Simulation advances in Coherent Synchrotron Radiation modeling

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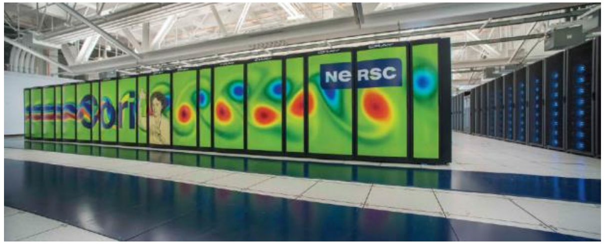
Acknowledgement

- CSR collaboration
  - CoSyR [1] development: Feiyu Li, Hoby Rakotoarivelo, Boqian Shen, Rao Garimella, Orion Yeung, Parker Pombrio, Gary Dilts, Jaryd Domine, Thomas Kwan, Bruce Carlsten, Robert Robey
  - CSR experiment @ AWA: Omkar Ramachandran, Xueying Lu, Gwanghui Ha, Qiang Ji, John Power
- Symplectic neural surrogate development
  - Qi Tang, Joshua Burby, Yuri K. Batygin, Sergey Kurennoy, Oleksii Beznosov, Anastasiia Kim, Yanzeng Zhang, George Quinn

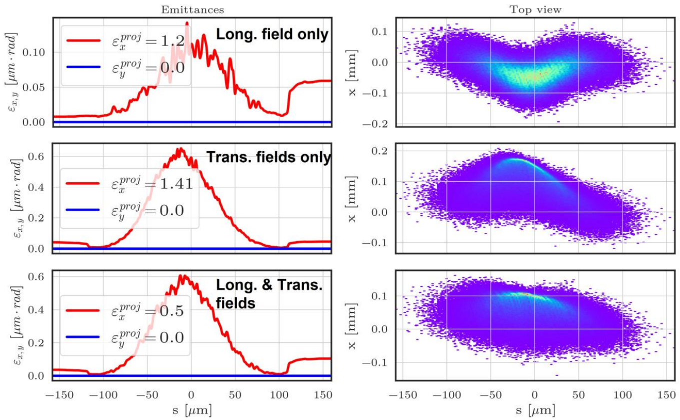
https://github.com/lanl/cosyr.git

[1]Huang et al., NIMA 1034 (2022) 166808

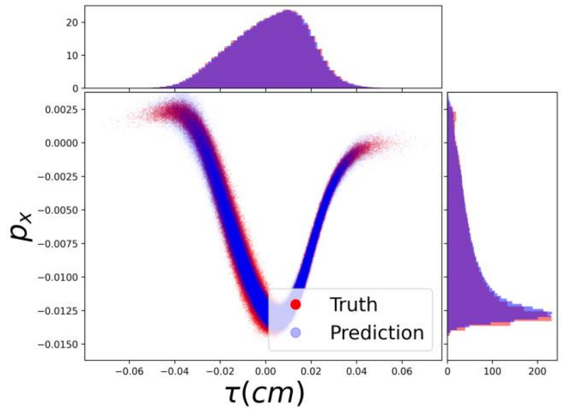


Outline

- Challenge: modeling/understanding of high brightness beams dynamics at the most detailed level and with fast turn-around time
- A new particle-mesh beam dynamics tool and neural surrogate models
- Some understandings from first beam dynamic simulations with accurate Coherent Synchrotron Radiation (CSR) treatment



Longitudinal and transverse CSR effects



Symplectic neural surrogate for CSR

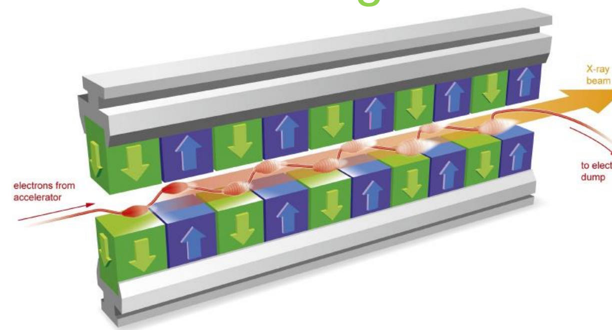
Near-field SR plays an important role in the dynamics of high-current beams

<u>Far field SR (x-ray) is generated in 3rd and 4th generation light sources</u>

3rd generation

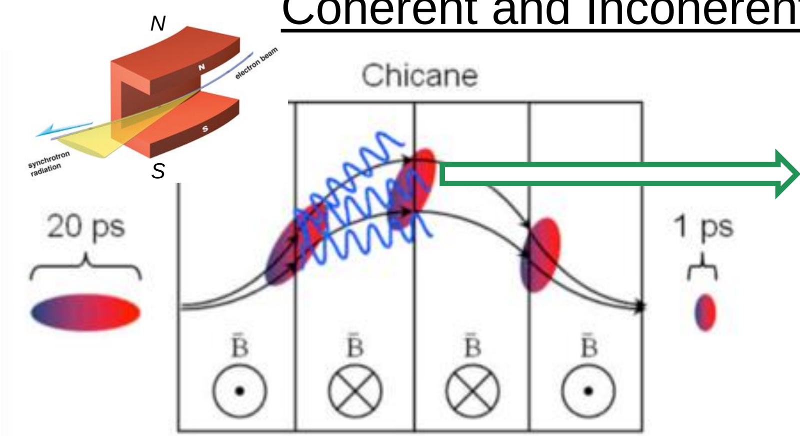
Synchrotron Soleil, Essone, France

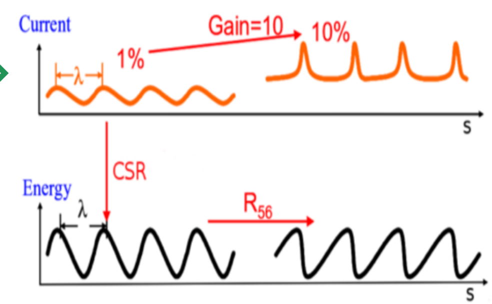
4th generation (FEL)



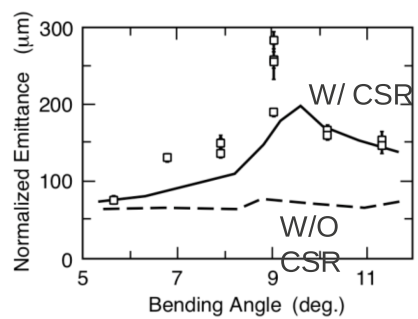
LCLS II, SLAC

<u>Coherent and Incoherent near-field SR interact with the beam itself</u>





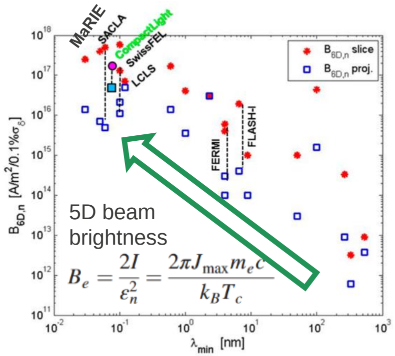
Microbunching



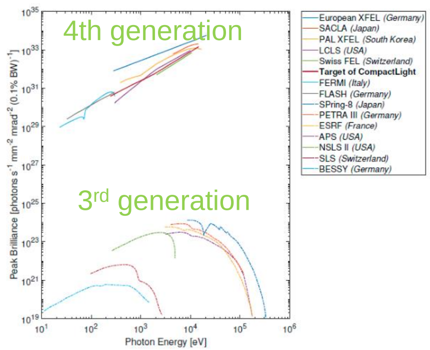
Emittance growth

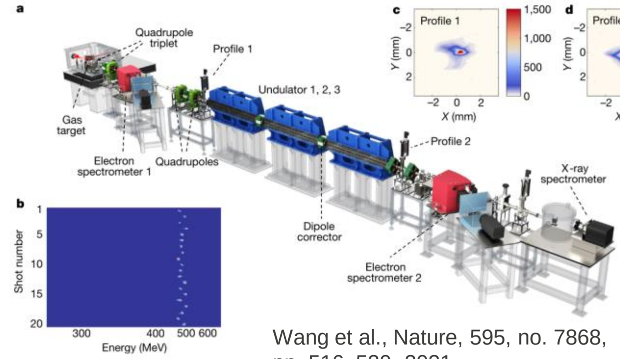
The quest for brighter beams with better control

Future accelerators will use brighter beams and stronger fields



S. Di Mitri *et al.*, *Photonics*, vol. 7, no. 4, 2020

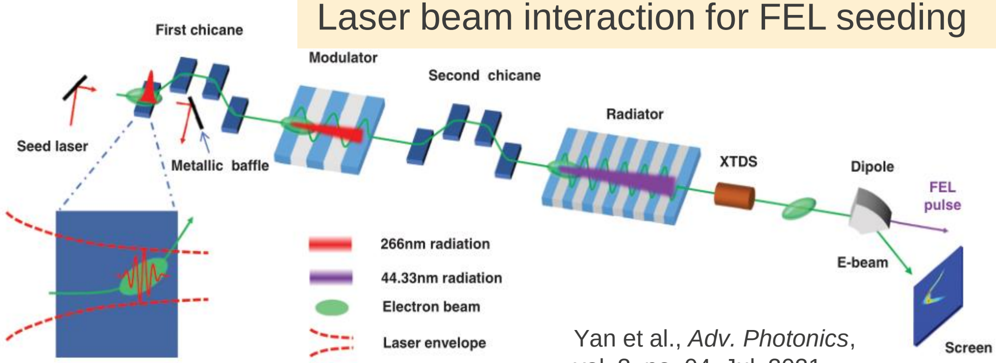




Wang et al., Nature, 595, no. 7868, pp. 516–520, 2021

Emerging laser-based beam manipulation techniques

Laser beam interaction for FEL seeding



Yan et al., *Adv. Photonics*, vol. 3, no. 04, Jul. 2021

Designer laser pulses

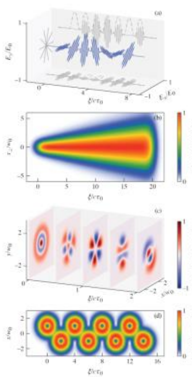
$$\left[\nabla^2 - \frac{1}{c^2}\partial_t^2\right] \mathbf{E}(\mathbf{x}, t) = 0.$$

$$\left[\nabla_\perp^2 + 2ik_0\partial_s\right] A(\mathbf{x}_\perp, s, \xi) \approx 0$$

$$A(\mathbf{x}_\perp, s, \xi) \approx B(\xi)C(\mathbf{x}_\perp, s)$$

arbitrarily structured laser

(arXiv:2207.13849v1)



Time dependency

Polarizatio
n

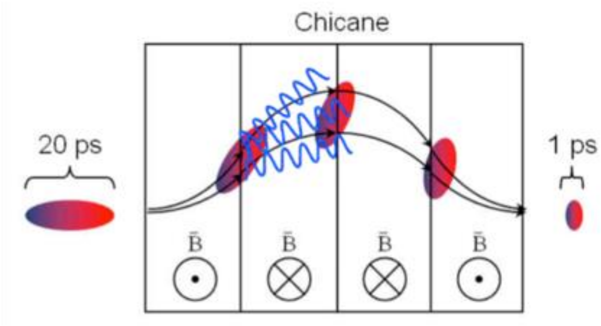
Spot size

Angular
momentu
m

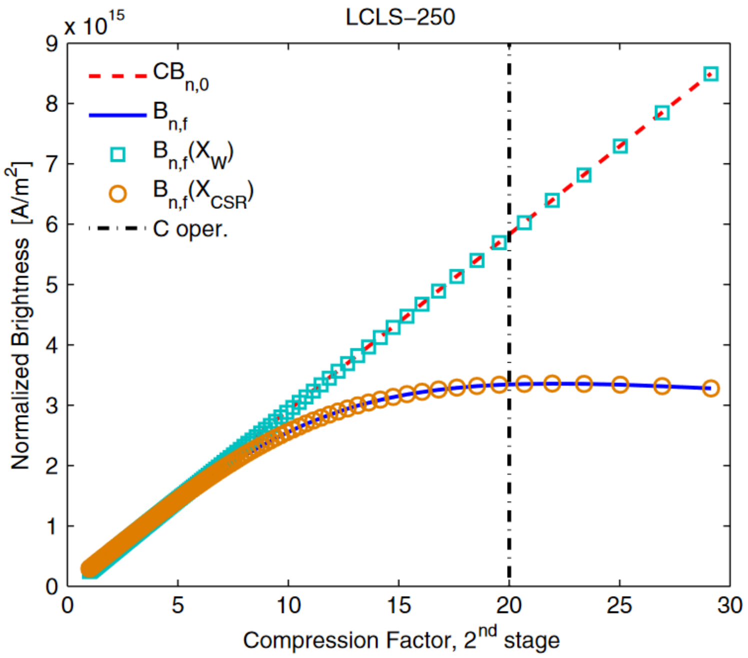
Optical
vortices

Increasing importance of collective beam dynamic effects from radiative interaction

- Beam brightness cannot be improved once generated from the source
- As a non-neutral plasma, the collective effects of a high brightness beam in the transport is critical for accelerator applications
  - Space charge (electrostatic interaction)
  - Intra-beam scattering (collisional interaction)
  - Wakefield (boundary interaction)
  - Coherent and incoherent synchrotron radiation (radiative interaction)

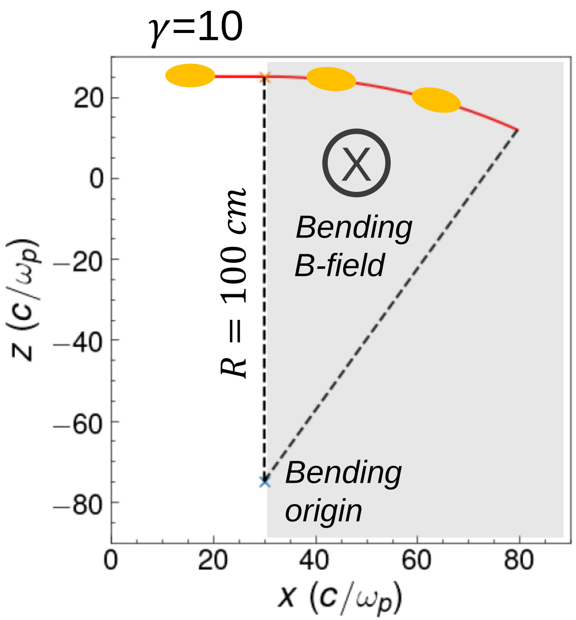


Degradation of brightness due to beam compression



S. Di Mitri et al., Phys. Rep., vol. 539, no. 1, 2014.

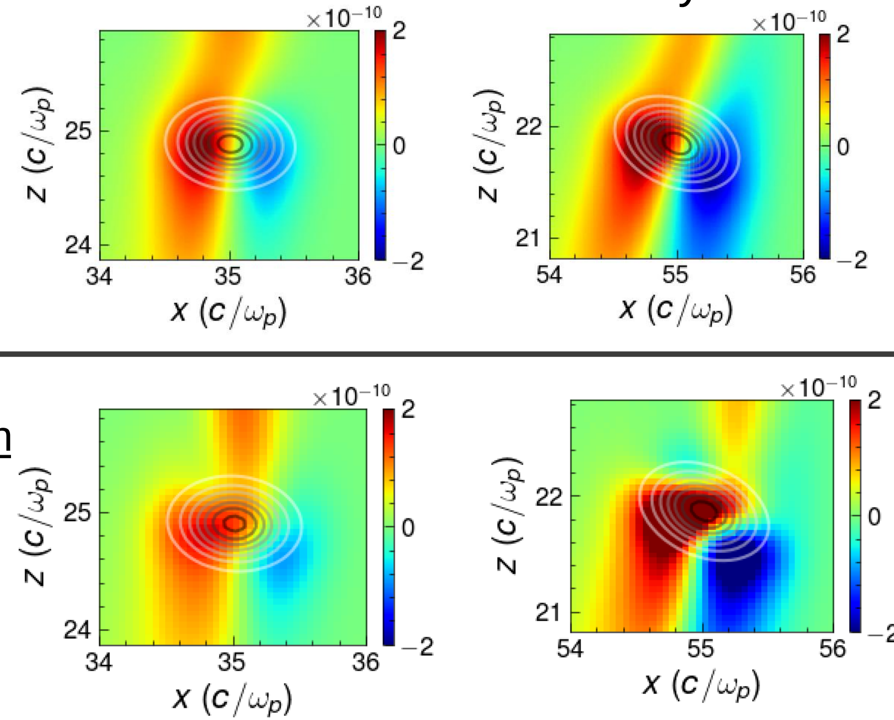
PIC simulation of beam dynamics from coherent synchrotron radiation : high resolution needed to reduce numerical errors



Transient state | Steady state

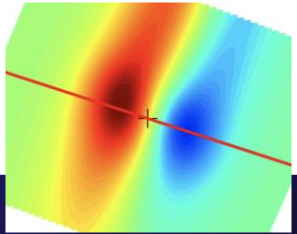
High resolution
($\sigma_x/dx = 25$)
3000 cpu-hrs

Low resolution
($\sigma_x/dx = 5$)



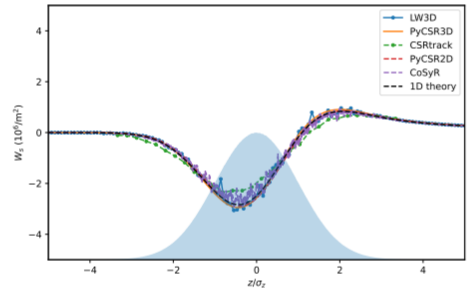
- High cost at fine resolution due to Courant condition
- Yee or high order solver
- Need moving window and proper PML
- Difficult to find suitable boosted frame
- Error in field cancellation from staggered grids (esp. for transverse force $\propto \gamma^{-2}$)

2D analytical steady-state model (LW-CSR)

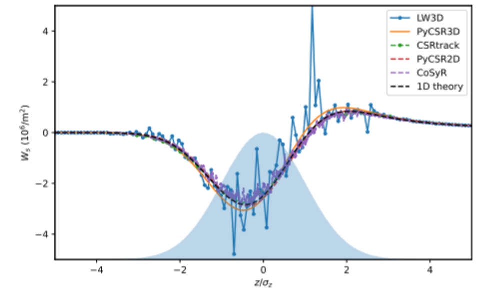


CSR simulation models

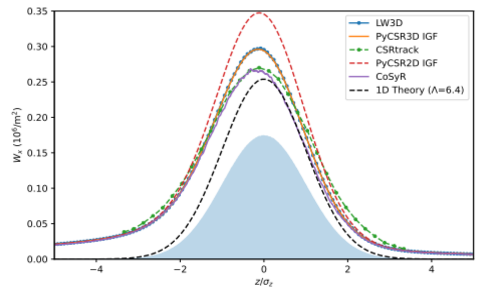
- 1D steady-state models
- 2D/3D steady-state models: pyCSR
- 2D self-similarity model based on Liénard-Wiechert (LW) equation
- 2D sub-bunch model: CSRtrack
- 3D model with prescribed particle trajectories: LW3D
- Particle-In-Cell (PIC) based on FDTD (Finite Difference Time Domain)
- Jefimenko's equation



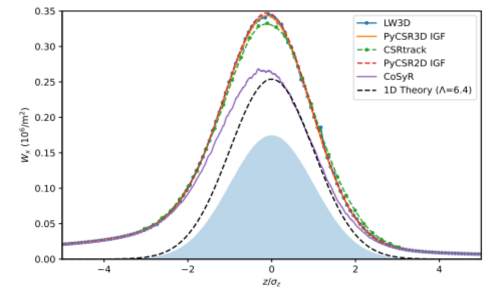
(e) Longitudinal wake at $x = 0$, $y = 0$, parameter set A



(f) Longitudinal wake at $x = 0$, $y = 0$, parameter set B



(a) Horizontal wake at $x = 0$, $y = 0$, parameter set A



(b) Horizontal wake at $x = 0$, $y = 0$, parameter set B

G. Bassi, T. Agoh, M. Dohlus, L. Giannessi, R. Hajima, A. Kabel, T. Limberg, and M. Quattromini, Overview of CSR Codes, Nucl. Instruments Methods Phys. Res. Sect. A Accel. Spectrometers, Detect. Assoc. Equip. 557, 189 (2006).
C. E. Mayes, *Computational Approaches to Coherent Synchrotron Radiation in Two and Three Dimensions*, J. Instrum. **16**, P10010 (2021).

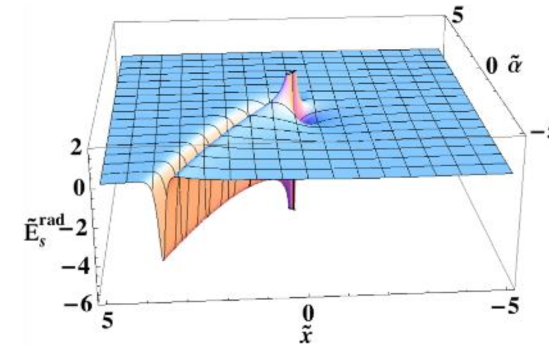
The need of a beam dynamics simulation tool for the extreme scale

- Modeling near-field SR **accurately & efficiently** is critical to understand the **self-consistent** dynamics of high brightness beams, e.g., for hard x-ray FELs.

- For CSR modeling, many methods and simulation tools have been developed. Yet **self-consistent multi-dimensional** simulation is lacking.

- We are developing a self-consistent Green's function-based dispersion-free particle-mesh method for the exascale inspired by Shintake's idea.

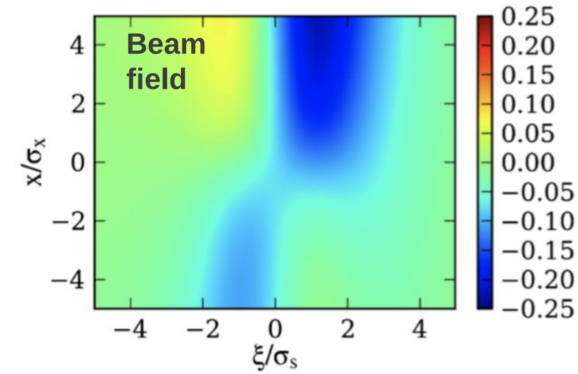
Longitudinal radiation field of a single particle



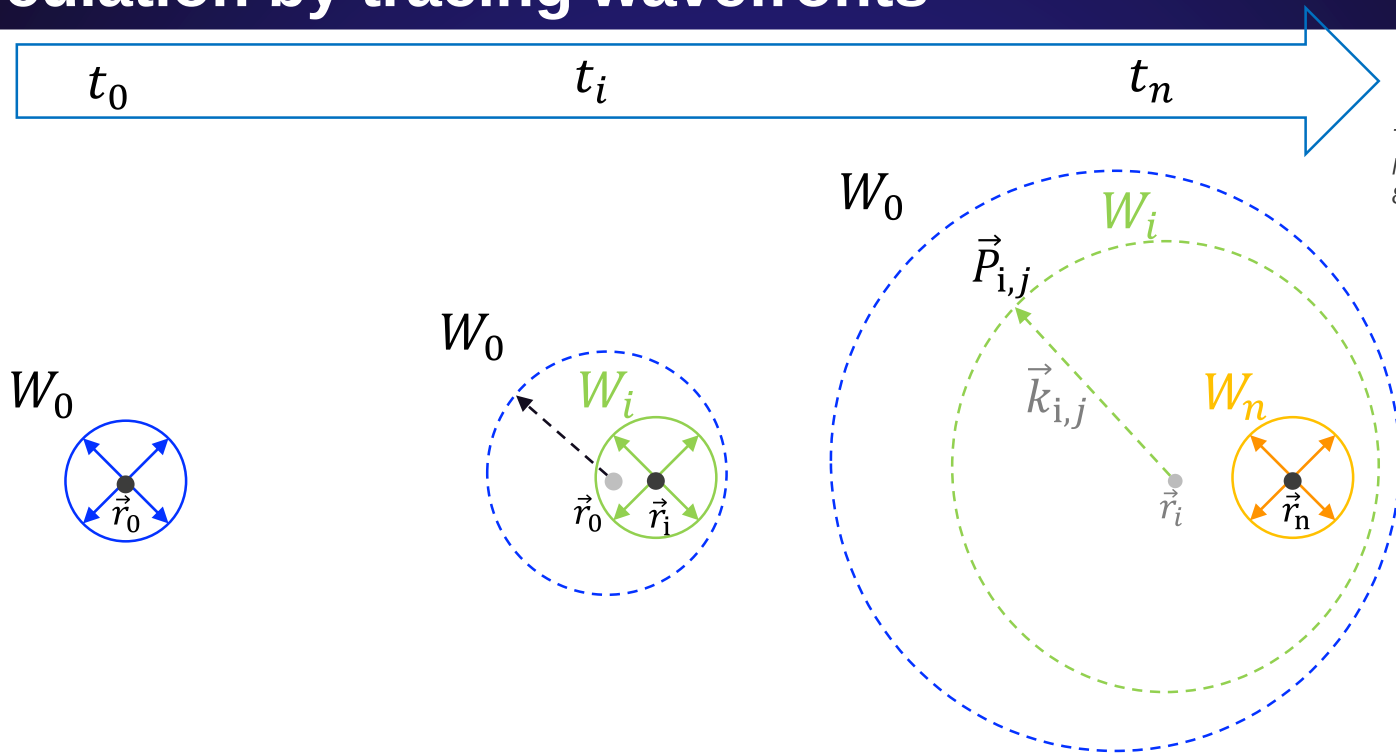
C.-K. Huang et al. PRST-AB (2013)

steady-state models

$$E_b(\xi, x) = \iint E_i(\xi - \xi_i, x - x_i)\rho(\xi_i, x_i)d\xi_i dx_i$$



A Lagrangian method that enables near-field radiation calculation by tracing wavefronts



T. Shintake Nucl. Instr. Methods Phys. Res. A 507, 89-92 (2003)

Wavelet emission

$$k_x = (\cos\theta' + \beta)/(1 + \beta\cos\theta')$$
$$k_y = \sin\theta'/\gamma(1 + \beta\cos\theta')$$

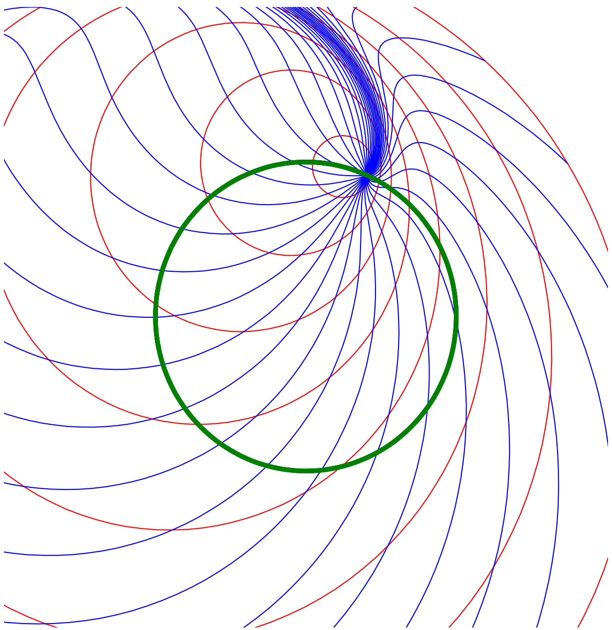
Wavelet update (no Courant condition!)

$$\vec{P}_{i,j}(N\Delta t) = \vec{r}_i + c(N-1)\Delta t\vec{k}_{i,j} \qquad (N > i)$$

Radiation field on an adaptive mesh from acceleration

- Electron trajectory
- Field lines
- Wavefronts

⇒ Adaptive mesh



Electric fields can be calculated in instantaneous electron rest frame

$$\vec{E}'_{\mathrm{vel}} = -e\,\frac{\hat{n}}{r'^2}$$

$$\vec{E}'_{\mathrm{acc}} = e\,\frac{\hat{n} \times (\hat{n} \times \vec{a}')}{r'}$$

⇐ *Missing term*

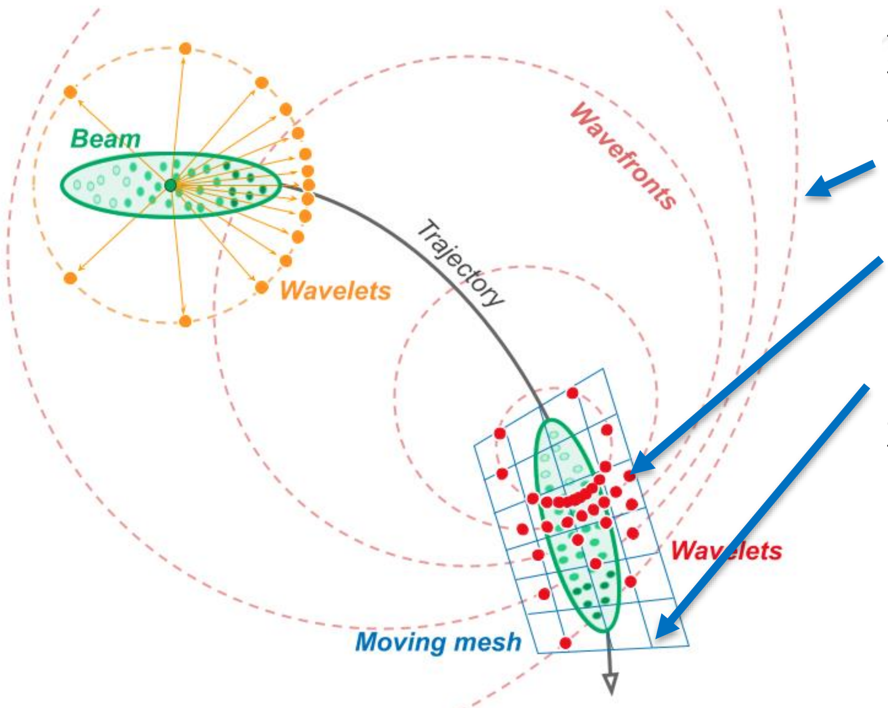
Li et al., Proc. 10th Int. Particle Accelerator Conf. 397-399 (2019)

⇓ Lorentz transformation

$$\vec{E} = \frac{e\left(\hat{n} - \vec{\beta'}\right)}{\gamma^2\rho^2\left(1 - \hat{n}\cdot\vec{\beta'}\right)^3} + \frac{e\hat{n} \times \left[\left(\hat{n} - \vec{\beta'}\right) \times \dot{\vec{\beta'}}\right]}{c\rho\left(1 - n\cdot\vec{\beta'}\right)^3}$$

Design of CoSyR: algorithm

C.-K. Huang, et al., NIM A, 1034, 166808, 2022



Algorithm

Algorithm 1: CSR Field Solver

Initialization;
for *each time step* **do**
  Emit the wavefronts;
  for *each wavefront and field direction* **do**
    if *the wavefront is inside the mesh* **then**
      Compute the field on the wavefront;
    end
  end
  Interpolate the fields from wacefronts to mesh (PORTAGE);
  Push the particles (VPIC);
end

field kernel

$$E_s^{rad} = \frac{e\beta'^2(1 - \hat{n}\cdot\vec{\beta}' - \gamma'^{-2})}{R^2\Psi(1 - \hat{n}\cdot\vec{\beta}')^3}(\hat{n}\cdot\hat{x})$$

$$E_x^{rad} = \frac{e\beta'^2(1 - \hat{n}\cdot\vec{\beta}' - \gamma'^{-2})}{R^2\Psi(1 - \hat{n}\cdot\vec{\beta}')^3}(\hat{n}\cdot\hat{s})$$

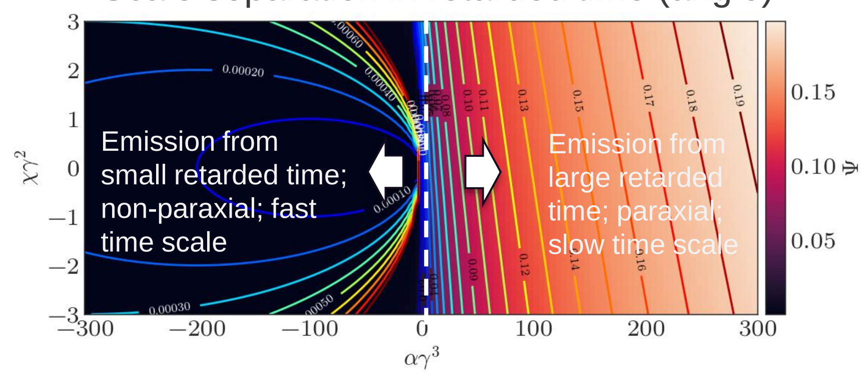
$$B^{rad} = \frac{e\beta'^2(1 - \hat{n}\cdot\vec{\beta}' - \gamma'^{-2})}{R^2\Psi(1 - \hat{n}\cdot\vec{\beta}')^3}$$

mixed-kernel

$$-\partial_s(\phi - \beta A_s)$$

$$-e(E_x^{rad} - \beta_j B^{rad})$$

Scale separation in retarded time (angle)



- The retarded Green's functions are collocated in this wavefront-wavelet approach
- General for other Green's functions, e.g., for the Jefimenko equation.

Combining subcycle and dynamic wavelets

C.-K. Huang, et al., NIM A, 1034, 166808, 2022

- Wavelets are generated at past dt's up to t=0 or up to the point the emitted wavefront outruns the mesh, whichever comes first

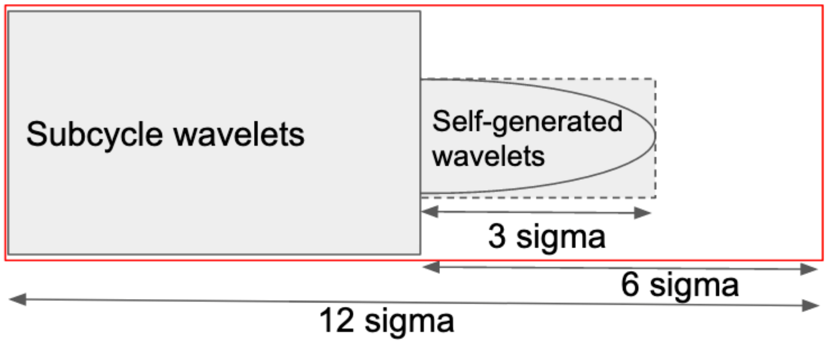
- Take left region as subcycle area (owing to their small retardation) defined by

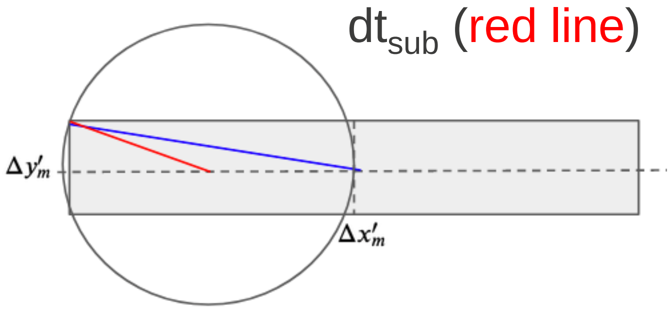
$$dt_{sub} \sim \frac{\Delta x'_m}{4} + \frac{{\Delta y'_m}^2}{2\Delta x'_m}$$

- Pre-calculate mesh points (from LW-CSR) with retarded angle(time) satisfying $\psi \leq dt_{sub}$ as subcycle wavelets, and shift according to particle's offset to mesh center (i.e., discrete convolution)

- Self-generated wavelets and shifted subcycle wavelets are interpolated to mesh

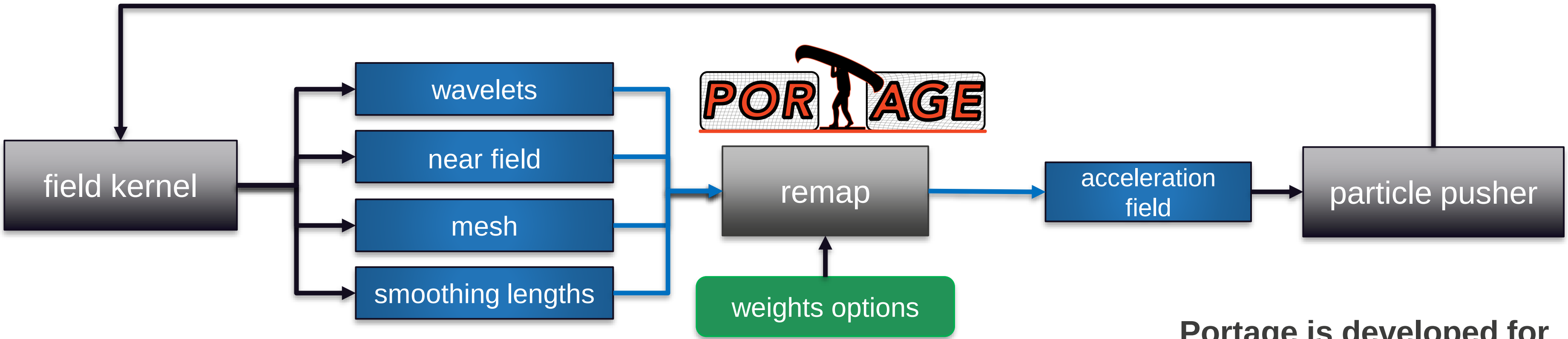
Code generated and preloaded wavelets



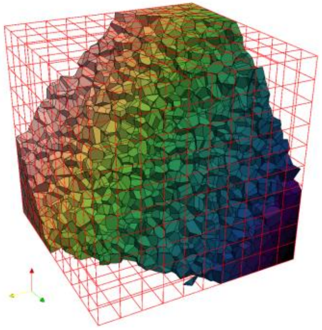


Mesh remapping with Portage

https://laristra.github.io/portage/



Portage is developed for remapping in fluid codes



- remap steps:
  1. redistribute points (optional)
  2. search: retrieve wavelet neighbors of a mesh point.
  3. accumulate: evaluate weight functions on each mesh point.
  4. estimate: do field estimation on mesh points.

Tailoring the search step for wavelets

- create dedicated search kernel in Portage:
  - assign a box to each mesh point based on the smoothing length.
  - create a helper grid that encloses those boxes (GPU-friendly).
  - bin wavelet points into cells by coordinates hashing.
  - queries: retrieve and scan cells that overlap with the box of the mesh point.

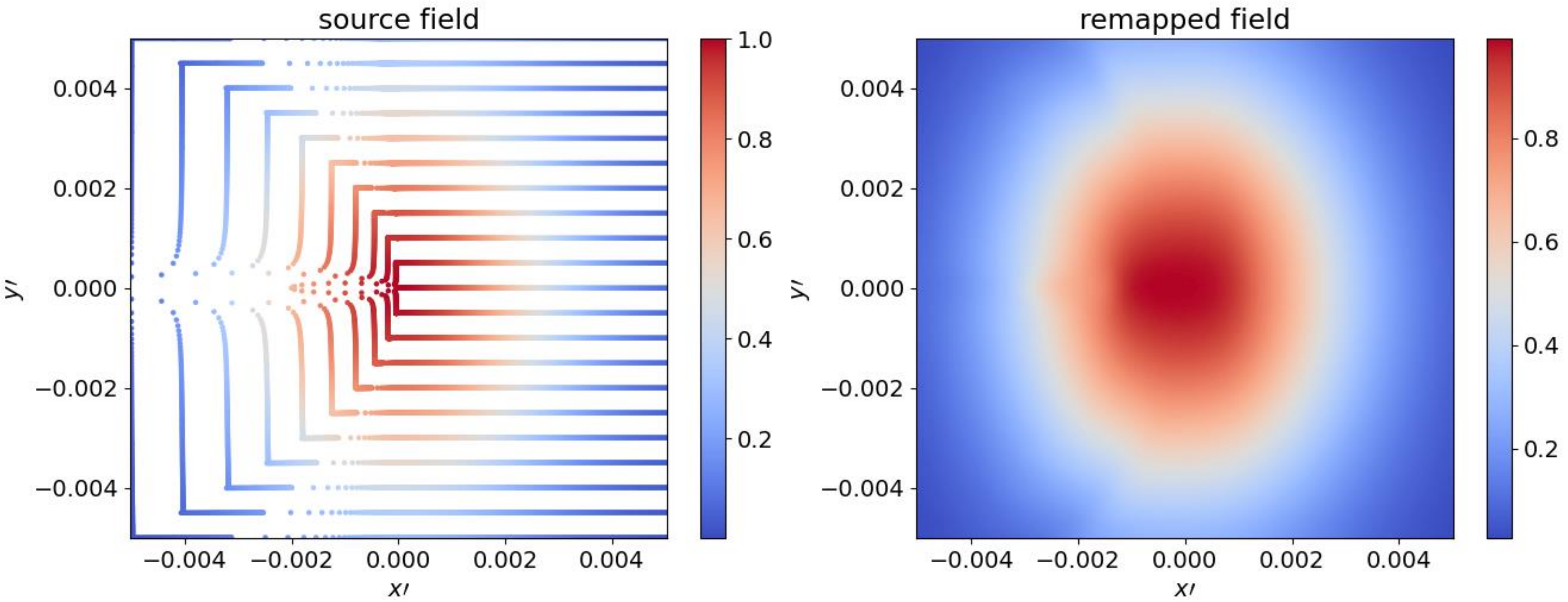
helper grid resolution

- h_{ij}: radius of p_i for j^{th} axis.
- mean radius: $h_j = \frac{\alpha}{n} \sum_{i=1}^{n} |h_{ij}|, \alpha > 0.$
- sides: $s_j = \min \left[\frac{x_{Mj} - x_{mj}}{h_j}, s_M \right]$ cached
- number of bins: $n_{bins} = s_1 s_2$ cached

binning

- point: $p_i = (x_{i1}, \cdots, x_{id})$
- cell: $c_{ij} = \min \left[s_j * \frac{x_{ij} - x_{mj}}{x_{Mj} - x_{mj}}, s_j - 1 \right]$
- bin: $b_i = c_{i1} + c_{i2} s_1$

Field remapping from wavelets

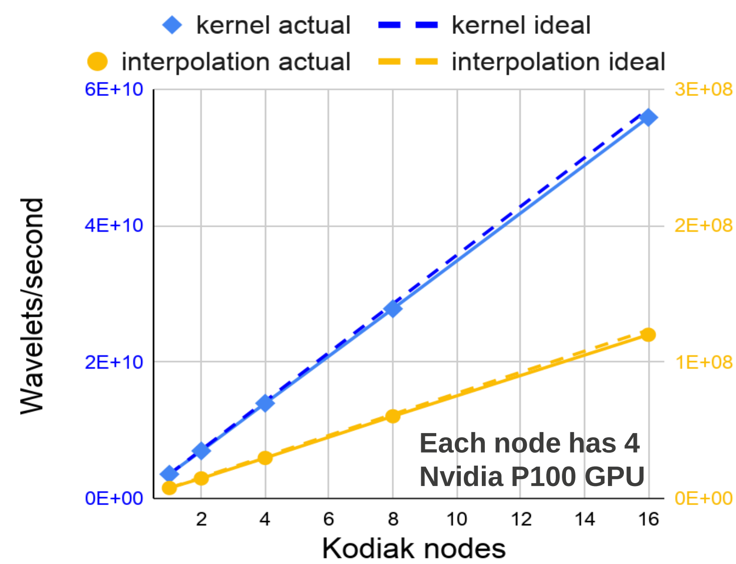


Design of CoSyR: parallelization

Parallelization

	CPU	GPU
Electron Beam	Cabana	
Wavefronts	Kokos	
Moving Mesh	Portage (OpenMP)	Portage (Kokkos)

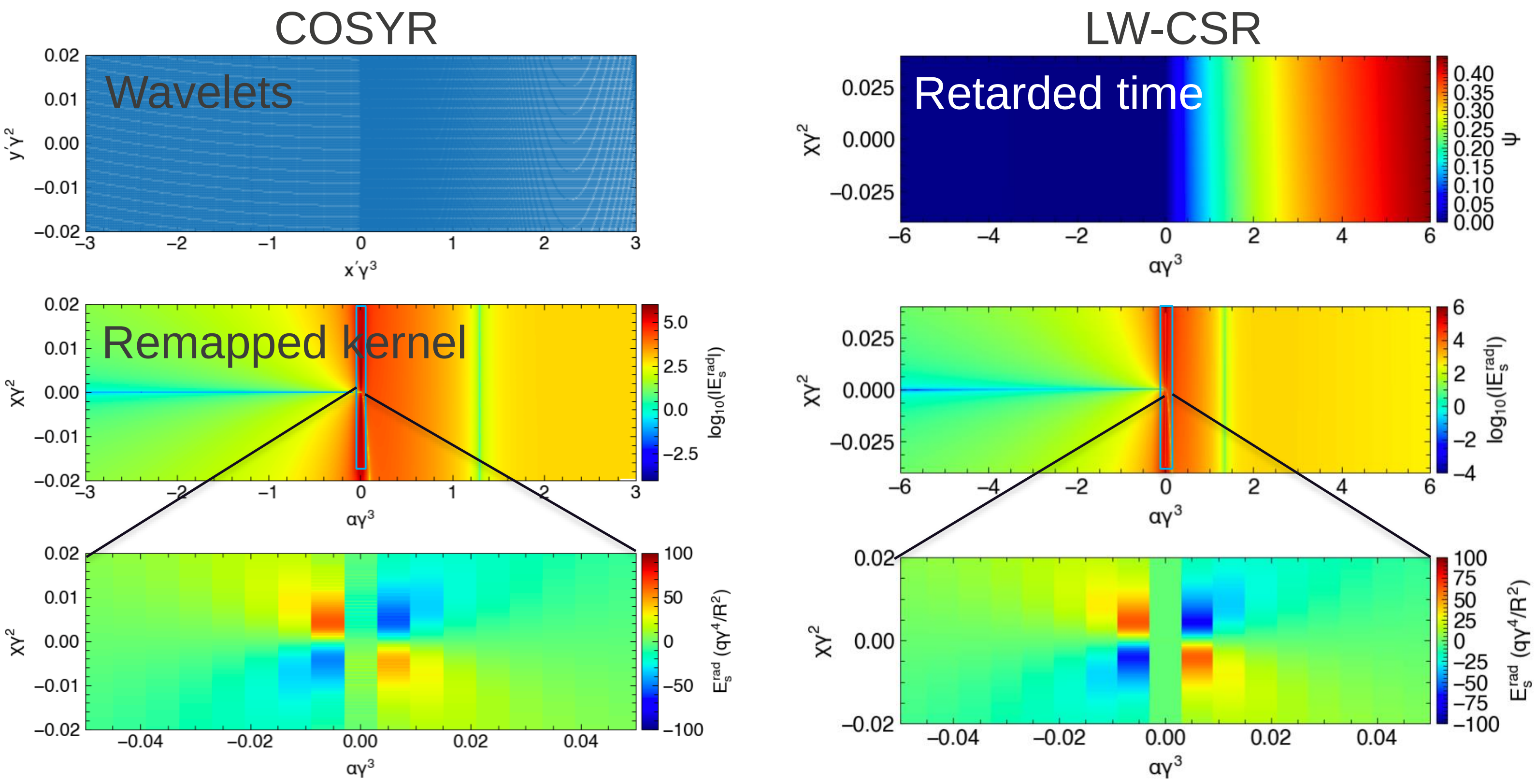
CoSyR weak scaling



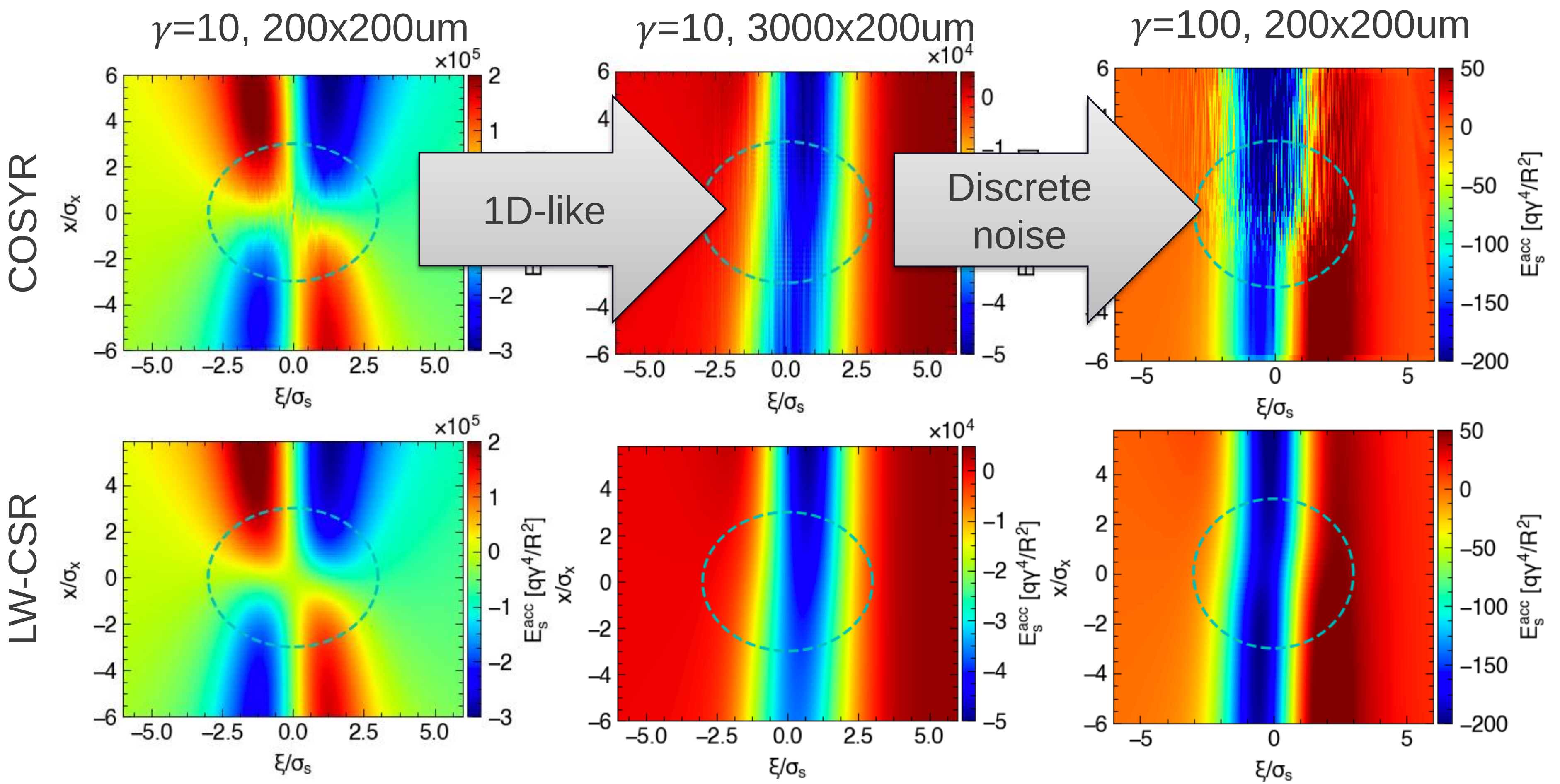
EXASCALE COMPUTING PROJECT

Kokkos, CoPA-Cabana are DOE ECP projects supporting major platforms (CPU/GPU/KNL/ROCm)

Single-particle kernel field benchmark (γ=10,3000x200um)



Benchmark and new understanding of beam CSR fields

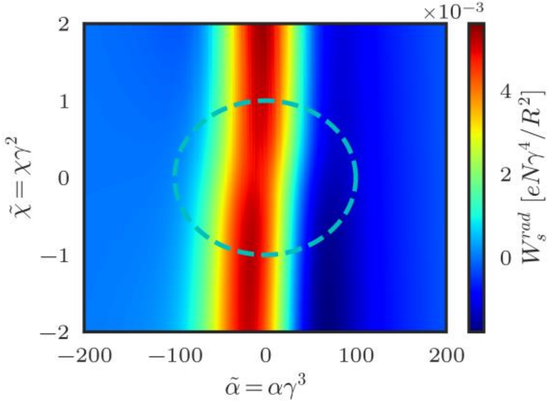


Benchmark and new understanding of beam CSR fields

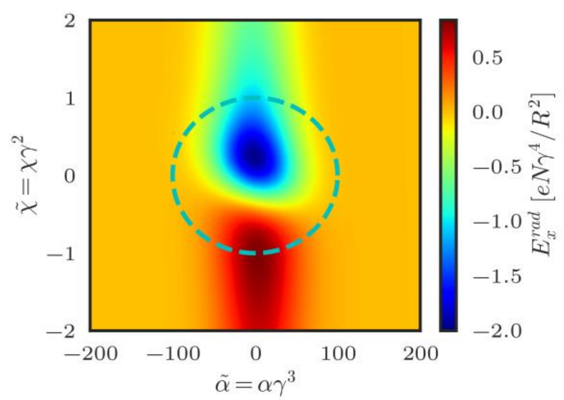
γ=100, 0.01nC, 200x200um

Subcycle wavelet only

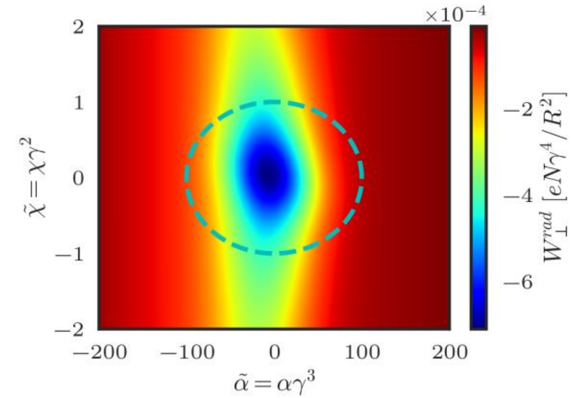
Longitudinal field



Transverse electric field



Transverse force

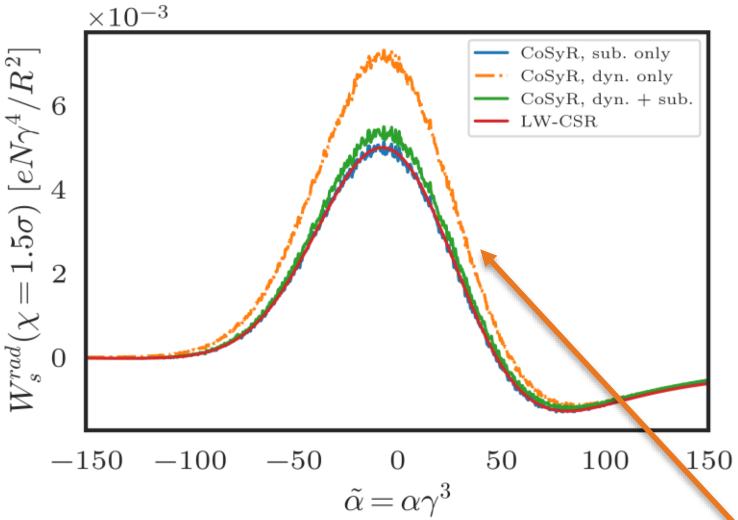


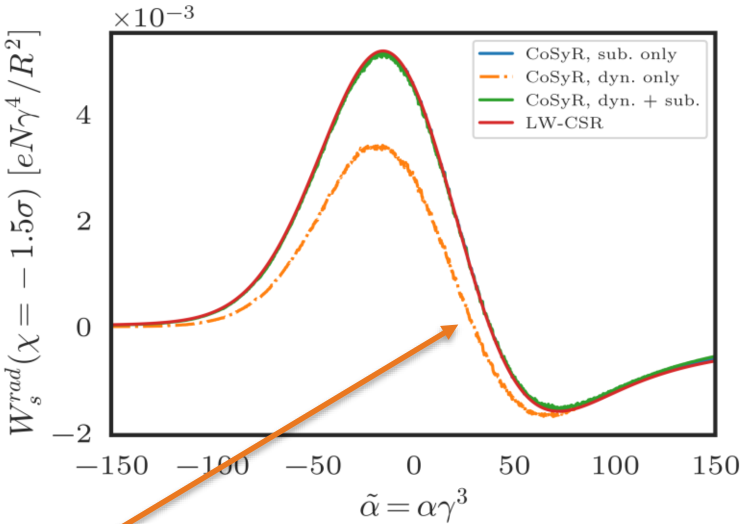
Large cancellation in transverse force, however, this force plays an important role in transverse dynamics and offsetting longitudinal energy loss

Benchmark and new understanding of beam CSR fields

Comparison between results with subcycle only and subcycle + dynamic wavelets

Longitudinal electric field



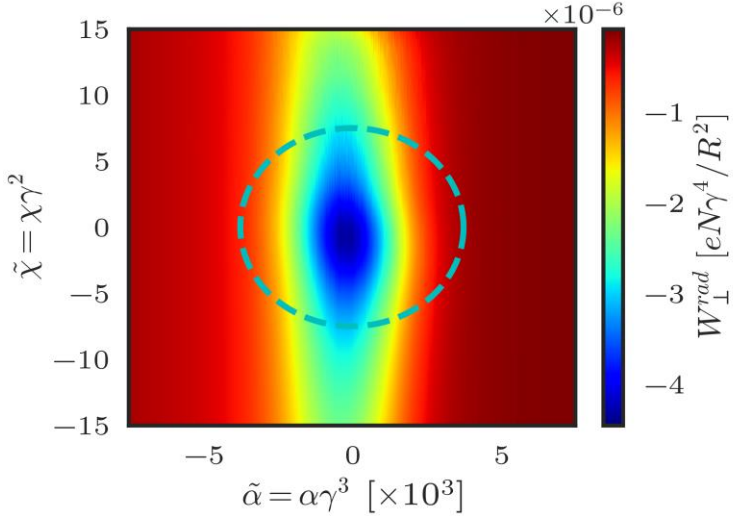


Dynamic components with slower time scale

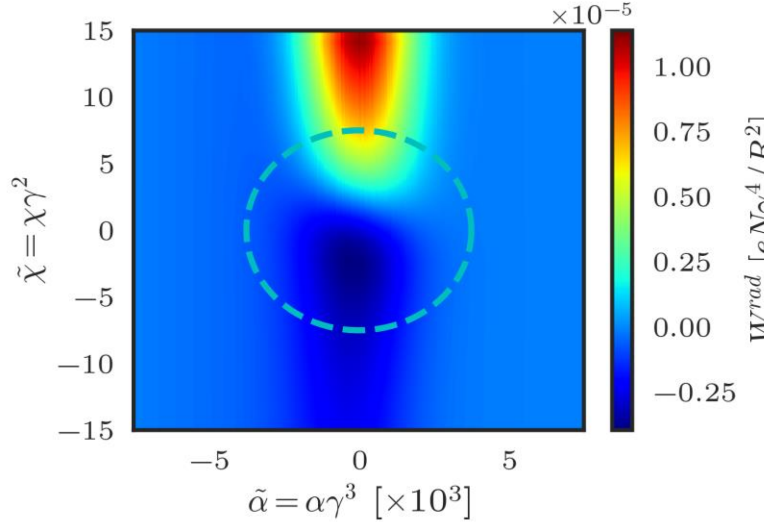
Benchmark and new understanding of beam CSR fields

Comparison between results with subcycle only and subcycle + dynamic wavelets

Transverse force



Subcycle

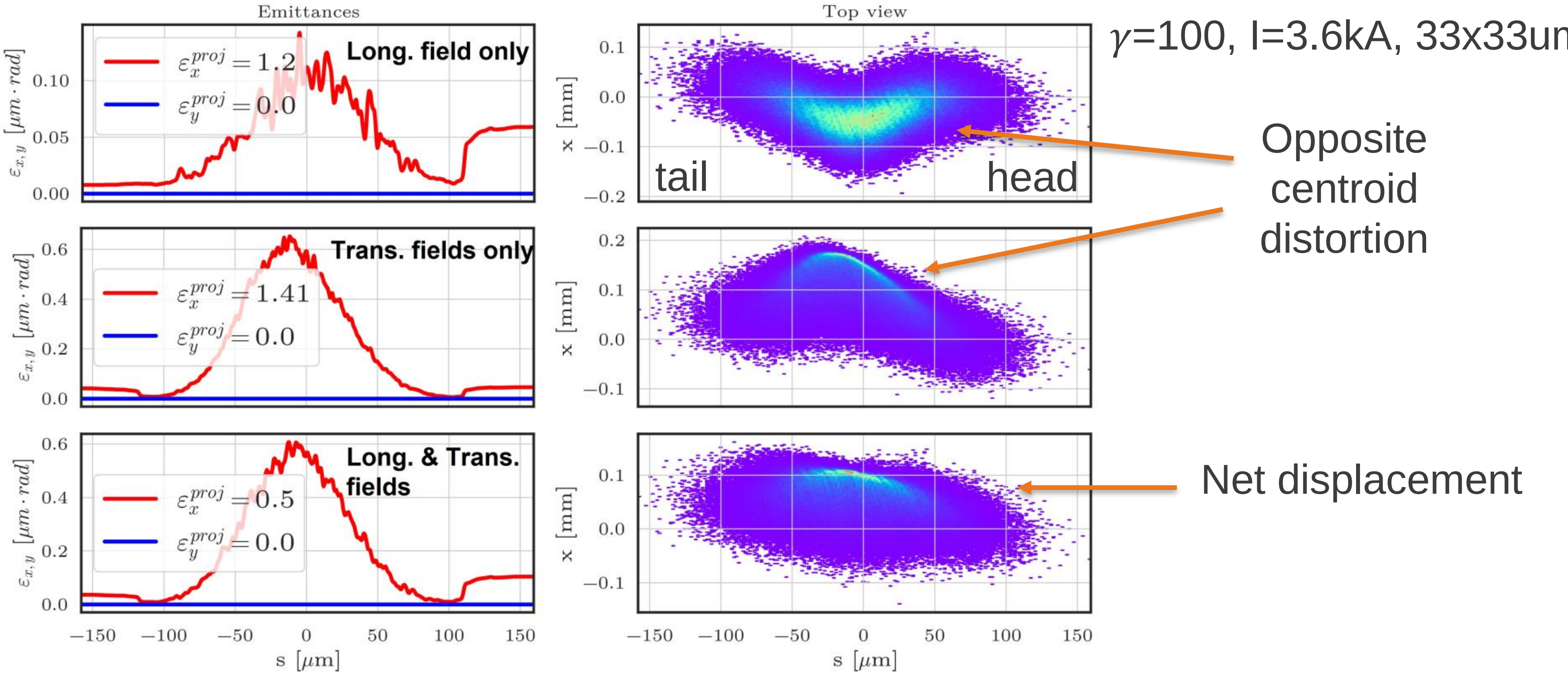


Subcycle + dynamic

γ=500, 0.01nC, 10x10um

CSR effects in a bending magnet

Comparison of beam dynamics from longitudinal and transverse CSR fields

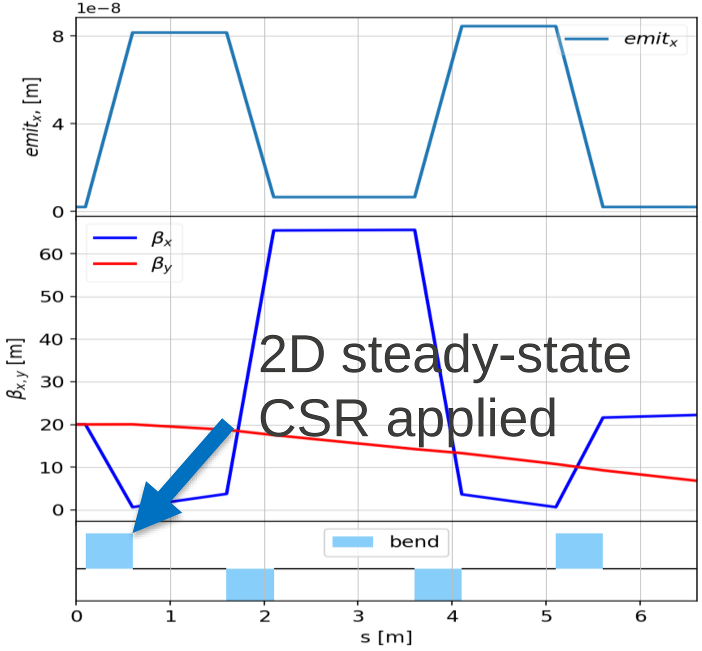


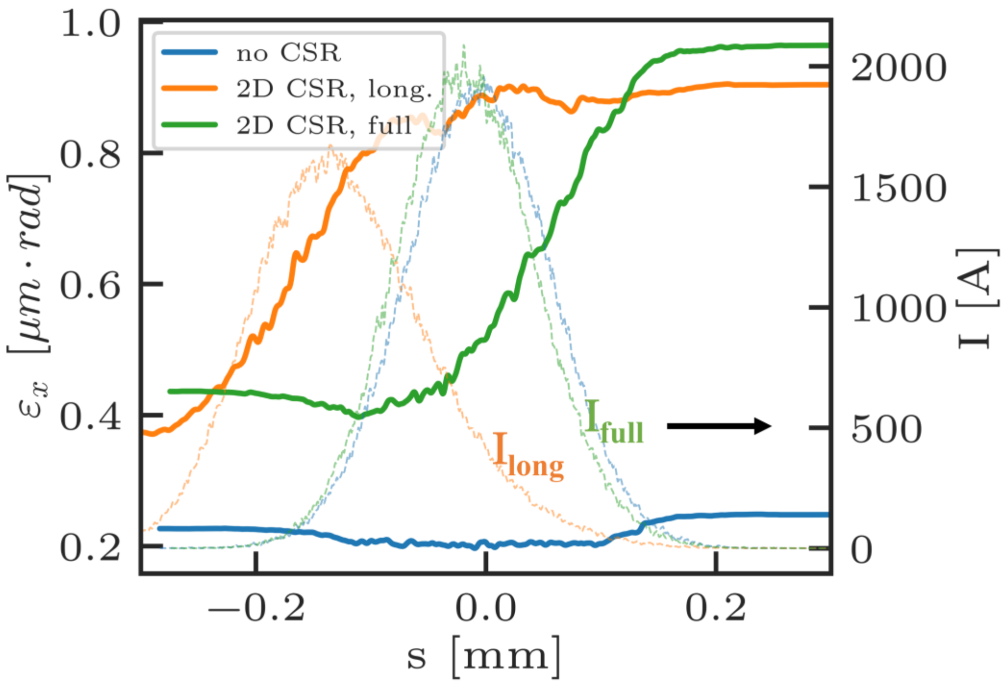
γ=100, I=3.6kA, 33x33um

1.5 hour on 6400 KNL cores @ NERSC Cori

CSR effects in a chicane compressor

Comparison of beam dynamics from longitudinal and transverse CSR fields

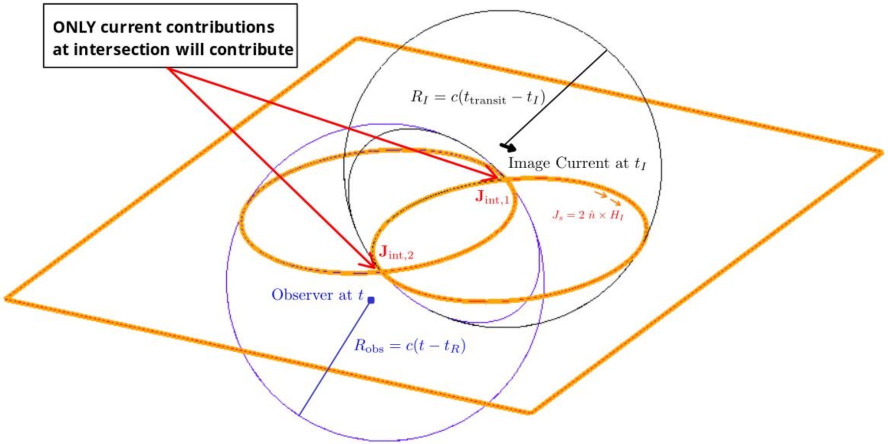




γ=100, I=0.6kA, compression rate ~3

Full 2D CSR simulation shows slice emittance growth, but substantially less than from the case with longitudinal field only

CSR shielding: Impressed Currents and Moving Mesh

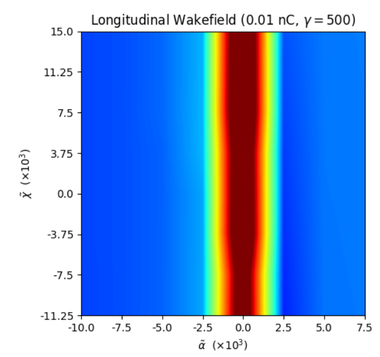


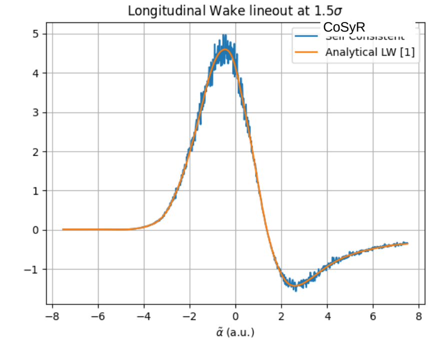
$$\mathbf{A}_{\mathrm{obs,Image\ I}}(\mathbf{r}, t) = \int_{\Omega} \frac{[\mathbf{J}_{\mathrm{int},1}(t_R)\delta(\mathbf{r}' - \mathbf{r}_{\mathrm{int},1}) + \mathbf{J}_{\mathrm{int},2}(t_R)\delta(\mathbf{r}' - \mathbf{r}_{\mathrm{int},2})]}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}'$$

t_{transit} can be swept so that all intersections at t_R can be found

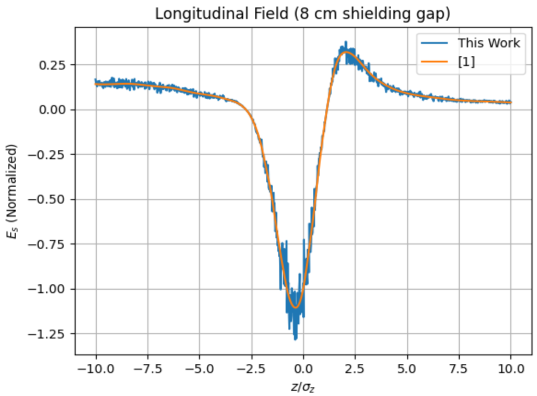
- To compute the CSR shielding effect, we employ an approach similar to a boundary element method (BEM).
- On all shielding walls, surface currents are computed in response to the moving bunch.
- The resulting radiating fields can be computed through a convolution.
- For parallel plates, this surface current can be computed analytically using image theory.
- Our eventual goal is to implement a complete BEM to generalize the method to arbitrary shielding surfaces.

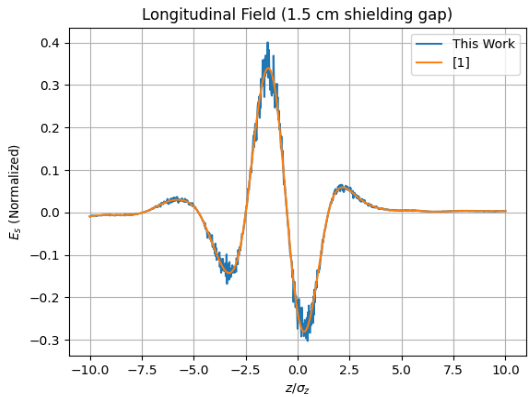
We validated the method by considering a bigaussian bunch (0.01 nC, gamma=500, 10 micron spot) being bent through a magnetic dipole.
We compare the longitudinal wake against CoSyR



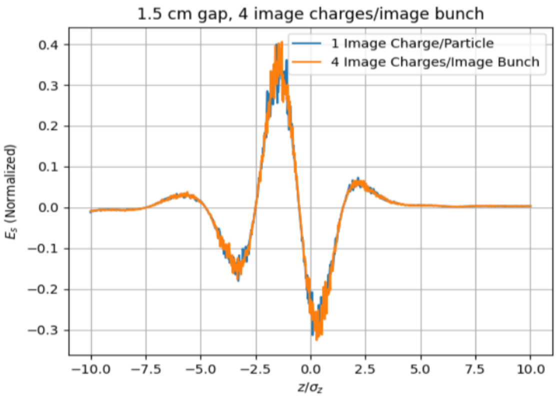


1D Shielding Results





- Next, we considered CSR shielding on a 1D Gaussian bunch moving through a dipole magnet.
- 1 nC, 0.3 mm length, 10 m bending radius
- We considered the steady state longitudinal field along the length of the bunch for different gap sizes.
- In each case, we see good agreement against the results from Sagan et al. 2008.
- We are currently working on validating shielding for a 2D bunch against CSRtrack.



- An interesting observation we made was that having image charges for each physical particle seems to be unnecessary.
- We note in the plot above that having 4 macroparticles approximating the image bunch gives use a virtually identical result.

Symplectic neutral surrogate model based on HenonNet

Basic idea: Hamiltonian dynamic of charge particle is symplectic, surrogate model can employ such constraint for robustness

The basic building block

Definition (Hénon layer)

Let $V_W : \mathbb{R}^N \to \mathbb{R}$, $W \in \mathcal{W}$, be a feed-forward neural network. The **Hénon layer** with potential V_W and bias $\eta \in \mathbb{R}^N$ is the layer $HL_{(W,\eta)} : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ given by

$$HL_{(W,\eta)} = H_{(W,\eta)} \circ H_{(W,\eta)} \circ H_{(W,\eta)} \circ H_{(W,\eta)},$$

where $H_{(W,\eta)} : (x, y) \mapsto (\overline{x}, \overline{y})$ is given by

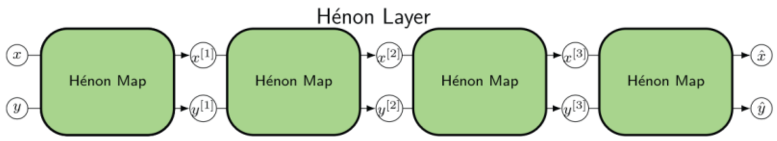
$$\overline{x} = y + \eta$$
$$\overline{y} = -x + \nabla V_W(y).$$

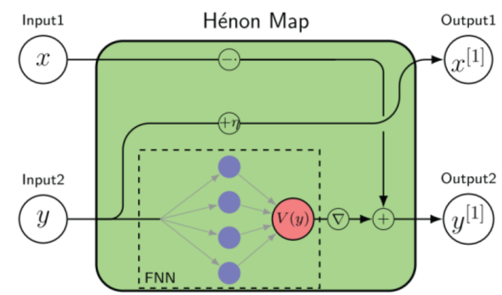
$HL_{(W,\eta)}$ is a symplectic map for any (W, η)!

Symplectic universal approximation theorem

$$|H[V,\eta]^{4N} - \mathcal{F}|_{C^r(U)} < \epsilon$$

Dmitry Turaev. Nonlinearity 16.1 (2002): 123.



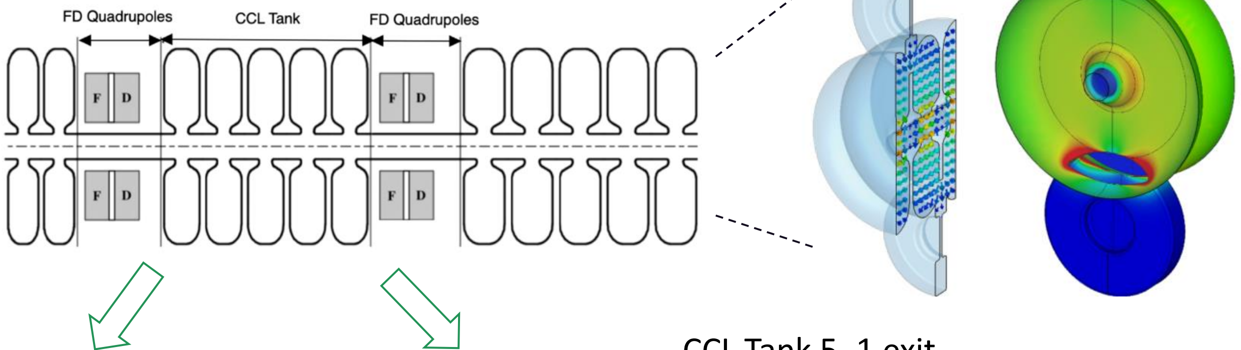


J. Burby J, Q. Tang Q and R. Maulik, Plasma Phys. Control. Fusion 63 024001, 2020

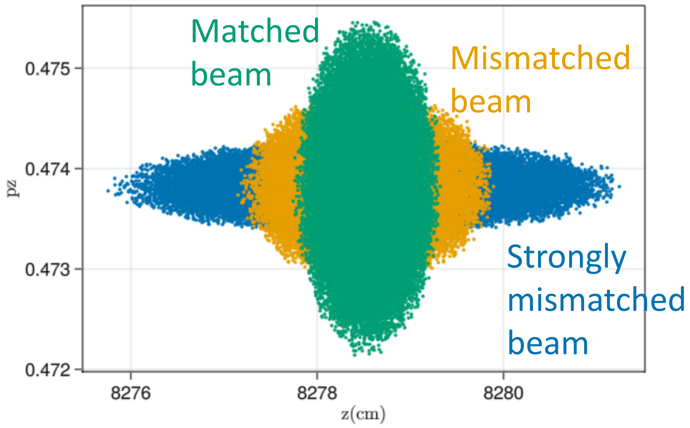
Symplectic neural surrogate for RF cavities

C.-K.Huang et al. 2022 Proc. North American Particle Accelerator Conference pp.462–464

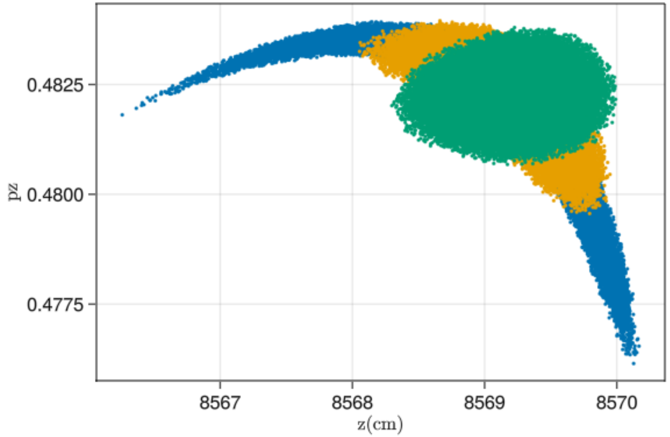
805MHz LANSCE Coupled-Cavity Linac (CCL)



CCL Tank 5_1 entrance



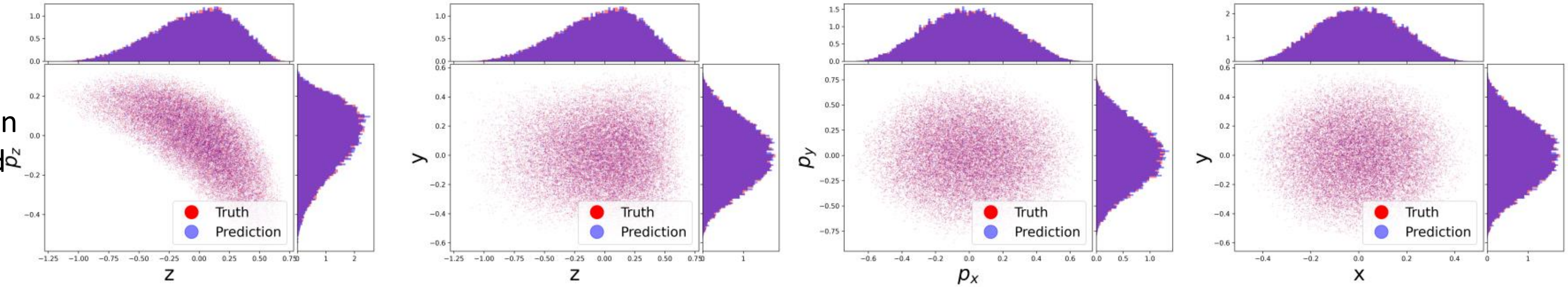
CCL Tank 5_1 exit



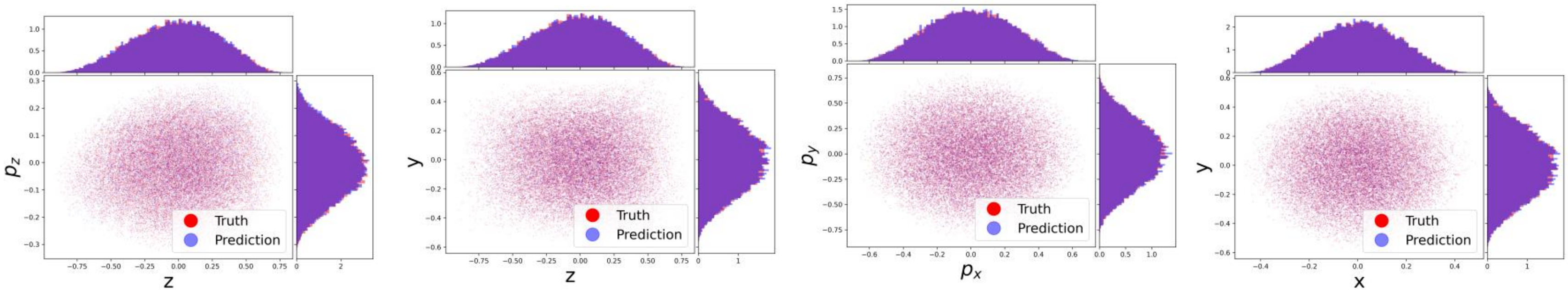
Synchronous particle defined by the average of the matched beam

Training and test on mismatched beams in CCL

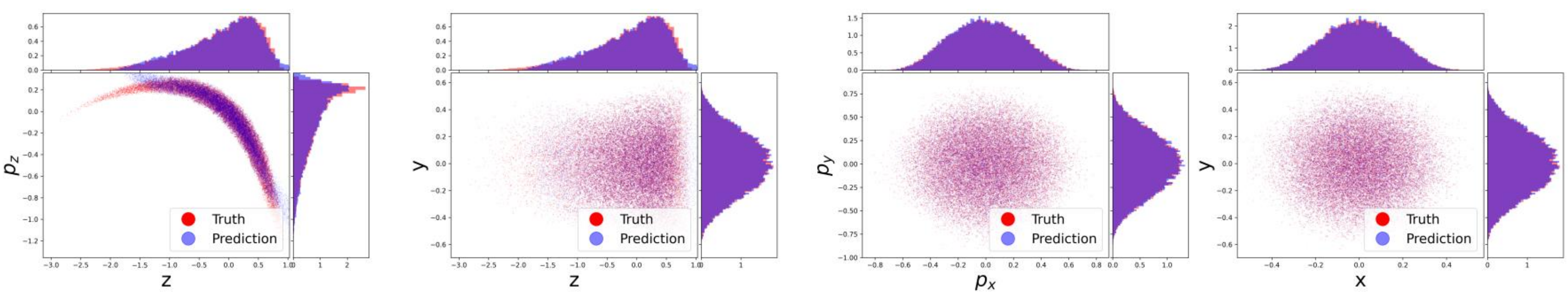
Train/ Validation on mismatched beam



Test on matched beam



Test on Strongly mismatched beam



Learned transfer matrix for the CCL RF tank

Analytic transfer matrix

$$\begin{pmatrix} x \\ p_x \\ y \\ p_y \\ V \\ p_V \end{pmatrix} = \begin{pmatrix} 1.2821 & 6.729 & 0 & 0 & 0 & 0 \\ 0.09569 & 1.2821 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.2821 & 6.729 & 0 & 0 \\ 0 & 0 & 0.09569 & 1.2821 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4955 & 4.108 \\ 0 & 0 & 0 & 0 & -0.18355 & 0.4955 \end{pmatrix} \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ V \\ p_V \end{pmatrix}_0$$

ML learned transfer matrix

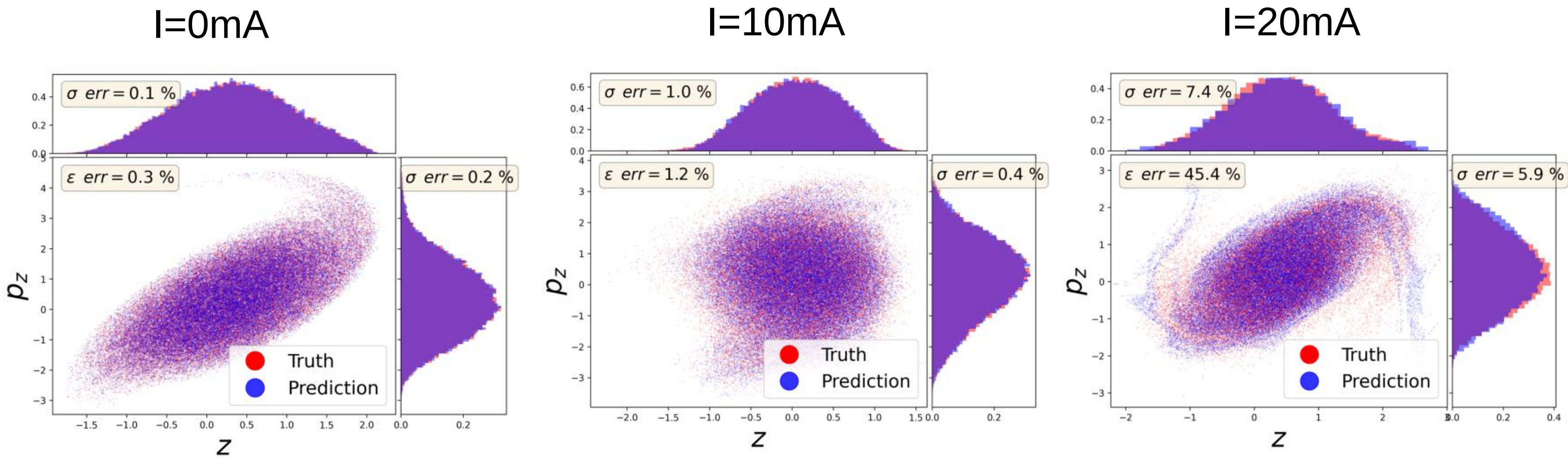
```
[ 1.256  6.659 -0.003  0.034 -0.003  0.022]
[ 0.084  1.241  0.000 -0.004  0.000 -0.000]
[ 0.013  0.120  1.245  6.663 -0.005 -0.012]
[ 0.001  0.013  0.085  1.259 -0.000  0.000]
[ 0.014  0.178 -0.000  0.053  0.576  4.338]
[-0.001 -0.012  0.002  0.028 -0.159  0.540]
```

- Demonstrated ~10% level accuracy
- Accuracy may be impacted by the nonlinear part of the beam
- We are exploring other techniques to improve this accuracy.

Parametric HenonNet can learn parameter-dependent collective beam dynamics

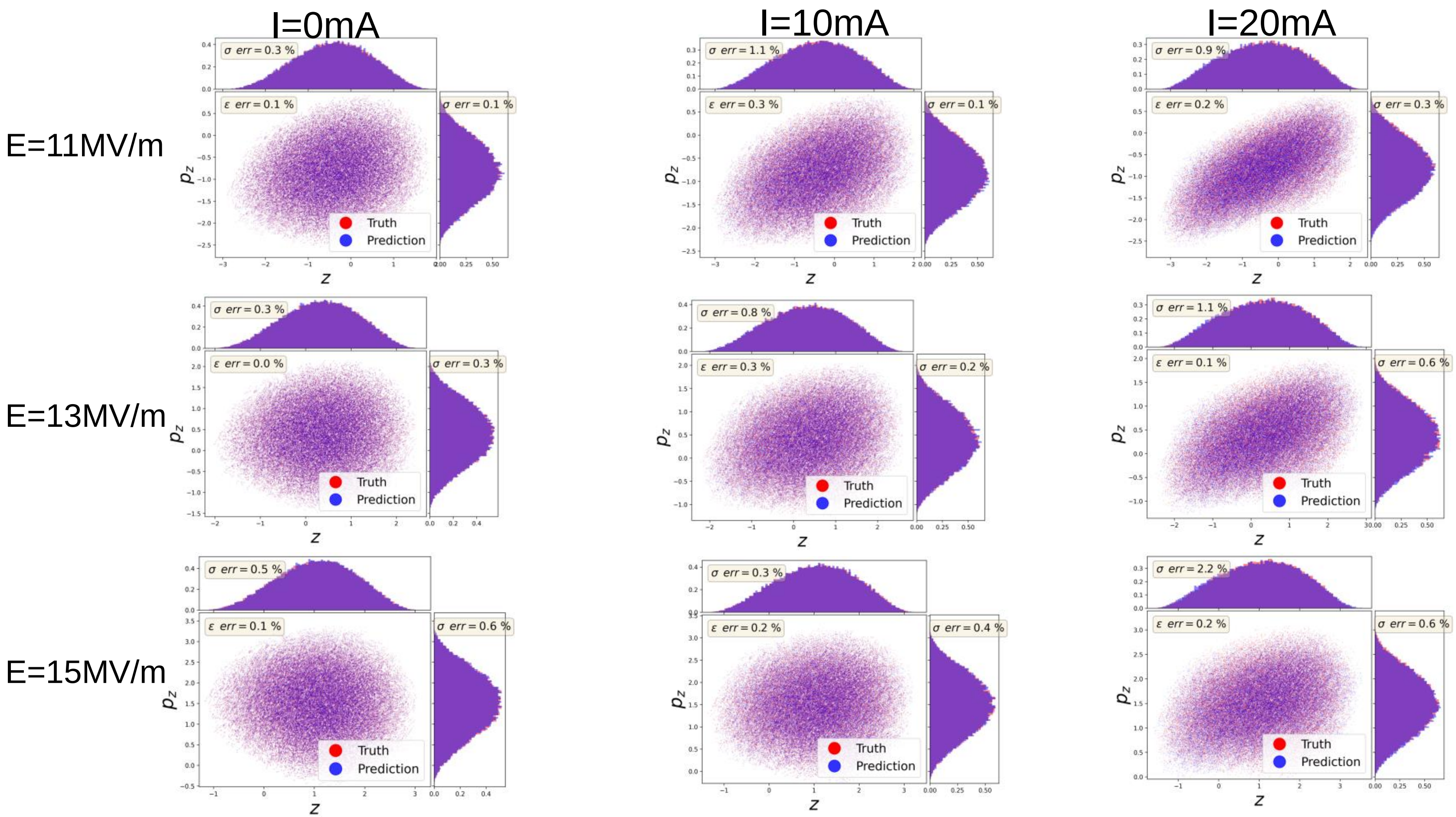
C.-K. Huang, et al., J. Phys. Conf. Ser. **2687**, 062026 (2024)

A single **P-HenonNet** learns longitude dynamics parameterized by beam current.

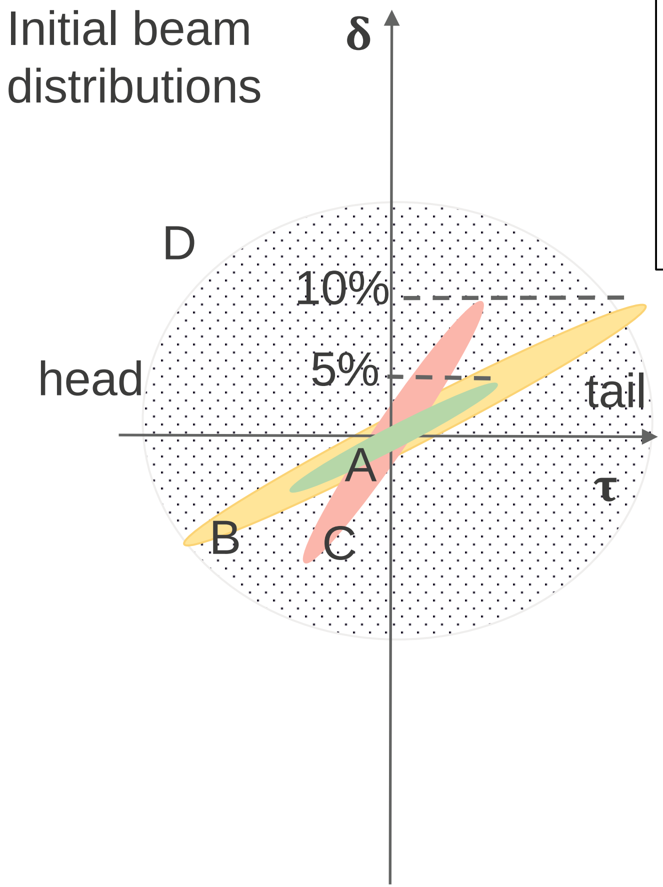


- Each plot predicts the dynamics of a LANSCE beam from 100 MeV to 800 MeV under various space charge conditions and only takes 0.015s on one GPU! (Data from each BEAMPATH simulation is generated in 20 hours using a high-end workstation)
- Error <~10% after transfer learning and fine tuning even for the most challenging case (I=20mA)

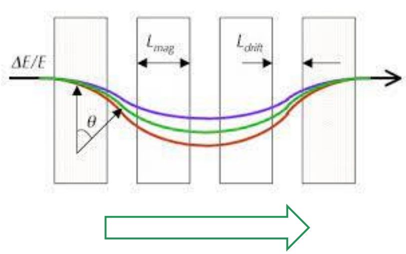
Parametric HenonNet can handle multiple parameters



Symplectic neutral surrogate for beam dynamics in chicane



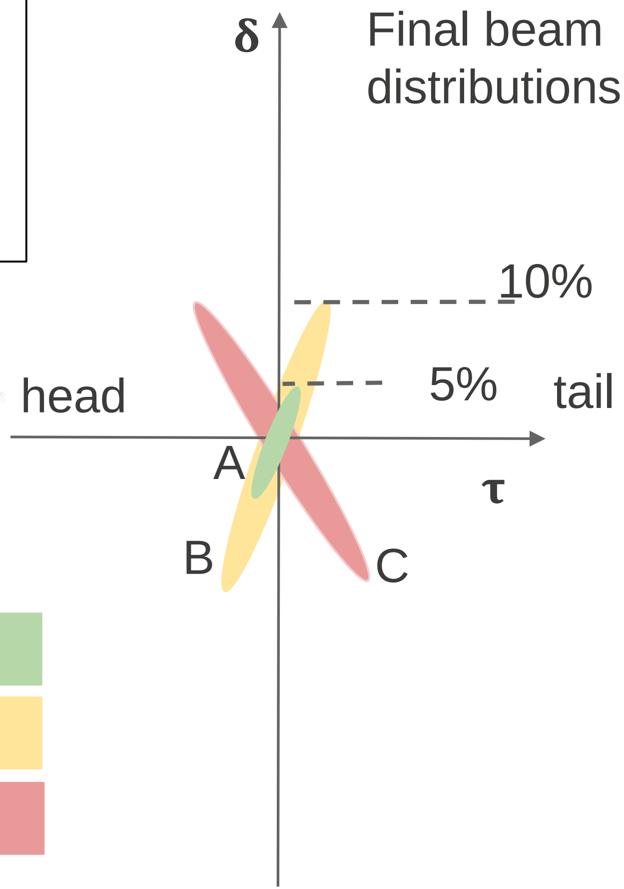
A: ideal compression
B: ideal compression (2x chirp/length)
C: over compression
D: uniform beam



Interpolation: Train & Test : A

Extrapolation: Train: A Test: B

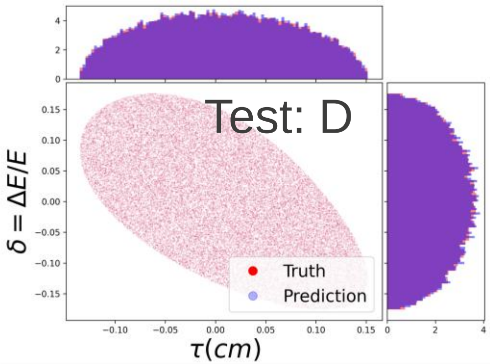
Extrapolation: Train:A Test: C

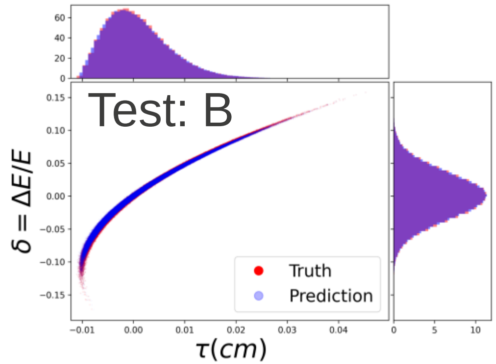


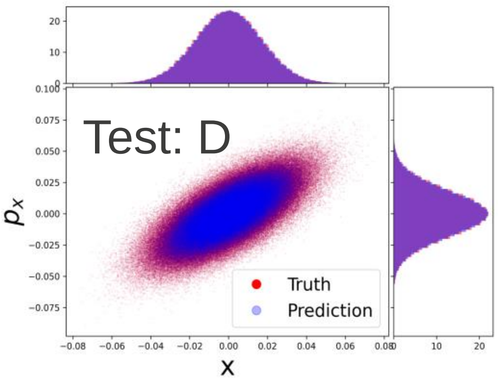
Symplectic neutral surrogate for beam dynamics in chicane

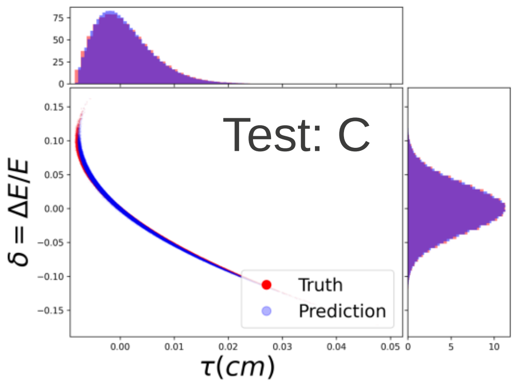
Interpolation

Train: D   Test: D, B, C





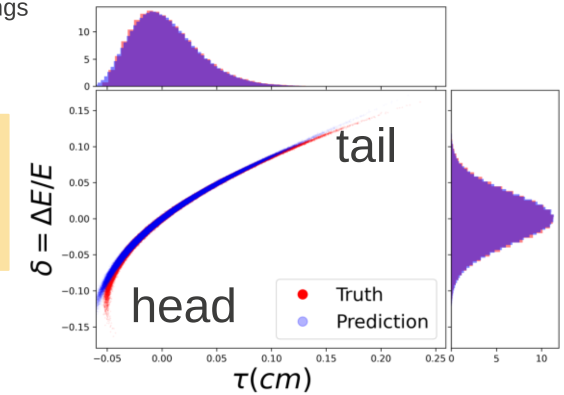




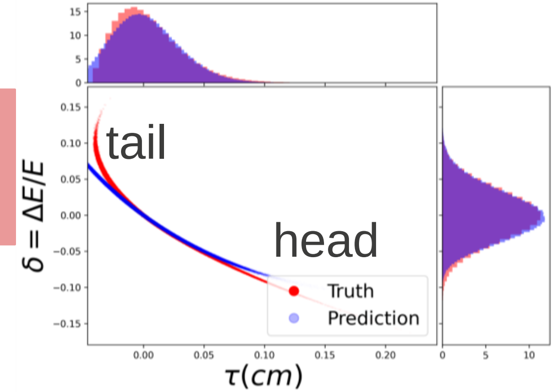
Extrapolation

Huang et al., proceedings of NA-PAC 2022.

Train: A
Test: B

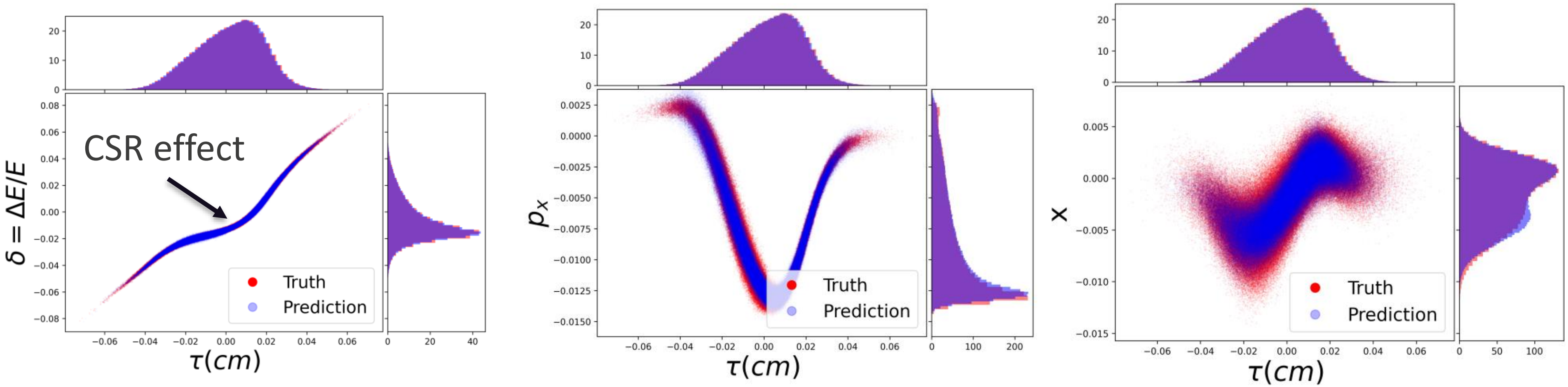


Train: A
Test: C



Symplectic neutral surrogate for beam dynamics in chicane

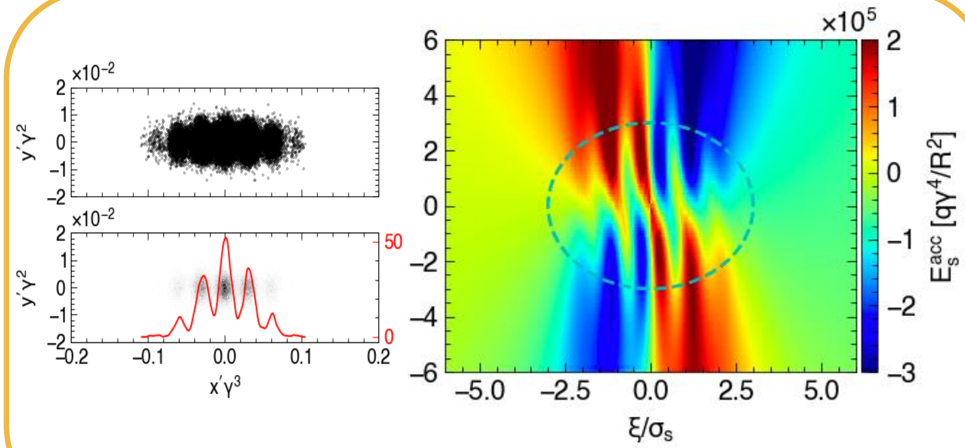
Linear chicane beam dynamics + CSR



Possible explanation: the network learns two maps, one for $\delta - \tau$, and the other one is the linear map of the chicane without self-field

Summary

- Challenge: modeling/understanding of high brightness beams dynamics at the most detailed level and with fast turn-around time
- Development and verification of a unique beam dynamics code (CoSyR) and ML model (HenonNet)
- First beam dynamic simulations with accurate CSR treatment, more to come!

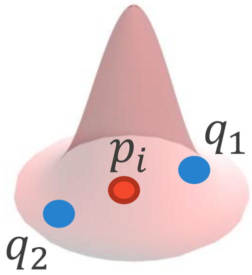


CSR calculation for a microbunched beam

- Future development: incoherent synchrotron radiation model; performance improvement for high beam energy and on Exascale platforms; neural surrogate models

Backup slides

Improving the weights computation

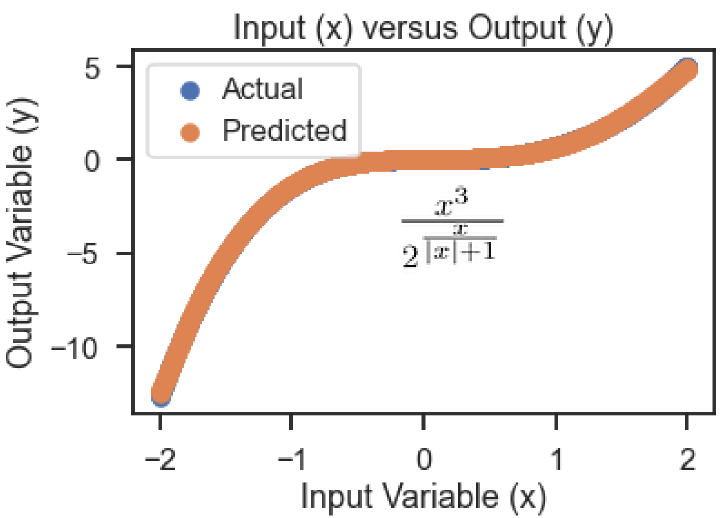


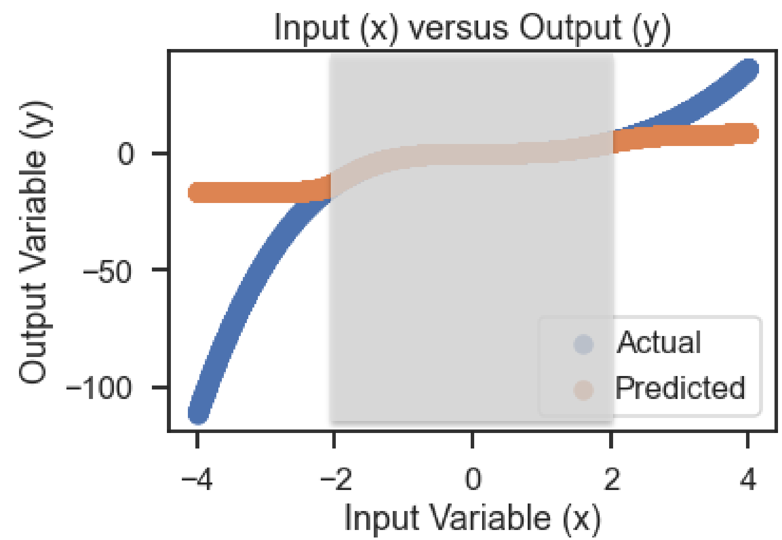
red: mesh point p_i
blue: wavelet points $q_{1\leq j\leq n}$

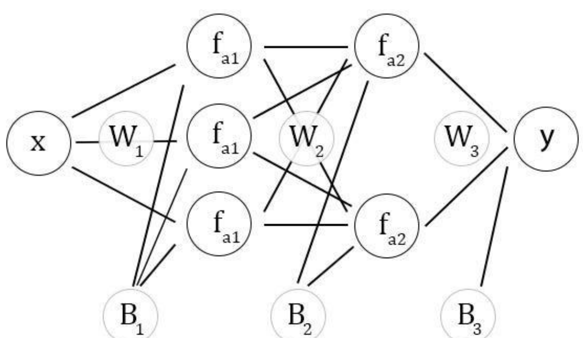
- basis vector: $b_i = (1, x, \frac{x^2}{2}, \cdots, \frac{x^n}{n!})$ in 1D
- moment vector: $m_{ij} = (b_i, \nabla b_i, \ldots)^{-1}\, b_j$
- scalar weight: $w_{ij} = \frac{K(d_{ij})}{K(0)}$ with kernel K.
- n_{basis} : number of components of b_i and m_{ij}.

- normal algorithm for each mesh point p_i:
  - build moment matrix $A_i = \left(a_{\tilde{\imath}\tilde{\jmath}}\right)_{0\leq\tilde{\imath},\tilde{\jmath}\leq n_{basis}-1}$ with each $a_{\tilde{\imath}\tilde{\jmath}} = \sum_{j=1}^{n} m_{ij|\tilde{\imath}} m_{ij|\tilde{\jmath}} w_{ij}$
  - for each wavelet neighbor q_j, solve $A_i\, X_{ij} = m_{ij}$, then deduce $\omega_{ij} = X_{ij}\, w_{ij}$.
- optimize it since field derivatives are not needed:
  - cache scalars $s_{ij} = m_{ij|0}$ and $a_i = \sum_{j=1}^{n} s_{ij}^2 w_{ij}$
  - for each wavelet neighbor q_j, compute $\omega_{ij} = \frac{s_{ij} w_{ij}}{a_i}$

Neural network as universal function approximator







$$y = -5.34\mathrm{f2}(0.53\mathrm{f1}(1.99 - 0.74x) + 0.36\mathrm{f1}(0.41x - 0.1) - 2.44\mathrm{f1}(1.24x + 2.52) + 0.14) - 7.22\mathrm{f2}(1.99\mathrm{f1}(1.99 - 0.74x) + 0.4\mathrm{f1}(0.41x - 0.1) - 2.15\mathrm{f1}(1.24x + 2.52) + 0.4) - 3.99$$

f1, f2: tanh

https://alasko.medium.com/convert-simple-neuron-network-to-mathematician-notation-58a0d72f0337

HenonNets have some good properties

- Collection of all HénonNets forms a group
- Closure of that set is the symplectomorphism group
- HénonNet is an invertible network. Inverse of a HénonNet is a HénonNet. Its inverse is easy to construct and fast to evaluate.
- Derivatives of a HénonNet are easy to compute using automatic differentiation (sensitivity analysis)
- HénonNet is a type of ResNet. We can prove each Hénon layer fits into the ResNet framework of

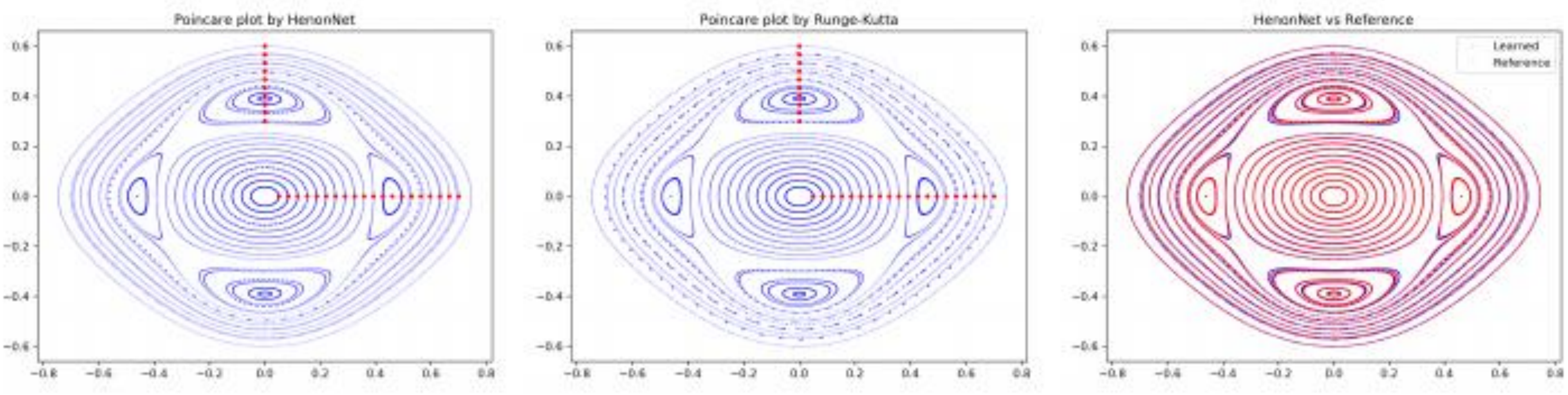
$$\bar{\mathbf{x}} = \mathbf{x} + \mathcal{F}(\mathbf{x}).$$

Symplectic neutral surrogate of Hamiltonian dynamics

Consider a perturbed pendulum

$$H_{\mathrm{pp}}(x, y, \phi) = \frac{1}{2}y^2 - \omega_0^2 \cos x - \epsilon \left[0.3\, xy \sin(2\phi) + 0.7\, xy \sin(3\phi) \right],$$

with $\omega_0 = 0.5$ and $\epsilon = 0.5$.



J. W. Burby, Q. Tang, and R. Maulik. ``Fast neural Poincaré maps for toroidal magnetic fields.'' Plasma Physics and Controlled Fusion 63.2 (2020): 024001.