[276/160]

RUHR UNIVERSITÄT BOCHUM RUB

Modern Techniques in Hadron Spectroscopy

July, 2024

QCD Phenomenology -- Exotics

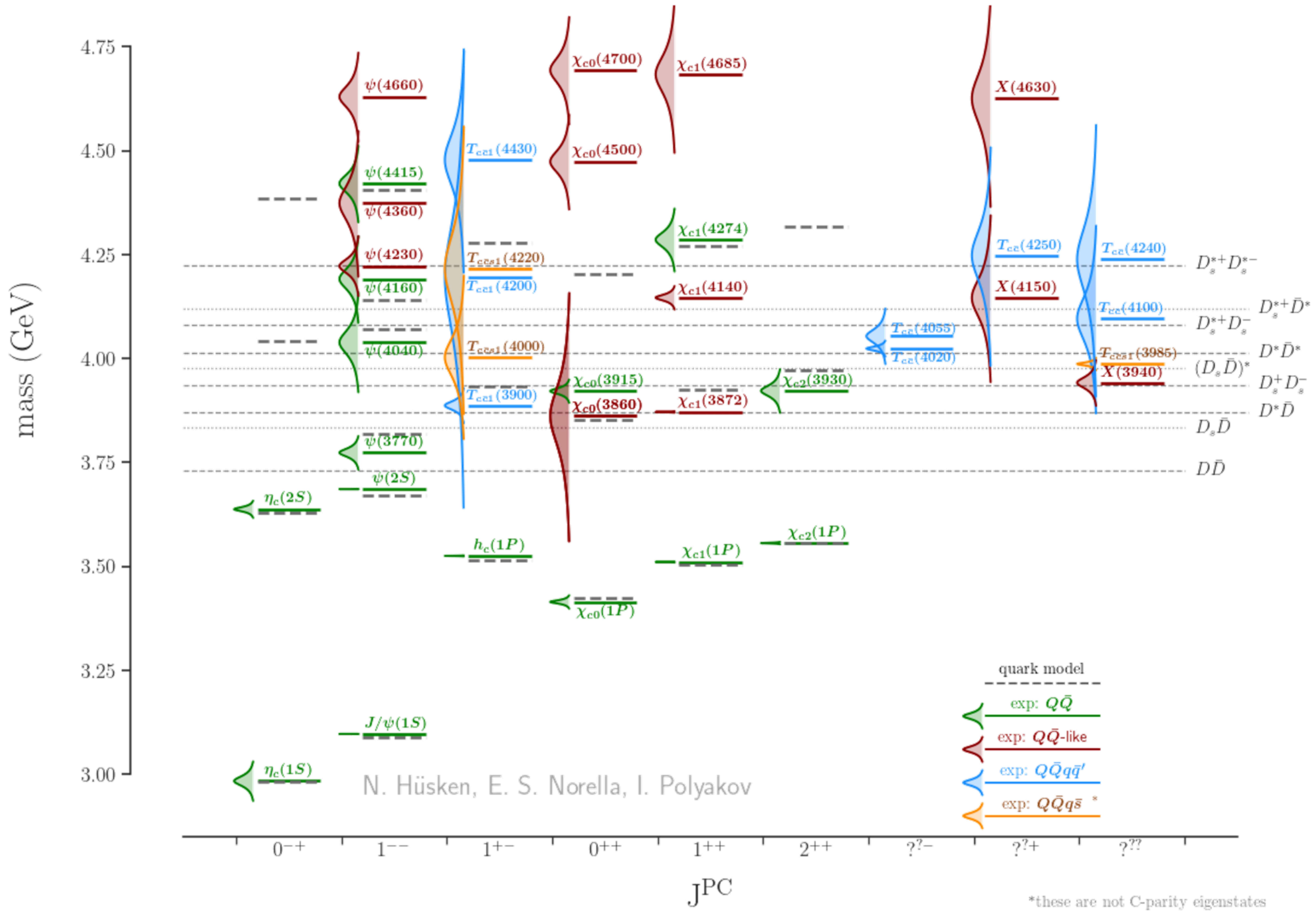
Eric Swanson

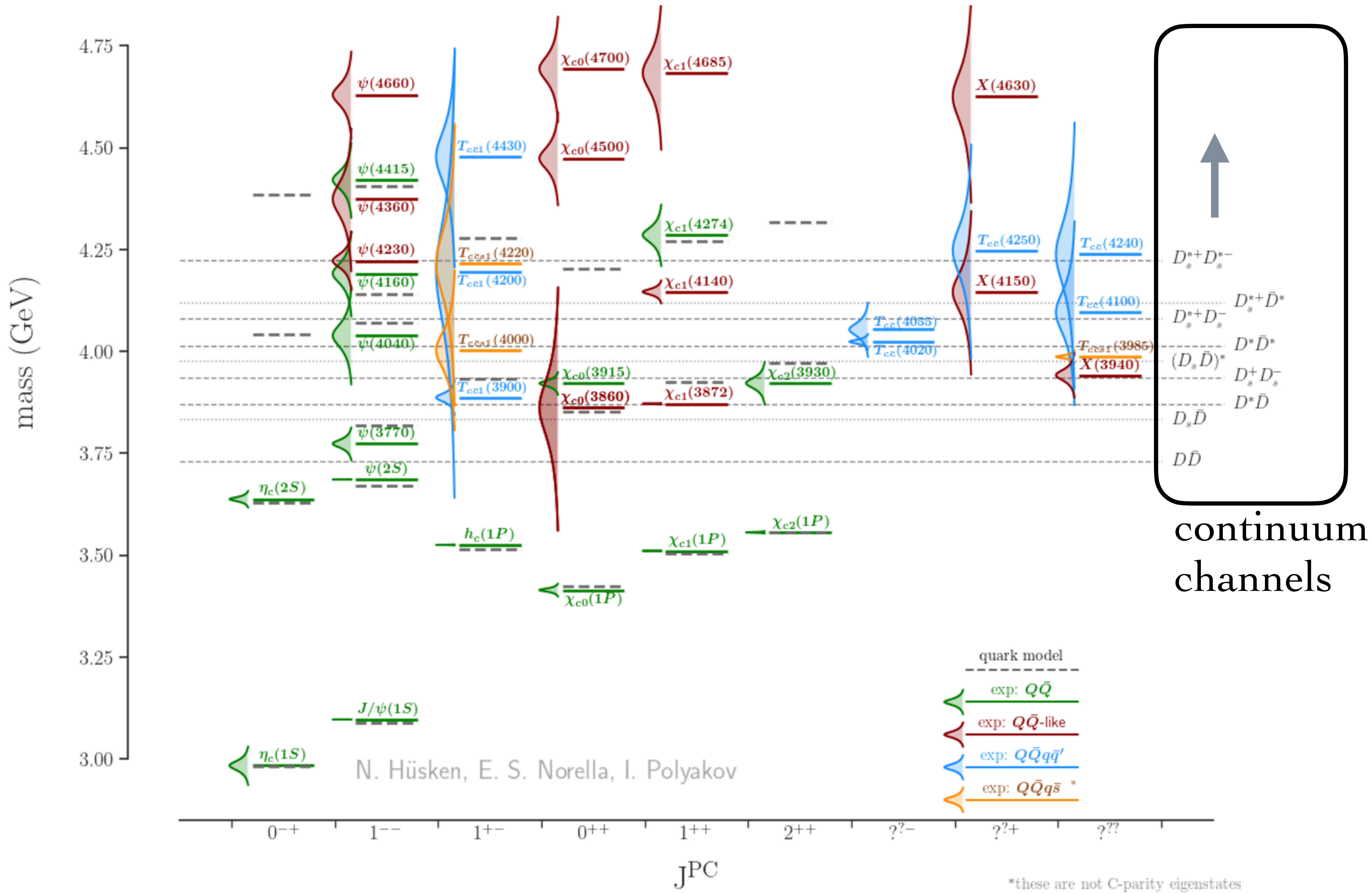
EXOHAD
EXOTIC HADRONS TOPICAL COLLABORATION

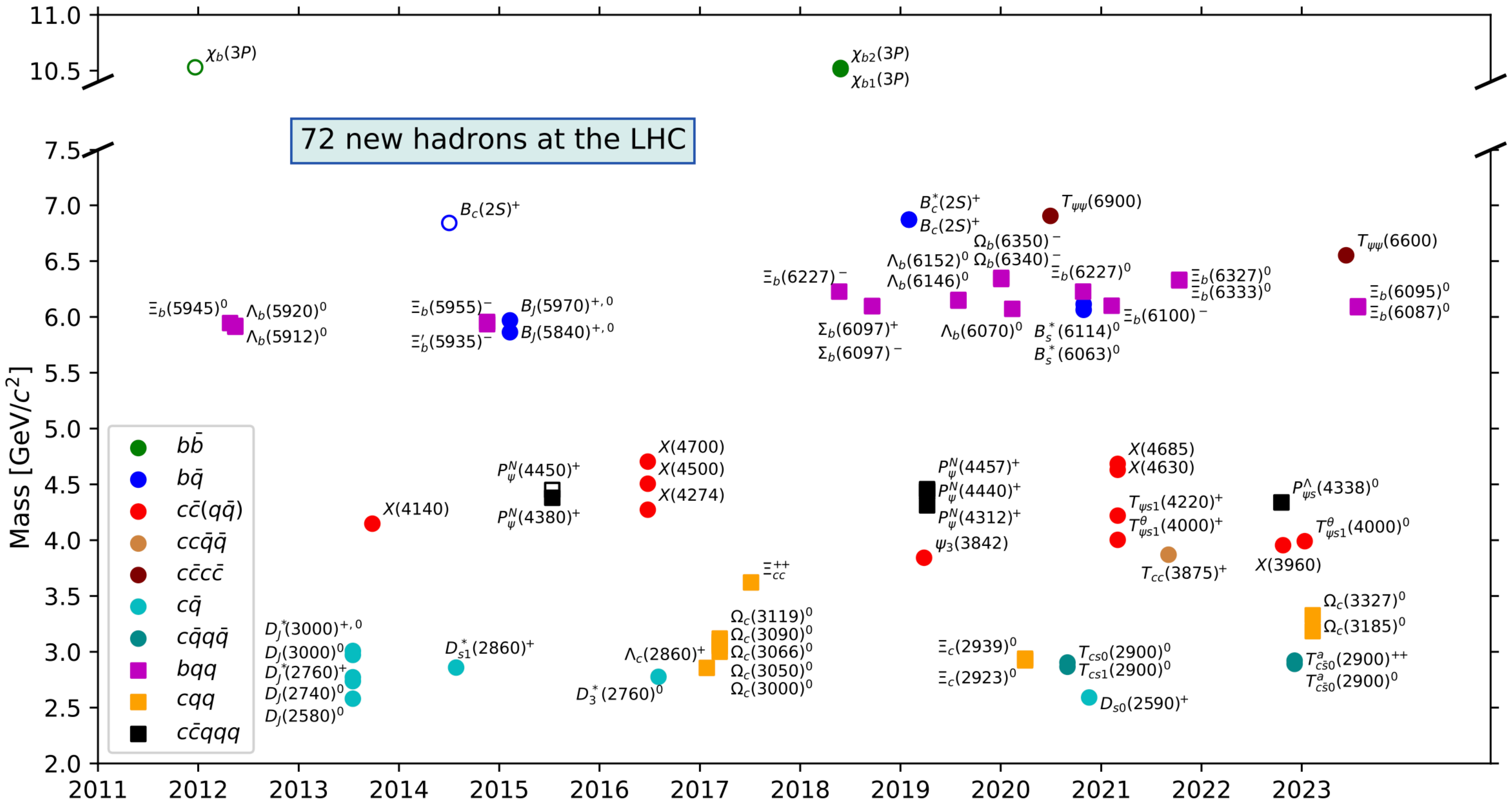
UNIVERSITY OF PITTSBURGH

things need not be as simple as $q\bar{q}$ or qqq.

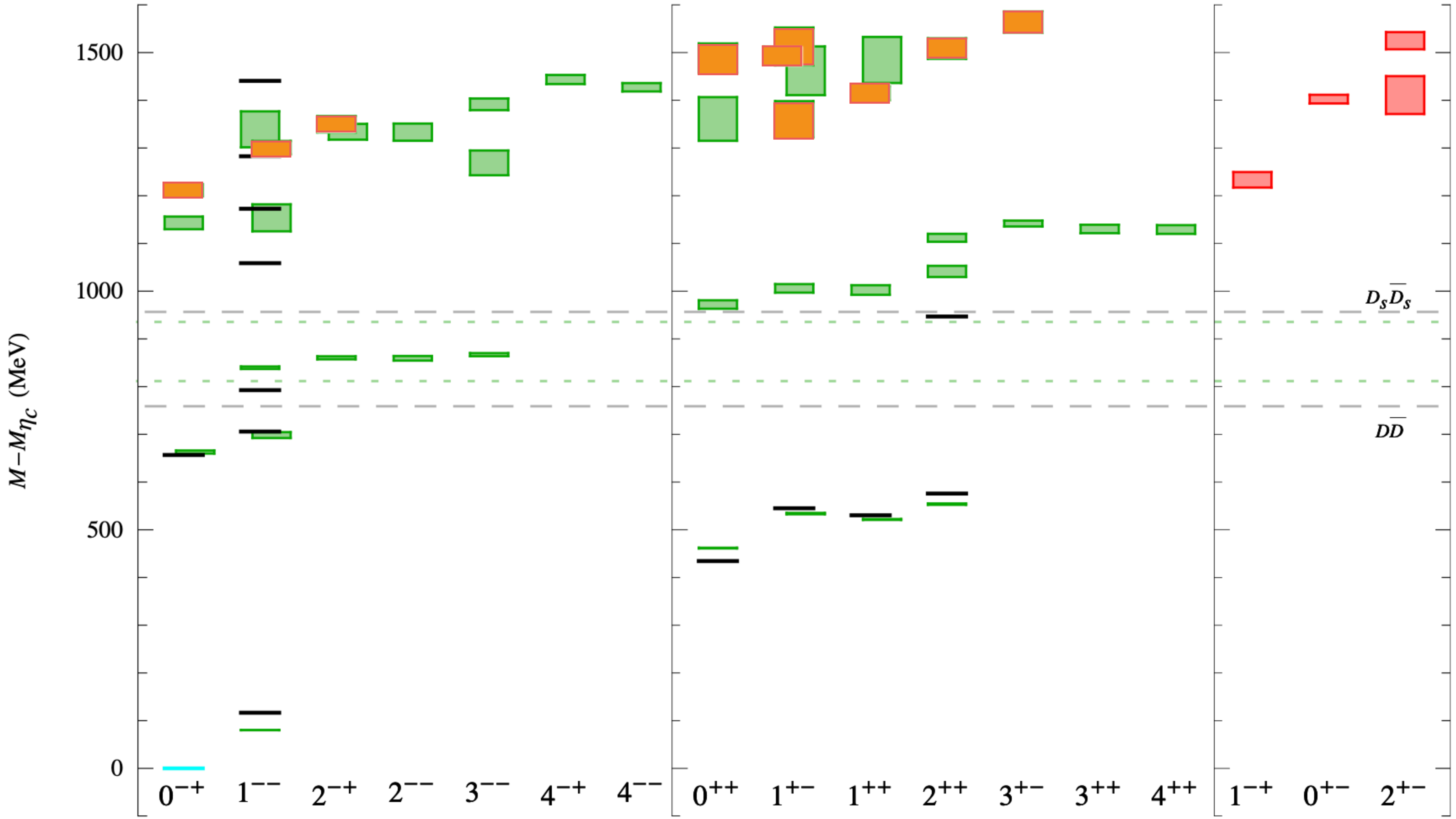
... and there is overwhelming evidence that they are not


mass (GeV)
4.75
4.50
4.25
4.00
3.75
3.50
3.25
3.00
$\psi(4660)$
$\psi(4415)$
$\psi(4360)$
$\psi(4230)$
$\psi(4160)$
$\psi(4040)$
$\psi(3770)$
$\psi(2S)$
$J/\psi(1S)$
$\eta_c(2S)$
$\eta_c(1S)$
$T_{c\bar{c}1}(4430)$
$T_{c\bar{c}s1}(4220)$
$T_{c\bar{c}1}(4200)$
$T_{c\bar{c}s1}(4000)$
$T_{c\bar{c}1}(3900)$
$h_c(1P)$
$\chi_{c0}(4700)$
$\chi_{c0}(4500)$
$\chi_{c0}(3915)$
$\chi_{c0}(3860)$
$\chi_{c0}(1P)$
$\chi_{c1}(4685)$
$\chi_{c1}(4274)$
$\chi_{c1}(4140)$
$\chi_{c1}(3872)$
$\chi_{c1}(1P)$
$\chi_{c2}(3930)$
$\chi_{c2}(1P)$
$T_{c\bar{c}}(4055)$
$T_{c\bar{c}}(4020)$
$X(4630)$
$T_{c\bar{c}}(4250)$
$X(4150)$
$T_{c\bar{c}}(4240)$
$T_{c\bar{c}}(4100)$
$T_{c\bar{c}s1}(3985)$
$X(3940)$
$D_s^{*+}D_s^{*-}$
$D_s^{*+}\bar{D}^*$
$D_s^{*+}D_s^-$
$D^*\bar{D}^*$
$(D_s\bar{D})^*$
$D_s^+D_s^-$
$D^*\bar{D}$
$D_s\bar{D}$
$D\bar{D}$
quark model
exp: $Q\bar{Q}$
exp: $Q\bar{Q}$-like
exp: $Q\bar{Q}q\bar{q}'$
exp: $Q\bar{Q}q\bar{s}$ *
N. Hüsken, E. S. Norella, I. Polyakov
0^{-+}
1^{--}
1^{+-}
0^{++}
1^{++}
2^{++}
$?^{?-}$
$?^{?+}$
$?^{??}$
J^{PC}
*these are not C-parity eigenstates


mass (GeV)
JPC
0−+
1−−
1+−
0++
1++
2++
??−
??+
???
N. Hüsken, E. S. Norella, I. Polyakov
quark model
exp: $Q\bar{Q}$
exp: $Q\bar{Q}$-like
exp: $Q\bar{Q}q\bar{q}'$
exp: $Q\bar{Q}q\bar{s}$ *
*these are not C-parity eigenstates
$\eta_c(1S)$
$\eta_c(2S)$
$J/\psi(1S)$
$\psi(2S)$
$\psi(3770)$
$\psi(4040)$
$\psi(4160)$
$\psi(4230)$
$\psi(4360)$
$\psi(4415)$
$\psi(4660)$
$h_c(1P)$
$T_{c\bar{c}1}(3900)$
$T_{c\bar{c}s1}(4000)$
$T_{c\bar{c}1}(4200)$
$T_{c\bar{c}s1}(4220)$
$T_{c\bar{c}1}(4430)$
$\chi_{c0}(1P)$
$\chi_{c0}(3860)$
$\chi_{c0}(3915)$
$\chi_{c0}(4500)$
$\chi_{c0}(4700)$
$\chi_{c1}(1P)$
$\chi_{c1}(3872)$
$\chi_{c1}(4140)$
$\chi_{c1}(4274)$
$\chi_{c1}(4685)$
$\chi_{c2}(1P)$
$\chi_{c2}(3930)$
$T_{c\bar{c}}(4020)$
$T_{c\bar{c}}(4055)$
$X(4150)$
$T_{c\bar{c}}(4250)$
$X(4630)$
$X(3940)$
$T_{c\bar{c}s1}(3985)$
$T_{c\bar{c}}(4100)$
$T_{c\bar{c}}(4240)$
$D\bar{D}$
$D_s\bar{D}$
$D^*\bar{D}$
$D_s^+D_s^-$
$(D_s\bar{D})^*$
$D^*\bar{D}^*$
$D_s^{*+}D_s^-$
$D_s^{*+}\bar{D}^*$
$D_s^{*+}D_s^{*-}$
continuum channels

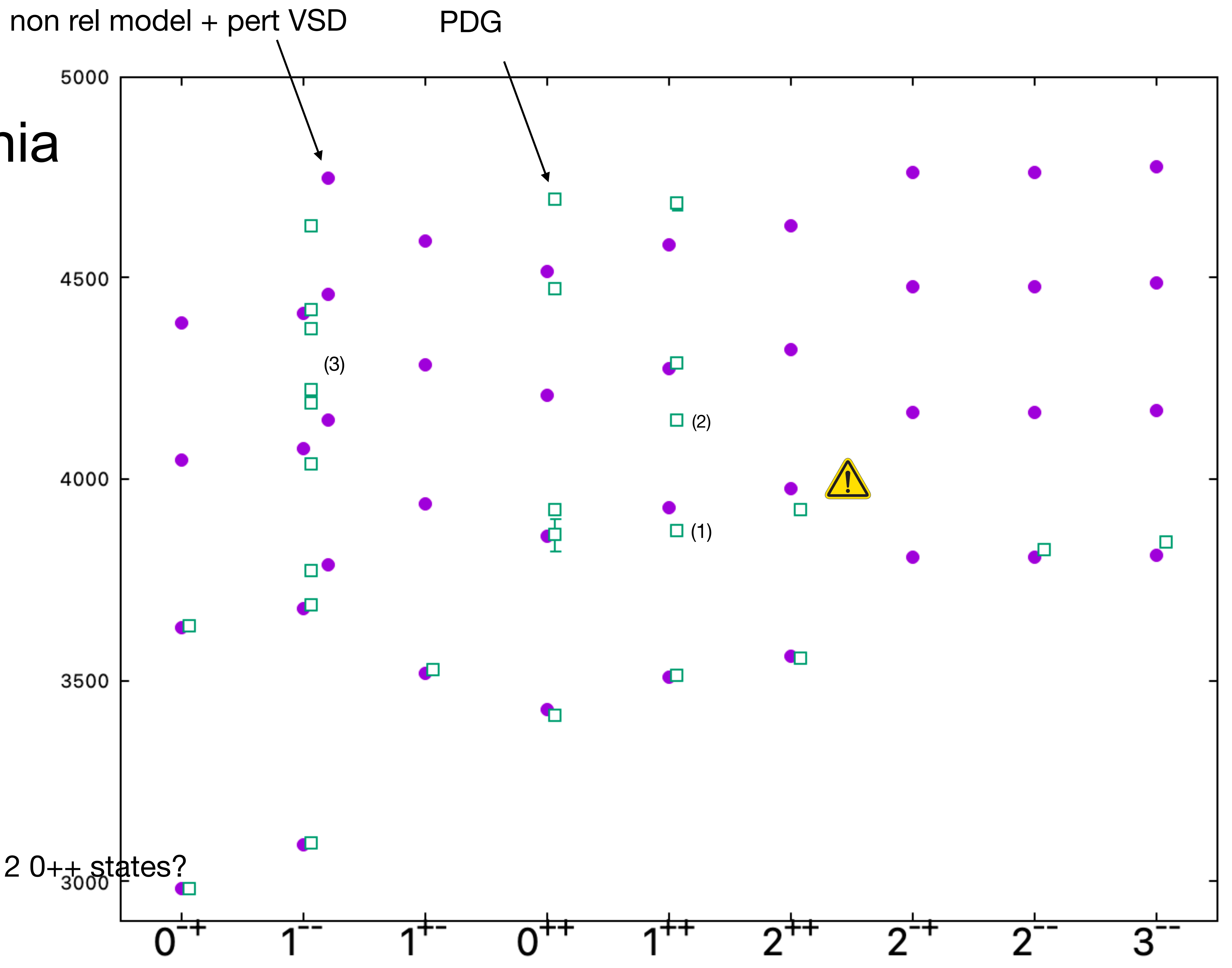

72 new hadrons at the LHC
Mass [GeV/c^2]
11.0
10.5
7.5
7.0
6.5
6.0
5.5
5.0
4.5
4.0
3.5
3.0
2.5
2.0
2011
2012
2013
2014
2015
2016
2017
2018
2019
2020
2021
2022
2023
$b\bar{b}$
$b\bar{q}$
$c\bar{c}(q\bar{q})$
$cc\bar{q}\bar{q}$
$c\bar{c}c\bar{c}$
$c\bar{q}$
$c\bar{q}q\bar{q}$
bqq
cqq
$c\bar{c}qqq$
$\chi_b(3P)$
$\chi_{b2}(3P)$
$\chi_{b1}(3P)$
$B_c(2S)^+$
$B_c^*(2S)^+$
$B_c(2S)^+$
$T_{\psi\psi}(6900)$
$T_{\psi\psi}(6600)$
$\Omega_b(6350)^-$
$\Lambda_b(6152)^0$ $\Omega_b(6340)^-$
$\Lambda_b(6146)^0$
$\Xi_b(6227)^-$
$\Xi_b(6227)^0$
$\Xi_b(6327)^0$
$\Xi_b(6333)^0$
$\Xi_b(6095)^0$
$\Xi_b(6087)^0$
$\Xi_b(5945)^0$
$\Lambda_b(5920)^0$
$\Lambda_b(5912)^0$
$\Xi_b(5955)^-$
$\Xi_b'(5935)^-$
$B_J(5970)^{+,0}$
$B_J(5840)^{+,0}$
$\Sigma_b(6097)^+$
$\Sigma_b(6097)^-$
$\Lambda_b(6070)^0$
$B_s^*(6114)^0$
$B_s^*(6063)^0$
$\Xi_b(6100)^-$
$P_\psi^N(4450)^+$
$P_\psi^N(4380)^+$
$X(4140)$
$X(4700)$
$X(4500)$
$X(4274)$
$P_\psi^N(4457)^+$
$P_\psi^N(4440)^+$
$P_\psi^N(4312)^+$
$\psi_3(3842)$
$X(4685)$
$X(4630)$
$T_{\psi s1}(4220)^+$
$T_{\psi s1}^\theta(4000)^+$
$T_{cc}(3875)^+$
$P_{\psi s}^\Lambda(4338)^0$
$T_{\psi s1}^\theta(4000)^0$
$X(3960)$
Ξ_{cc}^{++}
$D_J^*(3000)^{+,0}$
$D_J(3000)^0$
$D_J^*(2760)^+$
$D_J(2740)^0$
$D_J(2580)^0$
$D_{s1}^*(2860)^+$
$\Lambda_c(2860)^+$
$D_3^*(2760)^0$
$\Omega_c(3119)^0$
$\Omega_c(3090)^0$
$\Omega_c(3066)^0$
$\Omega_c(3050)^0$
$\Omega_c(3000)^0$
$\Xi_c(2939)^0$
$\Xi_c(2923)^0$
$T_{cs0}(2900)^0$
$T_{cs1}(2900)^0$
$D_{s0}(2590)^+$
$\Omega_c(3327)^0$
$\Omega_c(3185)^0$
$T_{c\bar{s}0}^a(2900)^{++}$
$T_{c\bar{s}0}^a(2900)^0$

Lattice Charmonia



HadSpec, 1204.5425

Model + PDG Charmonia



(1) 3872 = DD*, but 2++ is equally off, also 2 0++ states?

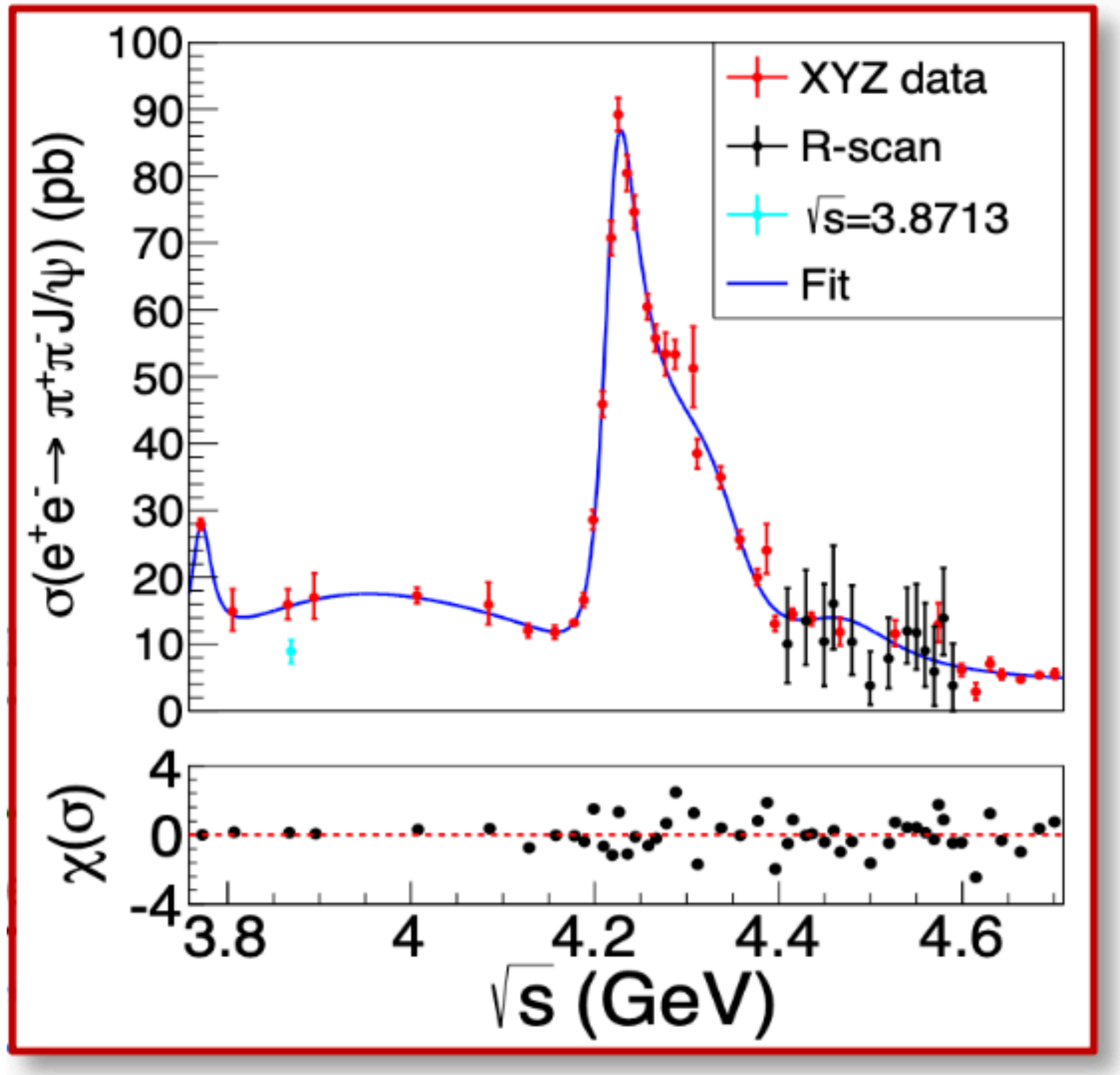
(2) could this be the 3P1(2P) shifted up?

(3) something of a fiasco here

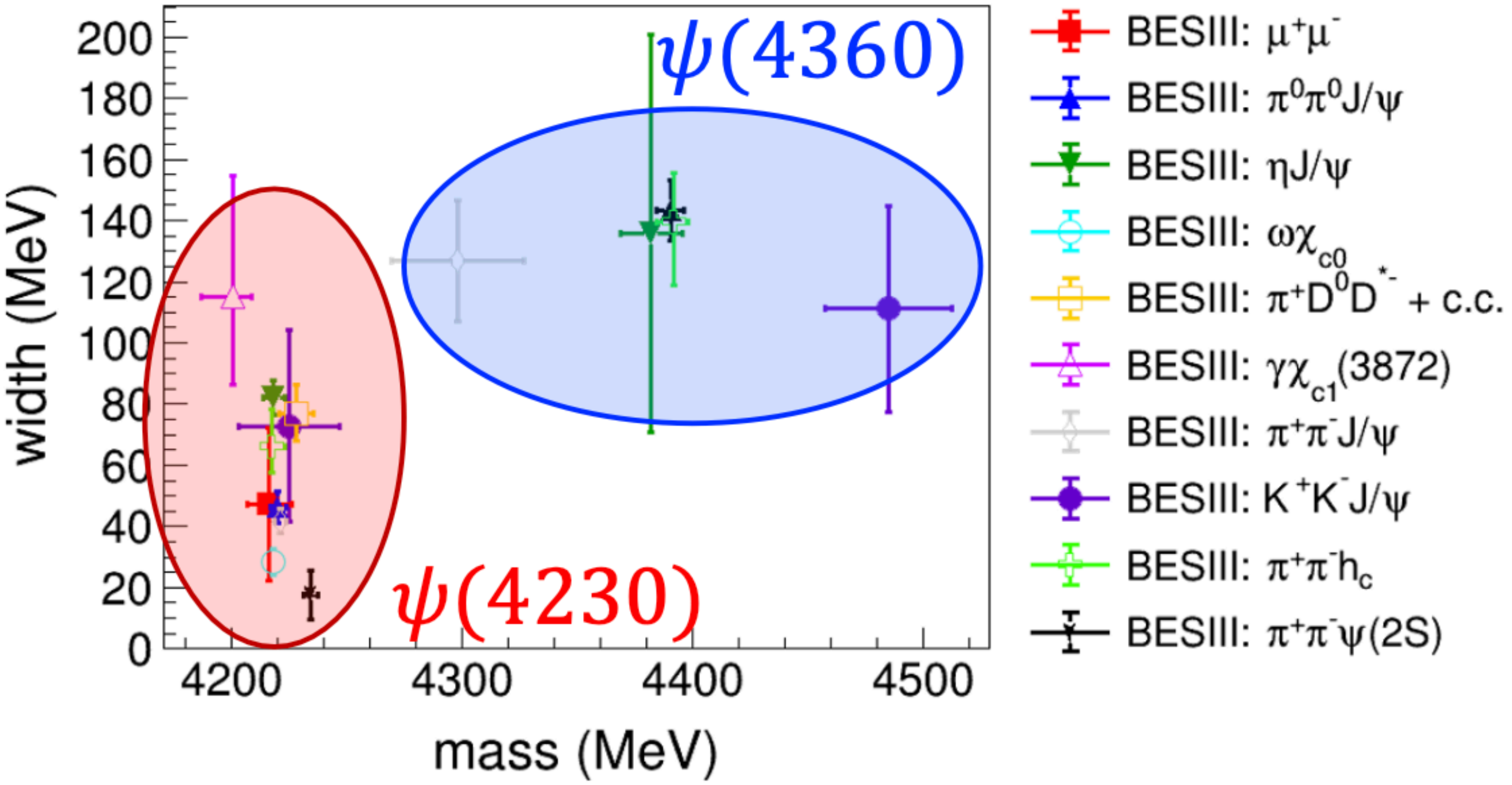
charmonia.dat / charmonia.plt ***hybrids/spectrum/bb/fitBBc.cpp***

an example:

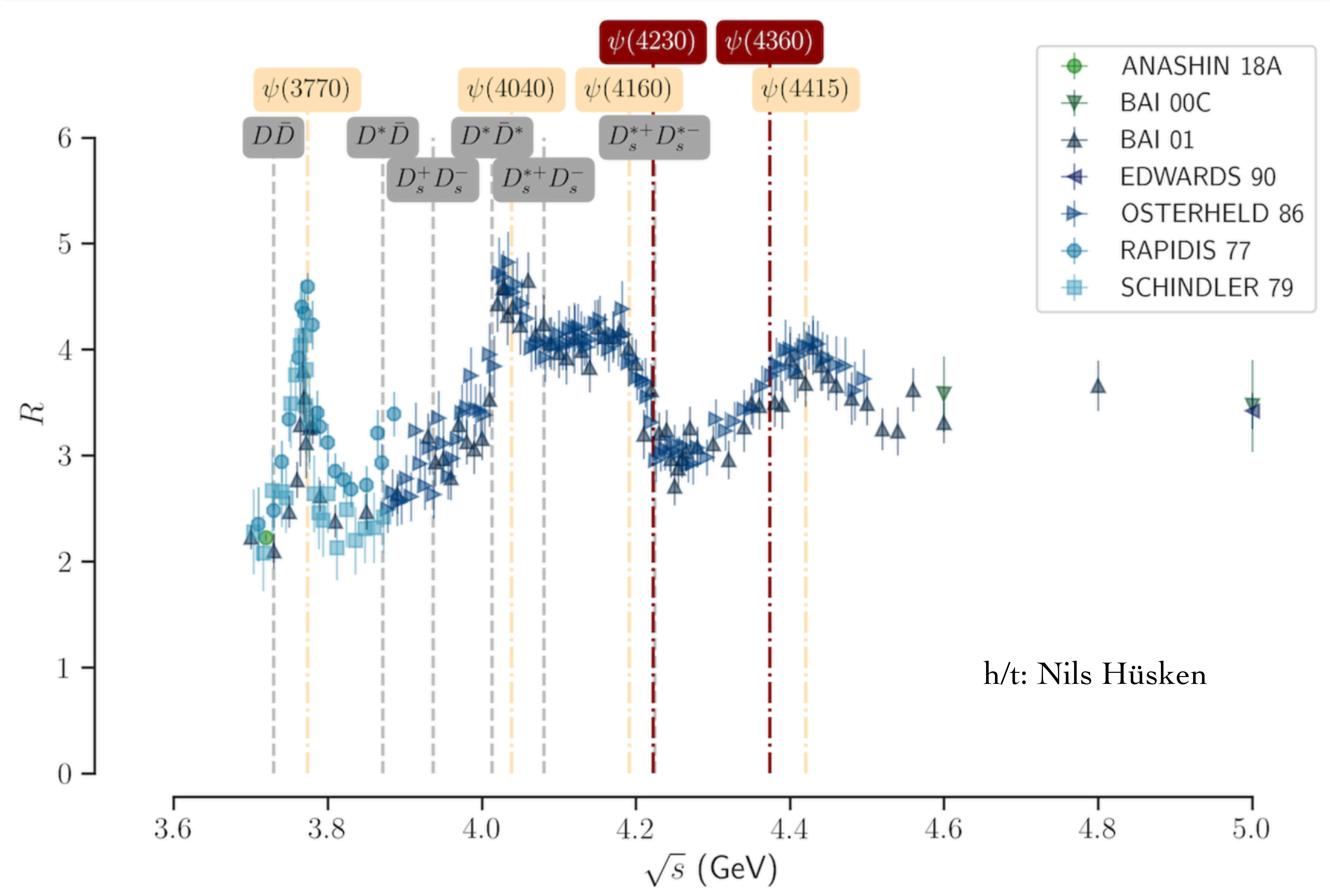
$\psi(4320)$ & $\psi(4360)$



PRD 106 (2022) 7, 072001



another example:



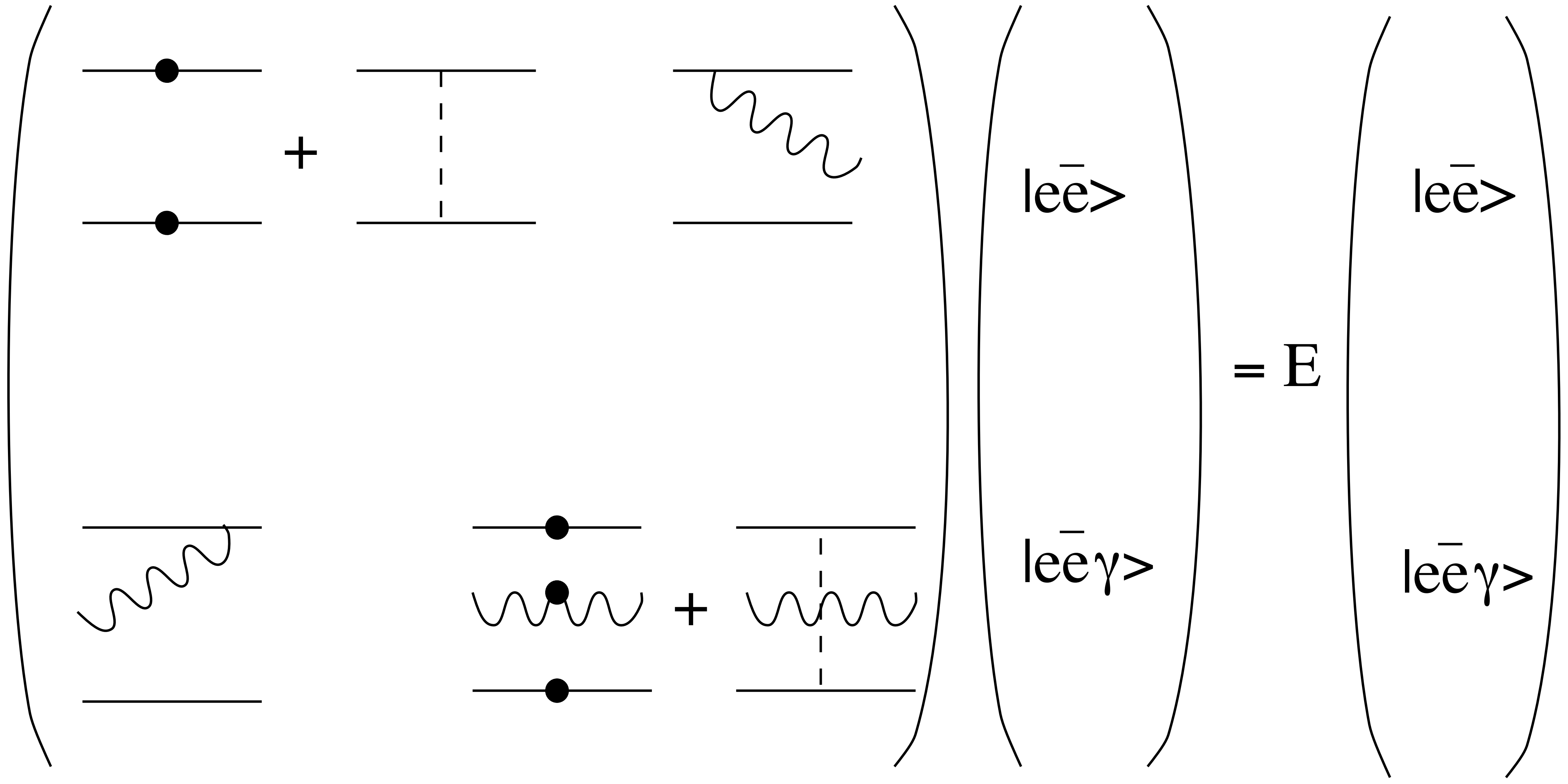
The situation is even worse: we rely on R for much information! This is not robust!

hybrids

$$|M\rangle = |q\bar{q}\rangle + \boxed{|q\bar{q}g >} + |q\bar{q}q\bar{q}\rangle + \ldots$$

what is a hybrid?

QED

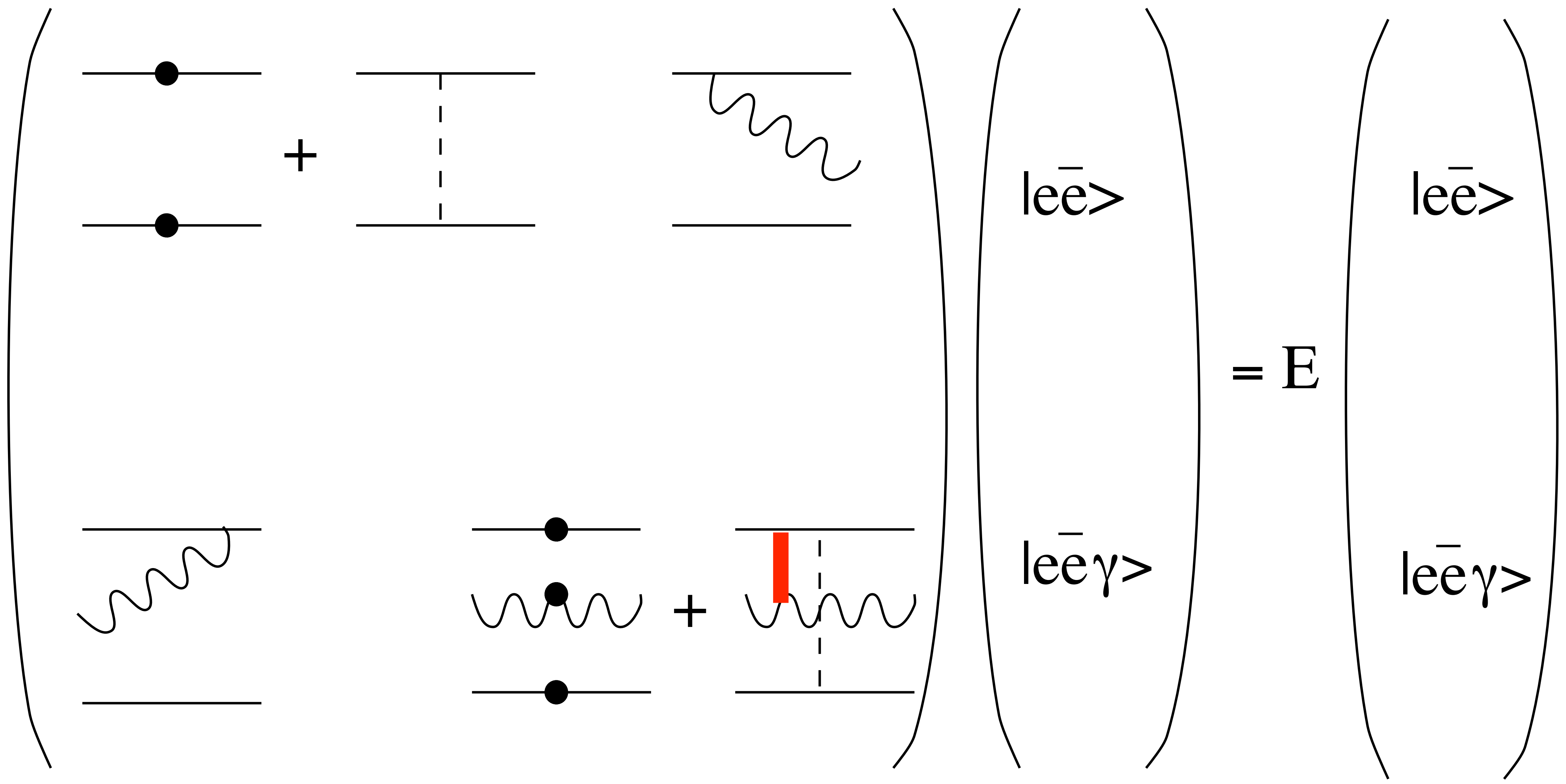


$$
\begin{aligned}
|0\rangle &= \sqrt{1-\epsilon^2}|e\bar{e}\rangle + \epsilon|e\bar{e}\gamma\rangle, \\
|1\rangle &= -\epsilon|e\bar{e}\rangle + \sqrt{1-\epsilon^2}|e\bar{e}\gamma\rangle.
\end{aligned}
$$

$\langle r|0\rangle \approx \phi(r)$
$\langle Rr|1\rangle \approx \phi(r)\psi(R)$

what is a hybrid?

QCD



$$
\begin{aligned}
|0\rangle &= \sqrt{1-\epsilon^2}|e\bar{e}\rangle + \epsilon|e\bar{e}\gamma\rangle, \\
|1\rangle &= -\epsilon|e\bar{e}\rangle + \sqrt{1-\epsilon^2}|e\bar{e}\gamma\rangle.
\end{aligned}
$$

$$
\langle r|0\rangle \approx \phi(r)
$$
$$
\langle Rr|1\rangle \approx \phi(r)\psi(R)
$$

Modelling: gluons

gluonic degrees of freedom must manifest in the spectrum somewhere.

Modelling: gluons

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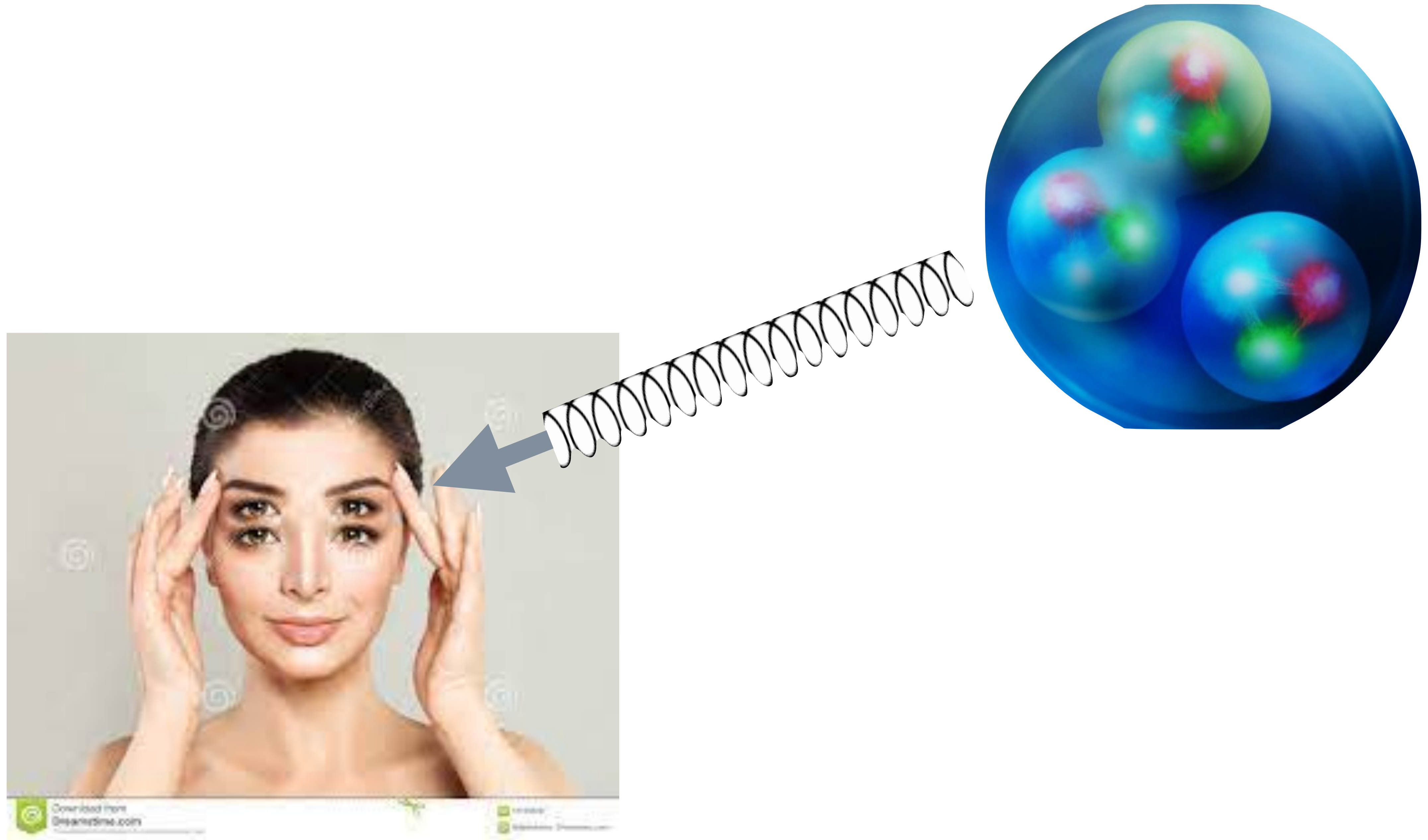


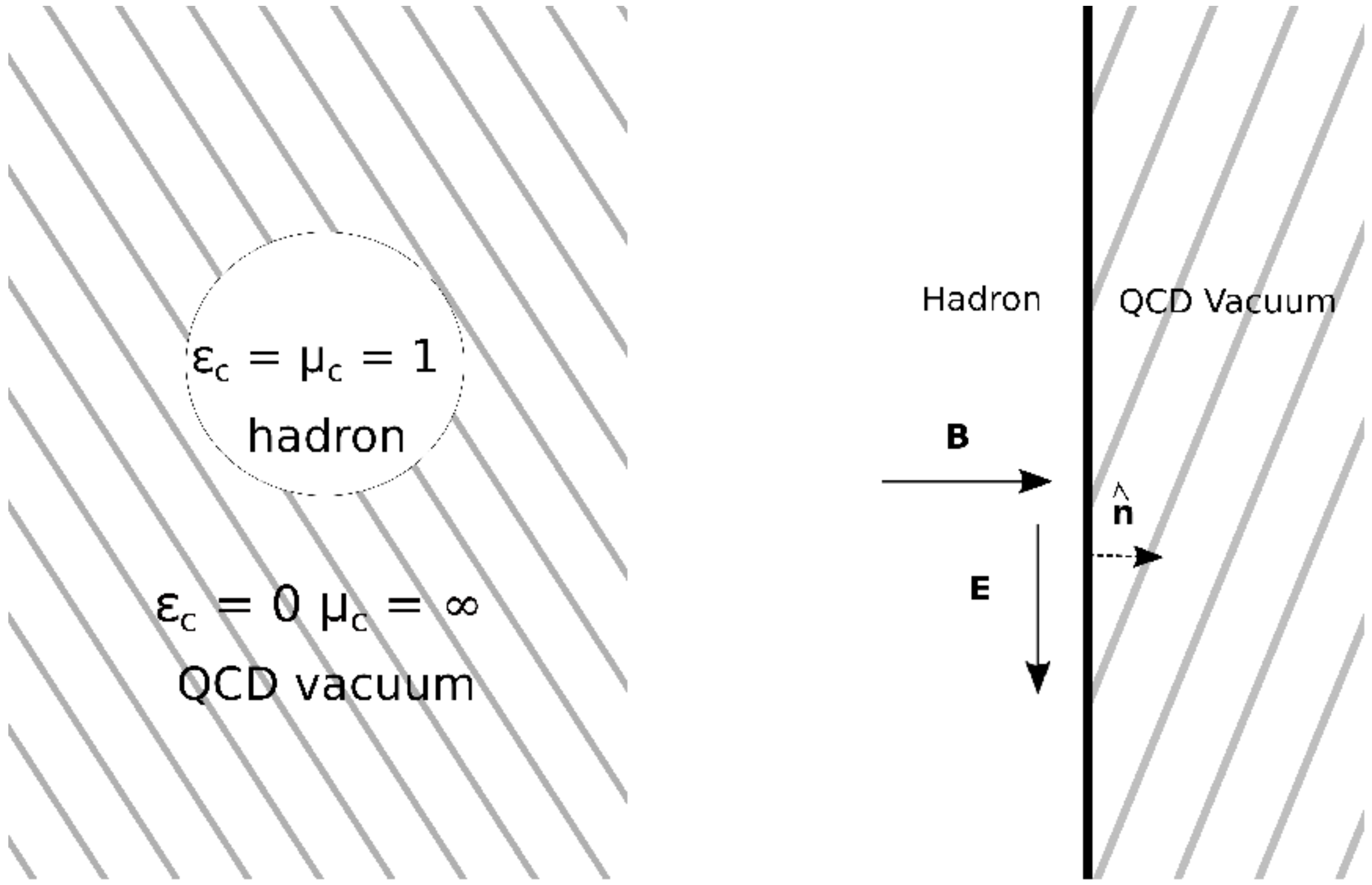
=> glueballs, hybrids, and mixed states.

Modelling: gluons

Bag Models

place quarks and gluons in a "bag" and allow them to interact perturbatively.

T. Barnes and F. E. Close, Phys. Lett. B 116, 365 (1982)
T. Barnes, F. E. Close and F. de Viron, Nucl. Phys. B 224, 241 (1983)
M. Chanowitz and S. Sharpe, Nucl. Phys. B 222, 211 (1983)



(a) The color permeability and permittivity of the bag-model.

(b) The color fields at the surface of the bag.

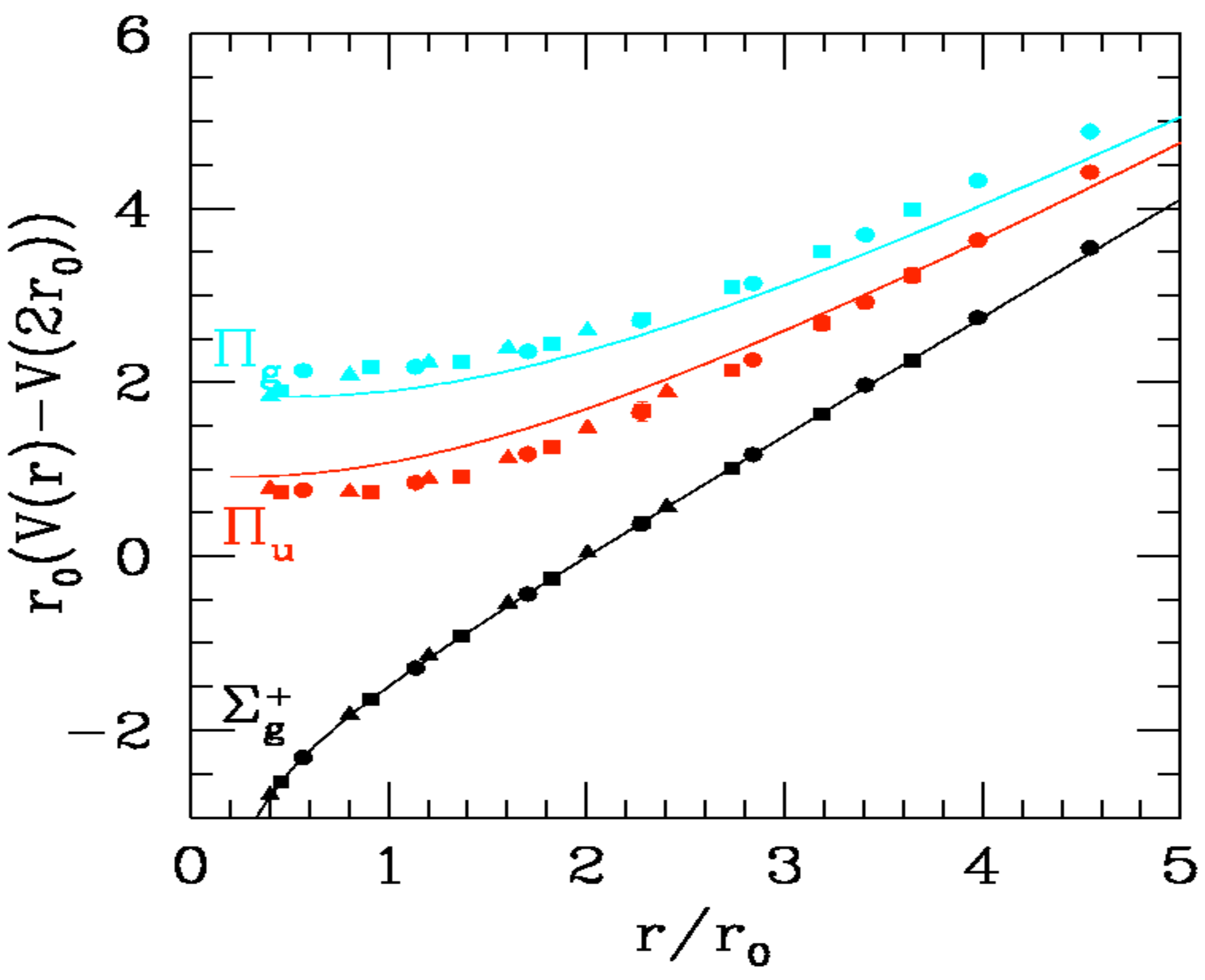
lowest mode is a 1+ TE gluon

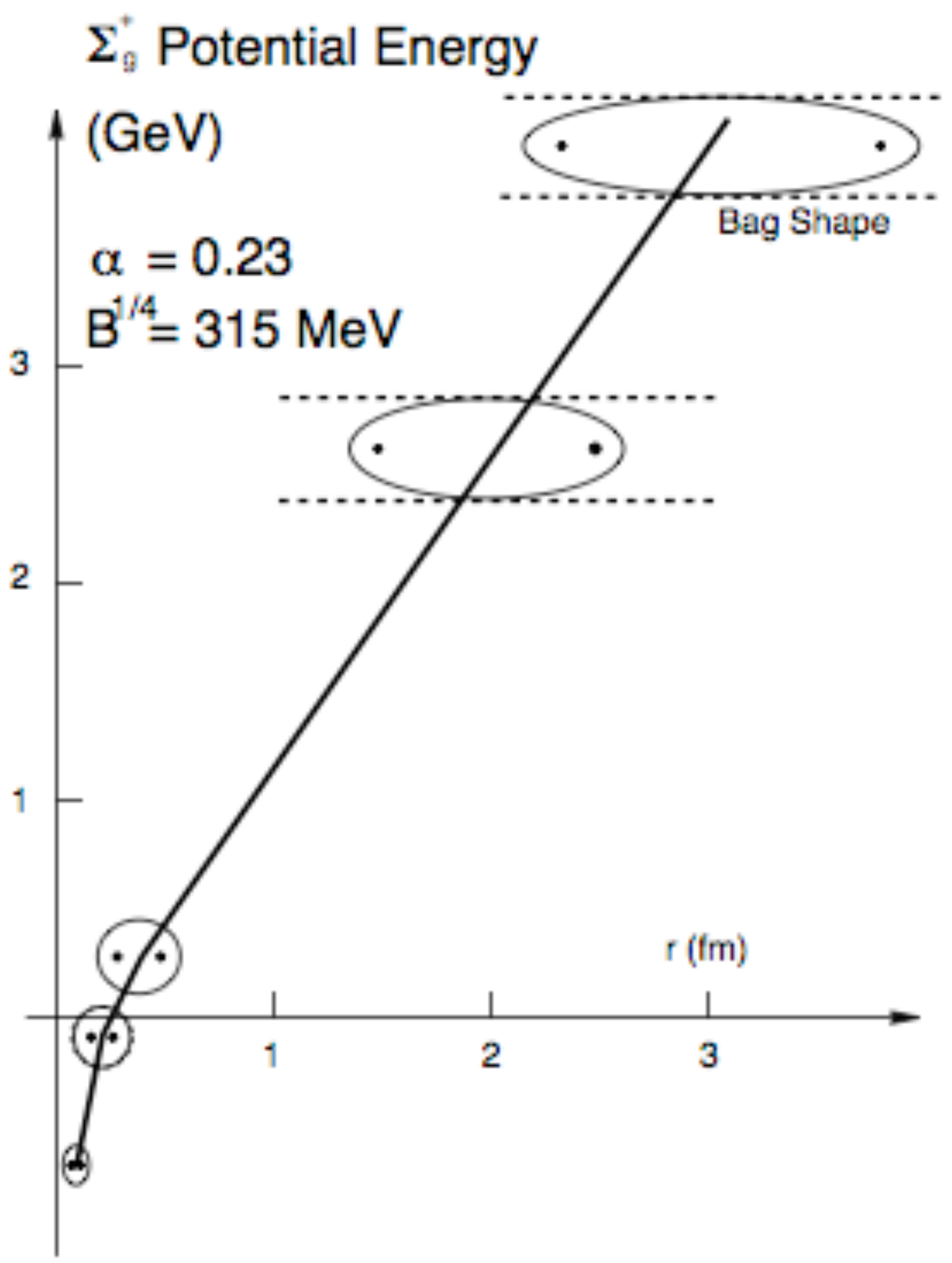
Modelling: gluons

Bag Models

Juge, Kuti, and Morningstar, (98)

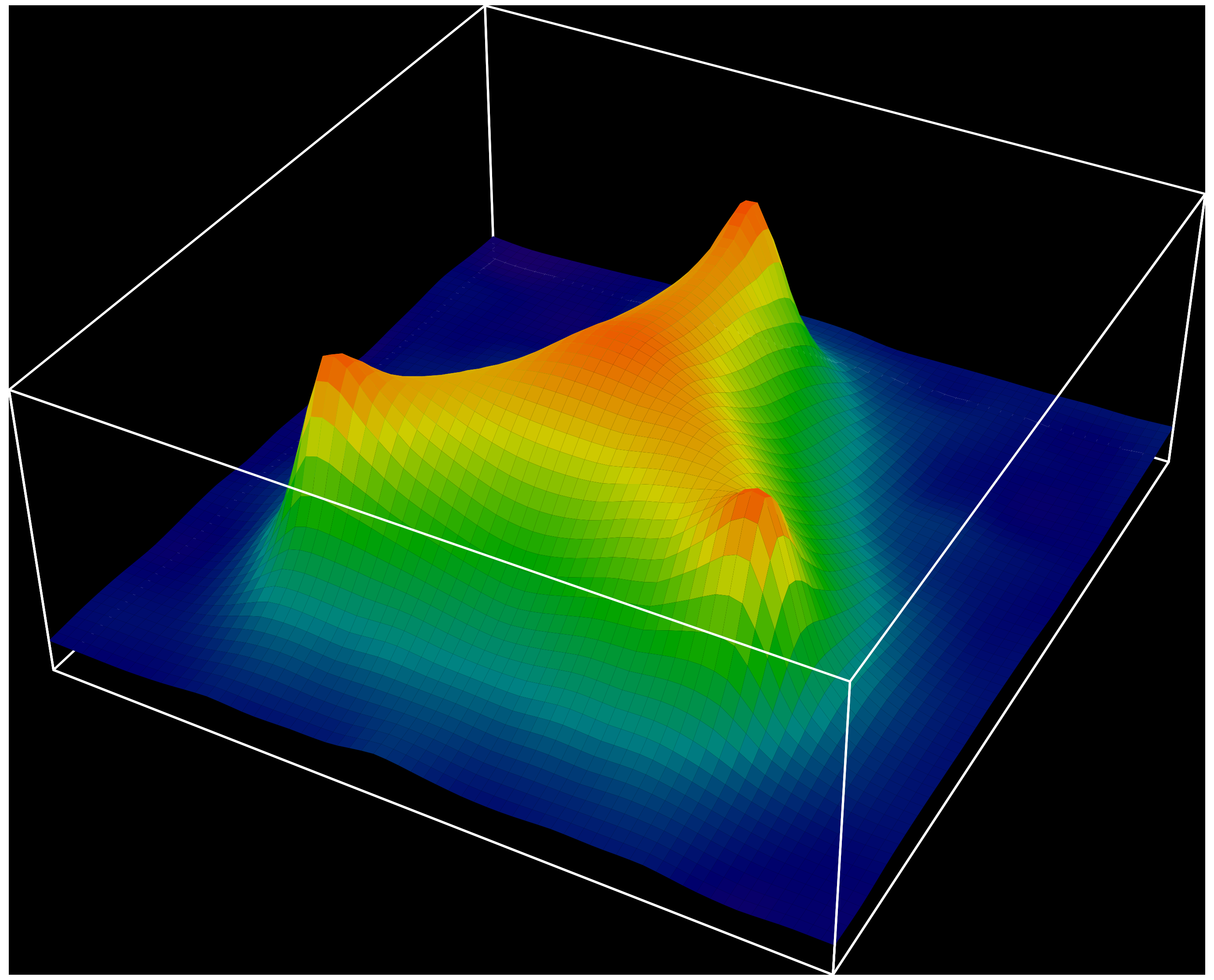
HHKR bag model computation

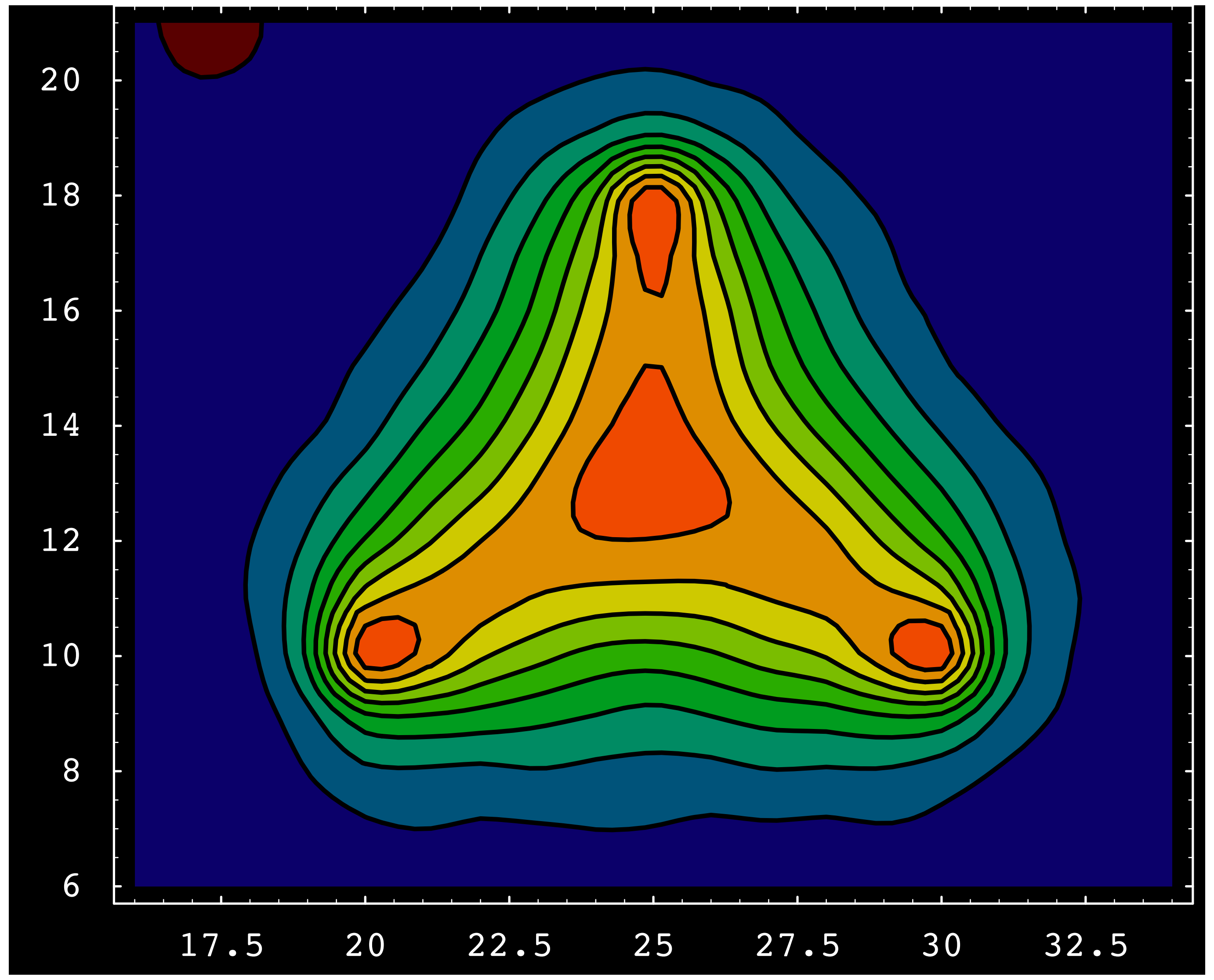




Modelling: gluons

Flux Tube Models





Modelling: gluons

Flux Tube Models

Giles and Tye, PRL37, 1175 (76).

T. Allen and M.G. Olsson, PLB434, 110 (98)

Coupled quarks to a relativistic 2d sheet... the "Quark Confining String Model".

"The presence of vibrational levels gives ... extra states in quantum mechanics. ... that are absent in the charmonium model."

$$V_N = \sigma r \left(1 + \frac{2N\pi}{\sigma r^2}\right)^{1/2}$$

GT

$$V_{NG} = \sigma r \left(1 - \frac{D-2}{12\sigma r^2} + \frac{2N\pi}{\sigma r^2}\right)^{1/2}$$

J.F. Arvis, PLB127, 106 (83); Luescher

16 APPLICATION OF THE QUARK-C

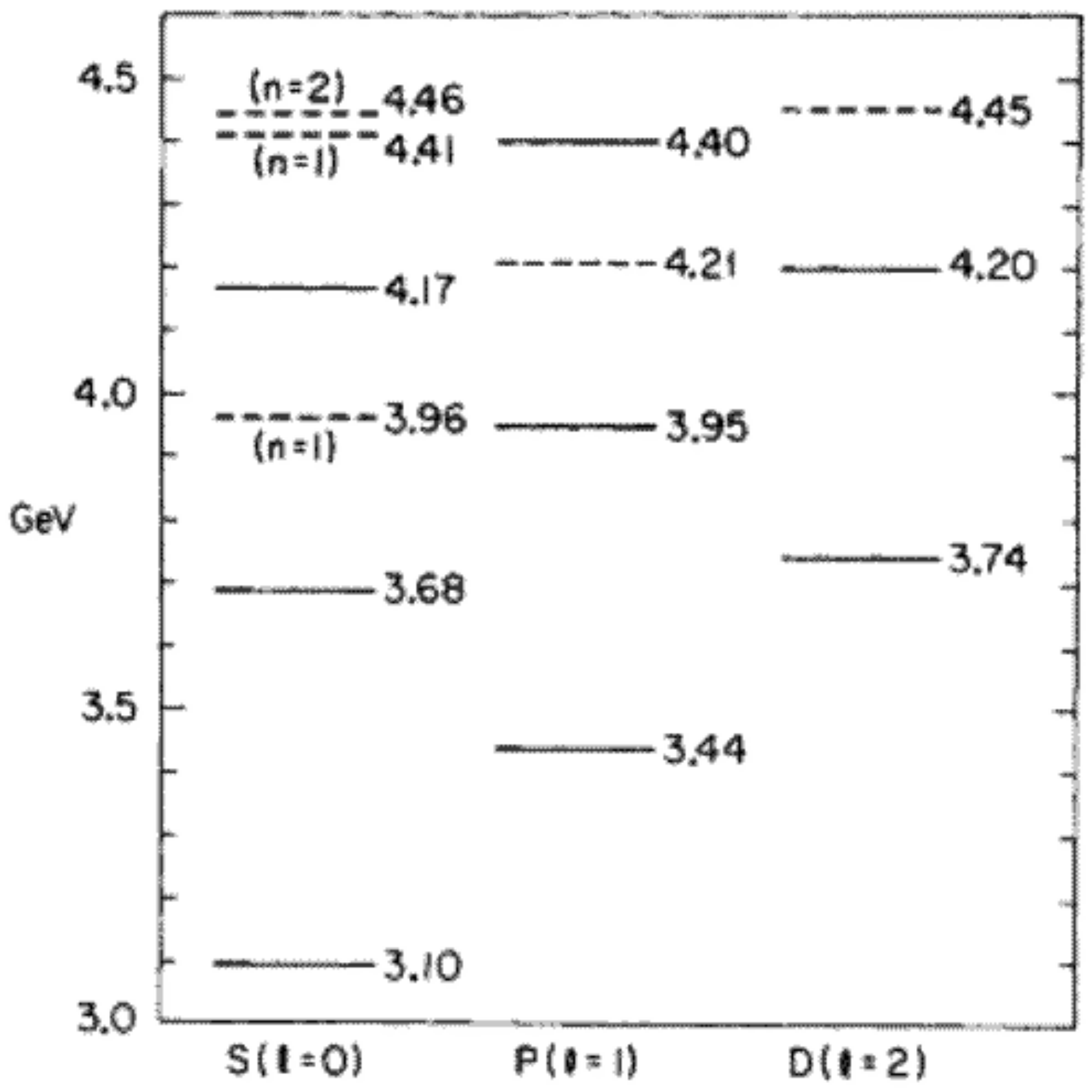


FIG. 4. The nonrelativistic spectroscopy of the charm string. $\psi(3.10)$ and $\psi(3.68)$ are fitted to obtain $M = 1.154$ GeV and $k = 0.21$ GeV^2. The dashed lines are the vibrational levels absent in the charmonium model. Levels

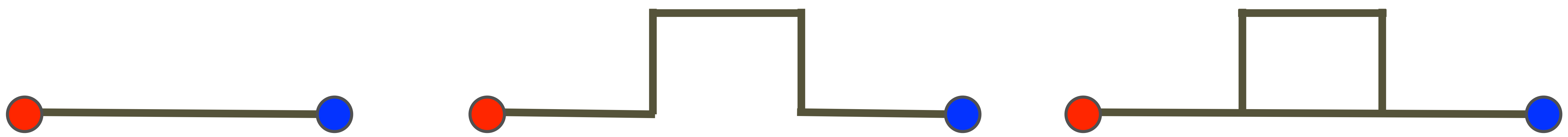
Modelling: gluons

Flux Tube Models

Isgur and Paton, PRD31, 2910 (85).

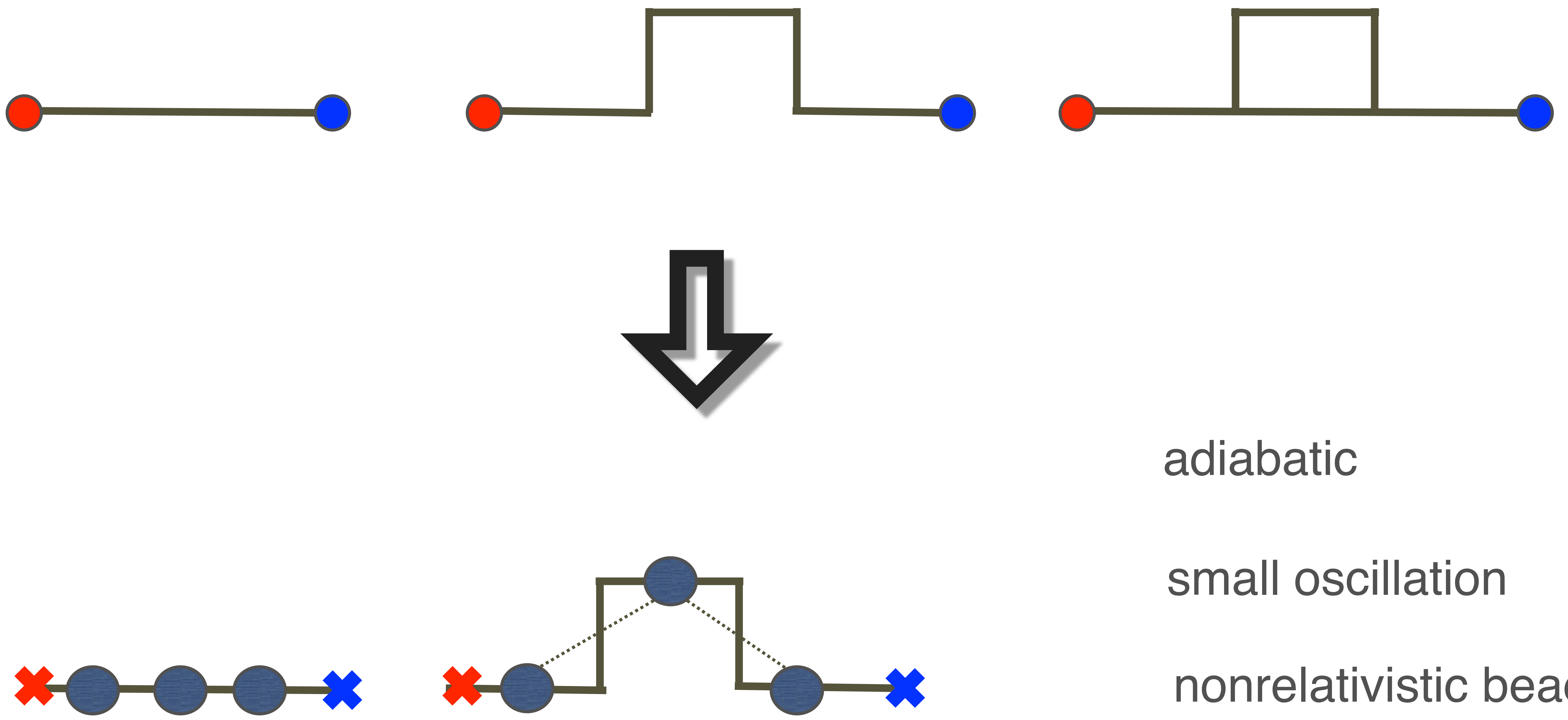
strong coupling Hamiltonian lattice gauge theory

$$H = \frac{g^2}{2a}\sum_\ell E^a_\ell E_{a\ell} + \sum_n m\bar{\psi}_n\psi_n + \frac{1}{a}\sum_{n,\mu}\psi^\dagger_n\alpha_\mu U_\mu(n)\psi_{n+\mu} + \frac{1}{ag^2}\sum_P \mathrm{tr}(N - U_P - U_P^\dagger)$$



Modelling: gluons

Flux Tube Models



adiabatic

small oscillation

nonrelativistic beads

$m_b = b\, a$

Modelling: gluons

Flux Tube Models

string Hamiltonian

$$H = b_0 R + \sum_n \left[\frac{p_n^2}{2ba} + \frac{b}{2a}(y_n - y_{n+1})^2 \right]$$

$$s_{m\lambda} = \sum_n^N y_n(\lambda) \sqrt{\frac{2}{N+1}} \sin \frac{nm\pi}{N+1}$$

$$y_n(\lambda) = \sum_m s_{m\lambda} \sqrt{\frac{2}{N+1}} \sin \frac{nm\pi}{N+1}$$

$$H = b_0 R + \sum_{n\lambda} \left[\frac{p_n^2}{2ba} + \frac{ba}{2} \omega_n^2 s_{n\lambda}^2 \right]$$

$$\omega_n = \frac{2}{a} \sin \frac{\pi n}{2(N+1)}$$

$$\alpha_{n\lambda} = \sqrt{\frac{b_0 \omega_n}{2}} s_{n\lambda} + i \frac{p_{n\lambda}}{\sqrt{b_0 a \omega_n}}$$

$$\omega_1 \to \frac{\pi}{R}$$

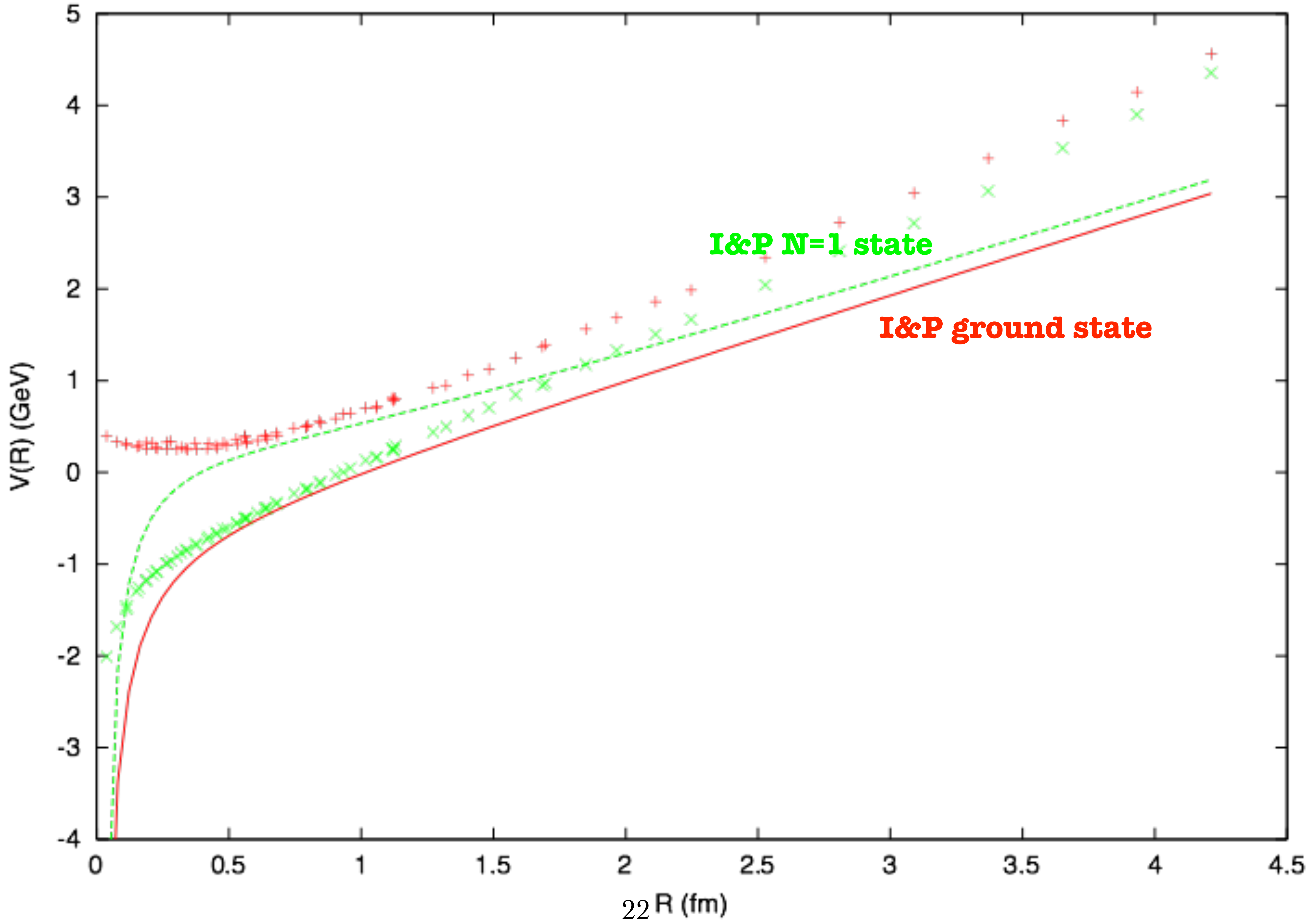
$$H = b_0 R + \sum_{n\lambda} \omega_n \left(\alpha_{n\lambda}^\dagger \alpha_{n\lambda} + \frac{1}{2} \right)$$

$$\boxed{H = b_0 R + \left(\frac{4}{\pi a^2} R - \frac{1}{a} - \frac{\pi}{12R} \right) + \sum_{n\lambda} \omega_n \alpha_{n\lambda}^\dagger \alpha_{n\lambda}}$$

Modelling: gluons

Flux Tube Models

compare to lattice



Modelling: gluons

Coulomb gauge QCD formalism

$$H_{QCD} = \int d^3x \left[\psi^\dagger \left(-i\alpha\cdot\nabla + \beta m\right)\psi + \frac{1}{2}\left(\mathcal{J}^{-1/2}\Pi\mathcal{J}\cdot\Pi\mathcal{J}^{-1/2} + B\cdot B\right) - g\psi^\dagger\alpha\cdot A\psi\right] + H_C$$

$$H_C = \frac{1}{2}\int d^3x\, d^3y\, \mathcal{J}^{-1/2}\rho^A(\mathbf{x})\mathcal{J}^{1/2}\hat{K}_{AB}(\mathbf{x},\mathbf{y};\mathbf{A})\mathcal{J}^{1/2}\rho^B(\mathbf{y})\mathcal{J}^{-1/2}$$

$$\mathcal{J} \equiv \det(\nabla\cdot D)$$

$$\rho^A(\mathbf{x}) = f^{ABC}\mathbf{A}^B(\mathbf{x})\cdot\mathbf{\Pi}^C(\mathbf{x}) + \psi^\dagger(\mathbf{x})T^A\psi(\mathbf{x})$$

$$\hat{K}^{AB}(\mathbf{x},\mathbf{y};\mathbf{A}) \equiv \langle\mathbf{x},A|\frac{g}{\nabla\cdot\mathbf{D}}(-\nabla^2)\frac{g}{\nabla\cdot\mathbf{D}}|\mathbf{y},B\rangle\,.$$

$$D^{AB} \equiv \delta^{AB}\nabla - gf^{ABC}A^C$$

Modelling: gluons

Coulomb gauge QCD formalism

Evaluate K with the aid of a nontrivial vacuum Ansatz

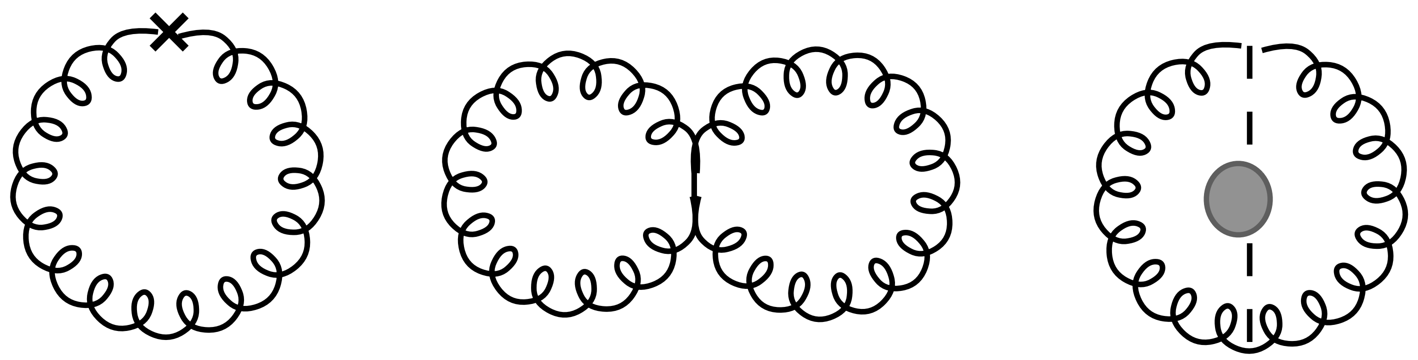
$$\langle A|\omega\rangle = \Psi_0[A] = \exp\left[-\frac{1}{2}\int \frac{d^3k}{(2\pi)^3} A^a(k)\,\omega(k)\,A^a(-k)\right]$$

Obtain ω by solving the gap equation

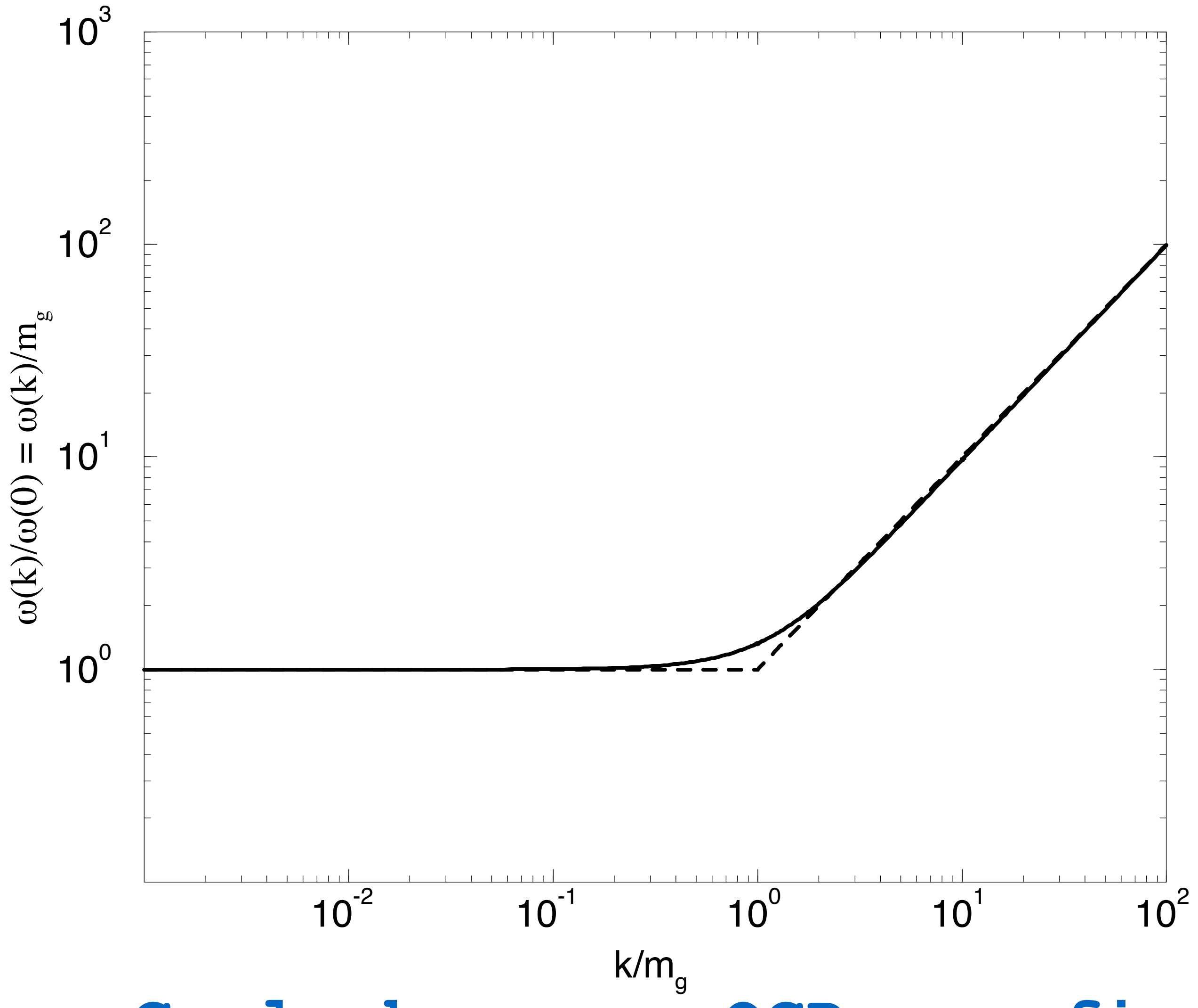
$$\frac{\delta}{\delta\omega}\langle\omega|H|\omega\rangle = 0$$

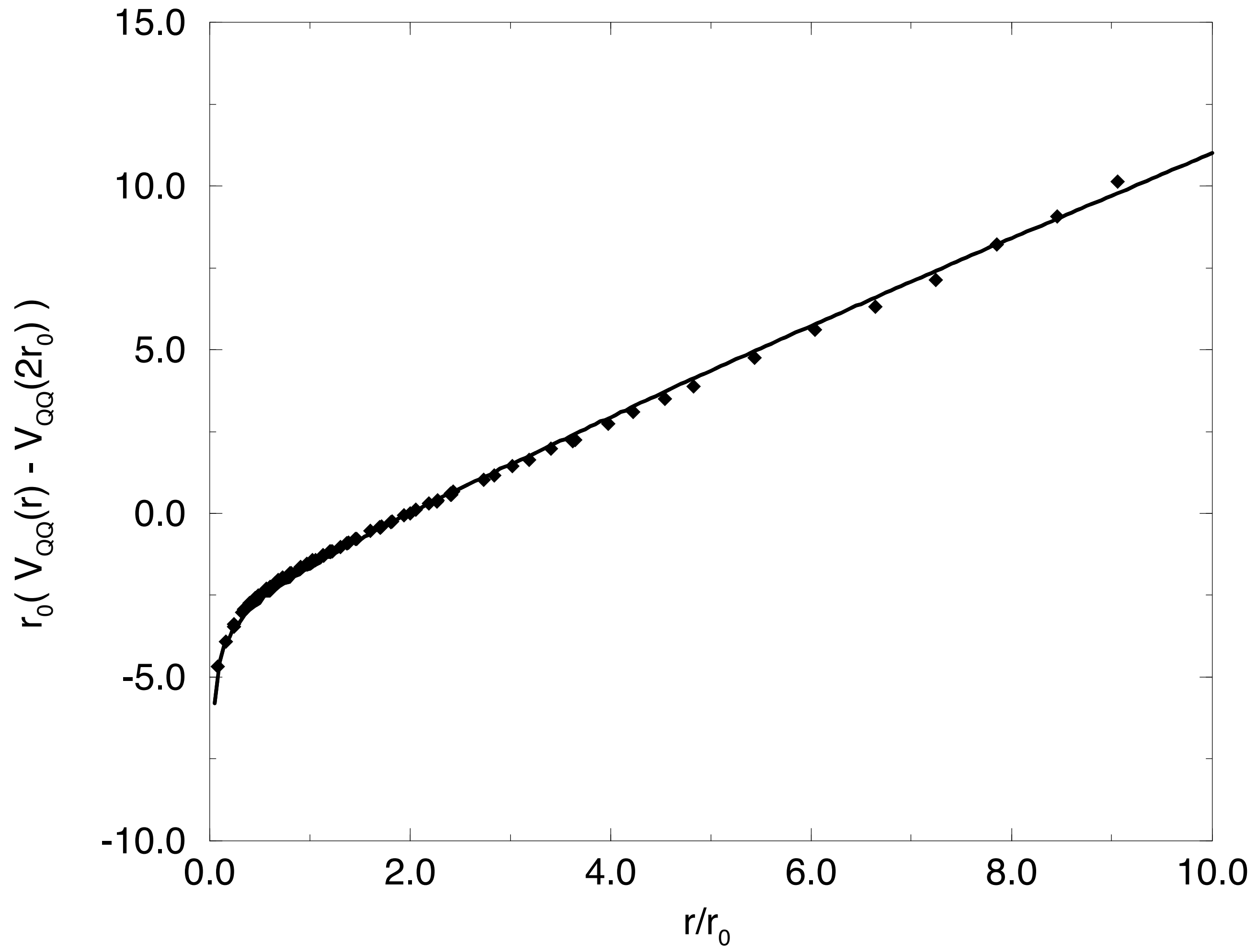
Modelling: gluons

Coulomb gauge QCD formalism



$$
\begin{aligned}
Z_\Pi^2(\Lambda)\omega^2(q;\Lambda) \quad = \quad & Z_A^2(\Lambda)q^2 + Z_m(\Lambda)\Lambda^2 + g^2(\Lambda)\frac{N_c}{4}\int^\Lambda \frac{d\mathbf{k}}{(2\pi)^3}\frac{(3-(\hat{\mathbf{k}}\cdot\hat{\mathbf{q}})^2)}{\omega(k;\Lambda)} + \\
& + \frac{N_c}{4}\int^\Lambda \frac{d\mathbf{k}}{(2\pi)^3}K^{(0)}(\mathbf{k}+\mathbf{q};\mathbf{A})(1+(\hat{\mathbf{k}}\cdot\hat{\mathbf{q}})^2)\,\frac{\omega^2(k;\Lambda)-\omega^2(q;\Lambda)}{\omega(k;\Lambda)}
\end{aligned}
$$





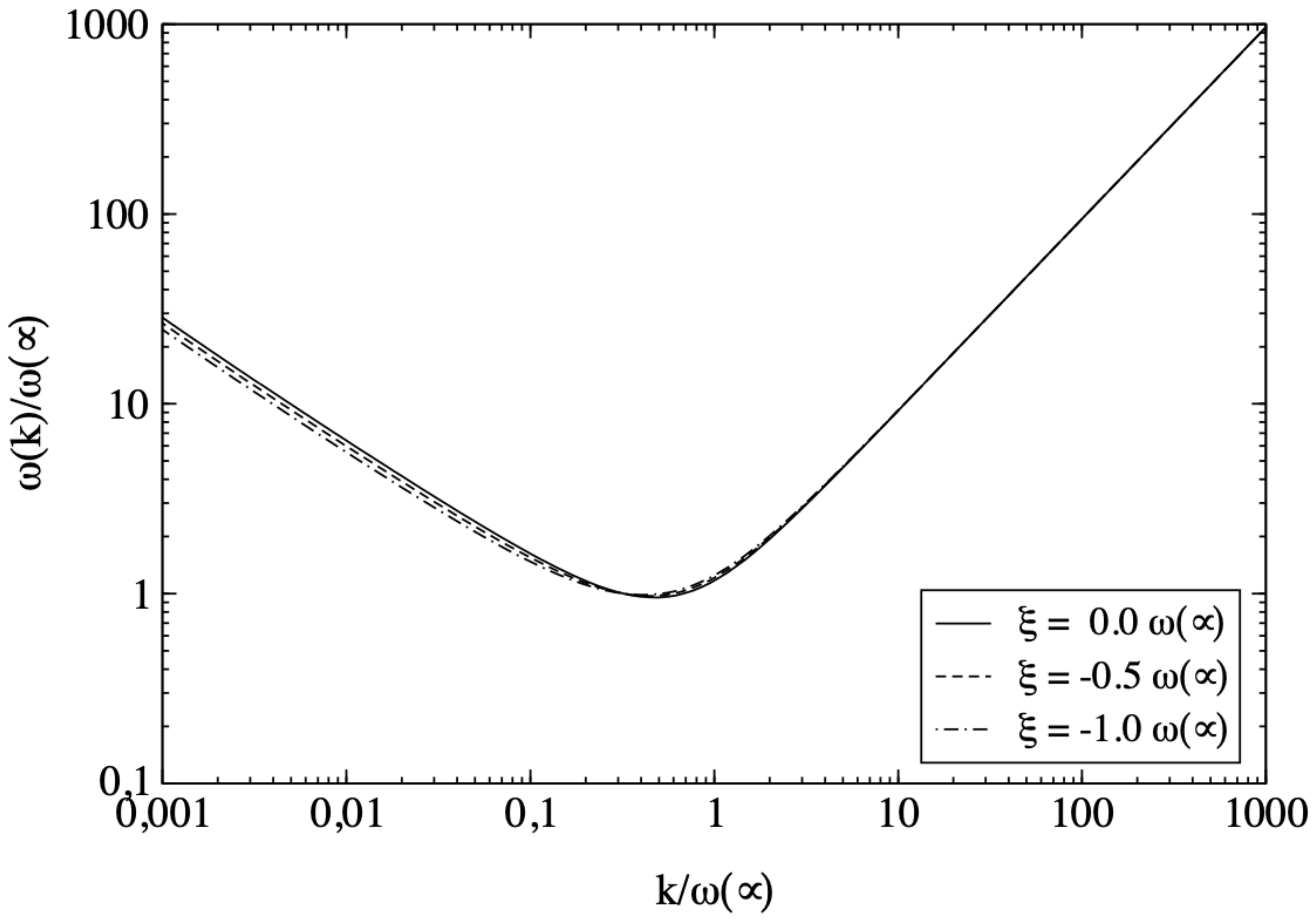
Coulomb gauge QCD, confinement, and the constituent representation, A.P. Szczepaniak & E.S. Swanson, *Phys.Rev.D* 65 (2001) 025012.

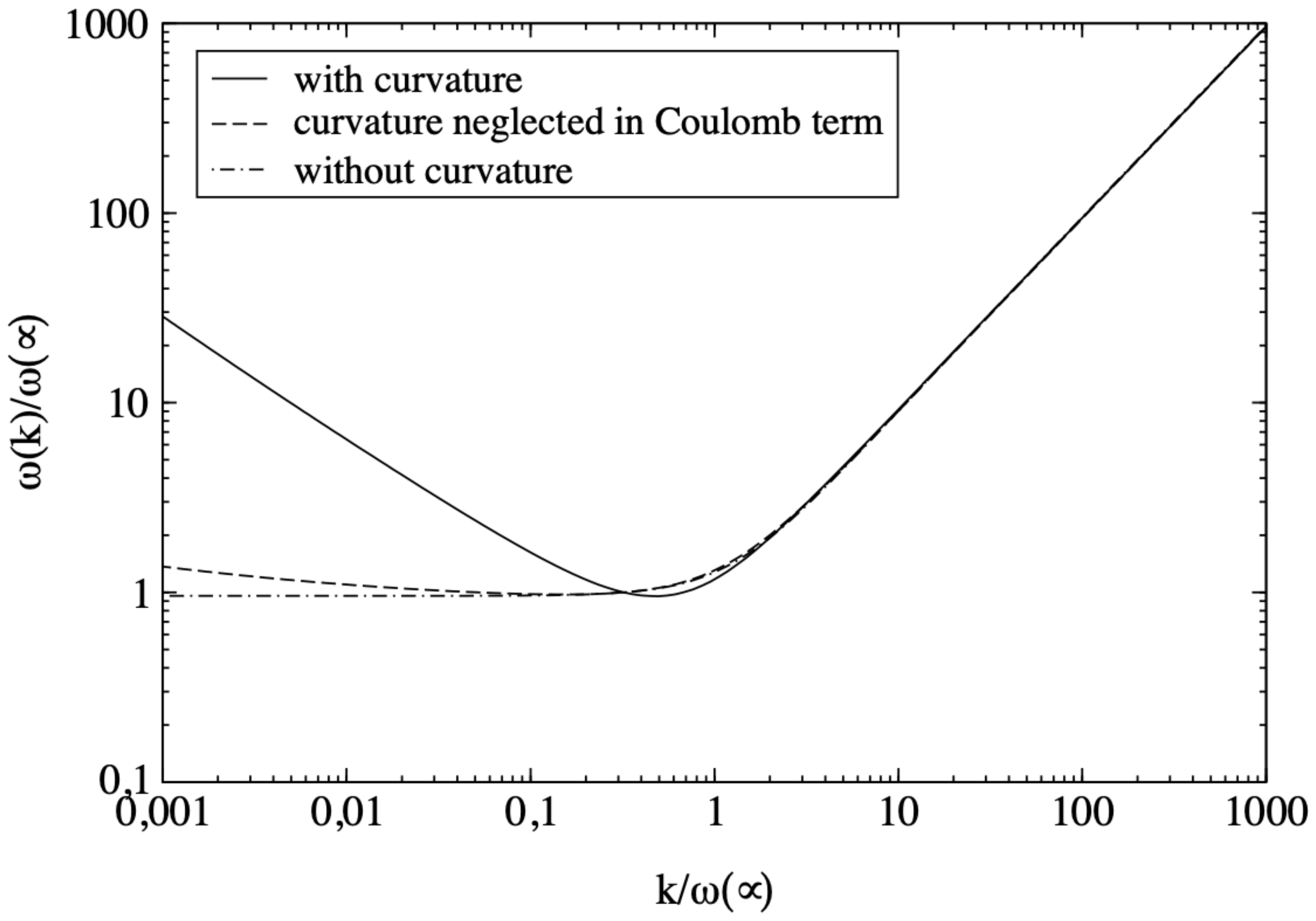
Modelling: gluons

Coulomb gauge QCD formalism

$$\tilde{\Psi}[A] = \langle A|\omega\rangle = \mathcal{N} \exp\left[-\frac{1}{2}\int d^3x \int d^3x' A_i^{\perp a}(\mathbf{x})\omega(\mathbf{x},\mathbf{x}')A_i^{\perp a}(\mathbf{x}')\right]$$

$$\Psi[A] = \mathcal{J}^{-\frac{1}{2}}[A]\tilde{\Psi}[A]$$

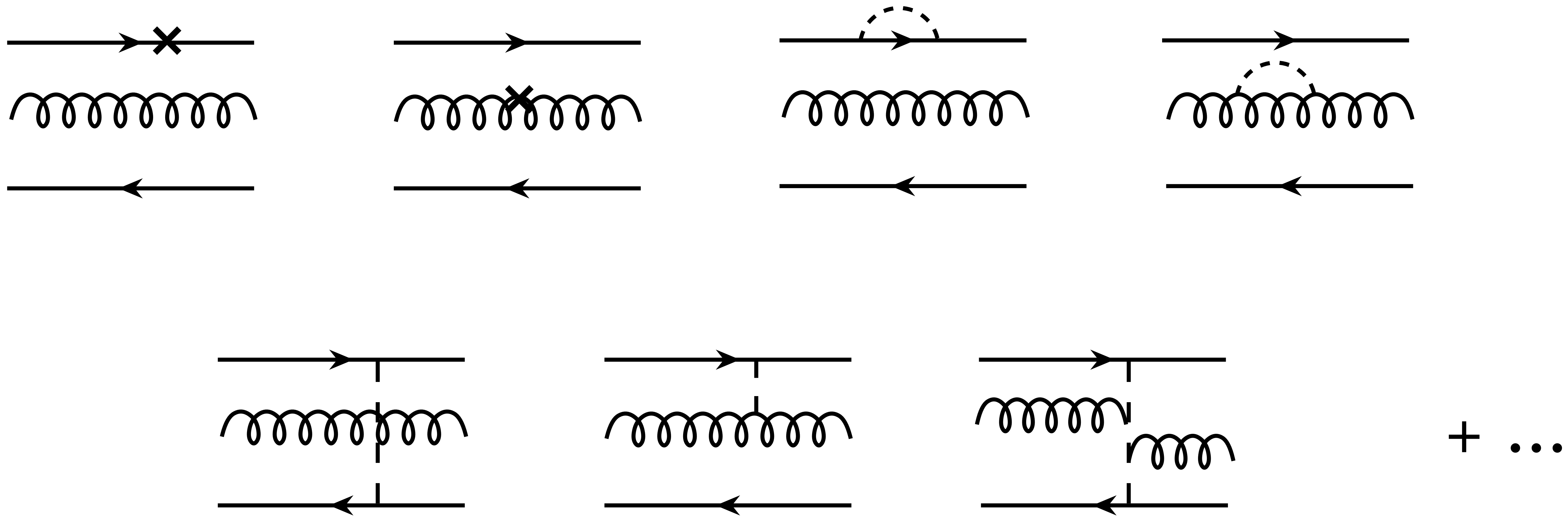




Variational solution of the Yang-Mills Schrödinger equation in Coulomb gauge, C. Feuchter & H. Reinhardt, hep-th/0408236.

Modelling: gluons

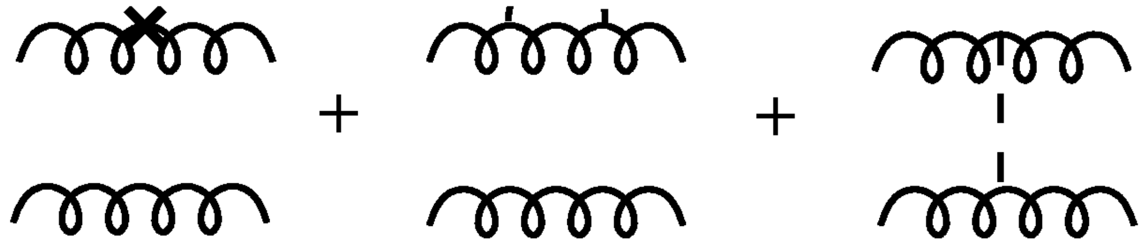
Coulomb gauge QCD formalism | Hybrids

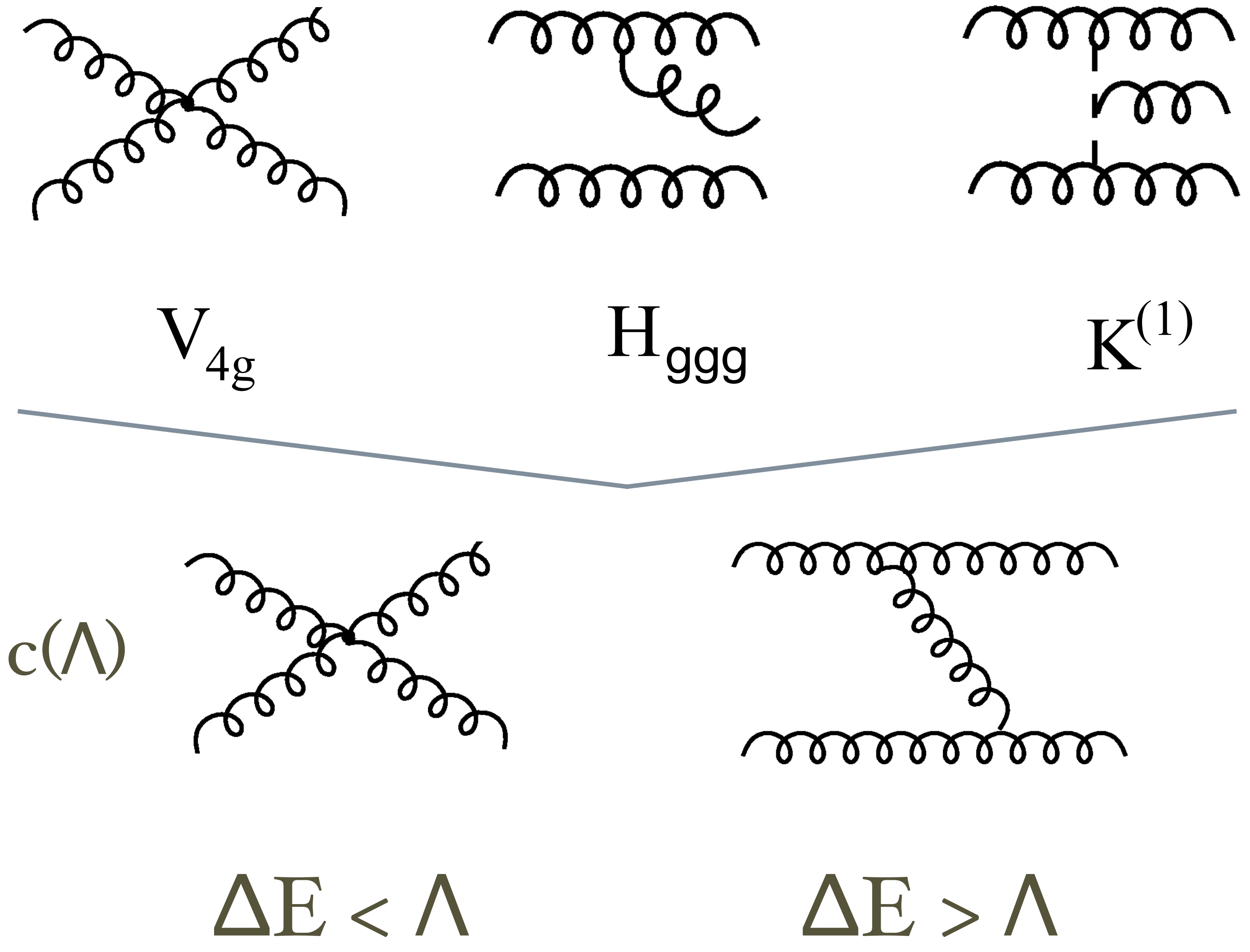


$$|JM[LS\ell j_g\xi]\rangle = \frac{1}{2}T^A_{ij}\int \frac{d^3q}{(2\pi)^3}\,\frac{d^3k}{(2\pi)^3}\,\Psi_{j_g;\ell m_\ell}(\mathbf{k},\mathbf{q})\sqrt{\frac{2j_g+1}{4\pi}}\,D^{j_g\,*}_{m_g\mu}(\hat{k})\,\chi^{(\xi)}_{\mu,\lambda}$$
$$\times\,\langle\frac{1}{2}m\frac{1}{2}\bar{m}|SM_S\rangle\,\langle\ell m_\ell, j_g m_g|LM_L\rangle\,\langle SM_S, LM_L|JM\rangle\,b^\dagger_{\mathbf{q}-\frac{\mathbf{k}}{2},i,m}\,d^\dagger_{-\mathbf{q}-\frac{\mathbf{k}}{2},j,\bar{m}}\,a^\dagger_{\mathbf{k},A,\lambda}|0\rangle.$$

Modelling: gluons

Coulomb gauge QCD formalism | Glueballs



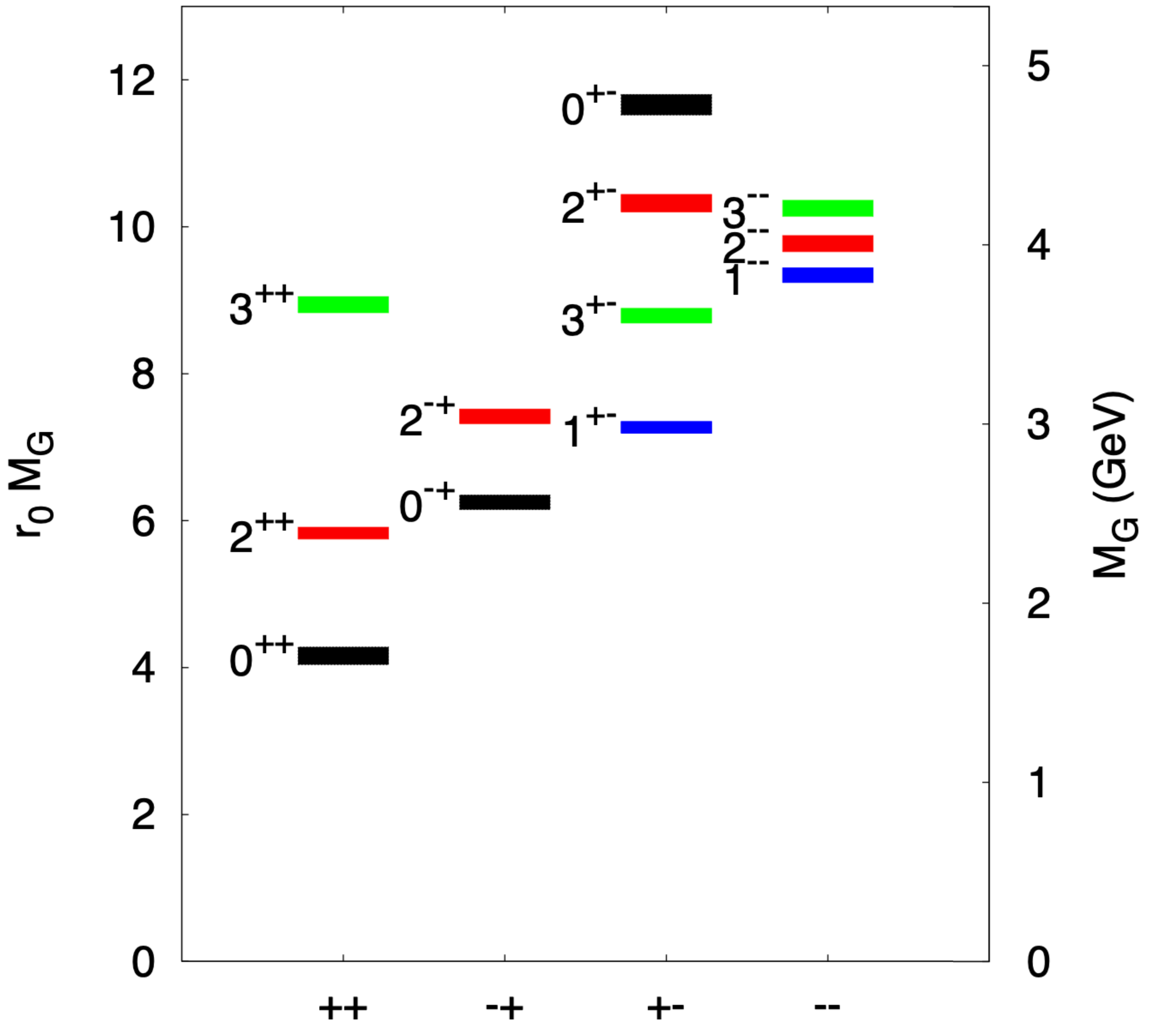


$$|JM;\lambda,\lambda'\rangle = \frac{1}{\sqrt{2(N_c^2-1)}}\sqrt{\frac{2J+1}{4\pi}}\int \frac{d^3k}{(2\pi)^3}\,\psi(k)\,D^{J*}_{M,\lambda-\lambda'}(\phi,\theta,-\phi)\,\Pi\,a^\dagger(k,\lambda,A)a^\dagger(-k,\lambda,A)\,|0\rangle$$

$$|JM;\eta\rangle = \frac{1}{\sqrt{2}}\left(|JM;\lambda,\lambda'\rangle + \eta\,|JM;-\lambda,-\lambda'\rangle\right)$$

Modelling: gluons

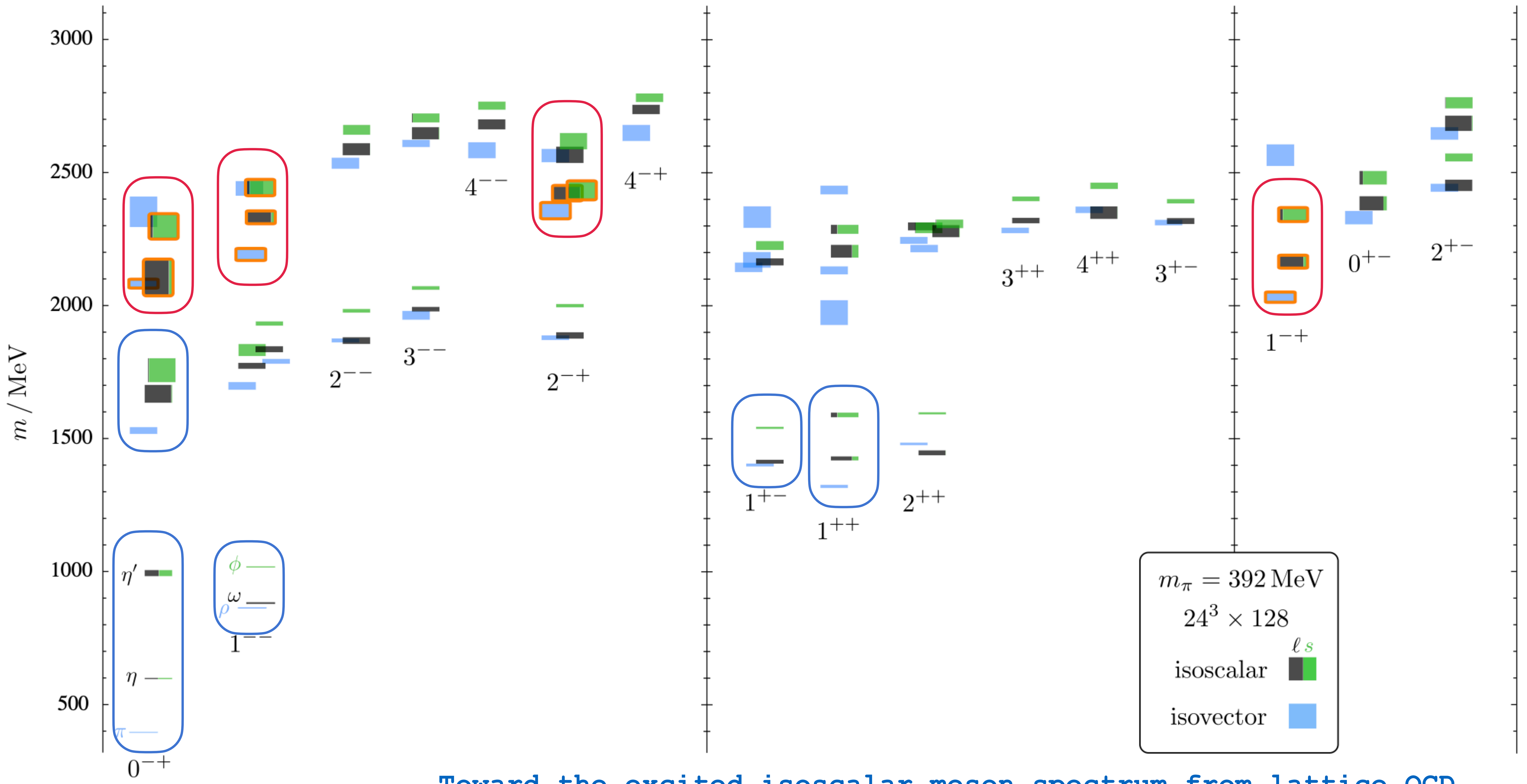
glueballs à la lattice



https://arxiv.org/pdf/hep-lat/0510074.pdf

Modelling: gluons

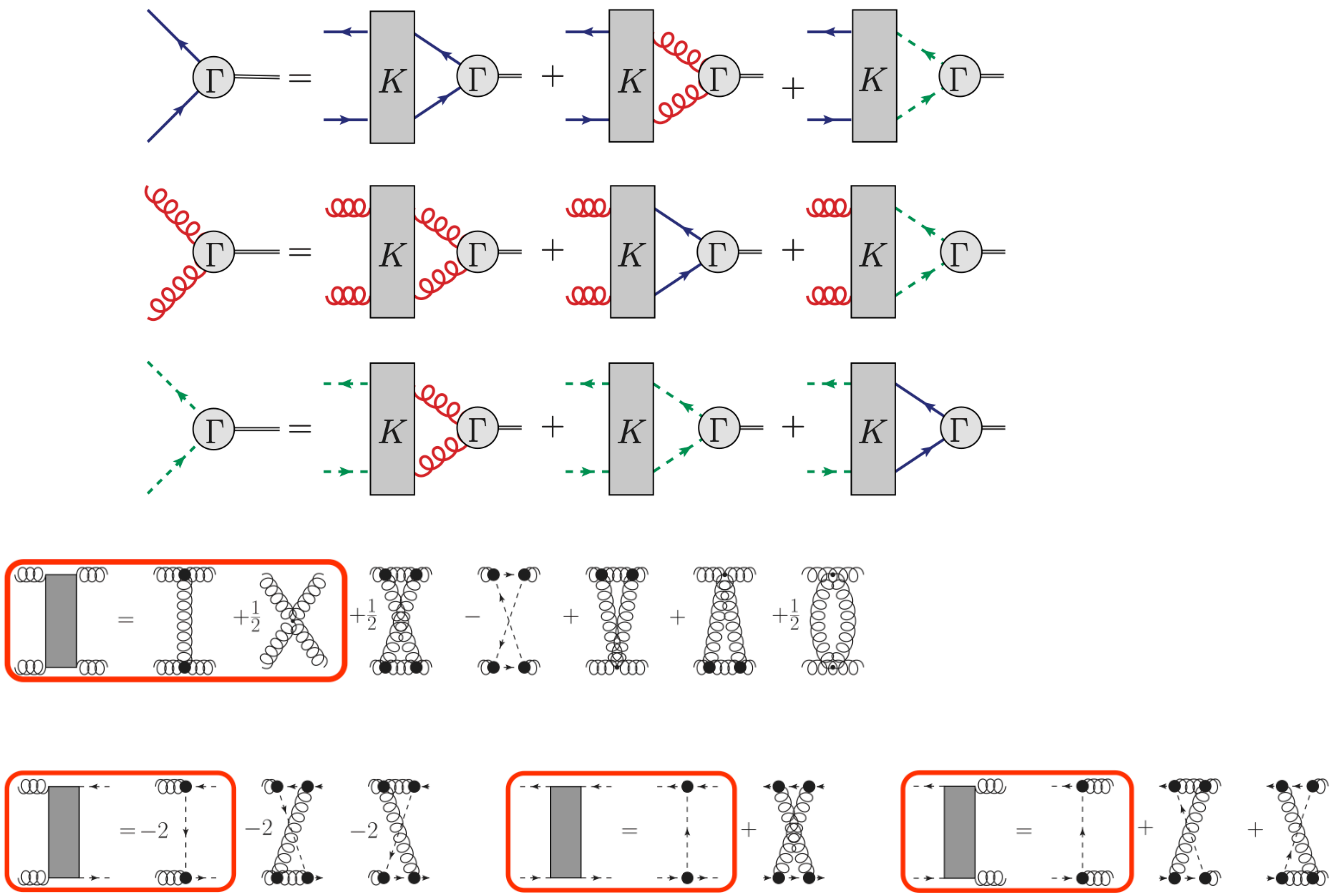
hybrids à la lattice



Toward the excited isoscalar meson spectrum from lattice QCD,
[HadSpec] J.J. Dudek et al. Phys.Rev.D 88 (2013) 9, 094505.

Modelling: gluons

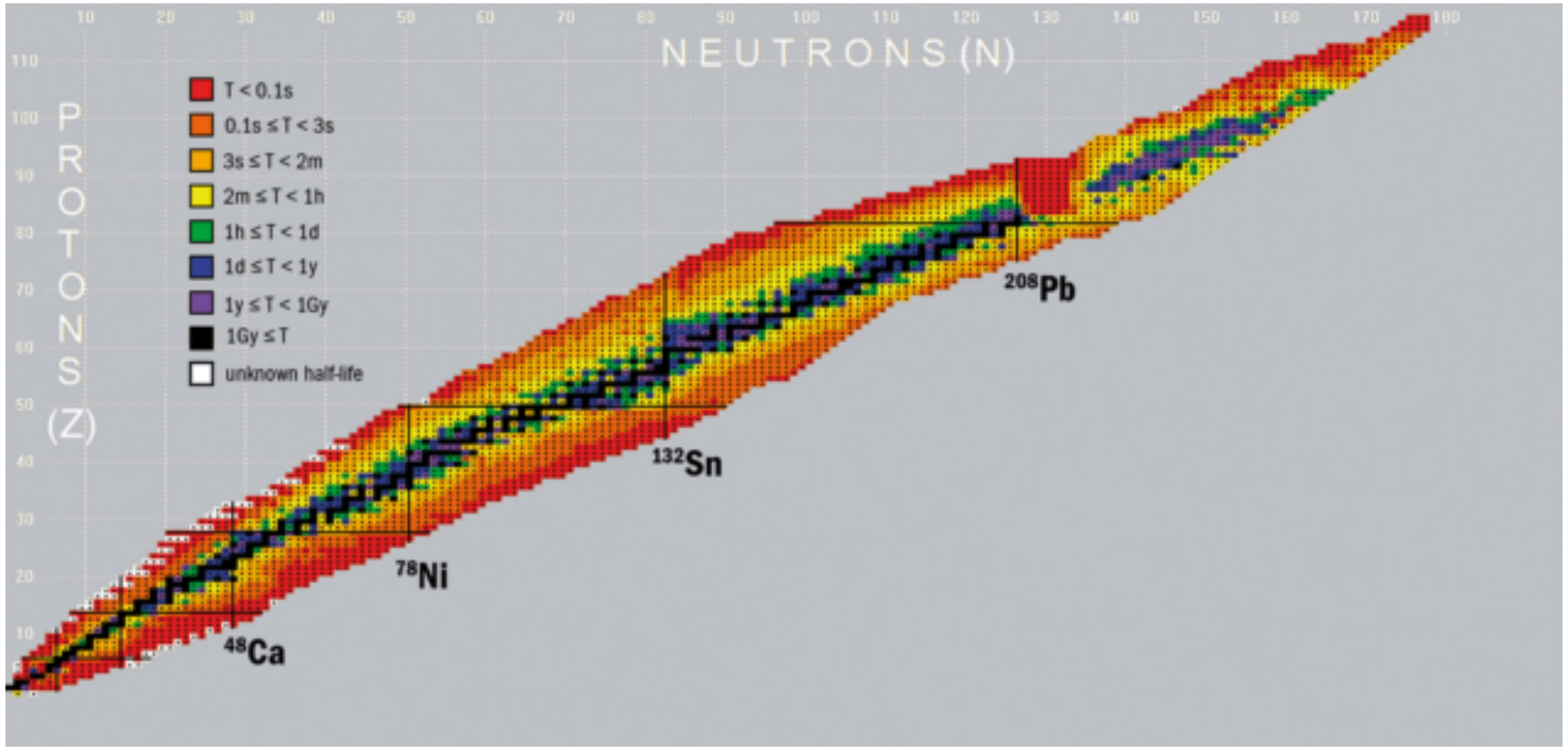
Bethe-Salpeter formalism



multiquarks

$$|M\rangle = |q\bar{q}\rangle + |q\bar{q}g > + \boxed{|q\bar{q}q\bar{q}\rangle} + \ldots$$

There are a lot of multiquarks!



Multi-quark States

"Vi har nu en model, der på smukke ste vis forklarer data og for første gang indeholder alle de begrænsninger, data giver," sagde fysikeren Tim Burns fra Swansea University ved offentliggørelsen.

Multi-electron States

1946: Wheeler suggests that Ps_2 might be bound

Wheeler, J. A. Polyelectrons. Ann. NY Acad. Sci. 48, 219–238 (1946).

1946: Ore proves it is unbound

1947: Hylleraas & Ore prove it is bound

Hylleraas, E. A. & Ore, A. Binding energy of the positronium molecule. Phys. Rev.71, 493–496 (1947).

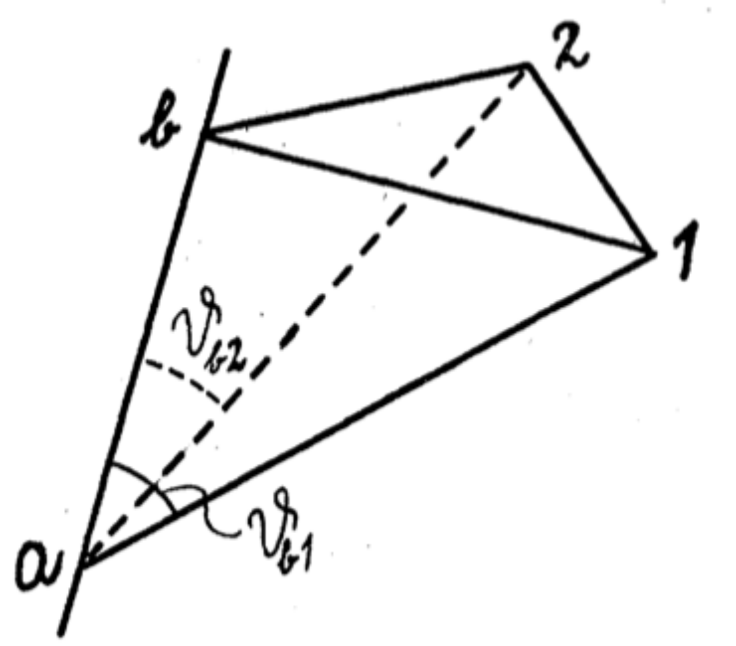


FIG. 1. Coordinate system for the positronium molecule.

2007: Ps_2 is observed

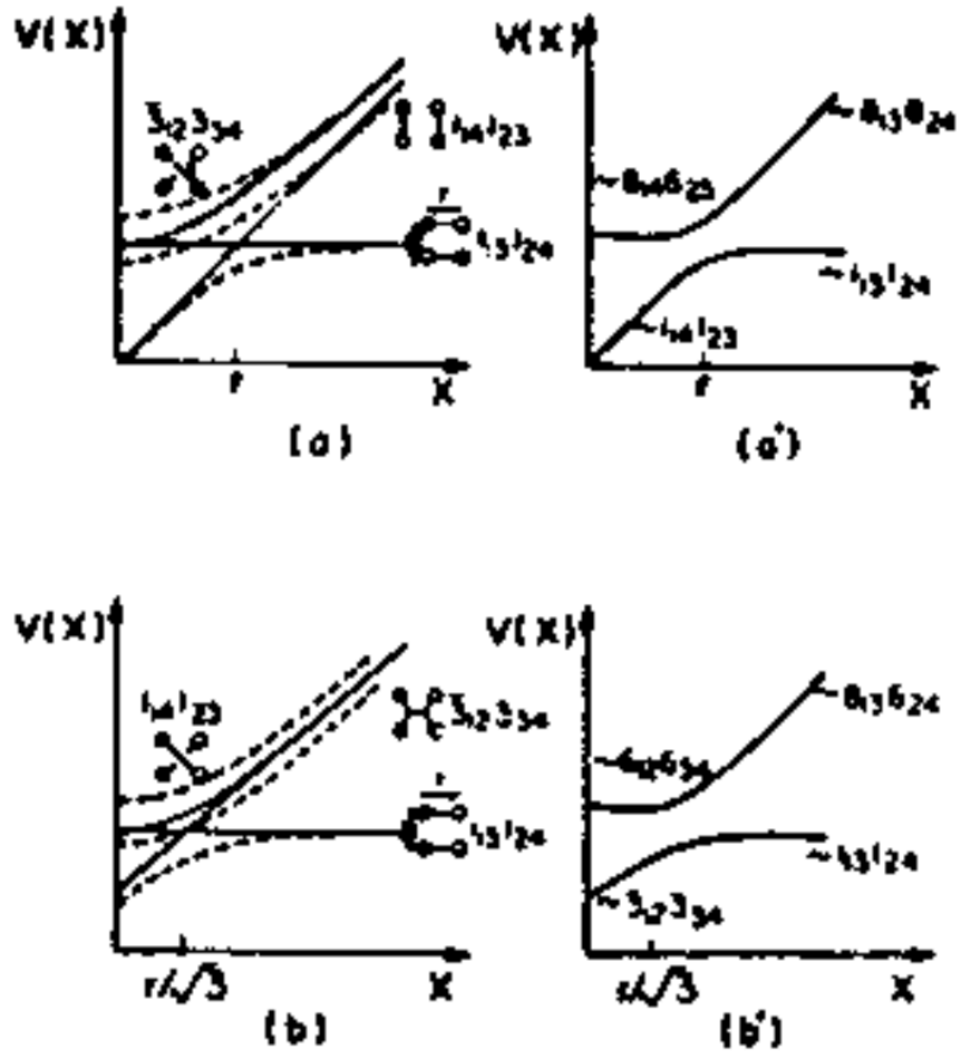
Cassidy, D.B.; Mills, A.P. (Jr.) (2007). "The production of molecular positronium". Nature 449 (7159): 195–197

Multi-quarks through the ages

B. The Multiquark Fiasco

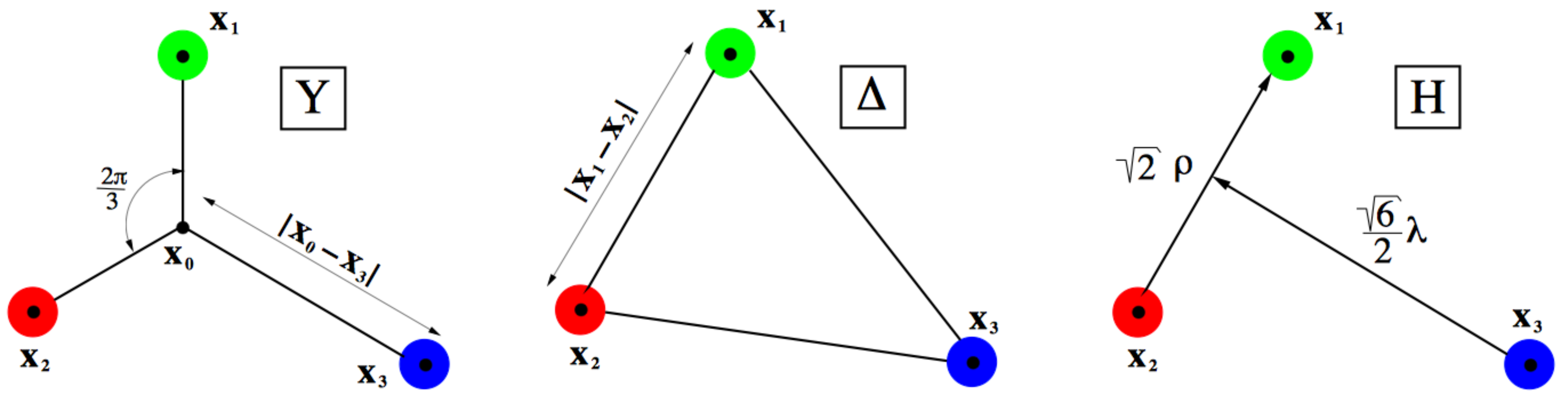
Multiquark physics has a somewhat unfortunate history. A confluence of dubious experimental results and dubious theoretical models in the late 1970's and early 1980's created, indeed, a multiquark fiasco. I am not competent to discuss what went wrong experimentally, but let me review the theoretical side of this fiasco in order to place it in perspective and thereby, I hope, point the way toward a better understanding of multiquark systems.

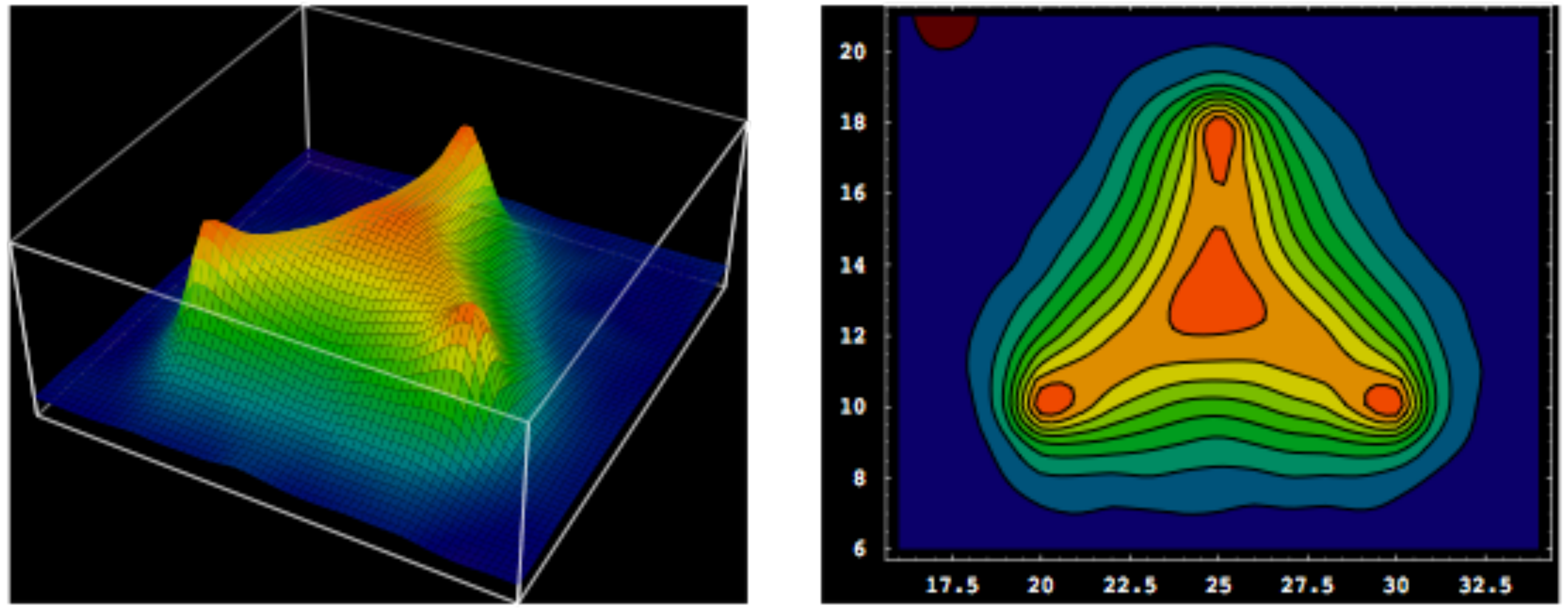
The story is basically one of throwing caution to the winds. Modelers from at least four different camps were, it seems to me, guilty:



UTPT-85-18
March,1985

What is the interaction?



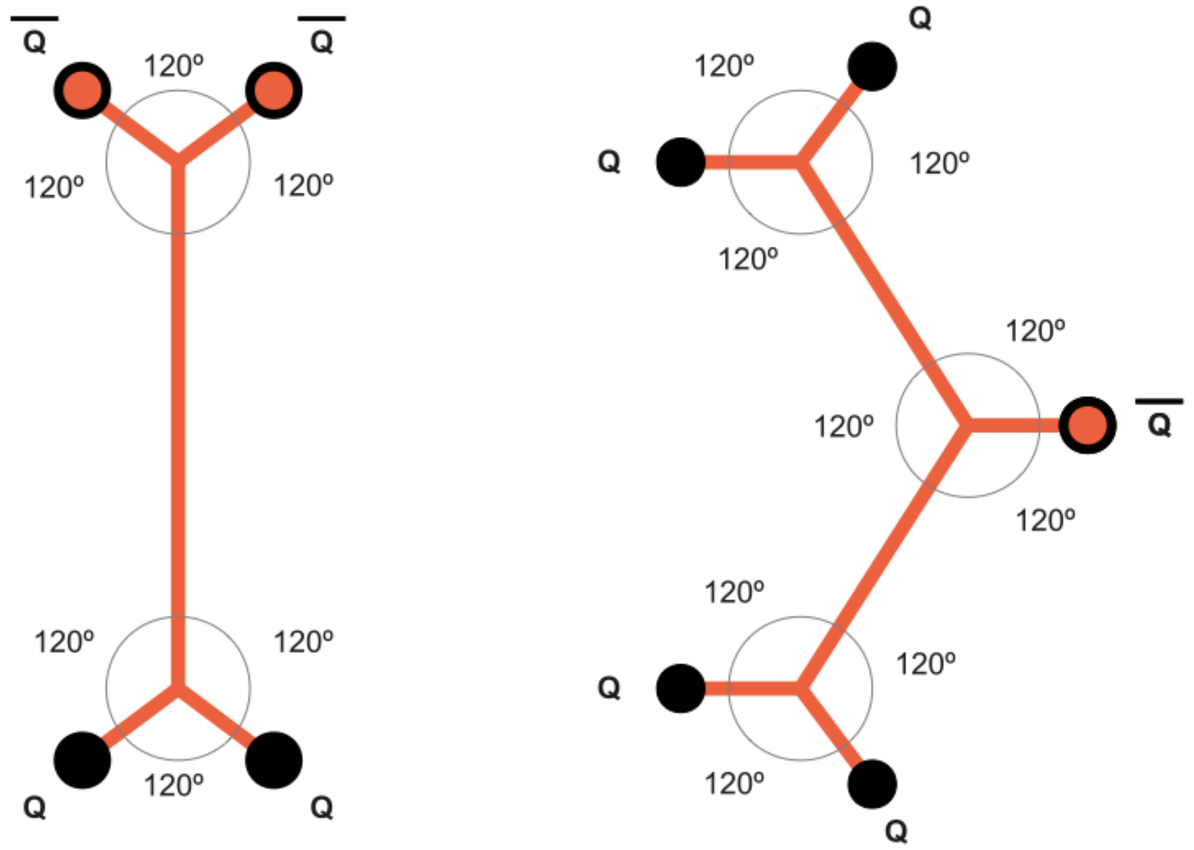


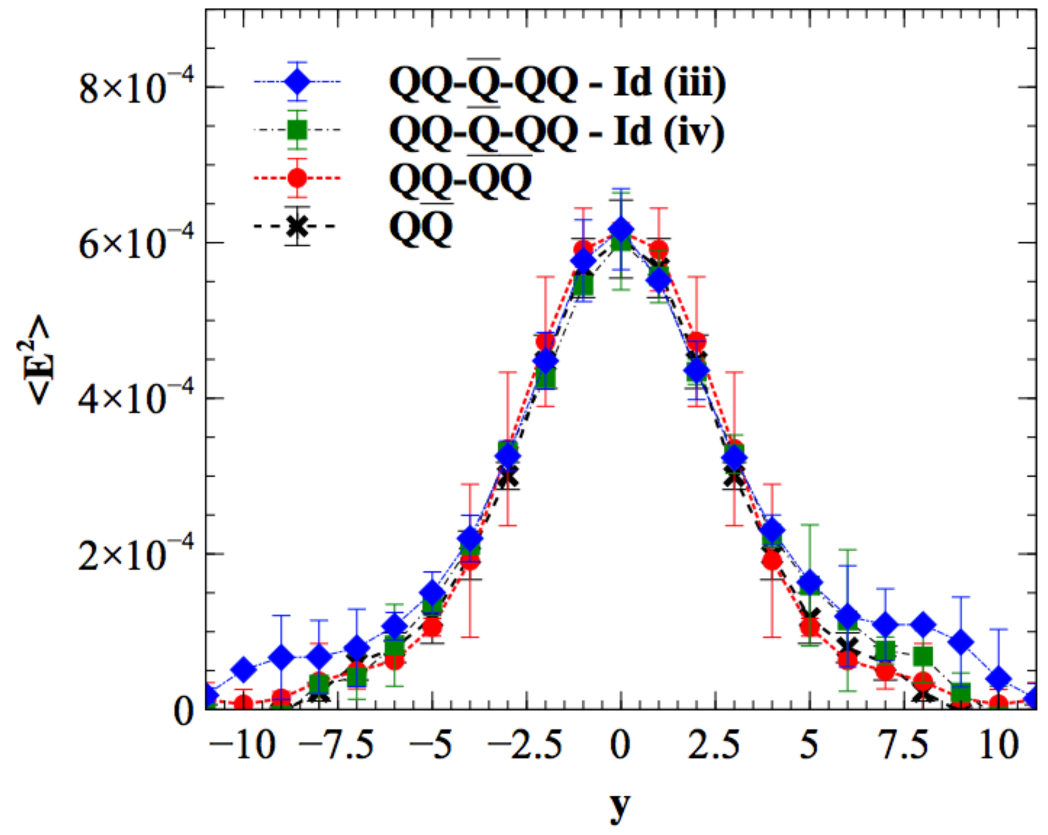
H. Ichie, V. Bornyakov, T. Streuer and G. Schierholz, "The flux distribution of the three quark system in SU(3)", arXiv:hep-lat/0212024.

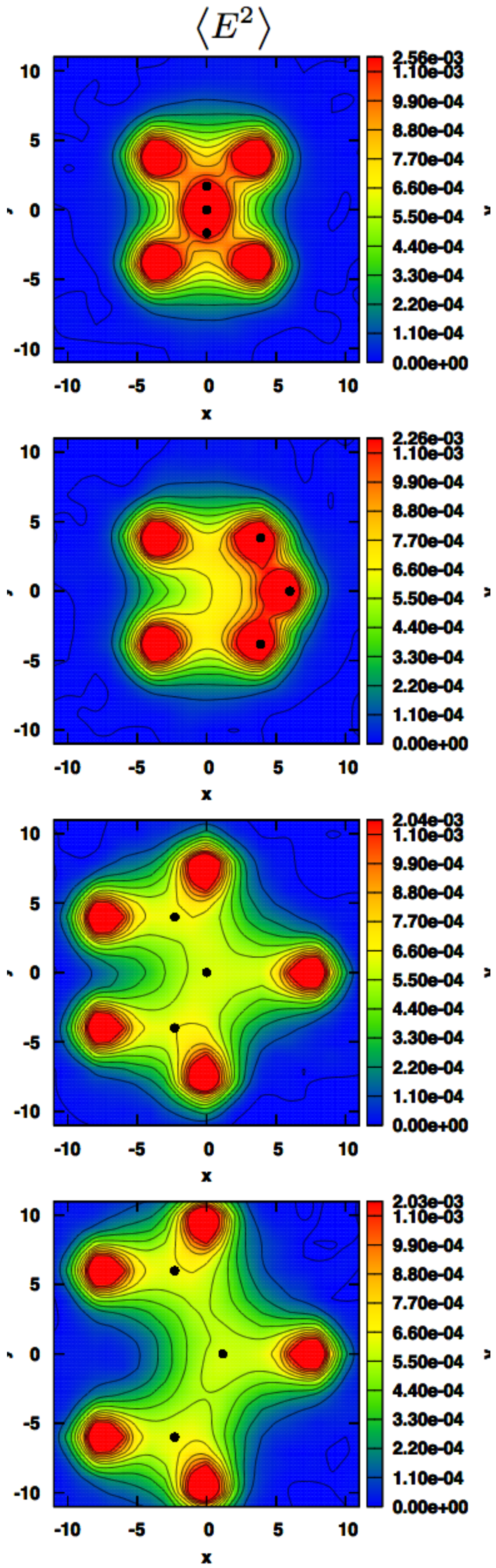
C. Alexandrou, P. De Forcrand and A. Tsapalis, Phys. Rev. D 65, 054503 (2002).

T. T. Takahashi, H. Suganuma, Y. Nemoto and H. Matsufuru, Phys. Rev. D 65, 114509 (2002).

What is the interaction?





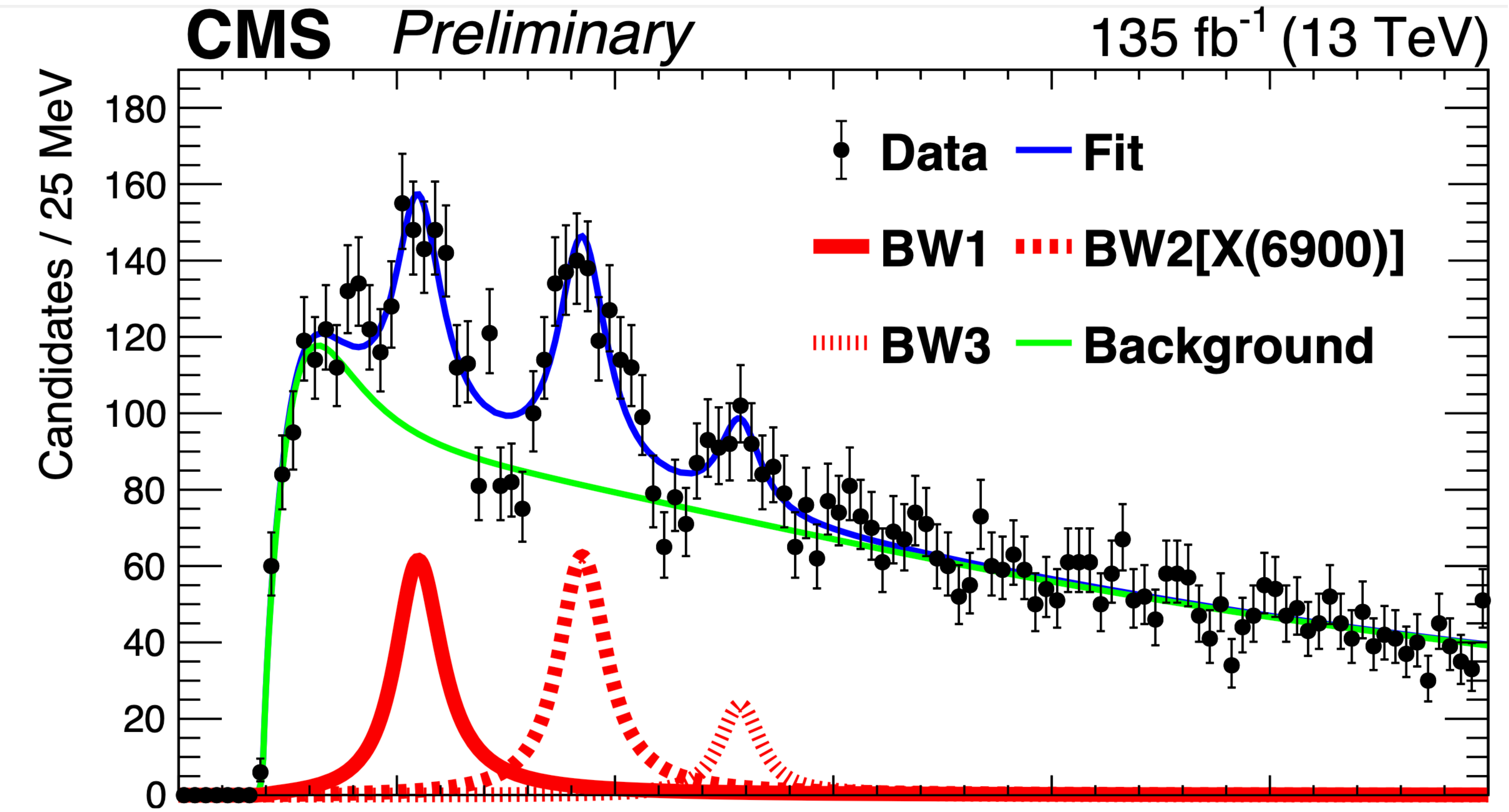


N. Cardoso and P. Bicudo,
`Color fields of the static pentaquark system computed in SU(3) lattice QCD,"
Phys. Rev. D {87}, no. 3, 034504 (2013)

S. Furui, A.M. Green and B. Masud,
``An analysis of four quark energies in SU(2) lattice Monte Carlo using the flux tube symmetry,"
Nucl. Phys. A {582}, 682 (1995)

an example application

Multiquark Exotics $cc\bar{c}\bar{c}$



https://cds.cern.ch/record/2815336

An Example Problem

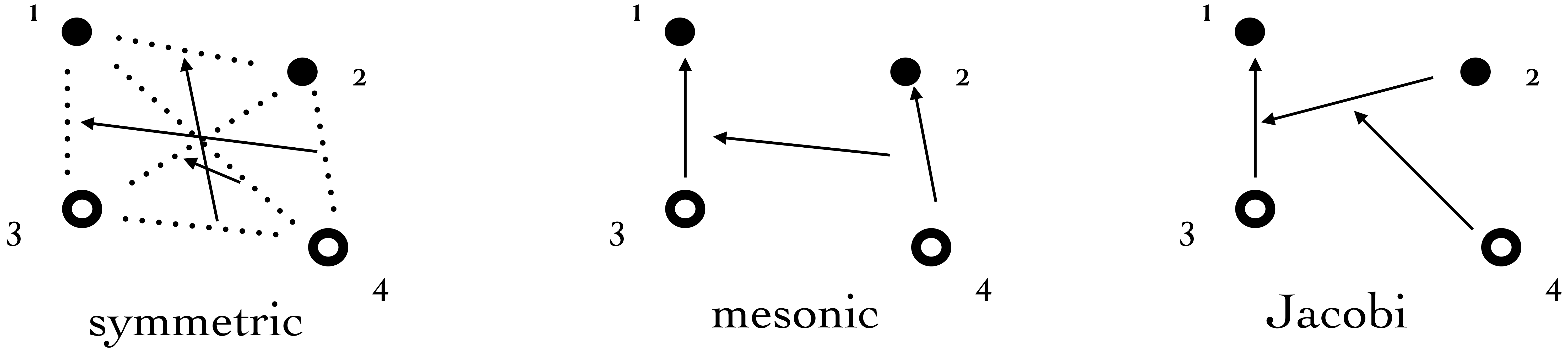
A simple nonrelativistic constituent model

$$H = -\sum_{i=1}^{4} \frac{\nabla_i^2}{2m_i} + \frac{1}{2}k \sum_{i<j} r_{ij}^2 \, \vec{\lambda}_i \cdot \vec{\lambda}_j$$

Possible colour states:

$$|1_{13}1_{24}\rangle, |1_{23}1_{14}\rangle \quad \text{or} \quad |\bar{3}3\rangle, |6\bar{6}\rangle \quad \text{or} \quad |1_{13}1_{24}\rangle, |8_{13}8_{24}\rangle.$$

Possible coordinate systems:



An Example Problem

Symmetric-3/6 form:

$$H = -\frac{1}{2}(\nabla_x^2 + \nabla_y^2 + \nabla_z^2) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\kappa}{2} \begin{pmatrix} 2x^2 + 2y^2 + \frac{4}{3}z^2 & -\sqrt{2}(x^2 - y^2) \\ -\sqrt{2}(x^2 - y^2) & x^2 + y^2 + \frac{10}{3}z^2 \end{pmatrix}$$

How does one solve this problem?

[We would like as full a spectrum as possible, including possible resonances.]

Computational Methods

Variational Resonating Group

J.A. Wheeler, PR52, 1107 (1937)
T. Kato, Prog Theor Phys 6, 394 (1951); L. Hulthen, Ark Mat Astr Fys A35, 25, (1948).

$$\psi = \sum_{c,I,S,\alpha,\beta} \mathscr{C}_c \phi_I \chi_S \, \psi_\alpha(\rho_1, \lambda_1) \, \psi_\beta(\rho_2, \lambda_2) \, F_{cIS\alpha\beta}(R)$$

$$\mathscr{L} u_L = \left(\frac{d^2}{dR^2} - \frac{L(L+1)}{R^2} + k^2 \right) u_L + \int W^{(L)}(R, R') \, u_L(R') \, dR' = 0$$

$$\delta \left(S_L + \frac{i}{2k} \int u_L \mathscr{L} u_L \right) = 0$$

$$u_L = \sum_i c_i \begin{cases} \gamma_i \chi_i^{(L)}, \; R < R_c \\ kR[h_L^{(-)}(kR) + s_i h_L^{(+)}(kR)], \; R > R_c \end{cases}$$

Computational Methods

Variational Resonating Group

Vary wrt the c_i to obtain
$$\sum_j \mathscr{M}_{ij} c_j = \mathscr{M}_i$$

$$\mathscr{M}_{ij} = K_{ij} - K_{i0} - K_{0j} + K_{00} \qquad \mathscr{M}_j = K_{00} - K_{j0} \qquad K_{ij} = \int \phi_i \mathscr{L}_L(\phi_j) dR$$

$$K_{ij} = \gamma_i \gamma_j \left[\int_0^{R_c} \chi_i(R) \left(\frac{d^2}{dR^2} - \frac{L(L+1)}{R^2} + k^2 \right) \chi_j(R) dR + \int_0^\infty \int_0^\infty \chi_i(R) W^{(L)}(R, R') \chi_j(R') \, dR \, dR' \right]$$

Choose the $\chi_i^{(L)}$ such that all integrals can be done analytically. Solve the linear equations, evaluate the functional at the stationary point, and obtain S.

Computational Methods

Lanczos Algorithm

1. Let $v_1 \in \mathbb{C}^n$ be an arbitrary vector with Euclidean norm 1.
2. Abbreviated initial iteration step:
   1. Let $w_1' = Av_1$.
   2. Let $\alpha_1 = w_1'^* v_1$.
   3. Let $w_1 = w_1' - \alpha_1 v_1$.
3. For $j = 2, \ldots, m$ do:
   1. Let $\beta_j = \|w_{j-1}\|$ (also Euclidean norm).
   2. If $\beta_j \neq 0$, then let $v_j = w_{j-1}/\beta_j$,

      else pick as v_j an arbitrary vector with Euclidean norm 1 that is orthogonal to all of $v_1, \ldots, v_j$
   3. Let $\boldsymbol{w_j' = Av_j}$.
   4. Let $\boldsymbol{\alpha_j = w_j'^* v_j}$.
   5. Let $w_j = w_j' - \alpha_j v_j - \beta_j v_{j-1}$.
4. Let V be the matrix with columns $v_1, \ldots, v_m$. Let $T = \begin{pmatrix} \alpha_1 & \beta_2 & & & & 0 \\ \beta_2 & \alpha_2 & \beta_3 & & & \\ & \beta_3 & \alpha_3 & \ddots & & \\ & & \ddots & \ddots & \beta_{m-1} & \\ & & & \beta_{m-1} & \alpha_{m-1} & \beta_m \\ 0 & & & & \beta_m & \alpha_m \end{pmatrix}$.

[work with a discrete (grid) basis -> simple to evaluate $H|\varphi\rangle$!]

[overcomes stability problems(!)]

Computational Methods

Lanczos Algorithm

Hydrogen naive 3d approach

```
./BoxHydrogen –N 320 –x 14    [33M x 33M !!]
       [slow convergence in Lanczos iterations...]

295: -0.4990305889 -0.1245724433 -0.0326180510
296: -0.4990305889 -0.1245748623 -0.0327418135
297: -0.4990305889 -0.1245771746 -0.0328625434
298: -0.4990305889 -0.1245793873 -0.0329804546
299: -0.4990305889 -0.1245815035 -0.0330955698
```

Helium   naive 6d Cartesian discretization

```
./BoxHeliumC –N 26 –x 12   (309M^2)

37: -1.9144083947 -1.4707632674 -1.3666560990
38: -1.9144083947 -1.4707925755 -1.3677339189
39: -1.9144083947 -1.4708129613 -1.3686194909
```

run into trouble as we need a larger grid

Computational Methods

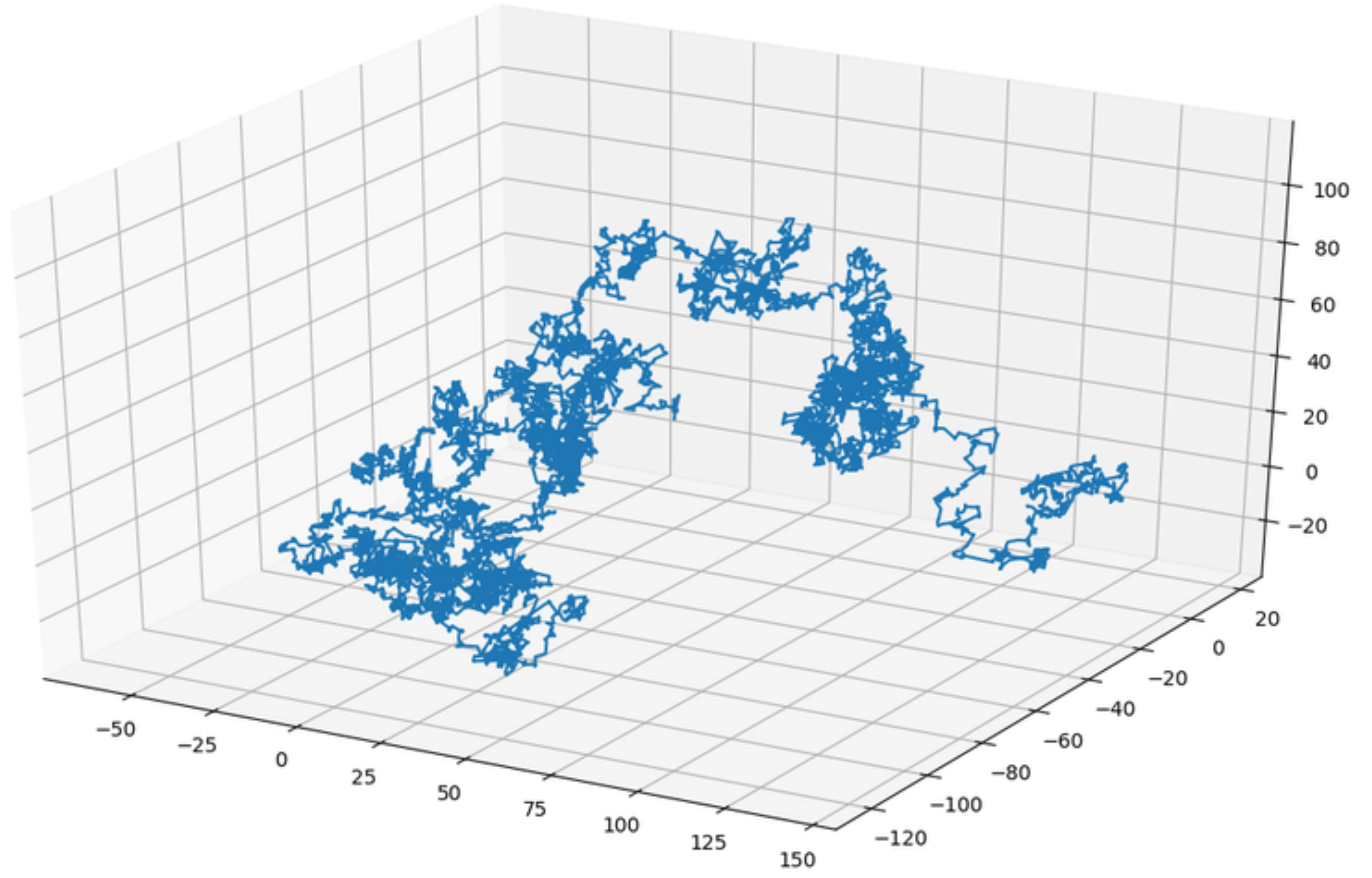
Guided Random Walks

T. Barnes, G.J. Daniell ,and D.Storey, Nucl. Phys. B265 [FS15] (1986) 253.
T. Barnes and E.S. Swanson, PRB37, 9405 (1988)
T. Barnes, F.E. Close, E.S Swanson, PRD52, 5242 (1995)

A simple and more direct version of the Greens function quantum Monte Carlo method, leveraging the mapping between random walks and the Euclidean time Schrödinger equation

Computational Methods

Guided Random Walks



$$-\dot{\psi}(x') = \langle x'|H\psi|x'\rangle = \langle x'|H_0|x'\rangle\,\psi(x') + \sum_{x\neq x'}\langle x'|H_I|x\rangle\,\psi(x)$$

Associate a weight with each walk

$$w(\tau) = \exp[-\int_0^\tau a(\tau)d\tau]\,.$$

$$-\dot{Q}(x',\tau) = \left[\sum_{x\neq x'} r(x,x') + a(\tau)\right] Q(x',\tau) - \sum_{x\neq x'} r(x,x')Q(x,\tau)$$

$$r(x \to x') = -\,\langle x'|H_I|x\rangle\frac{\varphi(x')}{\varphi(x)}$$

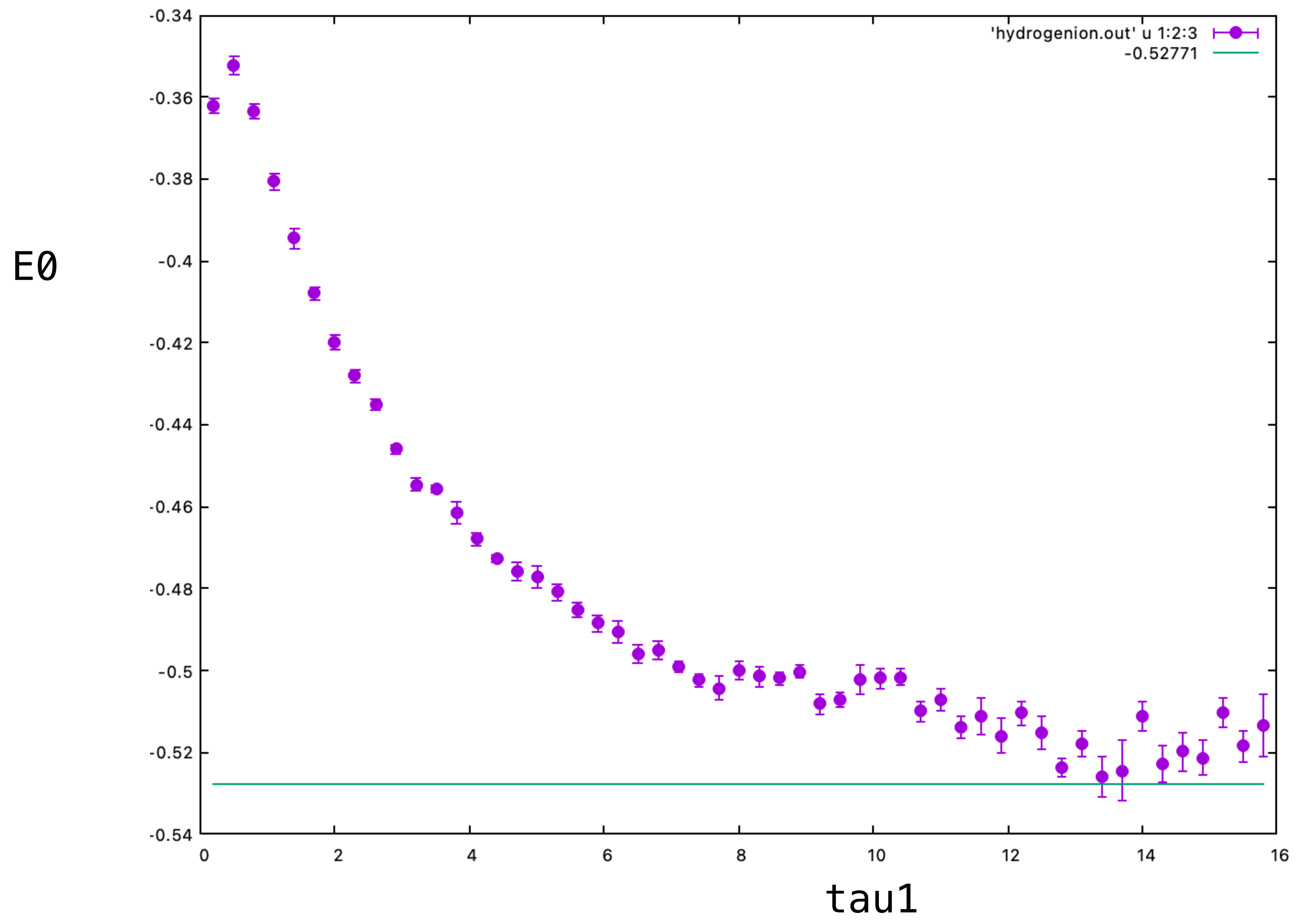
$$w_{trans} = \Pi_{x\to x'}\left[-\frac{\langle x'|H_I|x\rangle}{r(x,x')}\right]$$

Walk to large Euclidian time and extract the energy.

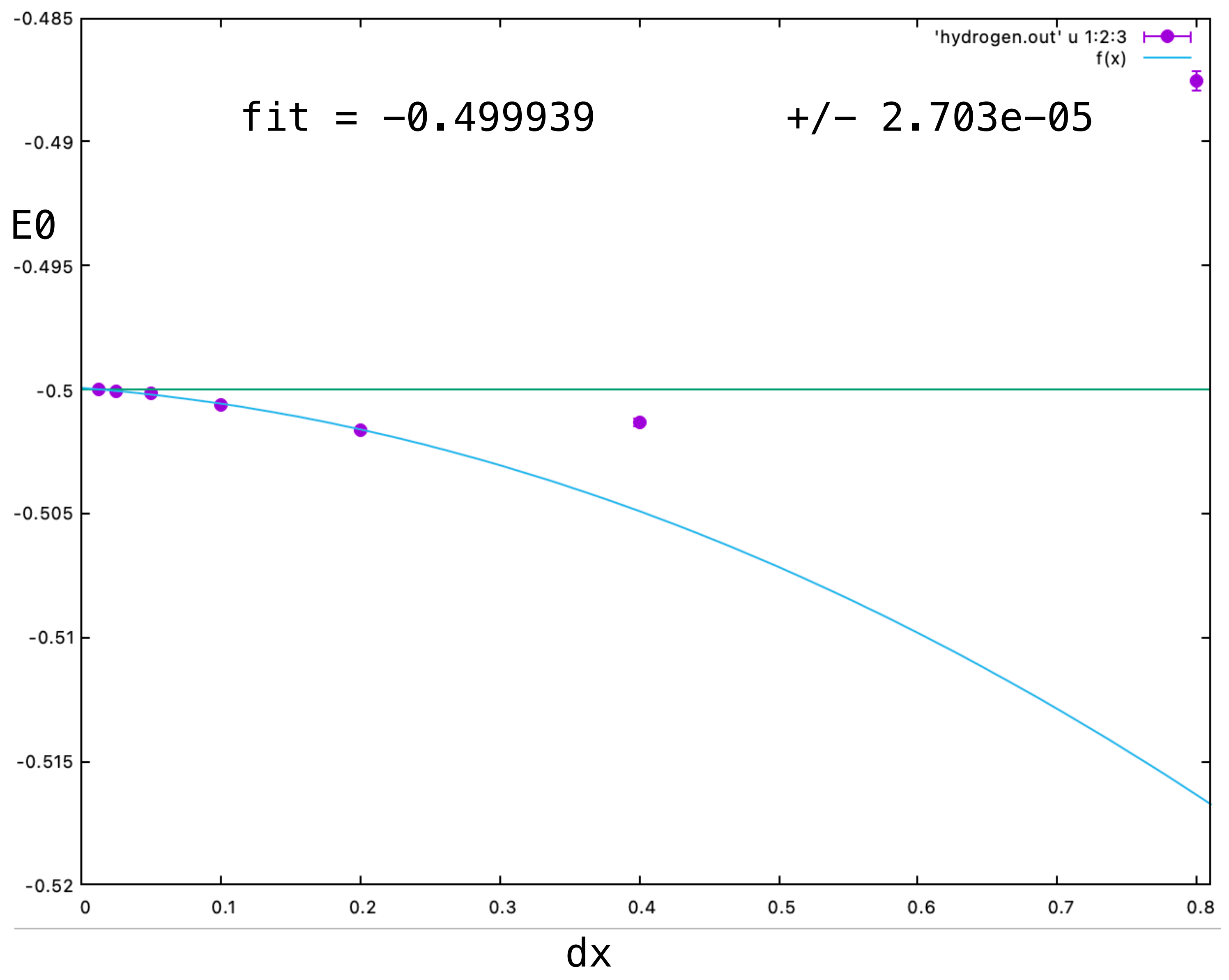
Computational Methods

Guided Random Walks

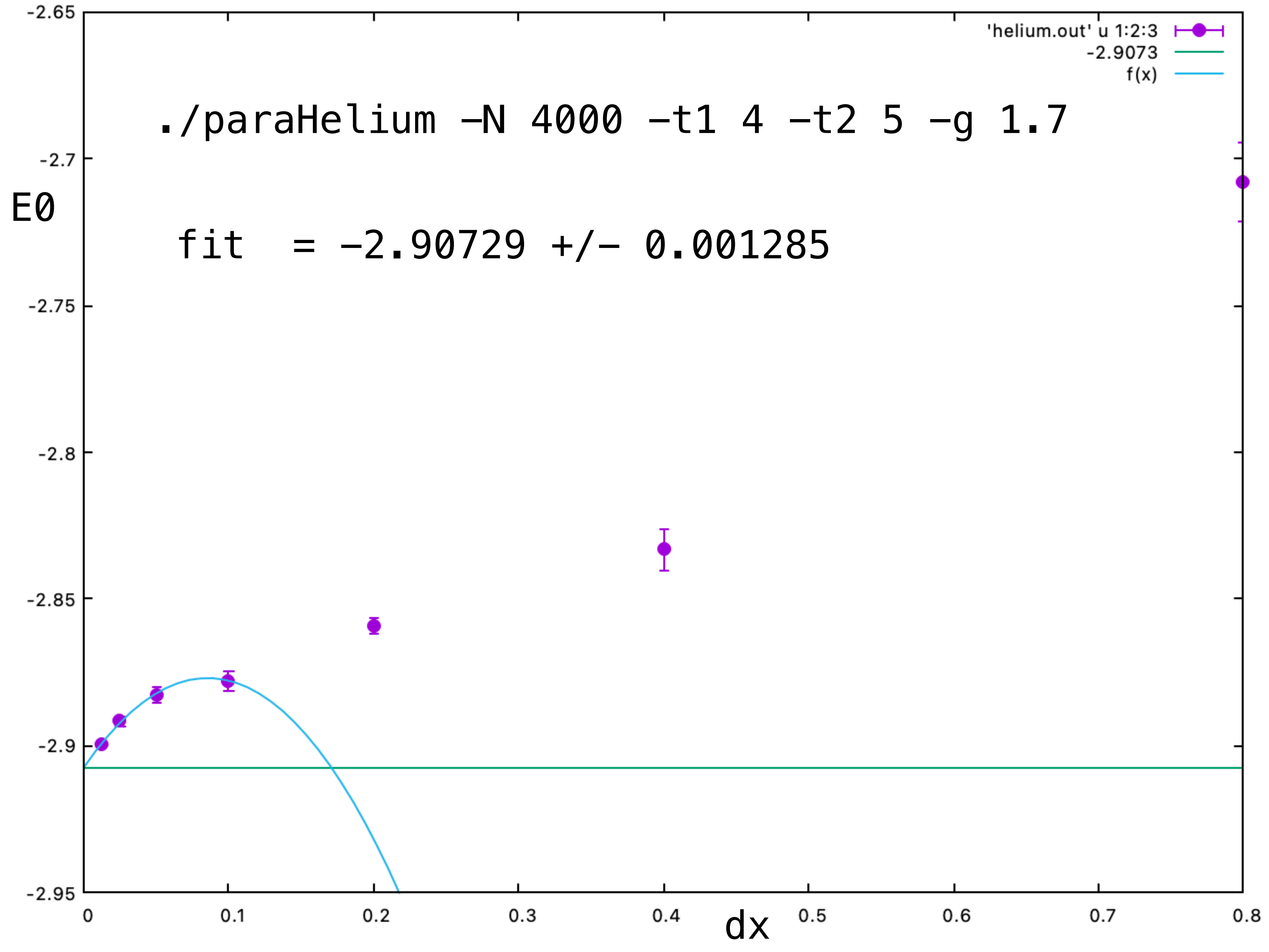
hydrogen ion



hydrogen



para helium, E0 = -2.9073



Computational Methods

Guided Random Walks

Ps2

```
SWANSONE@PHYAST-LWQG1H7VT0 GRW % time ./Ps2 -N 9000 -g .4 -t1 14 -t2 15 -d .1
-0.51502 +/- 0.00903268
./Ps2 -N 9000 -g .4 -t1 14 -t2 15 -d .1  317.81s user 0.28s system 688% cpu 46.204 total

SWANSONE@PHYAST-LWQG1H7VT0 GRW % time ./Ps2 -N 9000 -g .4 -t1 14 -t2 15 -d .05
-0.520122 +/- 0.0102292
./Ps2 -N 9000 -g .4 -t1 14 -t2 15 -d .05  1273.68s user 1.35s system 715% cpu 2:58.29 total

SWANSONE@PHYAST-LWQG1H7VT0 GRW % time ./Ps2 -N 9000 -g .4 -t1 14 -t2 15 -d .025
-0.513579 +/- 0.00678113
./Ps2 -N 9000 -g .4 -t1 14 -t2 15 -d .025  4996.82s user 11.28s system 735% cpu 11:21.18 total
```

Computational Methods

Variational

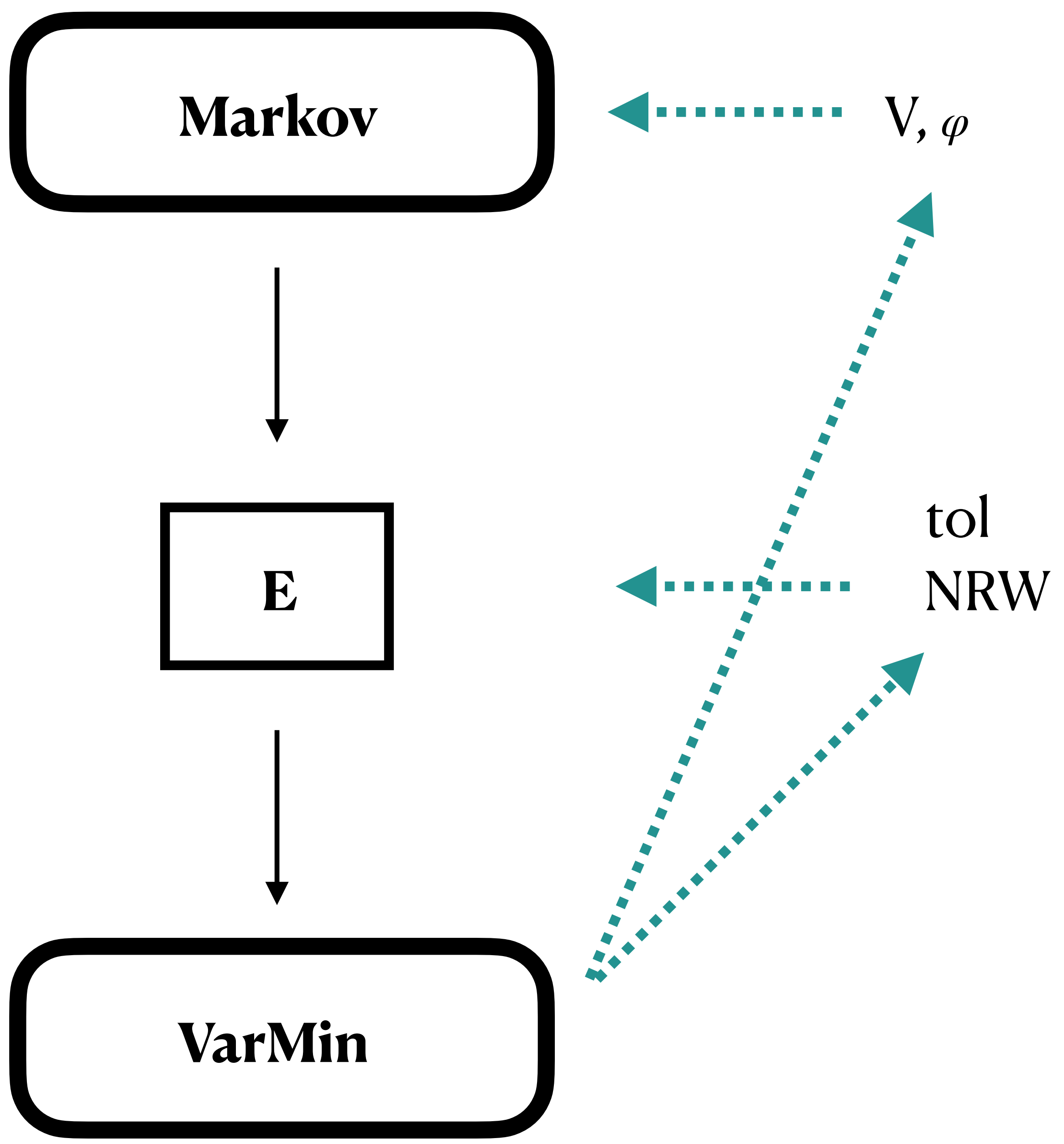
Once again, the grid is our friend

$$E_0 \sim \sum_{iC} |\psi_{iC}|^2 \cdot \left[\sum_{C'} V_i^{CC'} \frac{\psi_{iC'}}{\psi_{iC}} + \frac{2}{2m\delta^2} - \frac{1}{2m\delta^2} \frac{\psi_{i+1C}}{\psi_{iC}} - \frac{1}{2m\delta^2} \frac{\psi_{i-1C}}{\psi_{iC}} \right]$$

- evaluate with adaptive parallelized MCMC
- minimize the Anzatz with a variety of schemes (simulated annealing -> simplex walk -> differential evolution)
- minimizing a function with errors? -> adjust MCMC statistics as required as the minimum is approached
- user interface requires V and the Ansatz

Computational Methods

Variational



Ps2

Eo[Suzuki]= -0.51600

```
VAR % ./Ps2Var
 fcn eval time (ms): 102


 test eval -0.494706
 *** resetting NRW to 40000 [0.0728424| 0.00226158]
quick scan: -0.500108 +/- 0.00185215
 params: 0.52 0.07
 number of function evaluations: 80

 *** resetting NRW to 40000 [0.138703 | 0.00438151]
anneal: -0.500108 +/- 0.00185215
 params: 0.52 0.07
 number of function evaluations: 401

 *** resetting NRW to 40000 [0.14088| 0.00377591]
diff evo: -0.499023 +/- 0.00129373
 params: 0.517881 0.0758662
 number of function evaluations: 79

 *** resetting NRW to 40000 [0.247881| 0.00477687]
simplex: -0.499023 +/- 0.00129373
 params: 0.517881 0.0758662
 number of function evaluations: 6

 final min: -0.499023 +/- 0.00129373

 high stats estimate -0.500641 +/- 0.000200992
```

<< 3%

Computational Methods

Complex Scaling

The idea:

"complexify" coordinates:

$$U(\theta) r U^{\dagger}(\theta) = r \exp(i\ \theta)$$
$$U(\theta) p U^{\dagger}(\theta) = p \exp(-i\ \theta)$$

to reveal poles of the Schrödinger equation

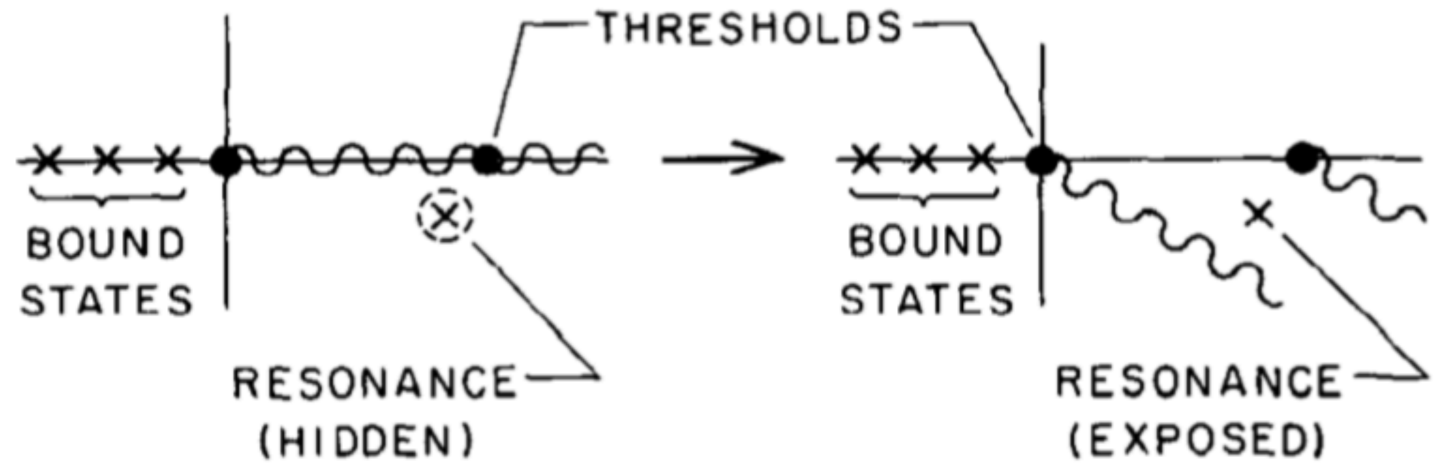
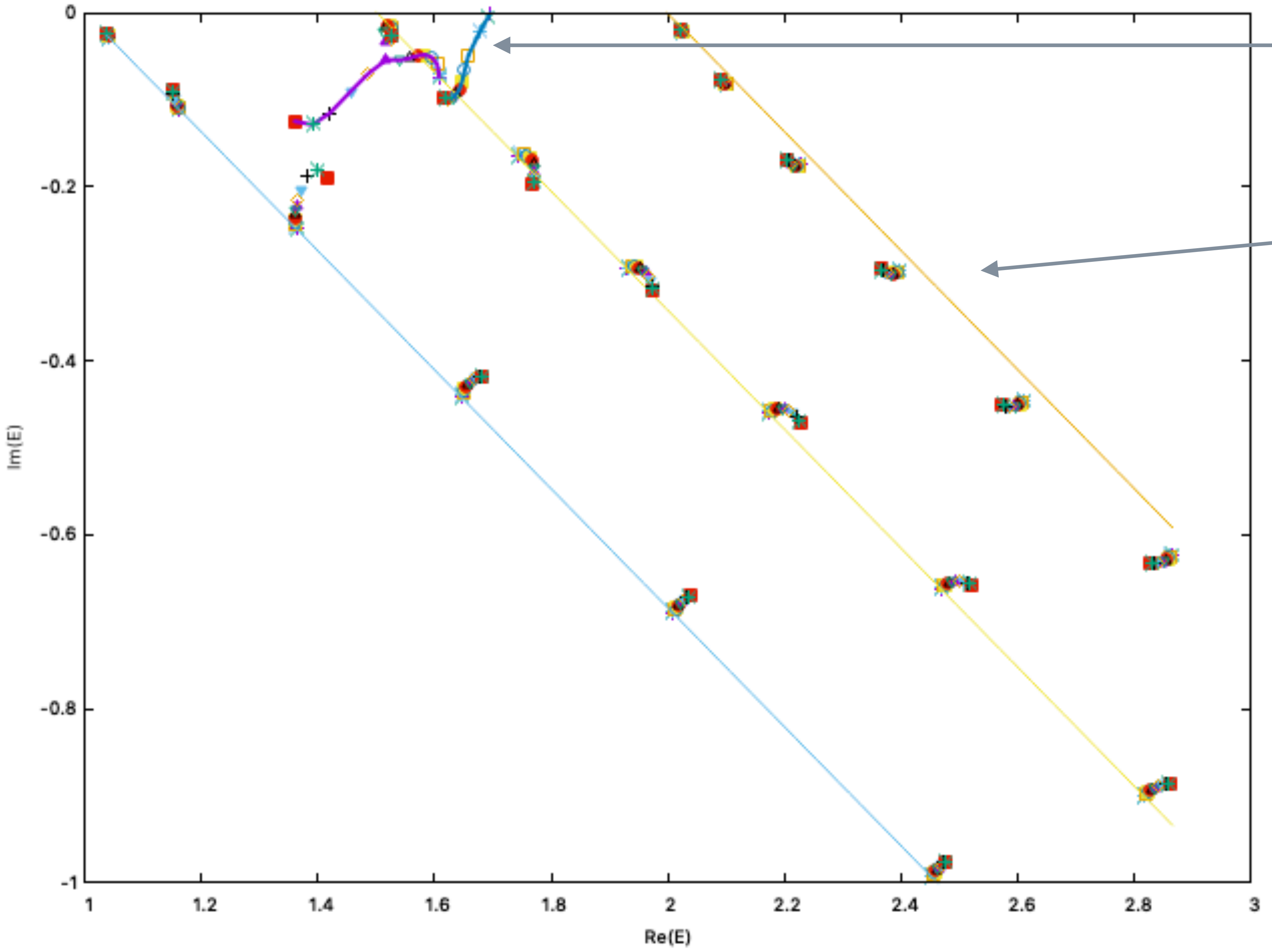


Figure 3 Effect of dilatation transformation on a many-body Hamiltonian. Again bound states and thresholds are invariant. However, as the continua rotate, complex resonance eigenvalues may be exposed. Such eigenvalues correspond to poles of the resolvent $R_{\phi}(z)$, but are "hidden" on a higher sheet if $\theta = 0$, and will be exposed if the cuts are appropriately moved.

Computational Methods

Complex Scaling

Ex. Poles in a three-channel model



resonance

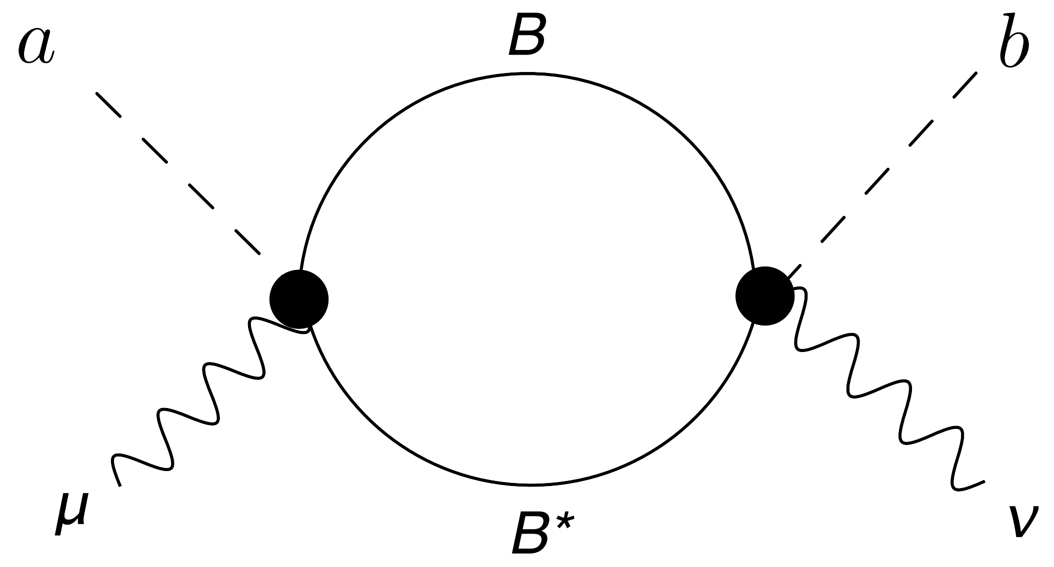
continuua

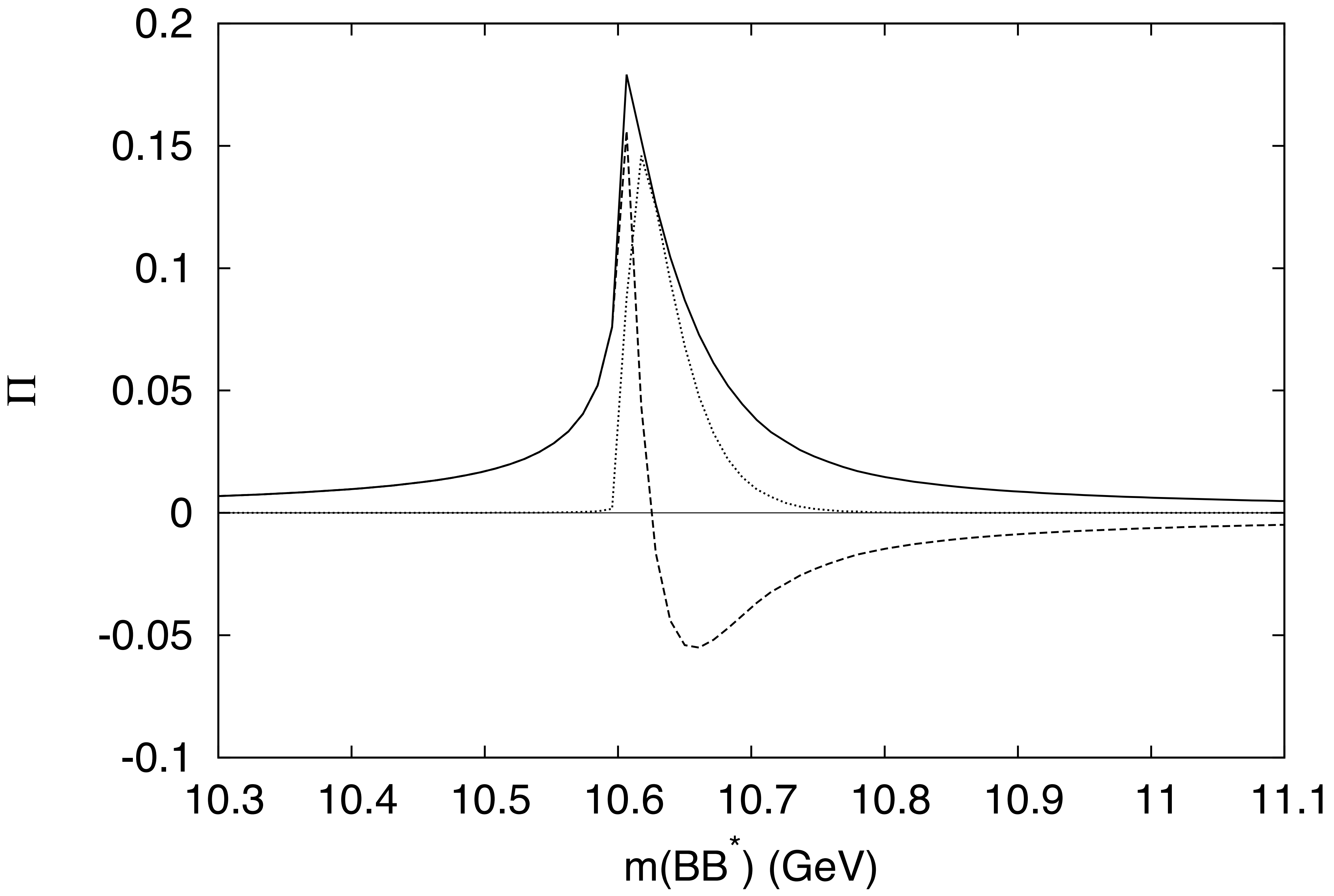
$$\mathbb{V} = \begin{pmatrix} 0 & 0 & \gamma \\ 0 & 0 & \gamma \\ \gamma & \gamma & -1.8 \end{pmatrix}$$

Real Scaling, System-in-a-box, Luescher,...

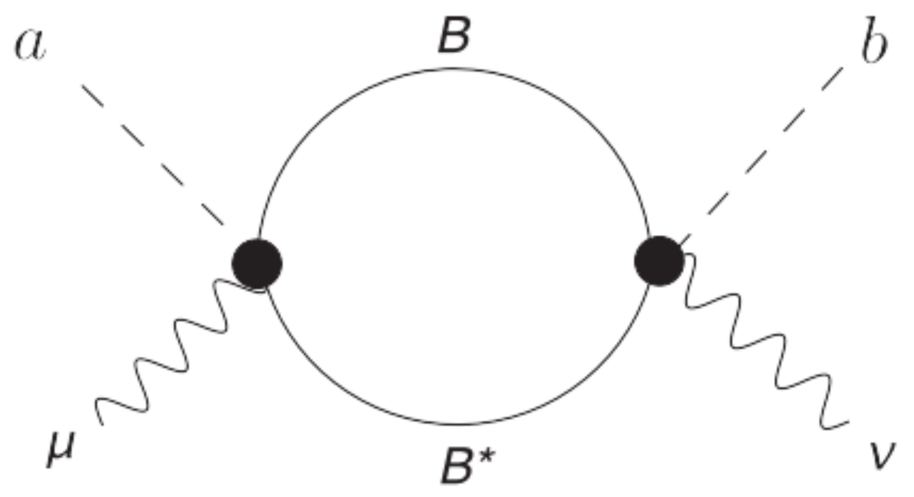
other things that can happen

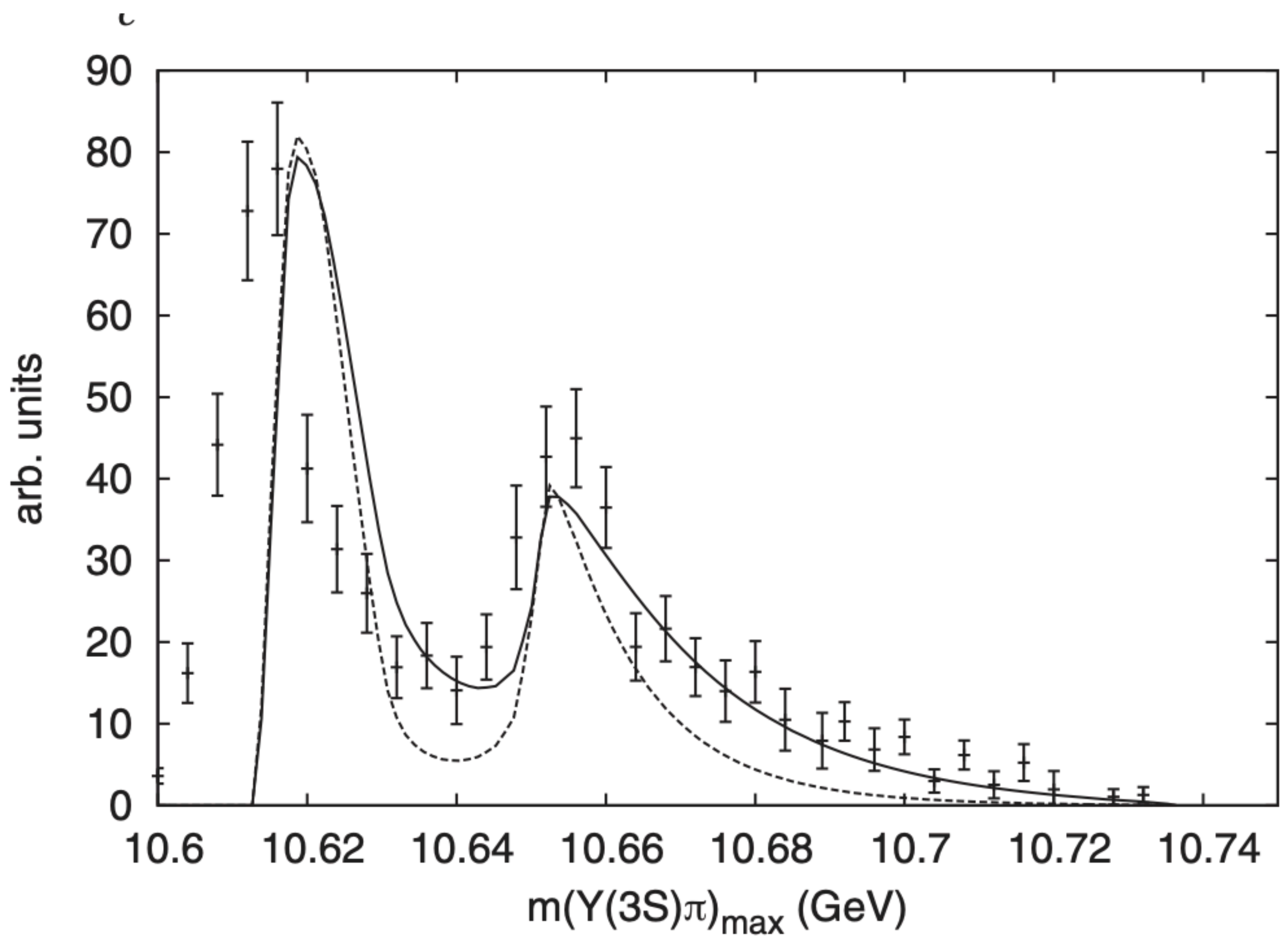
Loop diagrams can have sharp features





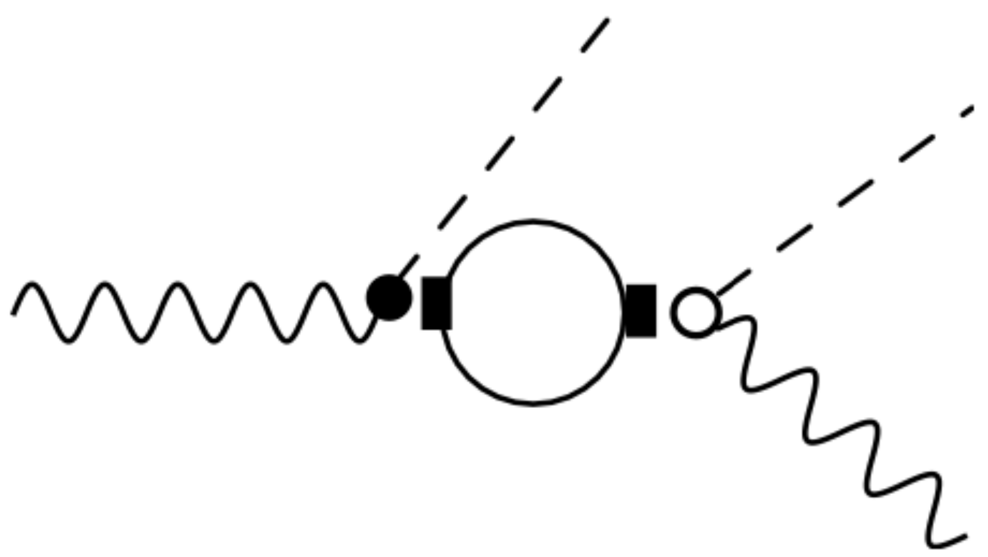
$\Upsilon(5s) \to \Upsilon(3S)\pi\pi$





Swanson, PRD91 034009 (2015)

$$Y \to \pi\pi J/\psi(h_c)$$



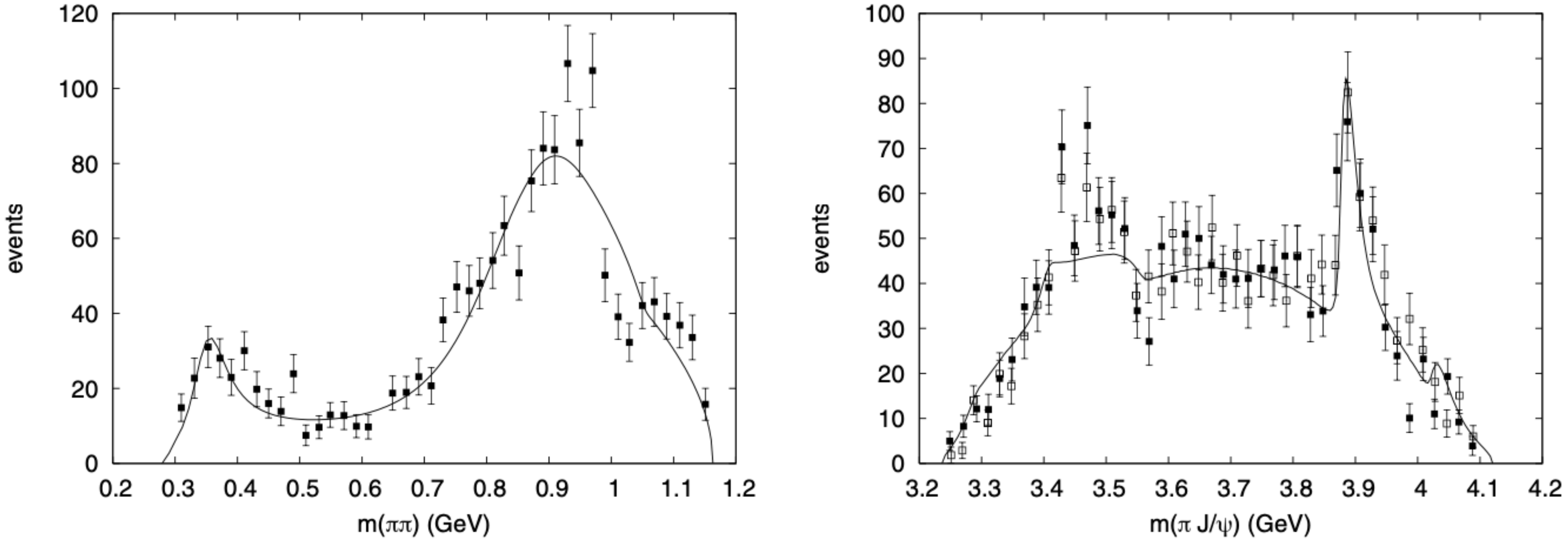


Fig. 4. $e^+e^-(\sqrt{s} = 4.26\,\text{GeV}) \to \pi\pi J/\psi$. Left panel: invariant $\pi\pi$ mass distribution. Right panel: invariant $\pi J/\psi$ mass distribution. Filled squares: $\pi^- J/\psi$; open squares: $\pi^+ J/\psi$. Data from Ref. 7.

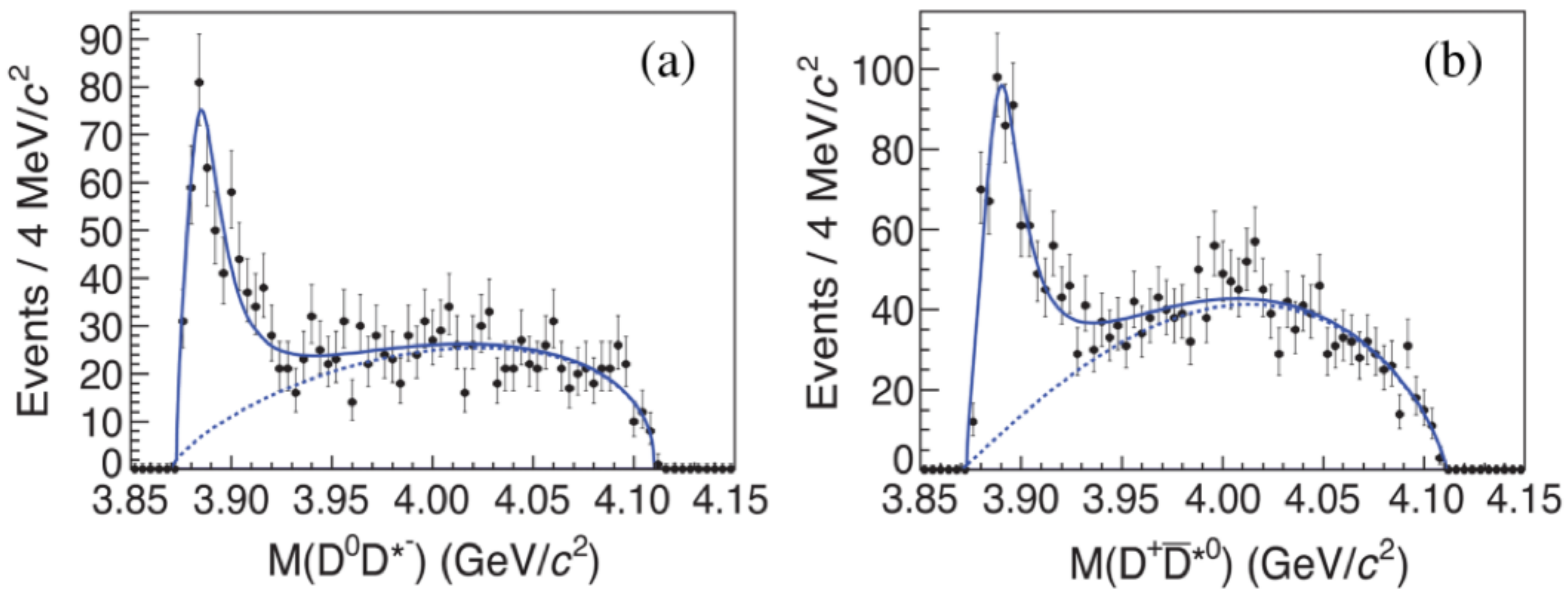
Swanson, IJMP E25, 1642010 (2016)

Zc(3900)

$$e^+e^- \to \pi D\bar{D}^* \qquad \sqrt{s} = 4.26$$

$$M = 3883.9 \pm 1.5 \pm 4.2$$

$$\Gamma = 24.8 \pm 3.3 \pm 11.0$$



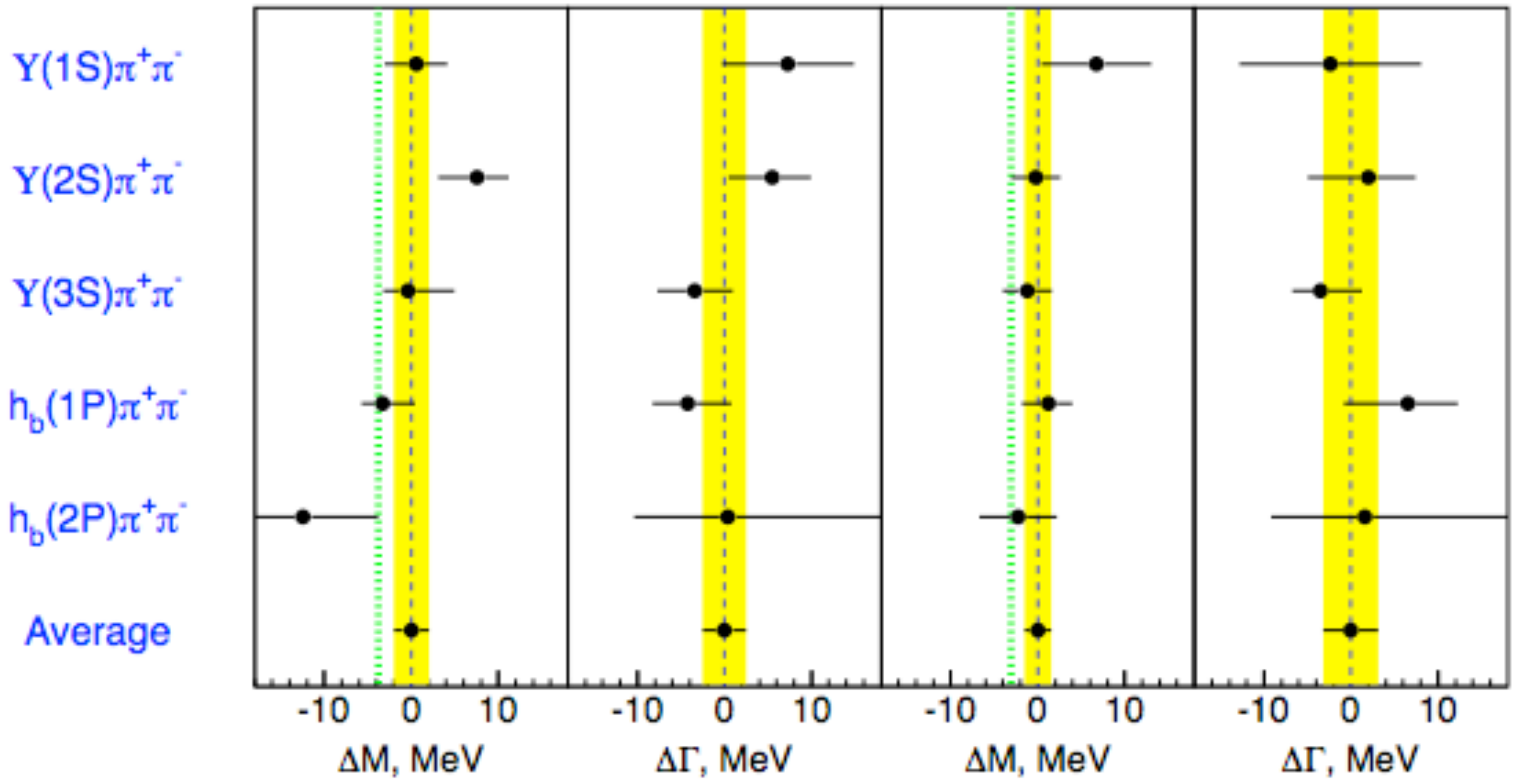
BESIII PRL112 022001 (14)

$Z_b^+(10610) \quad Z_b^+(10650)$

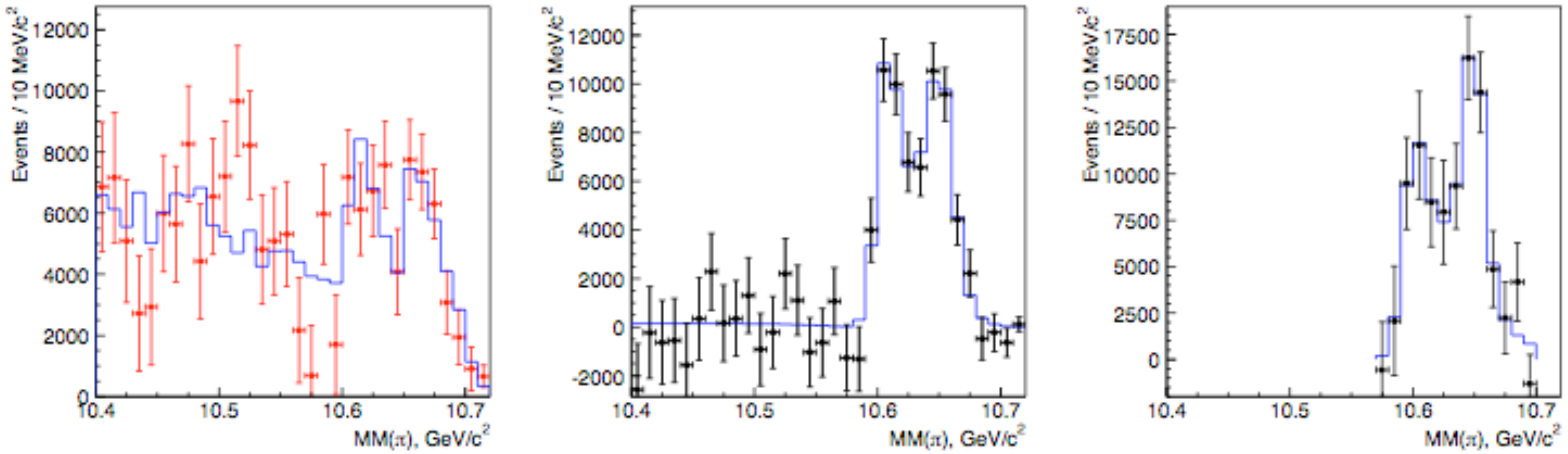
Adachi et al. [Belle] 1105.4583

$$I^G J^P = 1^+ 1^+$$

$B^* \bar{B}$ $B^* \bar{B}^*$

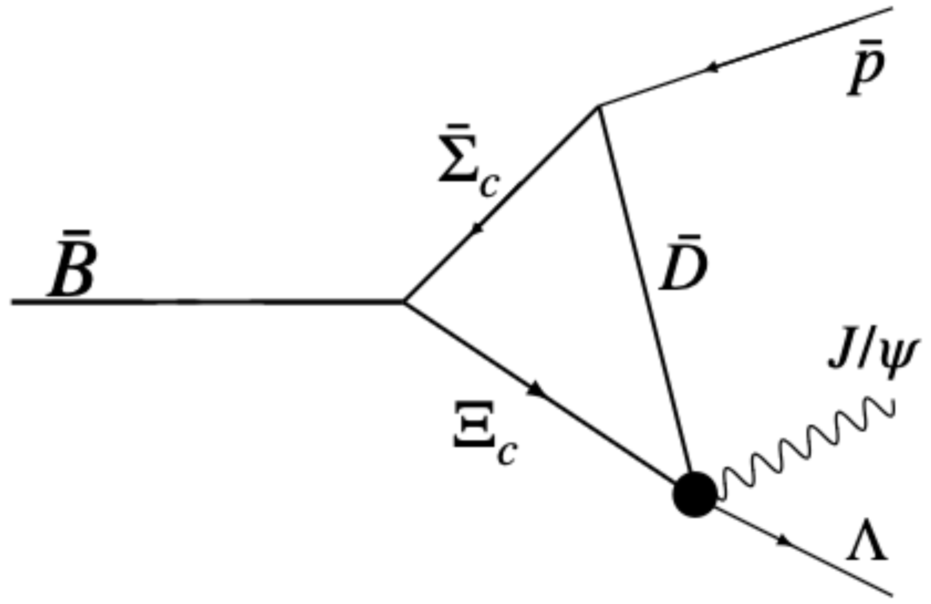


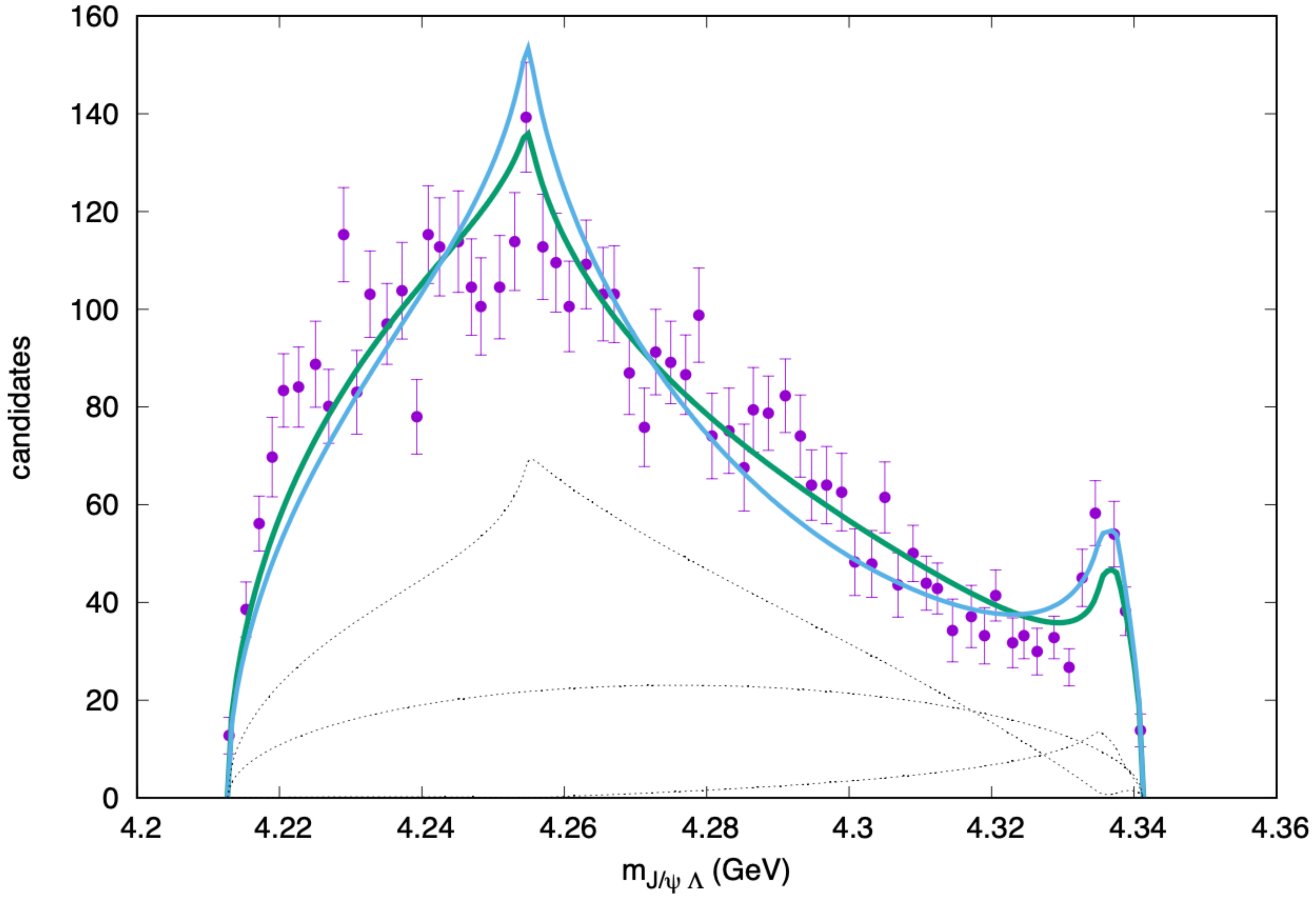
$\Upsilon(2S)$ $h_b(1P)$ $h_b(2P)$



The 'triangle' diagram has a log singularity in certain kinematical regions.

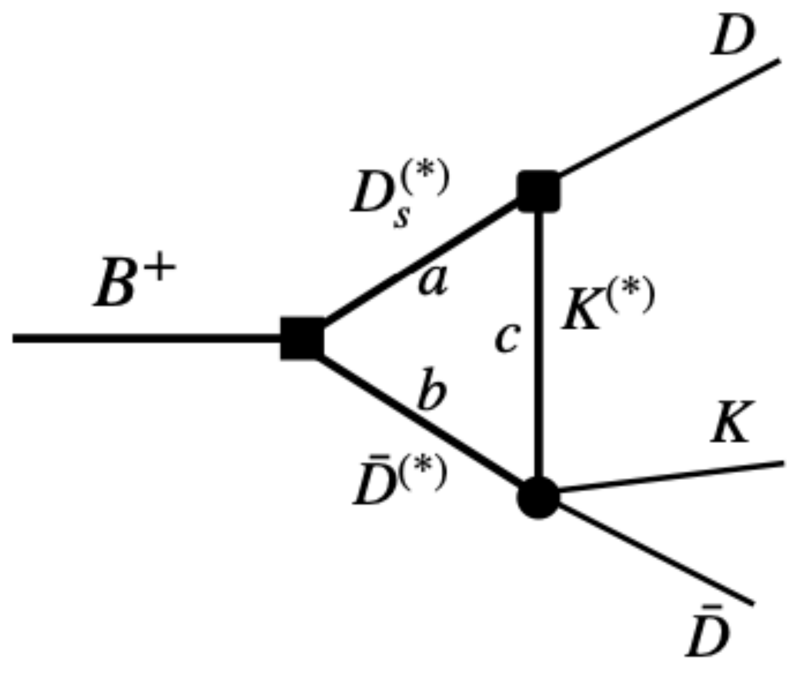
LHCb state $P_{\psi s}^{\Lambda}(4338)$

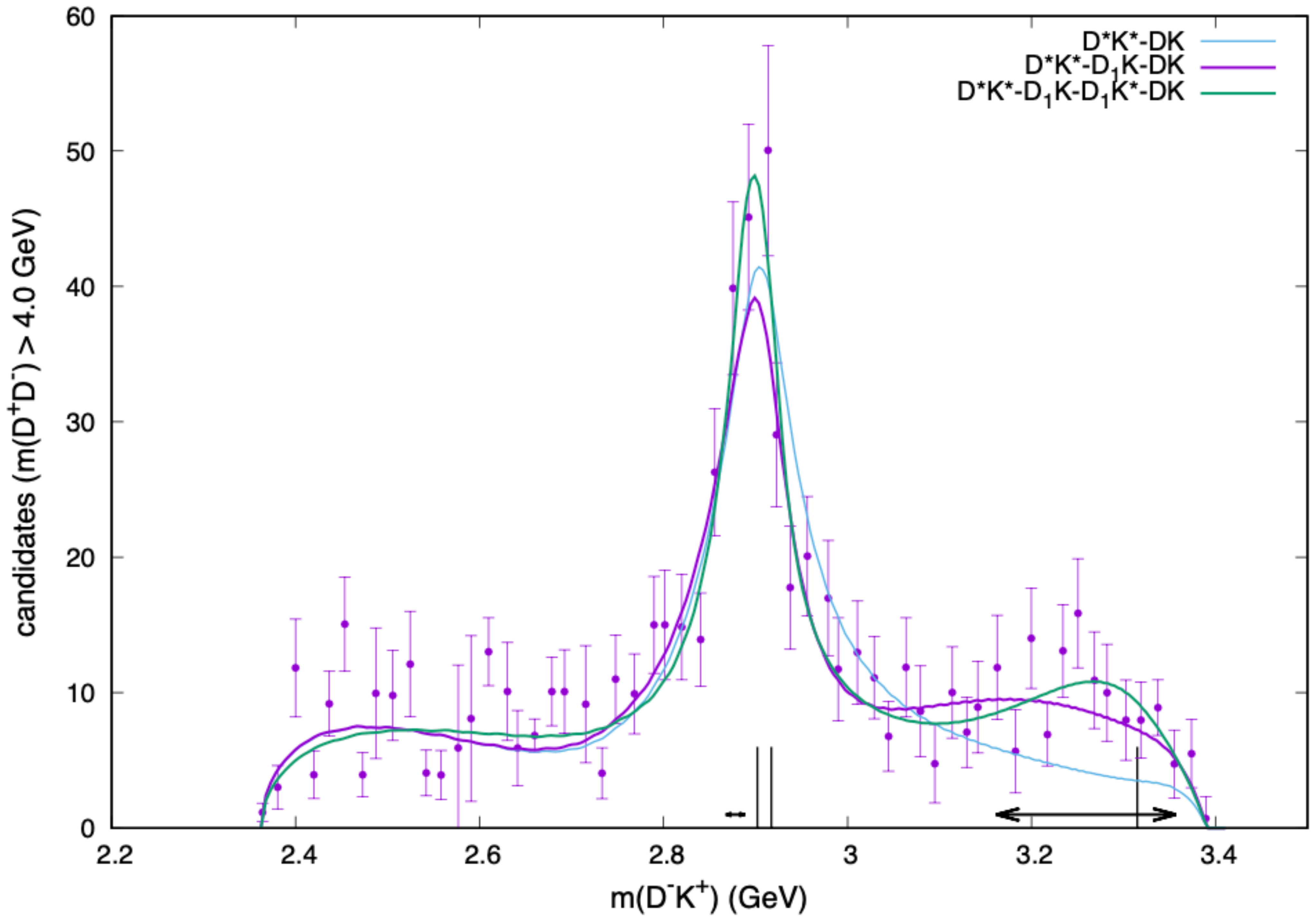




Burns & Swanson, PLB 838, 137715 (2023)

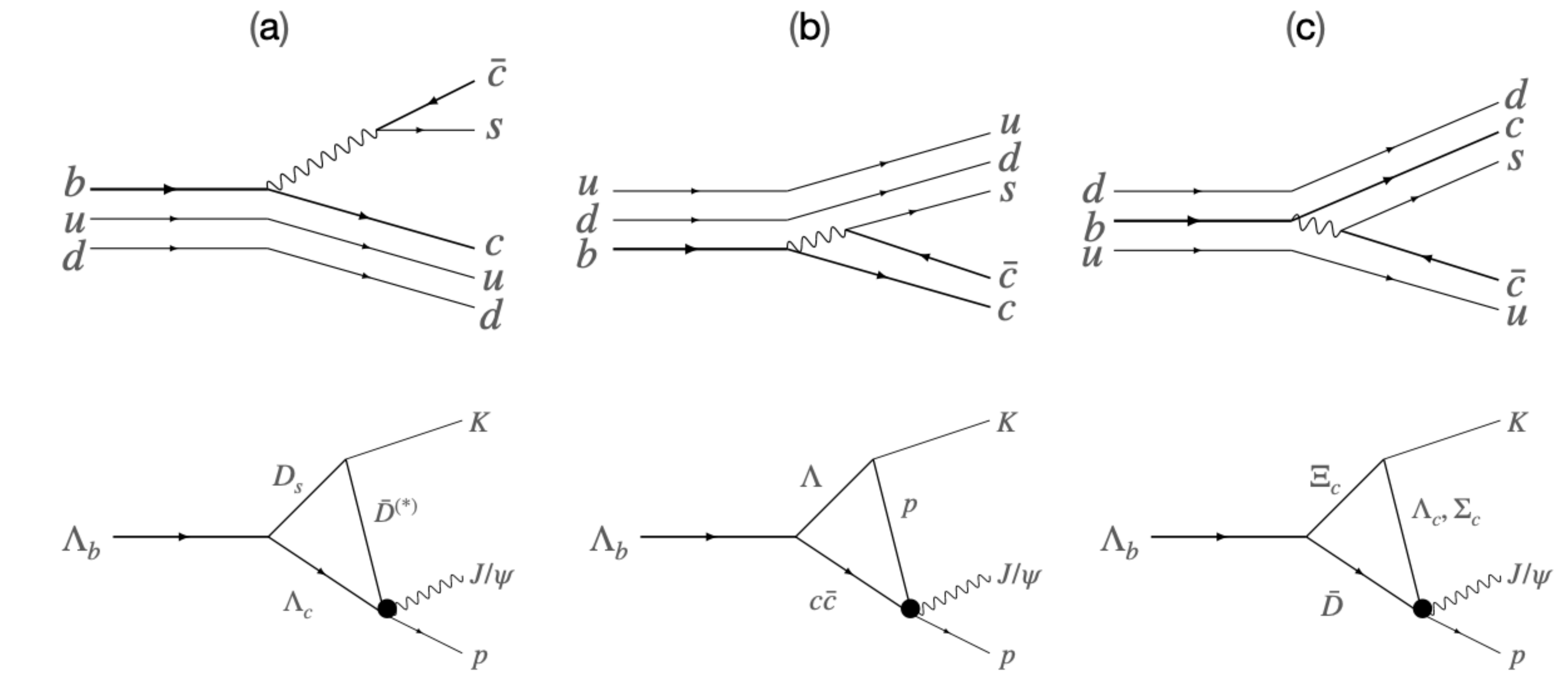
LHCb X(2900) $ud\bar{c}\bar{s}$



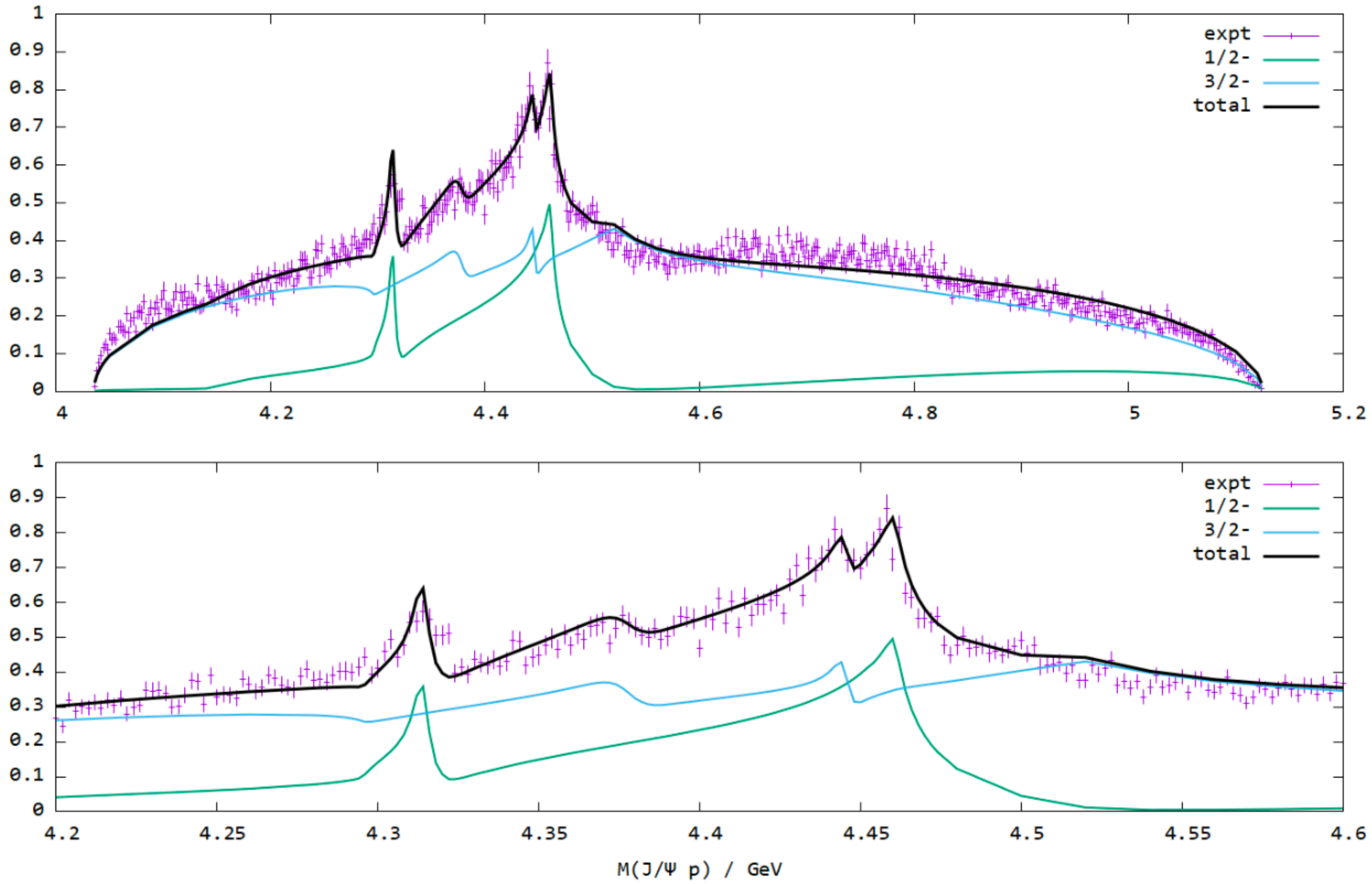


Burns & Swanson, PLB 813, 136057 (2021)

LHCb Pentaquark states $P_c(4312)$, $P_c(4380)$, $P_c(4440)$



	(a)	(b)	(c)
Λ_b vertex:	Large	Small	Small
P_c vertex:	Large	Small	Large



Burns & Swanson, PRD106, 054029 (2022)

What are the common features here?

- "state" lies just above thresholds
- S-wave quantum numbers
- tree level production is suppressed (eg, colour suppressed electroweak transitions)
- widths will depend on channel

Conclusions

We are fortunate to have QCD to guide the search for effective degrees of freedom and their interactions.

But this is not enough. Lattice field theory helps determine the *emergent* degrees of freedom and properties.

Old ideas (bag, flux tube) are giving way to approaches that are closer to QCD (Bethe-Salpeter, many-body, EFT, dispersion relations, Born-Oppenheimer).

The role of non-resonant interactions in generating amplitude "structure" is a recent (~2010) development.

The role of coupled channels has long been appreciated and still awaits a definitive treatment.

✢ ÆRIC MEC HEHT GEWYRCAN

Diquarks and the New Charmonia

Maiani, Piccinini,Polosa, Riquer; PRD71, 014028 (2005)

Bigi, Maiani, Piccinini,Polosa, Riquer; PRD72, 114016 (2005)

Maiani, Riquer, Piccinini, Polosa; PRD72, 031502 (2005)

Maiani, Polosa, Riquer; PRL99, 182003 (2007)

Maiani, Polosa, Riquer; arXiv:0708.3997

$$M([cq]_S) = 1933$$
$$M([cq]_V) = 1933$$

Assume a spin-spin interaction

$$
\begin{aligned}
|0^{++}\rangle &= |[cq]_S[\bar{c}\bar{q}]_S; J=0\rangle &(1)\\
|0^{++\prime}\rangle &= |[cq]_V[\bar{c}\bar{q}]_V; J=0\rangle &(2)\\
|1^{++}\rangle &= \frac{1}{\sqrt{2}}\left(|[cq]_S[\bar{c}\bar{q}]_V; J=1\rangle + |[cq]_V[\bar{c}\bar{q}]_S; J=1\rangle\right) &(3)\\
|1^{+-}\rangle &= \frac{1}{\sqrt{2}}\left(|[cq]_S[\bar{c}\bar{q}]_V; J=1\rangle - |[cq]_V[\bar{c}\bar{q}]_S; J=1\rangle\right) &(4)\\
|1^{+-\prime}\rangle &= |[cq]_V[\bar{c}\bar{q}]_V; J=1\rangle &(5)\\
|2^{++}\rangle &= |[cq]_V[\bar{c}\bar{q}]_V; J=2\rangle &(6)
\end{aligned}
$$