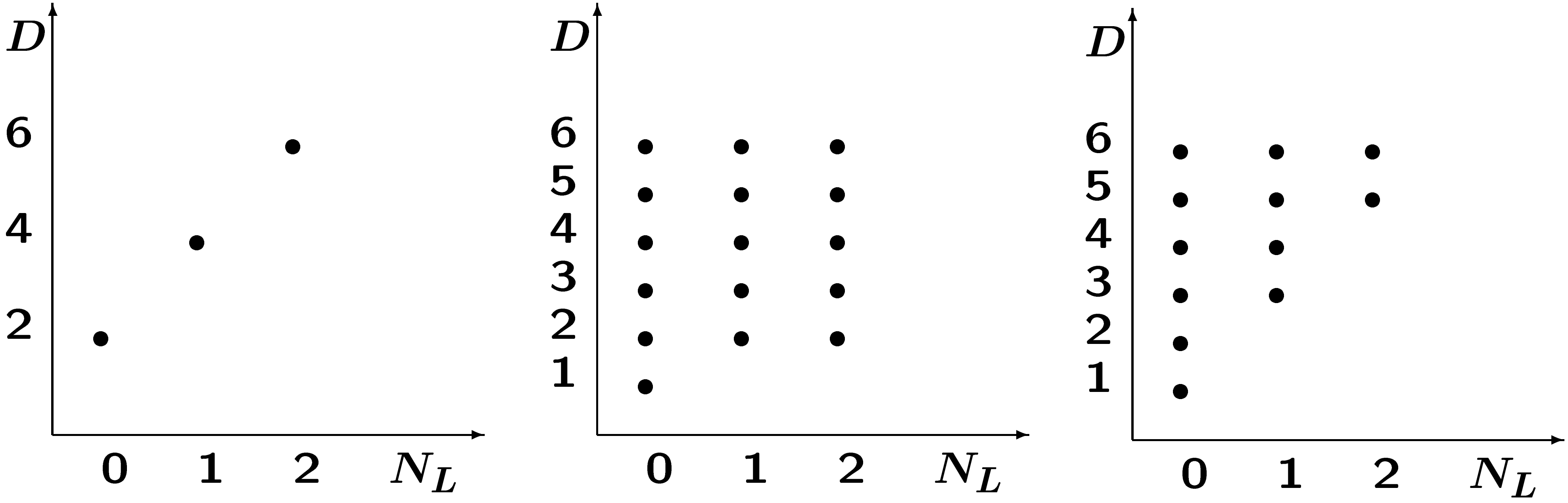
$$c_{1r} = c_1 + \delta c_1, \quad \delta c_1 = \frac{3m\mathrm{g}_A^2}{128\pi^2 F^2} + \cdots$$

Final result for the nucleon mass at order $\mathcal{O}(q^3)$

$$m_N = m - 4c_1 M^2 - \frac{3\mathrm{g}_A^2 M^3}{32\pi^2 F^2}$$

Chiral versus loop expansion

$\pi\pi$: $\overline{\text{MS}}$ $\qquad$ πN: $\overline{\text{MS}}$ $\qquad$ πN: EOMS, IR



More Applications

Mass of the nucleon at $\mathcal{O}(q^4)$ [18]

$$m_N = m + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln\left(\frac{M}{m}\right) + k_4 M^4 + O(M^5)$$

$$k_3 = \frac{3}{32\pi^2 F^2}\left(8c_1 - c_2 - 4c_3 - \frac{g_A^2}{m}\right), \quad k_4 = \cdots$$

$$m = [938.3 - 74.8 + 15.3 + 4.7 + 1.6 - 2.3]\,\text{MeV} = 882.8\,\text{MeV}$$

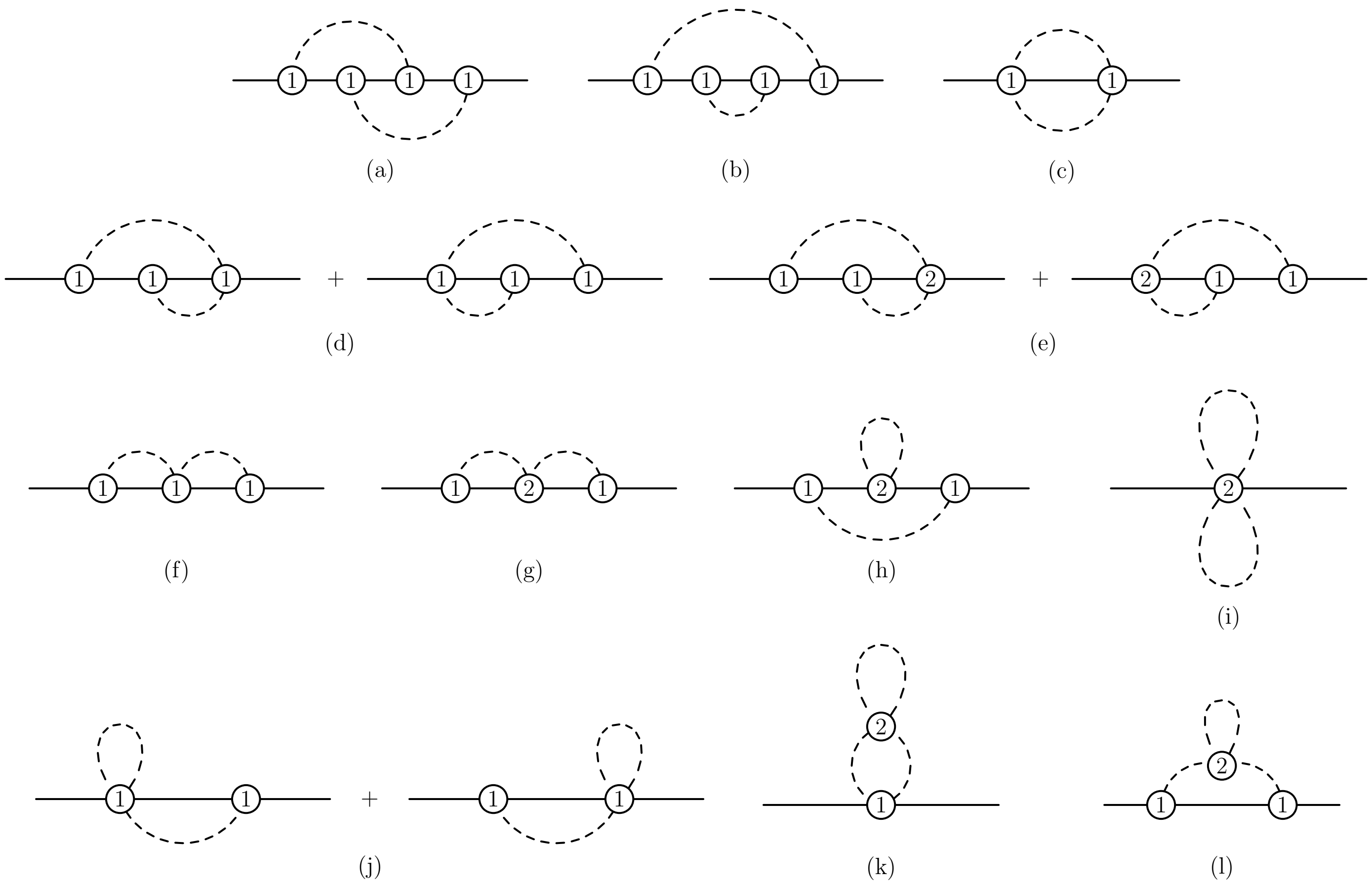
$$\boxed{\Delta m = 55.5\,\text{MeV}}$$

Remark: $m = m_N(m_u = 0, m_d = 0, m_s)$

[18] T. Fuchs, J. Gegelia, S. S., Eur. Phys. J. A 19, 35 (2004)

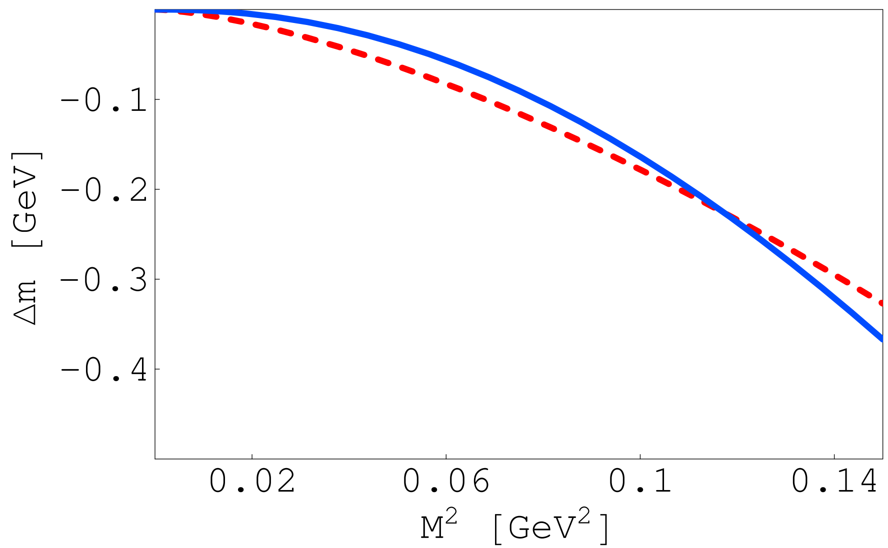
Mass of the nucleon at $\mathcal{O}(q^6)$ [19]

Two-loop contributions



[19] M. R. Schindler, D. Djukanovic, J. Gegelia, S. S., Phys. Lett. B 649, 390 (2007); Nucl. Phys. A 803, 68 (2008)

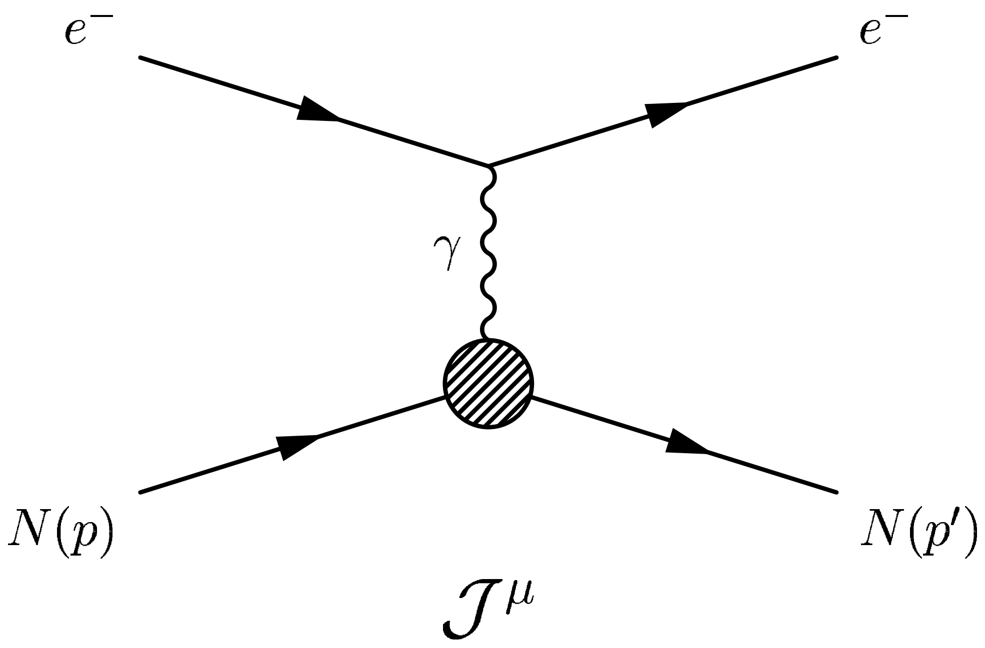
$$m_N = m + k_1 M^2 + k_2\, M^3 + k_3 M^4 \ln\frac{M}{\mu} + k_4 M^4$$
$$\underbrace{+k_5 M^5 \ln\frac{M}{\mu} + k_6 M^5 + k_7 M^6 \ln^2\frac{M}{\mu} + k_8 M^6 \ln\frac{M}{\mu} + k_9 M^6}_{\text{two loop}}$$



$M_0 \approx 360$ **MeV** **(convergence)**

At physical pion mass: $-\,4.8$ **MeV** $=$ **31% of** $k_2\, M^3$

Electromagnetic form factors



Electromagnetic current operator

$$\mathcal{J}^\mu(x) = \frac{2}{3}\,\bar{u}(x)\gamma^\mu u(x) - \frac{1}{3}\,\bar{d}(x)\gamma^\mu d(x) + \cdots = \bar{q}(x)\,Q\,q(x) + \cdots$$

Definition of Dirac and Pauli form factors

$$\langle N(p')|\mathcal{J}^\mu(0)|N(p)\rangle = \bar{u}(p')\left[\boxed{F_1^N(Q^2)}\gamma^\mu + i\frac{\sigma^{\mu\nu}q_\nu}{2m_p}\boxed{F_2^N(Q^2)}\right]u(p)$$

$$N = p, n, \quad q^\mu = p'^\mu - p^\mu, \quad Q^2 = -q^2$$

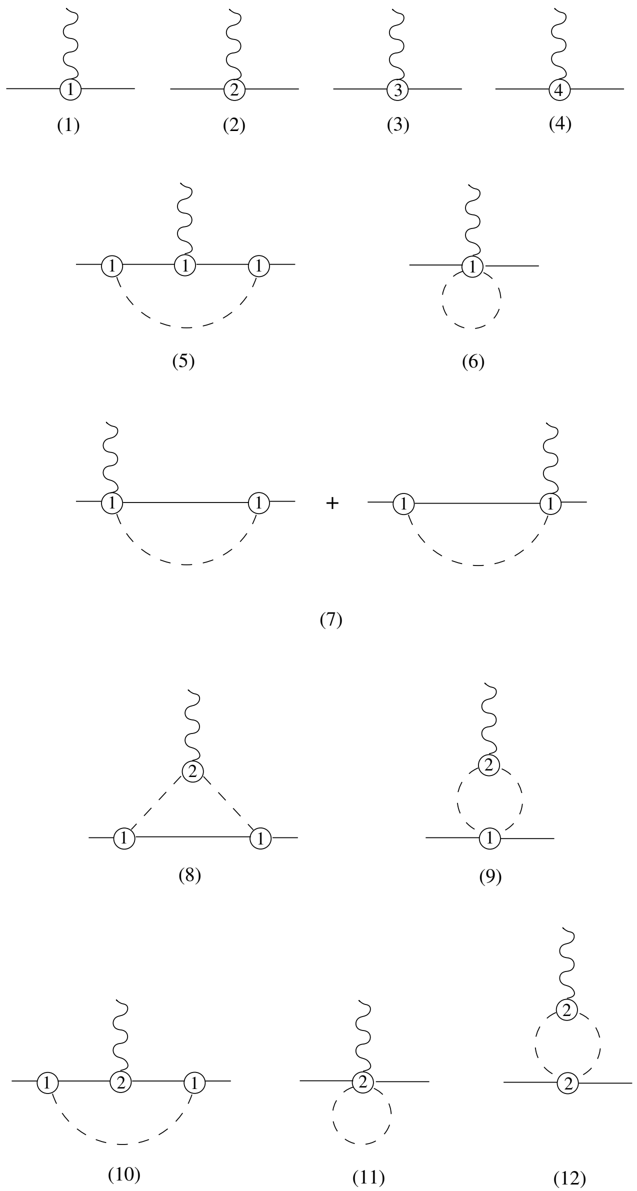
$$F_1^p(0) = 1, \; F_1^n(0) = 0, \; F_2^p(0) = 1.793, \; F_2^n(0) = -1.913.$$

Sachs form factors

$$G_E^N(Q^2) = F_1^N(Q^2) - \frac{Q^2}{4m_N^2} F_2^N(Q^2)$$

$$G_M^N(Q^2) = F_1^N(Q^2) + F_2^N(Q^2)$$

Diagrams at $\mathcal{O}(q^4)$



Diagrams potentially violating power counting: (5), (8), and (10).

EOMS subtractions

- Dirac form factor

$$\Delta F_1^{10} = \frac{g_A^2 m}{64\pi^2 F^2}\left(3c_7 - 2c_6\tau_3\right) t,$$

- Pauli form factor

$$\begin{aligned}
\Delta F_2^{5} &= -\frac{g_A^2 m_N (m - 4c_1 M^2)}{32\pi^2 F^2}\left(3 - \tau_3\right), \\
\Delta F_2^{8} &= \frac{g_A^2 m_N (m - 4c_1 M^2)}{8\pi^2 F^2}\tau_3, \\
\Delta F_2^{10} &= -\frac{g_A^2 m_N (m^2 - 8c_1 M^2 m)}{16\pi^2 F^2}\left(3c_7 - 2c_6\tau_3\right).
\end{aligned}$$

Parameters

	c_2	c_4	$\tilde{c}_6$	$\tilde{c}_7$	d_6	d_7	e_{54}	e_{74}
EOMS	2.66	2.45	1.26	-0.13	-0.57	-0.44	0.27	1.71
IR	2.66	2.45	0.47	-1.87	0.32	-0.89	0.33	1.65

The LECs c_i are given in units of GeV^{-1}, the d_i in units of GeV^{-2}, and the e_i in units of GeV^{-3}.

c_2 and c_4 from πN scattering;

$\tilde{c}_6$ and $\tilde{c}_7$ from anomalous magnetic moments;

d_6, d_7, e_{54}, and e_{74} from charge and magnetic radii: [20]

$r_E^p = 0.848\,\text{fm}$,

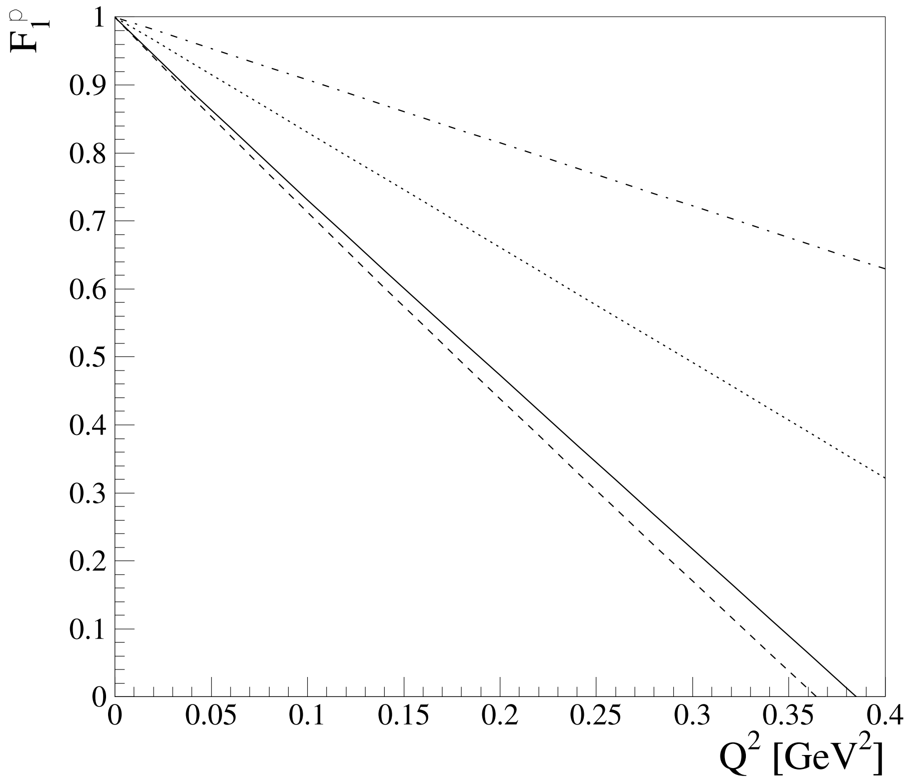
$r_M^p = 0.857\,\text{fm}$,

$r_E^n = 0.113\,\text{fm}$,

$r_M^n = 0.879\,\text{fm}$.

[20] H. W. Hammer and U.-G. Meißner, Eur. Phys. J. A 20, 469 (2004).

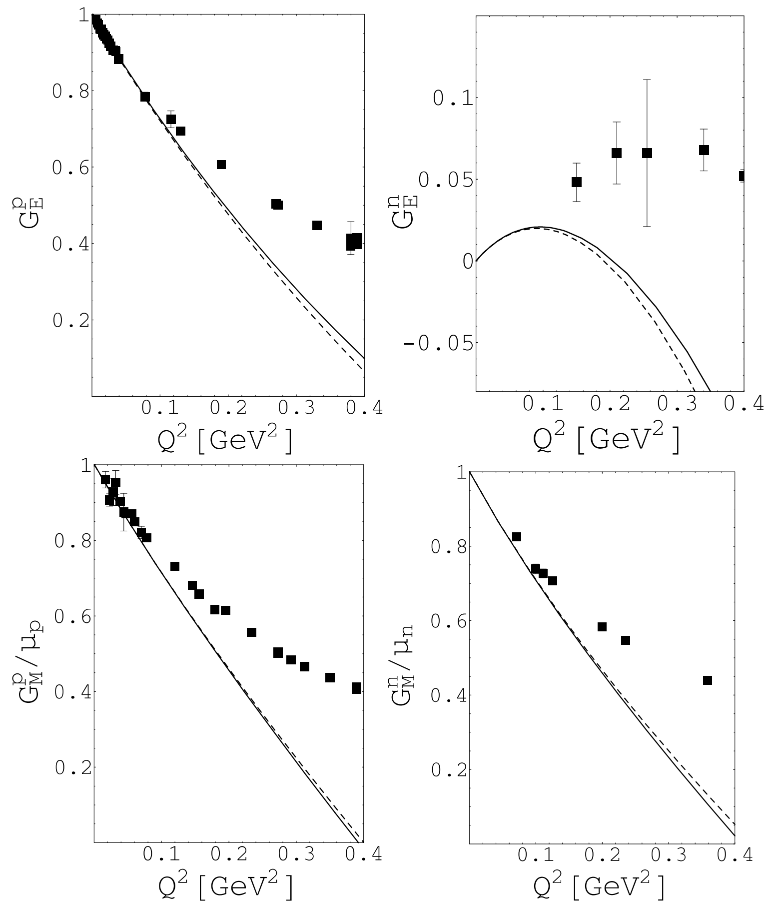
Dirac form factor of the proton at $\mathcal{O}(q^4)$ [21]



Solid line: EOMS; dashed line: infrared regularization; dotted line: EOMS without loop contribution; dashed-dotted line: infrared-regularization result without loop contribution.

[21] T. Fuchs, J. Gegelia, S. S., J. Phys. G 30, 1407 (2004)

Sachs form factors [22]



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[22] B. Kubis and U.-G. Meißner, Nucl. Phys. A679, 698 (2001); T. Fuchs, J. Gegelia, S. S., J. Phys. G 30, 1407 (2004); M. R. Schindler, J. Gegelia, S. S., Eur. Phys. J. A 26, 1 (2005); data taken from J. Friedrich and Th. Walcher, Eur. Phys. J. A 17, 607 (2003)

Vector meson dominance model $\rightarrow$ Important contributions to the electromagnetic form factors [23]

In standard ChPT: Vector meson contributions in low-energy constants

$$\frac{1}{q^2 - M_V^2} = -\frac{1}{M_V^2}\left[1 + \frac{q^2}{M_V^2} + \left(\frac{q^2}{M_V^2}\right)^2 + \mathcal{O}(q^6)\right]$$

Inclusion of vector mesons $\Rightarrow$ re-summation of higher-order contributions

Reformulated IR regularization and EOMS scheme allow for consistent inclusion of vector mesons

[23] B. Kubis and U.-G. Meißner, Nucl. Phys. A679, 698 (2001)

Inclusion of ρ, ω, and ϕ mesons [24]

Vector representation [25]

$$\mathcal{L}_{\pi V}^{(3)} = -f_\rho \mathrm{Tr}(\rho^{\mu\nu} f_{\mu\nu}^{+}) - f_\omega \omega^{\mu\nu} f_{\mu\nu}^{(s)} - f_\phi \phi^{\mu\nu} f_{\mu\nu}^{(s)} + \ldots,$$

$$\mathcal{L}_{NV}^{(0)} = \frac{1}{2} \sum_{V=\rho,\omega,\phi} g_V \bar{\Psi}\gamma^\mu V_\mu \Psi,$$

$$\mathcal{L}_{NV}^{(1)} = \frac{1}{4} \sum_{V=\rho,\omega,\phi} G_V \bar{\Psi}\sigma^{\mu\nu} V_{\mu\nu}\Psi.$$

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[24] M. R. Schindler, J. Gegelia, S. S., Eur. Phys. J. A 26, 1 (2005)

[25] G. Ecker, J. Gasser, H. Leutwyler, A. Pich, E. de Rafael, Phys. Lett. B 223, 425 (1989)

Values of the vector-meson coupling constants [26]

f_ρ	f_ω	f_ϕ	g_ρ	g_ω	g_ϕ	G_ρ [GeV^{-1}]	G_ω [GeV^{-1}]	G_ϕ [GeV^{-1}]
0.10	0.03	0.05	4.0	42.8	-20.6	13.0	0.96	-3.3

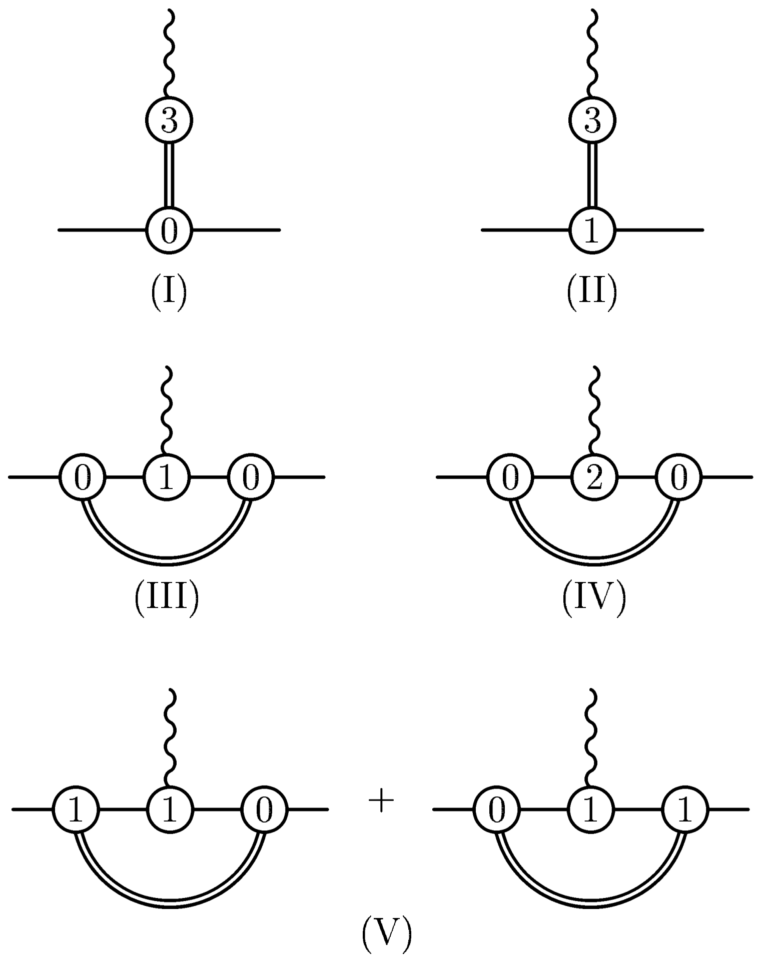
$\Rightarrow$ Modified couplings d_6, d_7, e_5 and e_{74}

	d_6	d_7	e_{54}	e_{74}
EOMS	1.21	1.30	-0.76	1.65
IR	0.98	0.24	-0.26	-0.90

Additional rules:

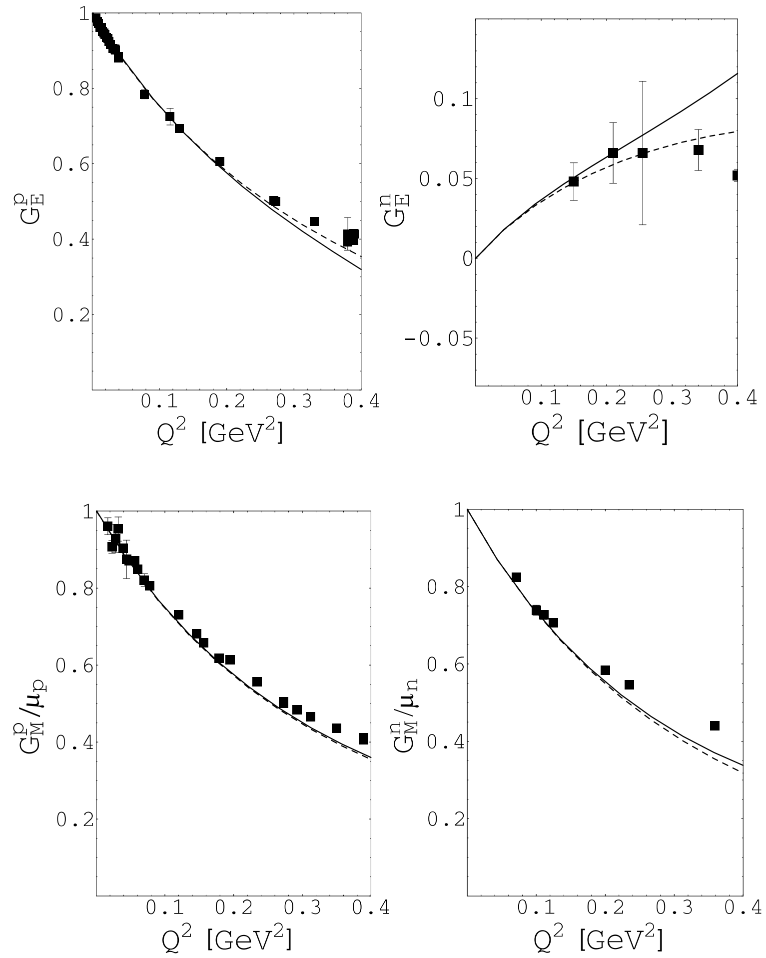
- Vector meson propagator $\sim \mathcal{O}(q^0)$
- Vertex from $\mathcal{L}_V^{(i)} \sim \mathcal{O}(q^i)$

[26] H. W. Hammer and U.-G. Meißner, Eur. Phys. J. A 20, 469 (2004).



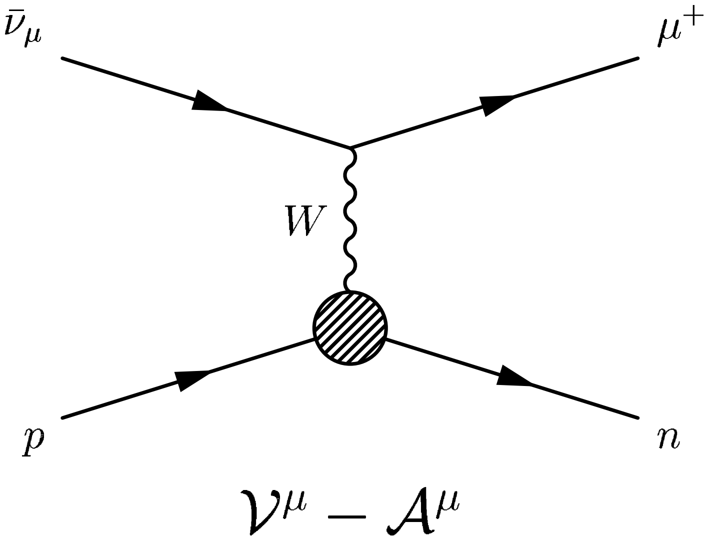
Feynman diagrams involving vector mesons contributing to the electromagnetic form factors up to and including $\mathcal{O}(q^4)$

E.m. form factors including vector mesons at $\mathcal{O}(q^4)$ [27]

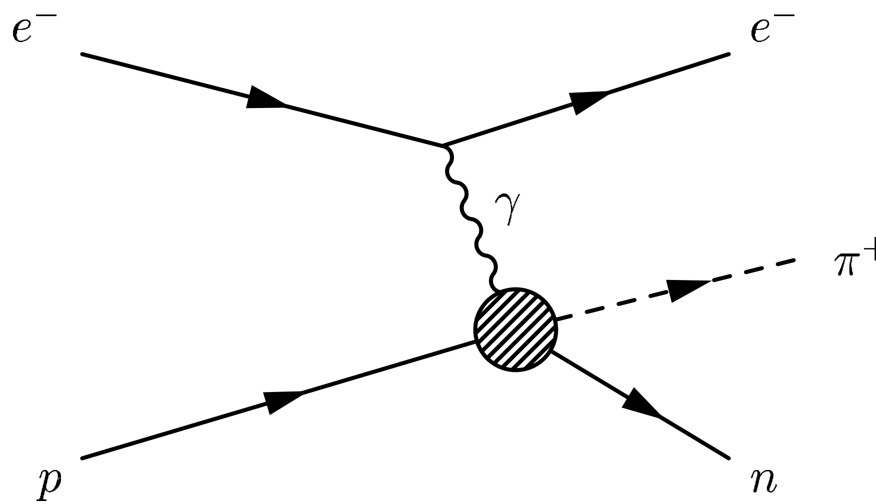


[27] M. R. Schindler, J. Gegelia, and S. S., Eur. Phys. J. A 26, 1 (2005); data taken from J. Friedrich and Th. Walcher, Eur. Phys. J. A 17, 607 (2003)

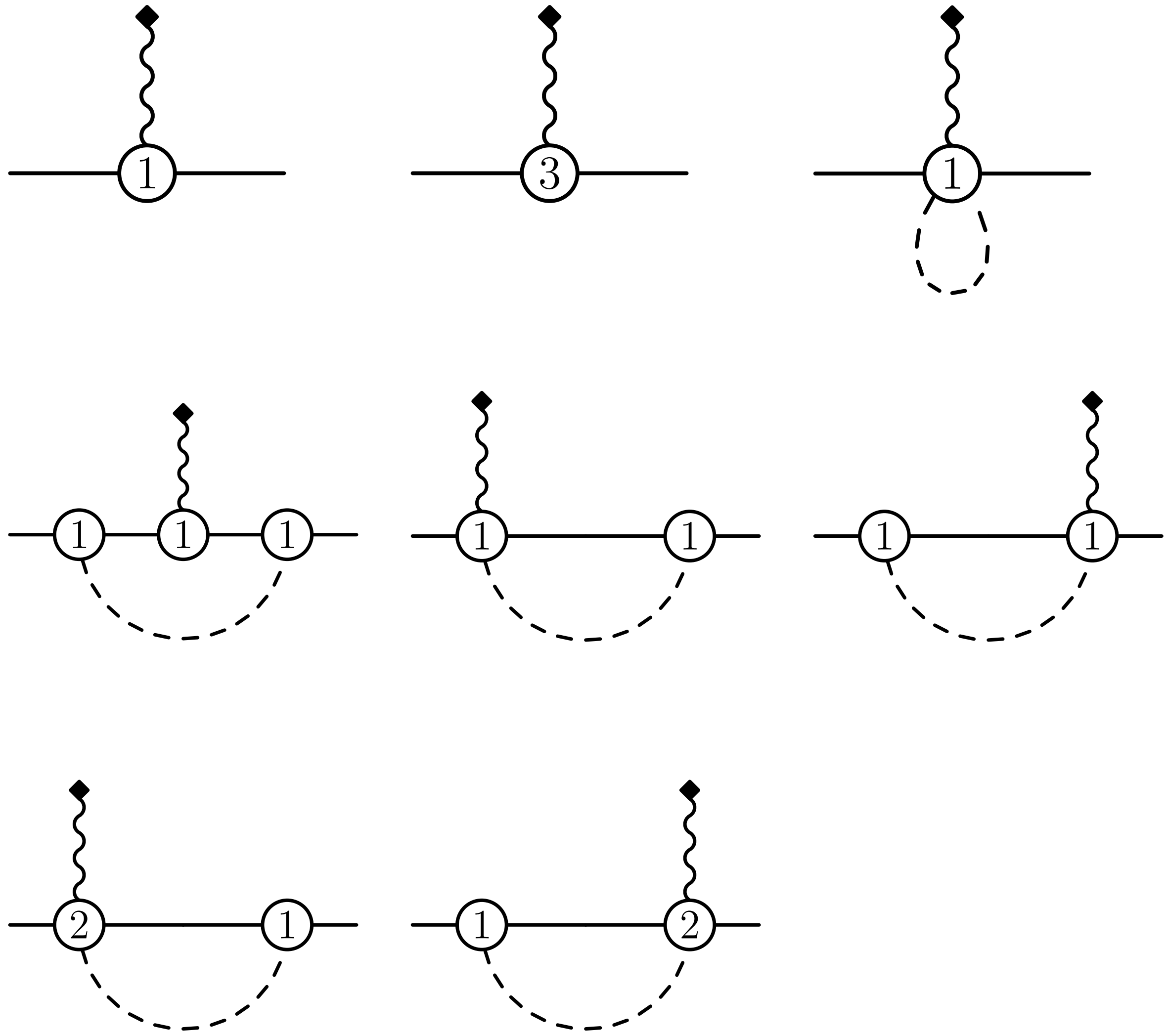
Axial and induced pseudoscalar form factors G_A and G_P



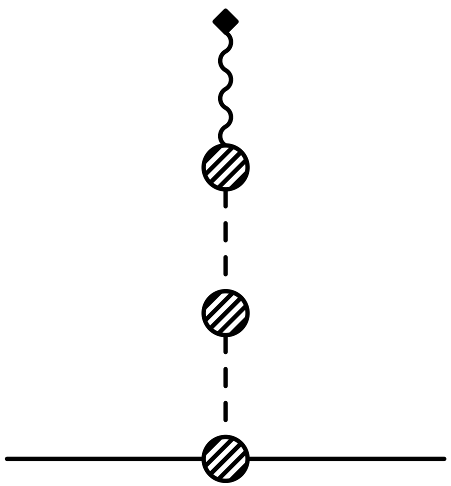
Partially
Conserved
Axial-vector
Current
hypothesis



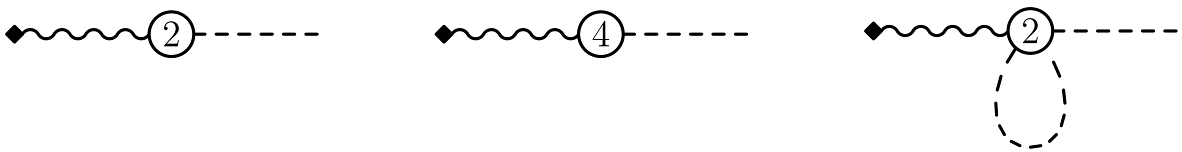
$$\langle n|\mathcal{A}^{\mu,-}(0)|p\rangle = \bar{u}(p')\left[\gamma^\mu\gamma_5 \boxed{G_A(Q^2)} + \frac{q^\mu}{2m_N}\gamma_5 \boxed{G_P(Q^2)}\right] u(p)$$



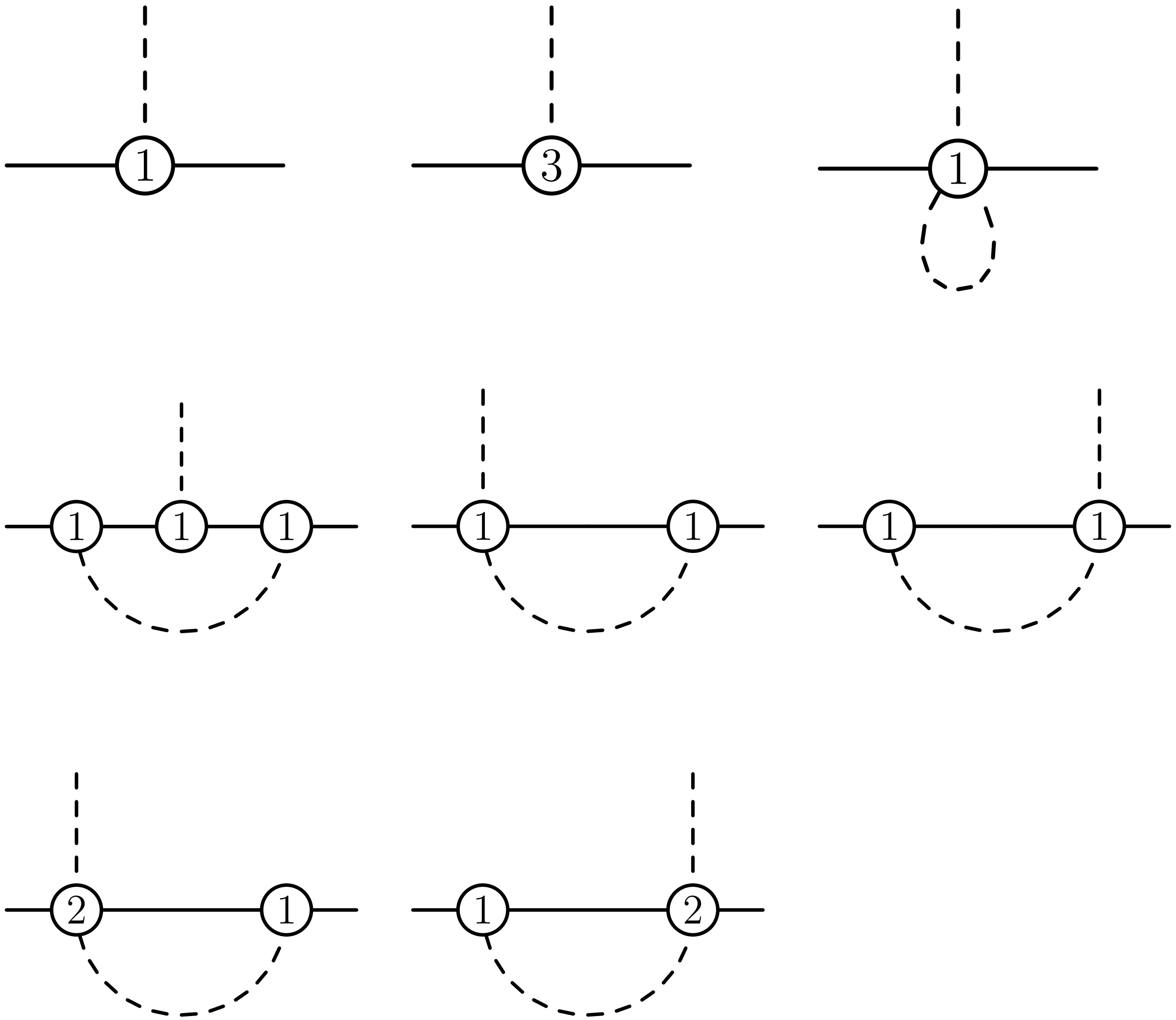
One-particle-irreducible diagrams contributing to the nucleon matrix element of the isovector axial-vector current.



Pion pole graph of the isovector axial-vector current.



Diagrams contributing to the coupling of the isovector axial-vector current to a pion up to $\mathcal{O}(q^4)$.



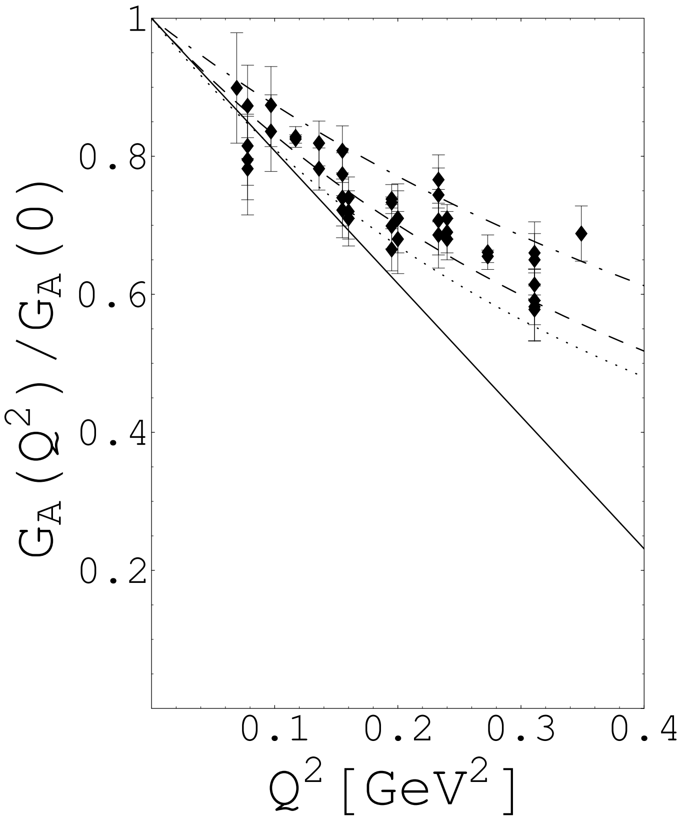
Diagrams contributing to the πN vertex up to $\mathcal{O}(q^4)$.

Result for G_A is of the form

$$G_A(Q^2) = g_A - \frac{1}{6} g_A \langle r_A^2 \rangle Q^2 + \frac{g_A^3}{4F^2} \bar{H}(Q^2).$$

$\langle r_A^2 \rangle$: axial mean-square radius (LEC)
$\bar{H}(Q^2)$: loop contributions
$\bar{H}(0) = \bar{H}'(0) = 0.$

Full line: result in infrared renormalization.
Again: No curvature!
Dashed line: Dipole, $M_A = 1.026$ GeV;
Dotted line: Dipole; $M_A = 0.95$ GeV;
Dashed-dotted line: Dipole $M_A = 1.20$ GeV,



Inclusion of $a_1(1260)$ meson [28]

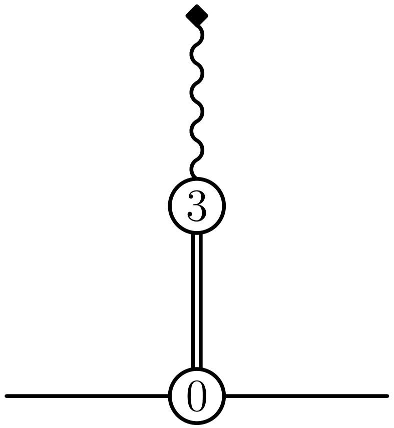
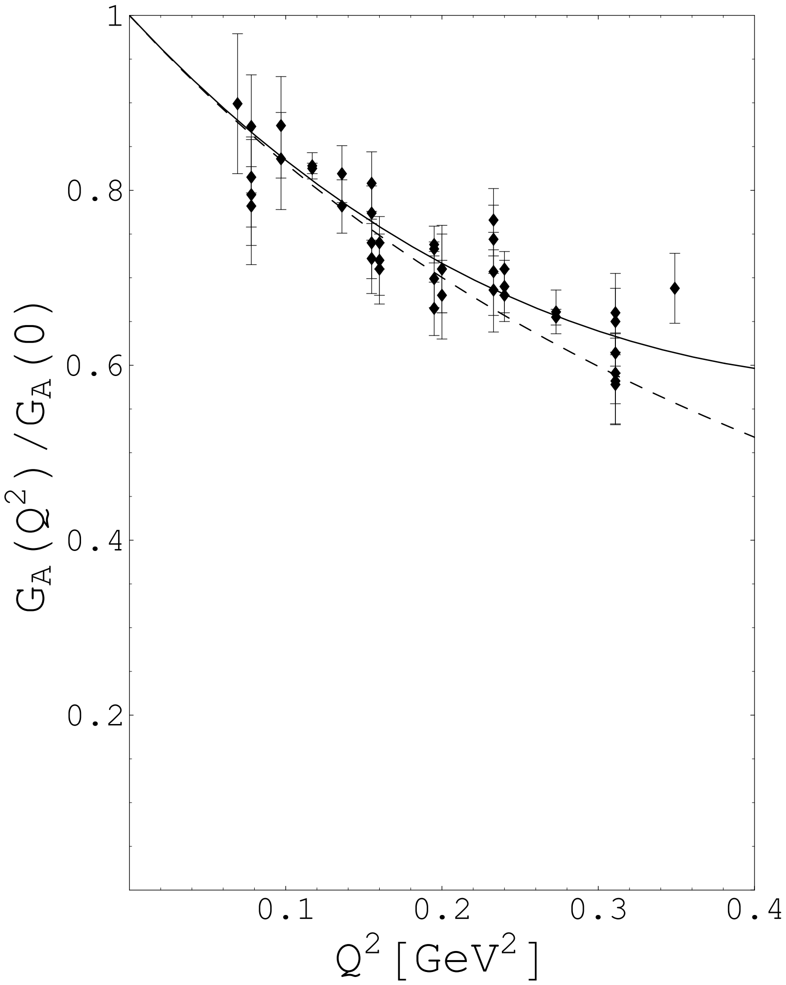


Diagram containing axial-vector meson (double line) contributing to the form factors G_A and G_P.

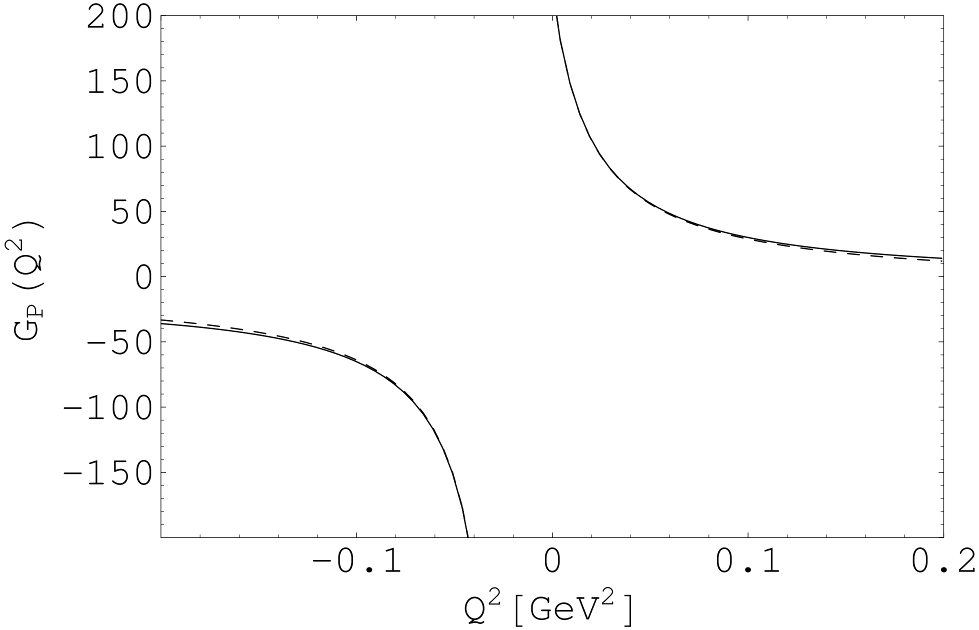
$$G_A^{AVM}(q^2) = -f_A g_{a_1} \frac{q^2}{q^2 - M_{a_1}^2},$$

$$f_A g_{a_1} \approx 8.70.$$

[28] M. R. Schindler, T. Fuchs, J. Gegelia, S. S, Phys. Rev. C 75, 025202 (2007)



G_A **including** a_1



G_P **at** $\mathcal{O}(q^4)$

Full line: result with axial-vector meson, dashed line: result without axial-vector meson.

Thanks to my collaborators

- Dr. Dalibor Djukanovic
- Dr. Thomas Fuchs
- Dr. Jambul Gegelia
- Dr. Björn C. Lehnhart
- Dr. Matthias R. Schindler

Thank You!

Infrared regularization in more detail.

Consider dimensionally regularized one-loop integral [29]

$$\begin{aligned} H(p^2, m^2, M^2; n) &\\ \equiv -i \int \frac{d^n k}{(2\pi)^n} & \frac{1}{[(p-k)^2 - m^2 + i0^+][k^2 - M^2 + i0^+]} \\ = -i \int \frac{d^n k}{(2\pi)^n} & \frac{1}{[k^2 - 2p\cdot k + \underbrace{(p^2 - m^2)}_{\mathcal{O}(q)} + i0^+][k^2 - \underbrace{M^2}_{\mathcal{O}(q^2)} + i0^+]}. \end{aligned}$$

Qualitative discussion:

- **Ultraviolet behavior:**

  Estimate of degree of divergence: For large values of k integrand behaves as k^{n-1}/k^4. $\Rightarrow$

---

[29]**Note the minus sign. Factor μ^{4-n} omitted.**

- $n = 4$: Logarithmic divergence (dim. reg.: $1/(n-4)$).
- $n < 4$: Integral converges.

- Infrared behavior: Consider limit $M^2 \to 0$.
  - $n = 4$: Integral is infrared regular for both $p^2 = m^2$ and $p^2 \neq m^2$, because, for small momenta, the integrand behaves as k^3/k^3 and k^3/k^2, respectively.
  - For $n = 3$ the integral is infrared regular for $p^2 \neq m^2$ but singular for $p^2 = m^2$.
  - For any smaller value of n it is infrared singular for arbitrary p^2.
  - Infrared singularity as $M^2 \to 0$ originates in the region, where the integration variable k is small, i.e., of the order $\mathcal{O}(q)$.

Counting powers of momenta, we (naively) expect this part to be of order $\mathcal{O}(q^{n-3})$.

- **Intermediate region:**

  On the other hand, for loop momenta of the order of and larger than the nucleon mass we expect power counting to fail, because the momentum of the nucleon propagating in loop integral is not constrained to be small.

Explicit evaluation of integral $H(p^2, m^2, M^2; n)$:

Feynman parameterization:

$$\frac{1}{ab} = \int_0^1 \frac{dz}{[az + b(1-z)]^2}$$

with $a = (p-k)^2 - m^2 + i0^+$ and $b = k^2 - M^2 + i0^+$.

Interchange the order of integrations:

$$H = -i \int_0^1 dz \int \frac{d^n k}{(2\pi)^n} \frac{1}{[k^2 - 2k \cdot pz + (p^2 - m^2)z + zM^2 + i0^+]^2}.$$

Perform the shift $k \to k + zp$:

$$H(p^2, m^2, M^2; n) = -i \int_0^1 dz \int \frac{d^n k}{(2\pi)^n} \frac{1}{[k^2 - A(z) + i0^+]^2},$$

where

$$A(z) = z^2 p^2 - z(p^2 - m^2 + M^2) + M^2.$$

Make use of (Exercise)

$$\int \frac{d^n k}{(2\pi)^n} \frac{(k^2)^p}{(k^2 - A)^q} = \frac{i(-)^{p-q}}{(4\pi)^{\frac{n}{2}}} \frac{\Gamma\left(p + \frac{n}{2}\right) \Gamma\left(q - p - \frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right) \Gamma(q)} A^{p + \frac{n}{2} - q}$$

in dim. reg.

Apply to H with $p = 0$ and $q = 2$. $\Rightarrow$

Intermediate result:

$$H(p^2, m^2, M^2; n) = \frac{1}{(4\pi)^{\frac{n}{2}}}\Gamma\left(2 - \frac{n}{2}\right) \int_0^1 dz[A(z) - i0^+]^{\frac{n}{2}-2}.$$

Discussion of relevant properties at the threshold:

$$\begin{aligned} p_{\rm thr}^2 &= (m+M)^2, \\ A_{\rm thr}(z) &= z^2(m+M)^2 - z[(m+M)^2 - m^2 + M^2] + M^2 \\ &= [z(m+M) - M]^2 \geq 0, \\ z_0 &= M/(m+M), \qquad A_{\rm thr}(z_0) = 0. \end{aligned}$$

Splitting integration interval into $[0, z_0]$ and $[z_0, 1]$, we have, for $n > 3$,

$$\begin{aligned} \int_0^1 dz[A_{\rm thr}(z)]^{\frac{n}{2}-2} &= \int_0^{z_0} dz[M - z(m+M)]^{n-4} \\ &\quad + \int_{z_0}^1 dz[z(m+M) - M]^{n-4} \\ &= \frac{1}{(n-3)(m+M)}(M^{n-3} + m^{n-3}). \end{aligned}$$

Analytic continuation for arbitrary n:

$$H((m+M)^2, m^2, M^2; n) = \frac{\Gamma\left(2-\frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}(n-3)} \left(\frac{M^{n-3}}{m+M} + \frac{m^{n-3}}{m+M}\right).$$

Discussion

- **The first term, proportional to M^{n-3}, is defined as the so-called infrared singular part I.**
- **As $M \to 0$, I behaves as in the qualitative discussion above.**
- **$M \to 0$ implies $p^2_{\mathrm{thr}} \to m^2$. I is singular for $n \leq 3$.**
- **The second term, proportional to m^{n-3}, is defined as the so-called infrared regular part R.**

- Can be thought of as originating from an integration region where k is of order m.

- For non-integer n the infrared singular part contains non-integer powers of M.

- Expansion of the regular part always contains non-negative integer powers of M only.

Formal definition of the infrared singular and regular parts (for arbitrary p^2).

Introduce the dimensionless variables

$$\begin{aligned}\alpha &= \frac{M}{m} = \mathcal{O}(q),\\ \Omega &= \frac{p^2 - m^2 - M^2}{2mM} = \mathcal{O}(q^0).\end{aligned}$$

Rewrite $A(z)$ as

$$A(z) = m^2[z^2 - 2\alpha\Omega z(1-z) + \alpha^2(1-z)^2] \equiv m^2 C(z).$$

$\Rightarrow H$ is now given by

$$H(p^2, m^2, M^2; n) = \kappa(m; n) \int_0^1 dz [C(z) - i0^+]^{\frac{n}{2}-2},$$

where

$$\kappa(m; n) = \frac{\Gamma\left(2 - \frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}} m^{n-4}.$$

- Infrared singularity originates from small values of z, where $C(z)$ goes to zero as $M \to 0$.

- Isolate divergent part by scaling integration variable $z \equiv \alpha x$. Upper limit $z = 1$ in Feynman parameterization corresponds to $x = 1/\alpha \to \infty$ as $M \to 0$.

- **Define** integral I having the same infrared singularity as H. To that end replace upper limit by ∞:

$$\begin{aligned} I &\equiv \kappa(m;n)\int_0^{\infty} dz[C(z)-i0^+]^{\frac{n}{2}-2} \\ &= \kappa(m;n)\alpha^{n-3}\int_0^{\infty} dx[D(x)-i0^+]^{\frac{n}{2}-2}, \end{aligned}$$

where

$$D(x) = 1 - 2\Omega x + x^2 + 2\alpha x(\Omega x - 1) + \alpha^2 x^2.$$

(The pion mass M is not sent to zero.)

- **Define** regular part of H as

$$R \equiv -\kappa(m;n)\int_1^{\infty} dz[C(z)-i0^+]^{\frac{n}{2}-2},$$

so that

$$H = I + R.$$

- Q: Do these definitions indeed reproduce the behavior for $p^2_{\rm thr}$?

  A: Yes!

  Verification: $\Omega_{\rm thr} = 1$.

- Threshold value of the infrared singular part:

$$I_{\rm thr} = \kappa(m;n)\alpha^{n-3} \int_0^\infty dx \left\{[(1+\alpha)x-1]^2 - i0^+\right\}^{\frac{n}{2}-2},$$

  which converges for $n < 3$.

  In order to continue the integral to $n > 3$, we write

$$\left\{[(1+\alpha)x-1]^2 - i0^+\right\}^{\frac{n}{2}-2} =$$
$$= \frac{(1+\alpha)x-1}{(1+\alpha)(n-4)} \frac{d}{dx} \left\{[(1+\alpha)x-1]^2 - i0^+\right\}^{\frac{n}{2}-2},$$

  and make use of a partial integration

$$\int_0^\infty dx \left\{[(1+\alpha)x-1]^2 - i0^+\right\}^{\frac{n}{2}-2} =$$

$$\left[\frac{(1+\alpha)x-1}{(1+\alpha)(n-4)}\left\{[(1+\alpha)x-1]^2-i0^+\right\}^{\frac{n}{2}-2}\right]_0^\infty$$
$$-\frac{1}{n-4}\int_0^\infty dx\left\{[(1+\alpha)x-1]^2-i0^+\right\}^{\frac{n}{2}-2}.$$

For $n<3$, the first expression vanishes at the upper limit and, at the lower limit, yields $1/[(1+\alpha)(n-4)]$.

Bringing the second expression to the left-hand side, we may then continue the integral analytically as

$$\int_0^\infty dx\left\{[(1+\alpha)x-1]^2-i0^+\right\}^{\frac{n}{2}-2}=\frac{1}{(n-3)(1+\alpha)},$$

so that we obtain for $I_{\rm thr}$

$$I_{\rm thr}=\kappa(m;n)\alpha^{n-3}\frac{1}{(n-3)(1+\alpha)}=\frac{\Gamma\left(2-\frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}(n-3)}\frac{M^{n-3}}{m+M}.$$

Agrees with the infrared singular part discussed above.

- Threshold value of the regular part obtained by analytic continuation from $n < 3$ to $n > 3$:

$$
\begin{aligned}
R_{\text{thr}} &= -\frac{\Gamma\left(2-\frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}}\int_1^\infty dz[z(m+M)-M_\pi]^{n-4} \\
&= -\frac{\Gamma\left(2-\frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}}\frac{1}{(n-3)(m+M)}(\infty^{n-3}-m^{n-3}) \\
&\overset{n \leq 3}{=} \frac{\Gamma\left(2-\frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}(n-3)}\frac{m^{n-3}}{m+M}.
\end{aligned}
$$

Again, agrees with the regular part discussed above.

- Distinction between I and R:

  For non-integer values of n, the chiral expansion of I gives rise to non-integer powers of small quantities.

  Regular part R may be expanded in an ordinary Taylor series.

  I satisfies power counting; R does not.

  Basic idea of the infrared regularization: Replace general integral H by its infrared singular part I, and drop the regular part R.

  In the low-energy region H and I have the same analytic properties.

  Contribution of R, which is of the type of an infinite series in the momenta, can be included by adjusting the coefficients of the most general effective Lagrangian.

- **Generalization to arbitrary one-loop graph.**
  - Reduce tensor integrals involving an expression of the type $k^{\mu_1}\cdots k^{\mu_2}$ in the numerator to scalar loop integrals of the form

$$-i\int\frac{d^nk}{(2\pi)^n}\frac{1}{a_1\cdots a_m}\frac{1}{b_1\cdots b_n},$$

    $a_i=(q_i+k)^2-M^2+i0^+$: Inverse meson propagators;

    $b_i=(p_i-k)^2-m^2+i0^+$ Inverse nucleon propagators;

    q_i: four-momenta of $\mathcal{O}(q)$;

    p_i: four-momenta which are not far off the nucleon mass shell, i.e., $p_i^2=m^2+\mathcal{O}(q)$.

  - Using the Feynman parameterization, combine all nucleon propagators separately and all pion propagators separately.

  - Write the result such that it is obtained by applying $(m-1)$ and $(n-1)$ partial derivatives with respect to M^2 and m^2, respectively, to a master formula.

Simple illustration:

$$\frac{1}{a_1 a_2} = \int_0^1 dz \frac{1}{[a_1 z + a_2(1-z)]^2} = \frac{\partial}{\partial M^2} \int_0^1 dz \frac{1}{a_1 z + a_2(1-z)},$$

where $a_i = (q_i + k)^2 - M^2 + i0^+$.

Expressions become more complicated for larger numbers of propagators!

Relevant property of the above procedure:

Result of combining the meson propagators is of the type $1/A$ with $A = (k+q)^2 - M^2 + i0^+$, where q is a linear combination of the m momenta q_i, with an analogous expression $1/B$ for the nucleon propagators.

– Finally, treat expression

$$-i \int \frac{d^n k}{(2\pi)^n} \frac{1}{AB}$$

in complete analogy to H: Combine denominators. Identify

infrared singular and regular pieces by writing

$$\int_0^1 dz \cdots = \int_0^\infty dz \cdots - \int_1^\infty dz \cdots .$$

– Q: Does the infrared regularization respect the constraints of chiral symmetry as expressed through the chiral Ward identities?

A: Yes

The argument is as follows.

∗ Total nucleon-to-nucleon transition amplitude is chirally symmetric.

(Invariant under a local transformation of the external fields.)

∗ Calculation within EFT:

Contribution from all the tree-level diagrams is chirally symmetric so that the loop contribution must also be chirally symmetric.

∗ Dim. reg.: Statement holds for an arbitrary n.

Now: Separation into infrared singular and regular parts amounts to distinguishing between contributions of non-integer and non-negative integer powers in the momentum expansion.

These powers do not mix for arbitrary n. $\Rightarrow$ Infrared singular and regular parts must be separately chirally symmetric.

Finally, regular part can be expanded in powers of either momenta or quark masses, and thus may as well be absorbed in the (modified) tree-level contribution.

σ term [30]

Definition of the so-called sigma commutator

$$\sigma^{ab}(x) \equiv [Q_A^a(x_0), [Q_A^b(x_0), \mathcal{H}_{\rm sb}(x)]], \quad a, b = 1, 2, 3$$

where

$$\mathcal{H}_{\rm sb} = \bar{q} M q = m_q(\bar{u}u + \bar{d}d)$$

Measure of explicit symmetry breaking

$$\sigma \equiv \frac{1}{2m_N} \langle p | \sigma^{11}(0) | p \rangle$$

[30] T. Fuchs, J. Gegelia, S. Scherer, Eur. Phys. J. A 19, 35 (2004)

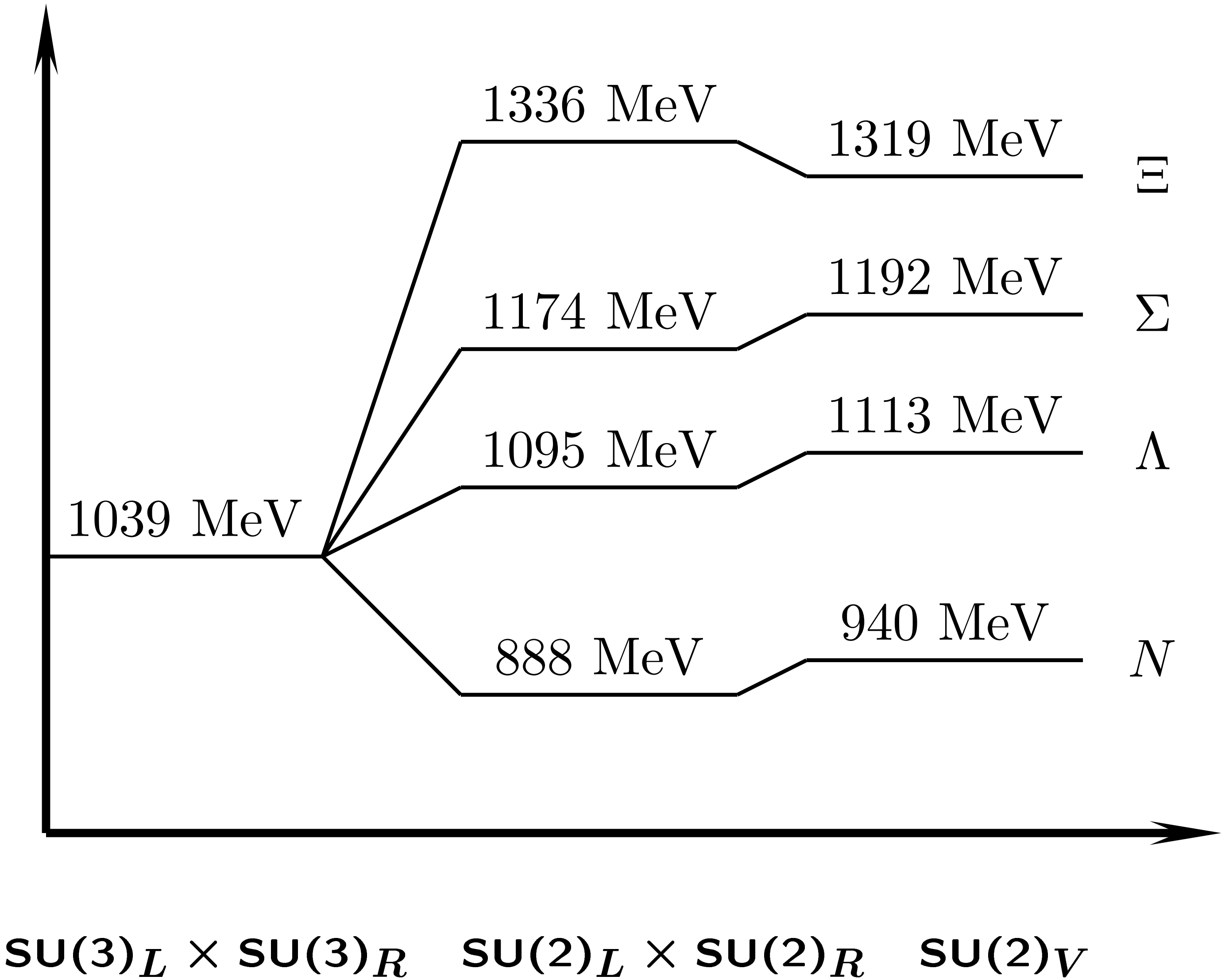
$$\sigma = \sigma_1 M^2 + \sigma_2 M^3 + \sigma_3 M^4 \ln\left(\frac{M}{m}\right) + \sigma_4 M^4 + O(M^5)$$

$$\begin{aligned}
\sigma_1 &= -4c_1 \\
\sigma_2 &= -\frac{9\mathrm{g}_A^2}{64\pi F^2} \\
\sigma_3 &= \frac{3}{16\pi^2 F^2}\left(8c_1 - c_2 - 4c_3 - \frac{\mathrm{g}_A^2}{m}\right) \\
\sigma_4 &= \frac{3}{8\pi^2 F^2}\left[\frac{3\mathrm{g}_A^2}{8m} + c_1(1+2\mathrm{g}_A^2) - \frac{c_3}{2}\right] + \alpha
\end{aligned}$$

$$\sigma = 45\,\text{MeV} = (74.8 - 22.9 - 9.4 - 2.0 + 4.5)\,\text{MeV}$$

Hellmann-Feynman theorem o.k.

Masses of the baryon octet at $\mathcal{O}(q^3)$ [31]



[31] B. C. Lehnhart, J. Gegelia, S. Scherer, J. Phys. G 31, 1 (2005)