$\eta \to 3\pi$ and the light quark mass determination

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Modern Techniques in Hadron Spectroscopy
Ruhr University Bochum, Germany, July 15 -27, 2024

Based on *Phys. Rev. Lett. 118 (2017) no.2, 022001,*
Eur.Phys.J. C78 (2018) no.11, 947
G. Colangelo, S. Lanz, E.P. and H. Leutwyler

Phys. Rept. 945 (2022), 1-105,
L. Gan, B. Kubis, E.P., S. Tulin

Outline

1. Introduction and Motivation
2. Why is it interesting to study $\eta \to 3\pi$?
3. Computation of the Amplitude
4. Fits to the Dalitz plots and Results
5. Conclusion and Outlook

1. Introduction and Motivation

1.1 Why is it interesting to study η and η' physics?

- Quantum numbers $I^G J^{PC} = 0^+ 0^{-+}$
  - *C*, *P* eigenstates, all additive quantum numbers are zero
  - flavour-conserving laboratory for symmetry tests

- η: pseudo-Goldstone boson, $M_\eta = 547.862(17)$ **MeV**, $\Gamma_\eta = 1.31$ keV

  All decay modes forbidden at leading order by *symmetries* (C, P, angular momentum, isospin/G-parity. . .)

- η': not a Goldstone boson due to $U(1)_A$ anomaly $M_{\eta'} = 957.78(6)$ **MeV**
  $\Gamma_{\eta'} = 196$ keV

- Theoretical methods:
  - (large-N_c) chiral perturbation theory, RChPT
  - dispersion relations to resum final state interactions
  - Vector-meson dominance

1.1 Why is it interesting to study η and η' physics?

- In the study of η and η' physics, large amount of data have been collected:

  ⇨ *CBall, WASA, KLOE & KLOEII, BESIII, A2@MAMI, CLAS, GlueX*

  More to come: *JEF, REDTOP (Elam et al'22), LHCb?, JLab@22GeV*

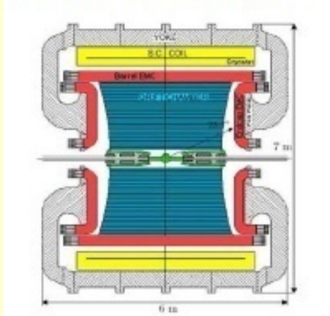
1.2 Experimental Facilities for studying η and η’

World competition in η decays

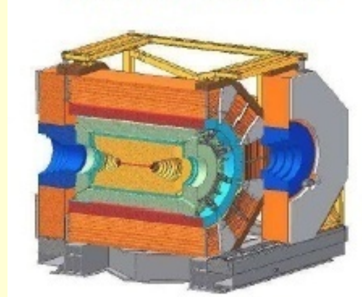
From L.Gan

e⁺e⁻ Collider

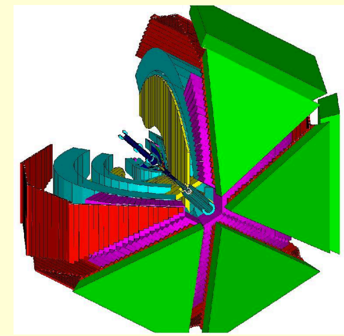
KLOE-2 at DAϕNE



BESIII at BEPCII

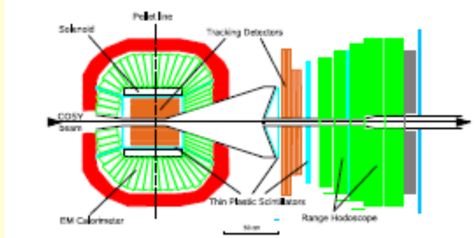


CLAS



Fixed-target

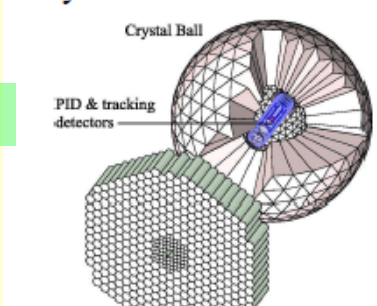
WASA at COSY



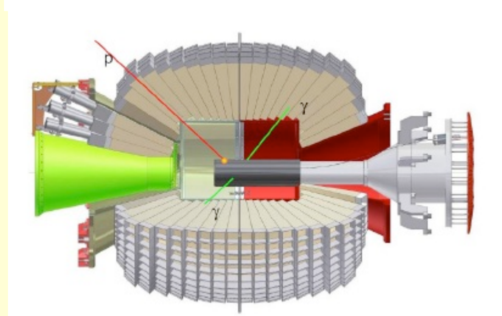
hadroproduction

photoproduction

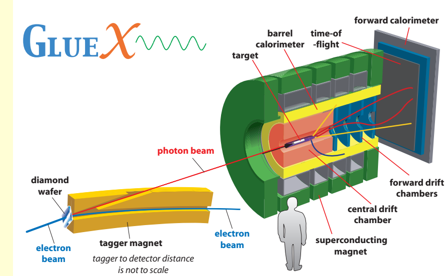
Crystall Ball at MAMI



CBELSA/TAPS at ELSA



JEF at Jlab



Why is it interesting to study η and η' physics?

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  More to come: *JEF, REDTOP (Elam et al'22), LHCb?, JLab@22GeV*

- Unique opportunity:
  - Test chiral dynamics at low energy
  - Extract fundamental parameters of the Standard Model:
    ex: light quark masses
  - Study of fundamental symmetries: P & CP and C & CP violation
  - Looking for beyond Standard Model Physics ⇨ Dark Sector

Rich physics program at η,η' factories

Standard Model highlights
- Theory input for light-by-light scattering for $(g\text{-}2)_\mu$
- Extraction of light quark masses
- QCD scalar dynamics

Fundamental symmetry tests
- P,CP violation
- C,CP violation

[Kobzarev & Okun (1964), Prentki & Veltman (1965), Lee (1965), Lee & Wolfenstein (1965), Bernstein et al (1965)]

Dark sectors (MeV—GeV)
- Vector bosons
- Scalars
- Pseudoscalars (ALPs)

(Plus other channels that have not been searched for to date)

Channel	Expt. branching ratio	Discussion
$\eta \to 2\gamma$	39.41(20)%	chiral anomaly, η–η' mixing
$\eta \to 3\pi^0$	32.68(23)%	$m_u - m_d$
$\eta \to \pi^0\gamma\gamma$	$2.56(22) \times 10^{-4}$	χPT at $O(p^6)$, leptophobic B boson, light Higgs scalars
$\eta \to \pi^0\pi^0\gamma\gamma$	$< 1.2 \times 10^{-3}$	χPT, axion-like particles (ALPs)
$\eta \to 4\gamma$	$< 2.8 \times 10^{-4}$	$< 10^{-11}$[52]
$\eta \to \pi^+\pi^-\pi^0$	22.92(28)%	$m_u - m_d$, C/CP violation, light Higgs scalars
$\eta \to \pi^+\pi^-\gamma$	4.22(8)%	chiral anomaly, theory input for singly-virtual TFF and $(g-2)_\mu$, P/CP violation
$\eta \to \pi^+\pi^-\gamma\gamma$	$< 2.1 \times 10^{-3}$	χPT, ALPs
$\eta \to e^+e^-\gamma$	$6.9(4) \times 10^{-3}$	theory input for $(g-2)_\mu$, dark photon, protophobic X boson
$\eta \to \mu^+\mu^-\gamma$	$3.1(4) \times 10^{-4}$	theory input for $(g-2)_\mu$, dark photon
$\eta \to e^+e^-$	$< 7 \times 10^{-7}$	theory input for $(g-2)_\mu$, BSM weak decays
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$\eta \to e^+e^-e^+e^-$	$2.40(22) \times 10^{-5}$	theory input for $(g-2)_\mu$
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$\eta \to \pi^+\pi^-\pi^0\gamma$	$< 5 \times 10^{-4}$	direct emission only
$\eta \to \pi^\pm e^\mp \nu_e$	$< 1.7 \times 10^{-4}$	second-class current
$\eta \to \pi^+\pi^-$	$< 4.4 \times 10^{-6}$	P/CP violation
$\eta \to 2\pi^0$	$< 3.5 \times 10^{-4}$	P/CP violation
$\eta \to 4\pi^0$	$< 6.9 \times 10^{-7}$	P/CP violation

Gan, Kubis, E. P., Tulin'22

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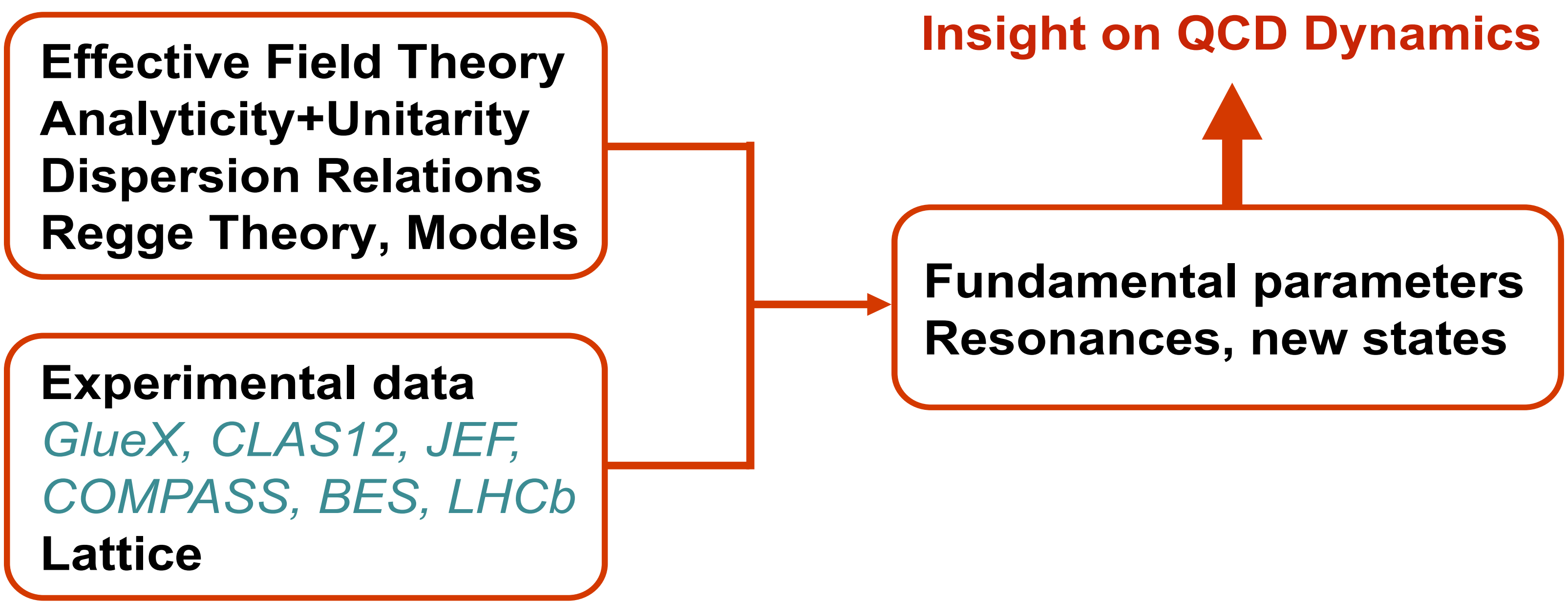
Gan, Kubis, E. P., Tulin'22

1.3 Analytical methods for light quark spectroscopy

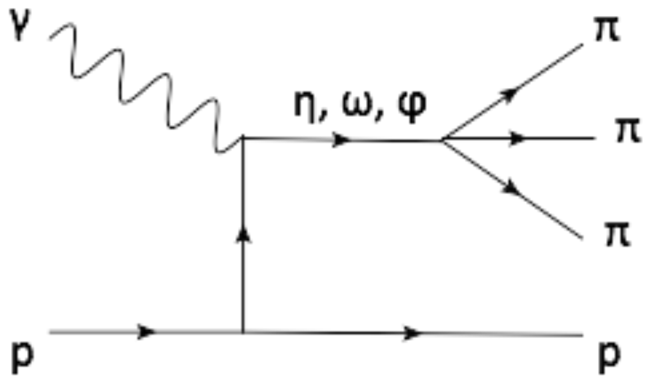
- In the study of hadron spectroscopy, large amount of very precise data on meson physics have been and will be collected:

  ⇨ *KLOE & KLOE-II, BES, A1, A2@MAMI, CLAS, GlueX, JEF, COMPASS, LHCb, PANDA,…*

  They are background for searches of new states



1.3 Analytical methods for studying light mesons



- If E > 1 GeV: ChPT not valid anymore to describe dynamics of the processes

  ➡ Resonances appear :

  For ππ: *I=1*: ρ(770), ρ(1450), ρ(1700), …,
  Especially true for ϕ (M_ϕ=1020 MeV)

- Use Isobar model to describe the data

  ➡ Improve to include FSI

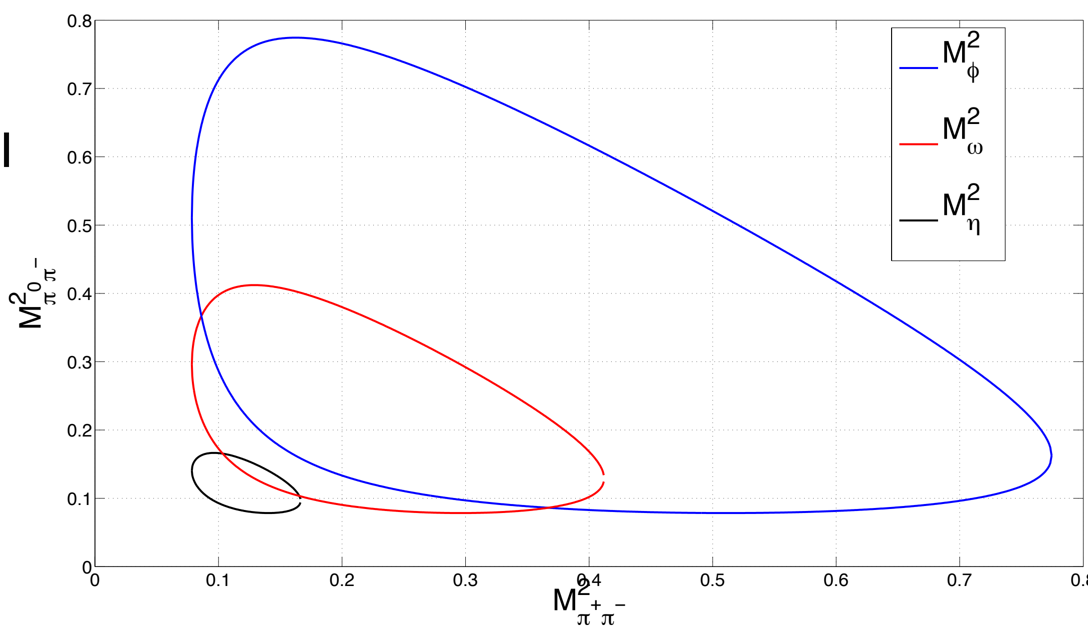
- Build an amplitude with physical properties:

  → Analyticity, Unitarity and Crossing Symmetry:

  ➡ *Dispersion Relations*

  → Chiral constraints at LE

  → Regge behavior at HE



2. Why is it interesting to study $\eta \rightarrow 3\pi$?

2.1 Light quark masses

- Fundamental unknowns of the the QCD Lagrangian
  In the following, consider the 3 light flavours u,d,s

- High precision physics at low energy as a key of new physics?
  m_d - m_u : small isospin breaking corrections but to be taken into account for high precision physics

- No direct access to the quarks due to confinement!

2.2 Meson masses from ChPT

- $m_{u,d,s} \ll \Lambda_{QCD}$: masses treated as small perturbations

  ⇨ *expansion in powers of m_q*

- *Gell-Mann-Oakes-Renner relations*:

  (meson mass)^2 = (spontaneous ChSB) x (explicit ChSB)

  ↗ $\langle \bar{q}q \rangle$ ↖ m_q

- From LO ChPT without e.m effects:

$$M^2_{\pi^+} = (m_u + m_d)\, B_0 + O(m^2)$$
$$M^2_{K^+} = (m_u + m_s)\, B_0 + O(m^2)$$
$$M^2_{K^0} = (m_d + m_s)\, B_0 + O(m^2)$$

- Electromagnetic effects: *Dashen's theorem*

$$\left(M^2_{K^+} - M^2_{K^0}\right)_{em} - \left(M^2_{\pi^+} - M^2_{\pi^0}\right)_{em} = O\left(e^2 m\right)$$

Dashen'69

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- *Gell-Mann-Oakes-Renner relations*:

  (meson mass)^2 = (spontaneous ChSB) x (explicit ChSB)

  $\langle \bar{q}q \rangle$ ↗ (spontaneous ChSB) ; m_q ↖ (explicit ChSB)

- From LO ChPT without e.m effects:

$$M^2_{\pi^+} = (m_u + m_d)\, B_0 + O(m^2)$$
$$M^2_{K^+} = (m_u + m_s)\, B_0 + O(m^2)$$
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Dashen'69

$$M^2_{\pi^0} = B_0\left(m_u + m_d\right)$$
$$M^2_{\pi^+} = B_0\left(m_u + m_d\right) + \Delta_{em}$$
$$M^2_{K^0} = B_0\left(m_d + m_s\right)$$
$$M^2_{K^+} = B_0\left(m_u + m_s\right) + \Delta_{em}$$

2 unknowns B_0 and Δ_{em}

2.2 Meson masses from ChPT

➡ Quark mass ratios *Weinberg'77*

$$\frac{m_u}{m_d} \stackrel{\text{LO}}{=} \frac{M_{K^+}^2 - M_{K^0}^2 + 2M_{\pi^0}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} = 0.56\,,$$

$$\frac{m_s}{m_d} \stackrel{\text{LO}}{=} \frac{M_{K^+}^2 + M_{K^0}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} = 20.2$$

2.3 Lattice QCD

- Compute the quark masses from first principles

  ➡ L_{QCD} on the lattice

  - QCD Lagrangian as input
  - Calculate the spectrum of the low-lying states for different quark masses
  - Tune the values of the quark masses such that the QCD spectrum is reproduced
  - Set the scale by adding an external input or extract quark mass ratios

- NB: computation in the isospin limit: $m_u = m_d = \hat{m}$ ← $\frac{m_u + m_d}{2}$

- To get $m_u - m_d$, needs handle on e.m. effects:
  - Input from phenomenology (e.g., Kaon mass difference)
  - Put photons on the lattice

➡ *See FLAG'21*

2.4 Extracting light quark masses from $\eta \to 3\pi$

- Decay forbidden by isospin symmetry $\eta(I^G = 0^+) \to 3\pi(I^G = 1^-)$

$$A = (m_u - m_d) A_1 + \alpha_{em} A_2$$

- α_{em} effects are small

  Sutherland'66, Bell & Sutherland'68
  Baur, Kambor, Wyler'96, Ditsche, Kubis, Meissner'09

- Decay rate measures the size of isospin breaking $(m_u - m_d)$ in the SM:

$$L_{QCD} \to L_{IB} = -\frac{m_u - m_d}{2}\left(\bar{u}u - \bar{d}d\right)$$

⇒ Unique access to $(m_u - m_d)$

Decays of η

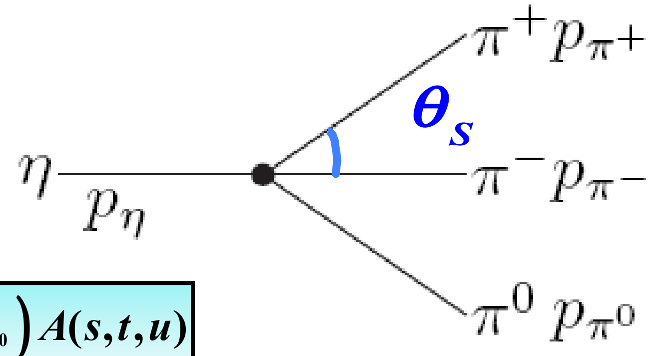
- η decay from PDG:

M_η = 547.862(17) MeV

η DECAY MODES

	Mode	Fraction (Γ_i/Γ)	Scale factor/ Confidence level
	Neutral modes		
Γ_1	neutral modes	(72.12±0.34) %	S=1.2
Γ_2	2γ	(39.41±0.20) %	S=1.1
Γ_3	$3\pi^0$	(32.68±0.23) %	S=1.1
	Charged modes		
Γ_8	charged modes	(28.10±0.34) %	S=1.2
Γ_9	$\pi^+\pi^-\pi^0$	(22.92±0.28) %	S=1.2
Γ_{10}	$\pi^+\pi^-\gamma$	(4.22±0.08) %	S=1.1

2.5 Definitions



- η decay: $\eta \rightarrow \pi^+ \pi^- \pi^0$

$$\langle \pi^+ \pi^- \pi^0{}_{out} | \eta \rangle = i(2\pi)^4 \delta^4 \left(p_\eta - p_{\pi^+} - p_{\pi^-} - p_{\pi^0}\right) A(s,t,u)$$

- Mandelstam variables $s = \left(p_{\pi^+} + p_{\pi^-}\right)^2,\ t = \left(p_{\pi^-} + p_{\pi^0}\right)^2,\ u = \left(p_{\pi^0} + p_{\pi^+}\right)^2$

  ⇨ only two independent variables

$$s + t + u = M_\eta^2 + M_{\pi^0}^2 + 2M_{\pi^+}^2 \equiv 3s_0$$

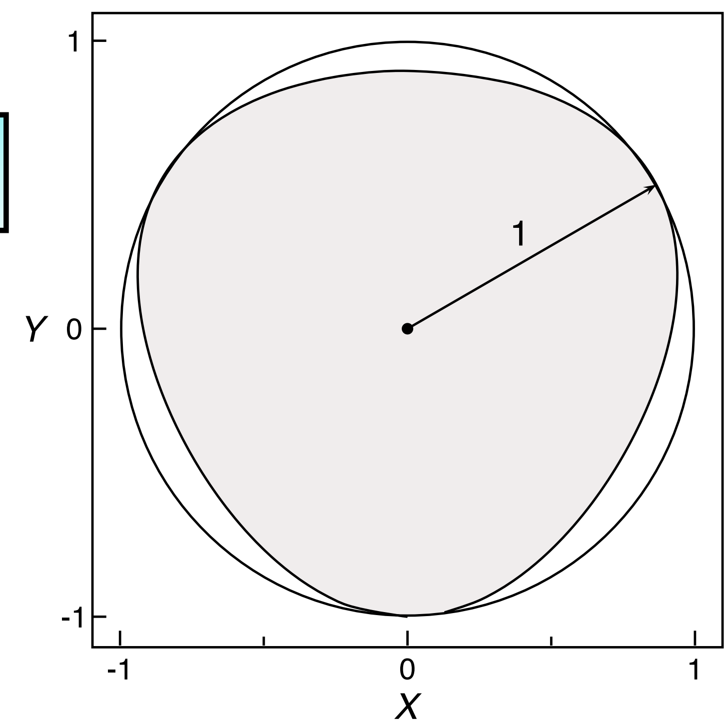
- 3 body decay ⇨ Dalitz plot

$$\left|A(s,t,u)\right|^2 = N\left(1 + aY + bY^2 + dX^2 + fY^3 + \ldots\right)$$

Expansion around X=Y=0

$$X = \sqrt{3}\,\frac{T_+ - T_-}{Q_c} = \frac{\sqrt{3}}{2M_\eta Q_c}(u - t)$$

$$Y = \frac{3T_0}{Q_c} - 1 = \frac{3}{2M_\eta Q_c}\left(\left(M_\eta - M_{\pi^0}\right)^2 - s\right) - 1$$



$$Q_c \equiv M_\eta - 2M_{\pi^+} - M_{\pi^0}$$

2.6 Quark mass ratio

- In the following, extraction of Q from $\eta \rightarrow \pi^+ \pi^- \pi^0$

$$\Gamma_{\eta\to\pi^+\pi^-\pi^0} = \frac{1}{Q^4}\frac{M_K^4}{M_\pi^4}\frac{\left(M_K^2 - M_\pi^2\right)^2}{6912\pi^3 F_\pi^4 M_\eta^3}\int_{s_{\min}}^{s_{\max}} ds \int_{u_-(s)}^{u_+(s)} du \left|M(s,t,u)\right|^2$$

Determined from experiment

Determined from:
- Dispersive calculation
- ChPT

Fit to Dalitz distr.

$$\left[Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} \right] \qquad \left[\hat{m} \equiv \frac{m_d + m_u}{2} \right]$$

- Aim: Compute M(s,t,u) with the *best accuracy*

2.6 Quark mass ratio

- Mass formulae to second chiral order *Gasser & Leutwyler'85*

$$\frac{M_K^2}{M_\pi^2} = \frac{m_s + \hat{m}}{2\hat{m}}\left[1 + \Delta_M + \mathcal{O}(m^2)\right]$$

$$\frac{M_{K^0}^2 - M_{K^+}^2}{M_K^2 - M_\pi^2} = \frac{m_d - m_u}{m_s - \hat{m}}\left[1 + \Delta_M + \mathcal{O}(m^2)\right]$$

with $\Delta_M = \frac{8(M_K^2 - M_\pi^2)}{F_\pi^2}(2L_8 - L_5) + \chi\text{-logs}$

- The same O(m) correction appears in both ratios $\left[\hat{m} \equiv \frac{m_d + m_u}{2}\right]$

  ⇨ Take the double ratio

$$Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} = \frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{\left(M_{K^0}^2 - M_{K^+}^2\right)_{QCD}}\left[1 + O(m_q^2, e^2)\right]$$

Very Interesting quantity to determine since Q^2 does not receive any correction at NLO!

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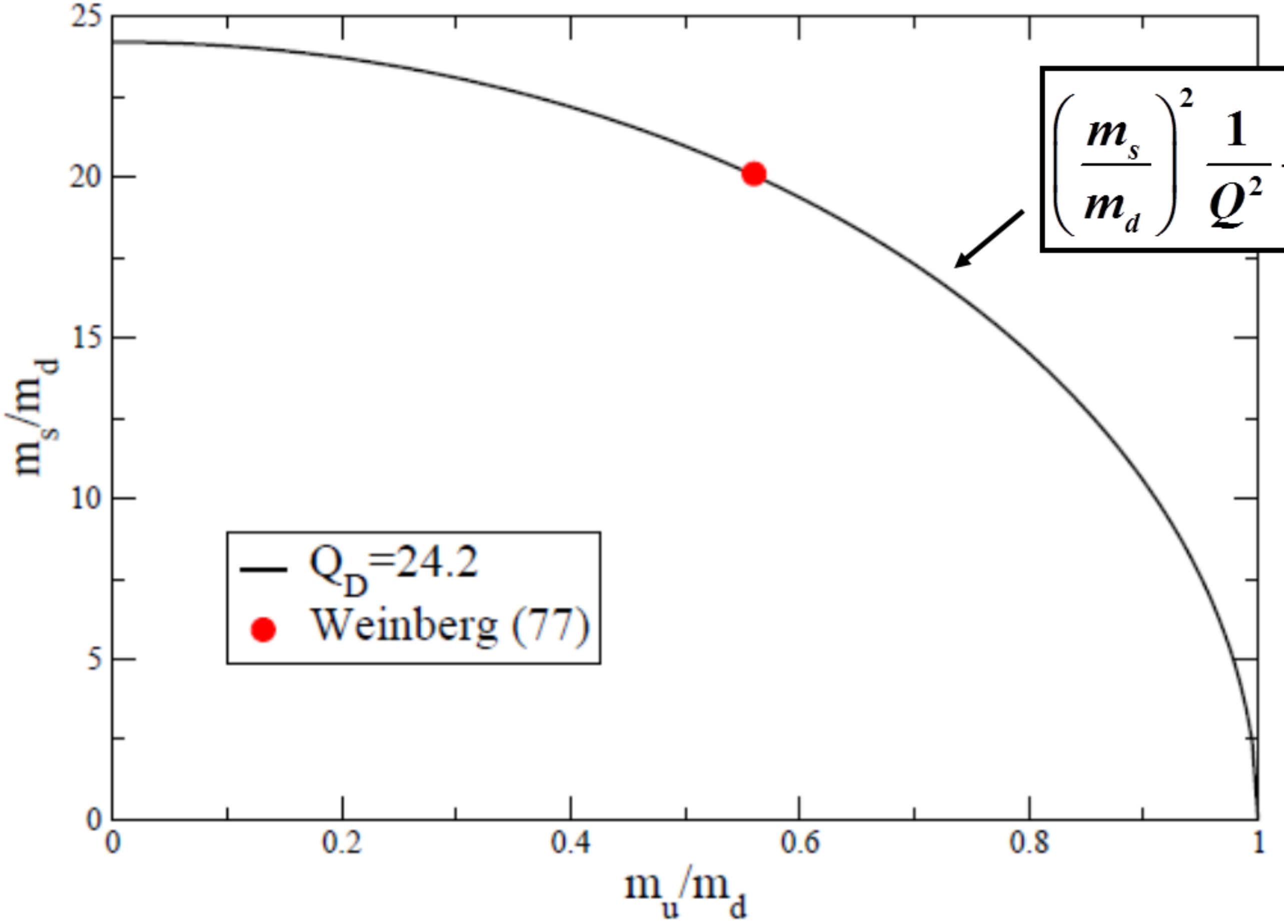
- Using Dashen's theorem and inserting Weinberg LO values

$$Q_D^2 \equiv \frac{(M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2 + M_{\pi^0}^2)(M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2 - M_{\pi^0}^2)}{4M_{\pi^0}^2(M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2 - M_{\pi^0}^2)}$$

⇨ $Q_D = 24.2$

2.6 Quark mass ratio

- From Q ⇨ Ellipse in the plane m_s/m_d, m_u/m_d *Leutwyler's ellipse*



$$\left(\frac{m_s}{m_d}\right)^2 \frac{1}{Q^2} + \left(\frac{m_u}{m_d}\right)^2 = 1$$

2.6 Quark mass ratio

- Estimate of Q: $$B_0\left(m_u - m_d\right) = \frac{1}{Q^2}\frac{M_K^2\left(M_K^2 - M_\pi^2\right)}{M_\pi^2} + O(M^3)$$

   - From corrections to the Dashen's theorem

     $$\Rightarrow \quad B_0\left(m_d - m_u\right) = \left(M_{K^+}^2 - M_{K^0}^2\right) - \left(M_{\pi^+}^2 - M_{\pi^0}^2\right) + O\left(e^2 m\right)$$

     The corrections can be large due to $e^2 m_s$ corrections, difficult to estimate due to LECs

   - From $\eta \to \pi^+ \pi^- \pi^0$: $$A(s,t,u) = -\frac{1}{Q^2}\frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{3\sqrt{3}F_\pi^2}M(s,t,u)$$

     $$\Rightarrow \quad \Gamma_{\eta\to 3\pi} \propto \int \left|A(s,t,u)\right|^2 \propto Q^{-4}$$

- In the following, compute the normalized amplitude M(s,t,u) with the best accuracy ⇒ *extraction of Q*

2.6 Quark mass ratio

- Use Q to determine m_u and m_d from lattice determinations of m_s and $\hat{m}$

$$m_u = \hat{m} - \frac{m_s^2 - \hat{m}^2}{4\hat{m}Q^2}$$ and $$m_d = \hat{m} + \frac{m_s^2 - \hat{m}^2}{4\hat{m}Q^2}$$

- From lattice determinations of m_s and $\hat{m}$ + Q

Light quark masses: m_u, m_d, m_s

3. Computation of the Amplitude

3.1 Introduction

- What do we know?

- Compute the amplitude using ChPT : the effective theory that describe dynamics of the Goldstone bosons (kaons, pions, eta) at low energy

- Goldstone bosons interact weakly at low energy and $m_u, m_d \ll m_s < \Lambda_{QCD}$
  Expansion organized in external momenta and quark masses

Weinberg's power counting rule

$$\mathcal{L}_{eff} = \sum_{d \geq 2} \mathcal{L}_d \;,\; \mathcal{L}_d = \mathcal{O}\left(p^d\right) \;,\; p \equiv \left\{q, m_q\right\}$$

$$\mathbf{p} << \Lambda_H = 4\pi F_\pi \sim 1 \text{ GeV}$$

3.2 Chiral Perturbation Theory

- What do we know?
- Compute the amplitude using ChPT :

$$\Gamma_{\eta\to 3\pi} = \left(66 + 94 + ... + ...\right)\text{eV} = \left(300 \pm 12\right)\text{eV}$$

LO NLO NNLO

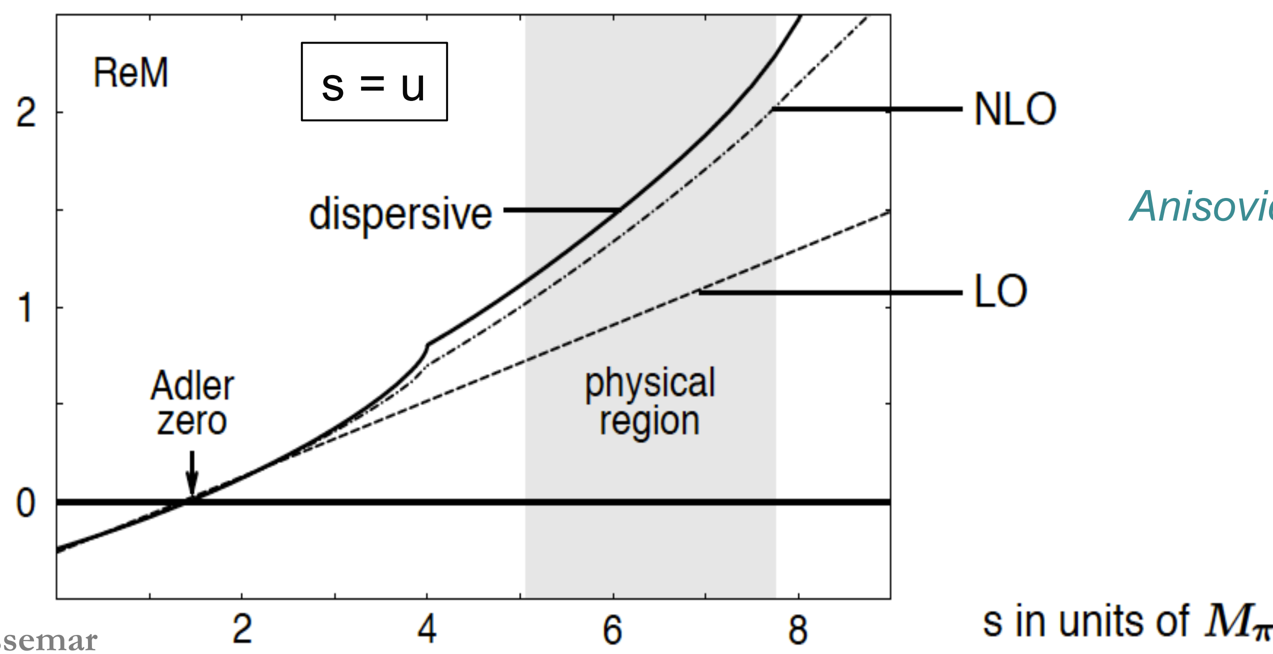
PDG'16

LO: *Osborn, Wallace'70*

NLO: *Gasser & Leutwyler'85*

NNLO: *Bijnens & Ghorbani'07*

The Chiral series has convergence problems

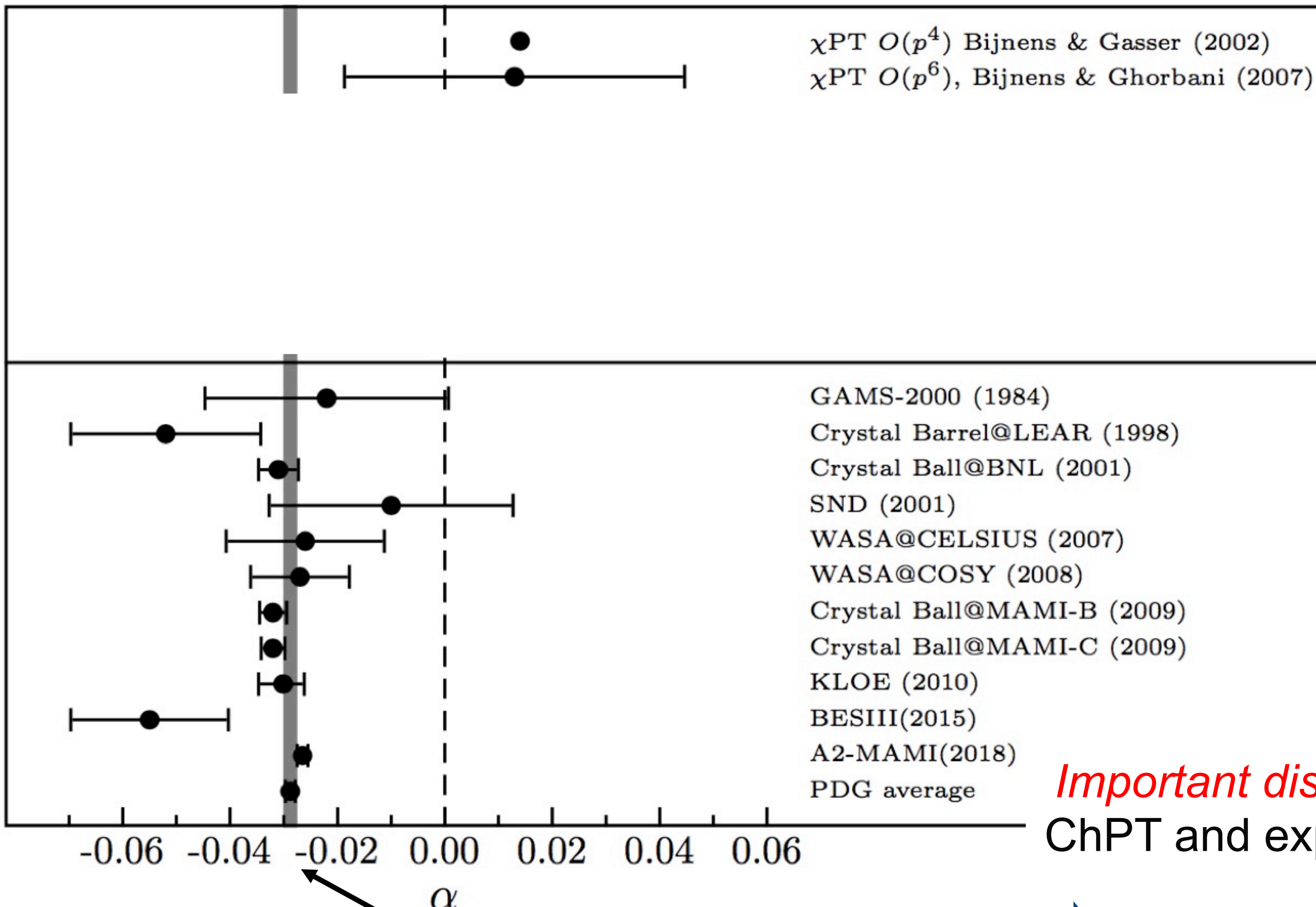


Anisovich & Leutwyler'96

3.3 Neutral Channel : $\eta \to \pi^0 \pi^0 \pi^0$

$Q_n \equiv M_\eta - 3M_{\pi^0}$

- Decay amplitude $\Gamma_{\eta\to 3\pi} \propto |\bar{A}|^2 \propto 1 + 2\alpha Z$ with $Z = \frac{2}{3}\sum_{i=1}^{3}\left(\frac{3T_i}{Q_n} - 1\right)^2$



$\alpha \sim 0.015$

$\alpha = -0.0288 \pm 0.0012$

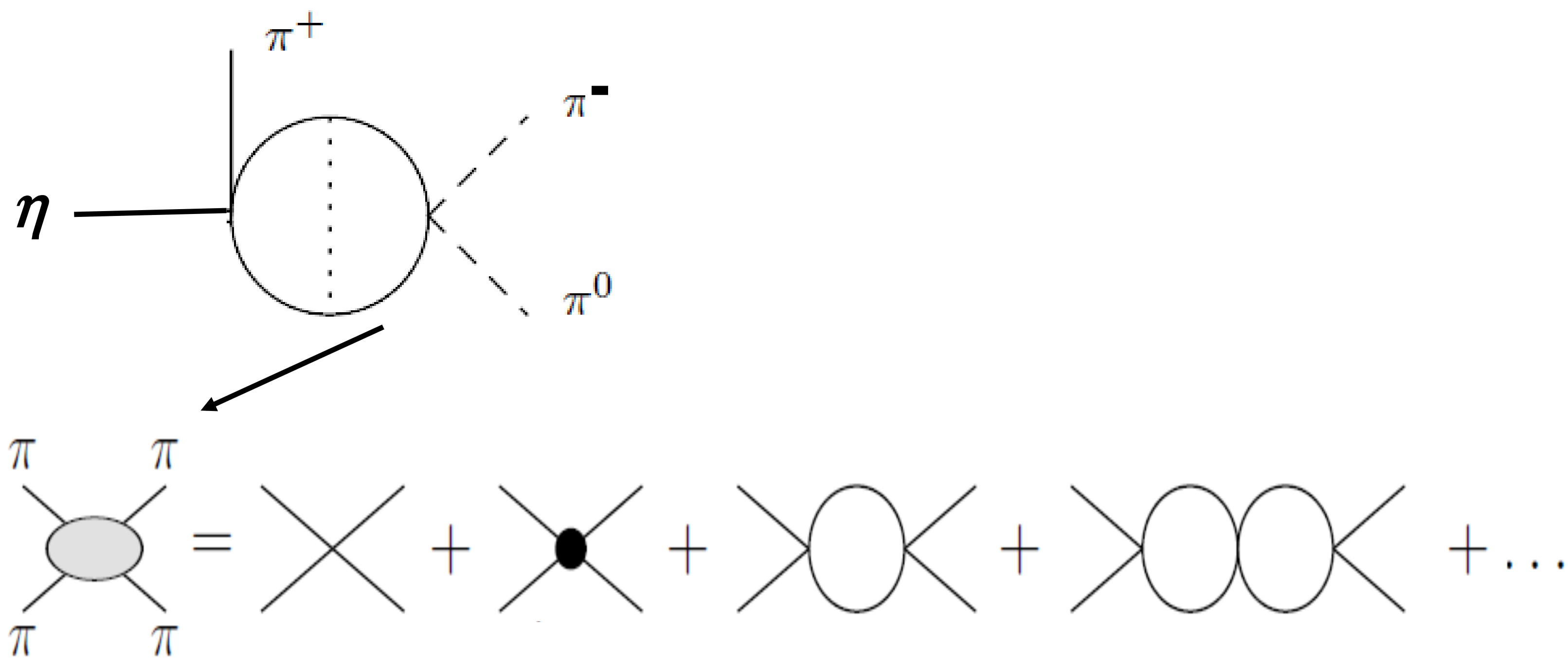
Important discrepancy between ChPT and experiment!

⇨ Help of a dispersive treatment?

3.4 Dispersive treatment

- The Chiral series has convergence problems

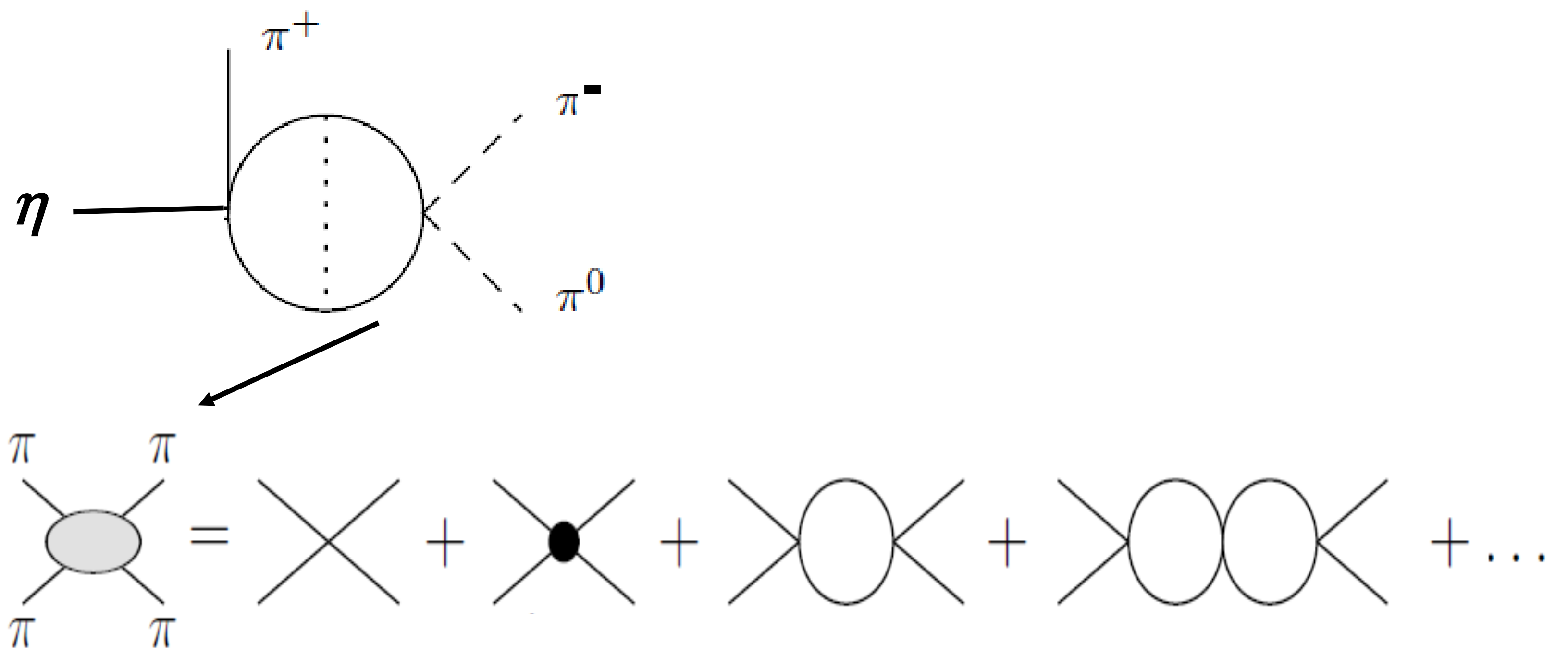
  ⇒ Large *ππ final state interactions* *Roiesnel & Truong'81*



3.4 Dispersive treatment

- The Chiral series has convergence problems

⇨ Large *ππ final state interactions* *Roiesnel & Truong'81*



- *Dispersive treatment :*
  - analyticity, unitarity and crossing symmetry
  - Take into account all the rescattering effects

3.5 Why a new dispersive analysis?

- Several new ingredients:
  - New inputs available: extraction $\pi\pi$ phase shifts has improved

    Ananthanarayan et al'01, Colangelo et al'01

    Descotes-Genon et al'01

    Kaminsky et al'01, Garcia-Martin et al'09

  - New experimental programs, precise Dalitz plot measurements

    TAPS/CBall-MAMI (Mainz), WASA-Celsius (Uppsala), WASA-Cosy (Juelich)

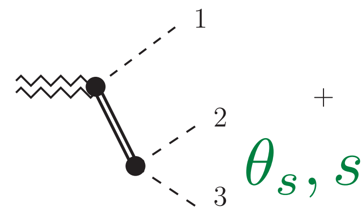
    CBall-Brookhaven, CLAS, GlueX (JLab), KLOE I-II (Frascati)

    BES III (Beijing)

  - Many improvements needed in view of very precise data: inclusion of
    - Electromagnetic effects ($\mathcal{O}(e^2 m)$) *Ditsche, Kubis, Meissner'09*
    - Isospin breaking effects

3.6 Method

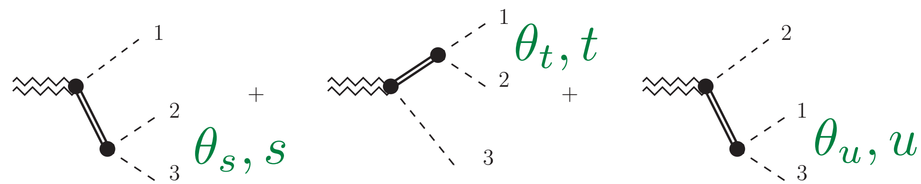
- S-channel partial wave decomposition



$$A_\lambda(s,t) = \sum_J^\infty (2J+1) d^J_{\lambda,0}(\theta_s) A_J(s)$$

- One truncates the partial wave expansion : ➡ Isobar approximation

$$A_\lambda(s,t) = \sum_J^{J_{\max}} (2J+1) d^J_{\lambda,0}(\theta_s) f_J(s) + \sum_J^{J_{\max}} (2J+1) d^J_{\lambda,0}(\theta_t) f_J(t) + \sum_J^{J_{\max}} (2J+1) d^J_{\lambda,0}(\theta_u) f_J(u)$$

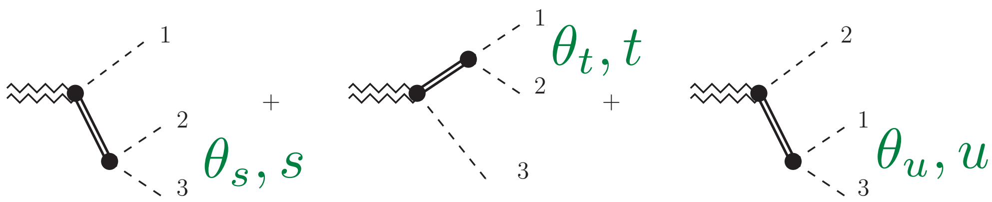


3 BWs (ρ^+, ρ^-, ρ^0) + background term

➡ Improve to include final states interactions

3.6 Method

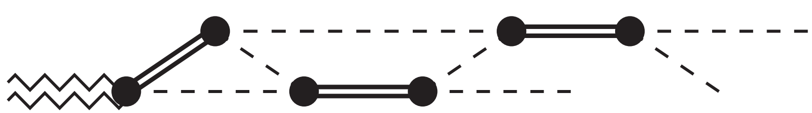
- Decompose amplitide in partial waves:



- Usual assumption: 3 BWs (ρ^+, ρ^-, ρ^0) + background term

  ⇨ Improve to include final states interactions + restore unitarity

- Use a *Khuri-Treiman* approach or *dispersive* approach
  Restore 3 body unitarity and take into account the final state interactions in a systematic way



3.7 Representation of the amplitude

- Decomposition of the amplitude as a function of isospin states

$$M(s,t,u) = M_0(s) + (s-u)M_1(t) + (s-t)M_1(u) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s)$$

Fuchs, Sazdjian & Stern'93
Anisovich & Leutwyler'96

  - M_I isospin *I* rescattering in two particles
  - Amplitude in terms of S and P waves ⇨ exact up to NNLO ($\mathcal{O}(p^6)$)
  - Main two body rescattering corrections inside M_I

3.7 Method: Representation of the amplitude

- Decomposition of the amplitude as a function of isospin states

$$M(s,t,u) = M_0(s) + (s-u)M_1(t) + (s-t)M_1(u) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s)$$

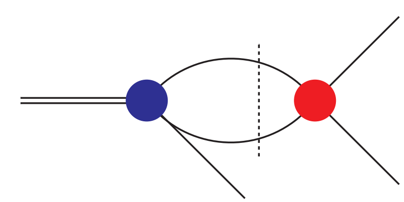
Fuchs, Sazdjian & Stern'93

Anisovich & Leutwyler'96

  - M_I isospin *I* rescattering in two particles
  - Amplitude in terms of S and P waves ⇨ exact up to NNLO ($\mathcal{O}(p^6)$)
  - Main two body rescattering corrections inside M_I

- Functions of only one variable with only right-hand cut of the partial wave ⇨ $disc\left[M_I(s)\right] \equiv disc\left[f_1^I(s)\right]$

- Elastic unitarity *Watson's theorem*



$$disc\left[f_1^I(s)\right] \propto t_1^*(s) f_1^I(s)$$

with $t_1(s)$ partial wave of elastic $\pi\pi$ scattering

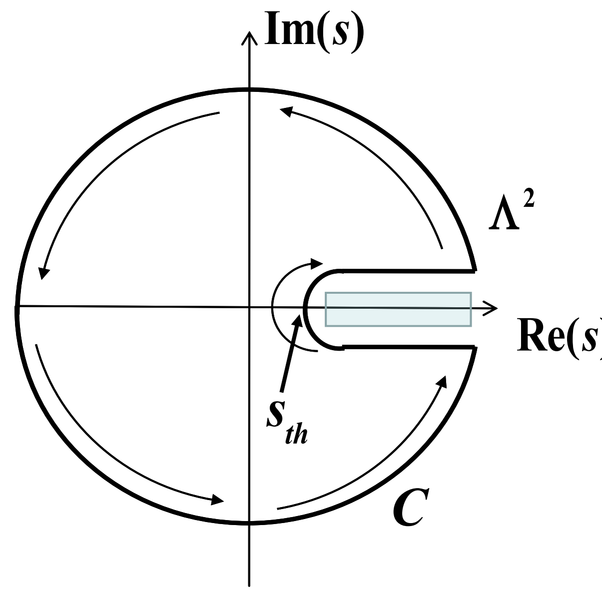
3.7 Method: Representation of the amplitude

- Knowing the discontinuity of M_I ⇨ write a dispersion relation for it

- Cauchy Theorem and Schwarz reflection principle

⇨
$$M_I(s) = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \frac{disc\left[M_I(s')\right]}{s'-s-i\varepsilon} ds'$$

M_I can be reconstructed everywhere from the knowledge of $disc\left[M_I(s)\right]$



- If M_I doesn't converge fast enought for $|s| \to \infty$ ⇨ subtract the dispersion relation

$$M_I(s) = P_{n-1}(s) + \frac{s^n}{\pi}\int_{4M_\pi^2}^{\infty} \frac{ds'}{s'^n} \frac{disc\left[M_I(s')\right]}{(s'-s-i\varepsilon)}$$

$P_{n-1}(s)$ polynomial

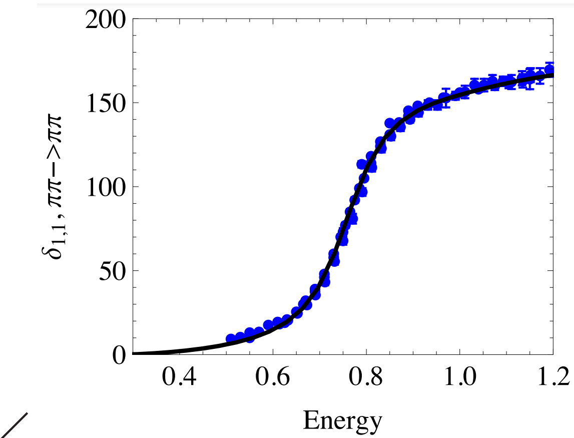
3.7 Representation of the amplitude

- Decomposition of the amplitude as a function of isospin states

$$M(s,t,u) = M_0^0(s) + (s-u)M_1^1(t) + (s-t)M_1^1(u) + M_0^2(t) + M_0^2(u) - \frac{2}{3}M_0^2(s)$$

$$disc\left[M_I(s)\right] \equiv disc\left[f_\ell^I(s)\right]$$

Roy analysis
Colangelo et al.'01



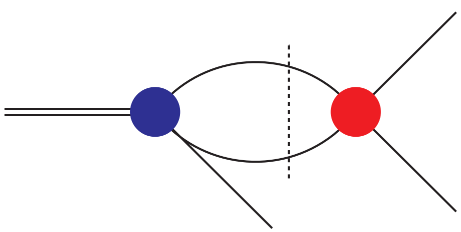
input

- Unitarity relation:

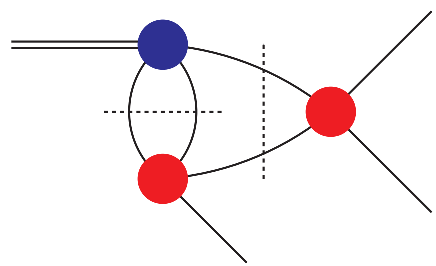
$\pi\pi \to \pi\pi$

$$disc\left[M_\ell^I(s)\right] = \rho(s)t_\ell^{*}(s)\left(M_\ell^I(s) + \hat{M}_\ell^I(s)\right)$$

right-hand cut



left-hand cut



$$f_1^I(s) = M_I(s) + \hat{M}_I(s)$$

3.7 Representation of the amplitude

- Decomposition of the amplitude as a function of isospin states

$$M(s,t,u) = M_0(s) + (s-u)M_1(t) + (s-t)M_1(u) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s)$$

- Unitarity relation:

$$disc\left[M_\ell^I(s)\right] = \rho(s)t_\ell^*(s)\left(M_\ell^I(s) + \hat{M}_\ell^I(s)\right)$$

- Relation of dispersion to reconstruct the amplitude everywhere:

$$M_I(s) = \Omega_I(s)\left(P_I(s) + \frac{s^n}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^n}\frac{\sin\delta_I(s')\,\hat{M}_I(s')}{\left|\Omega_I(s')\right|\left(s'-s-i\varepsilon\right)}\right)$$

$$\left[\Omega_I(s) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}ds'\frac{\delta_I(s')}{s'(s'-s-i\varepsilon)}\right)\right]$$

Omnès function

Gasser & Rusetsky'18

- $P_I(s)$ determined from a fit to NLO ChPT + experimental Dalitz plot

4. Fits to the Dalitz plots and Results

4.1 Isospin breaking corrections

- Dispersive calculations in the isospin limit ⇨ to fit to data one has to include isospin breaking corrections

- $$M_{c/n}(s,t,u) = M_{disp}(s,t,u)\,\frac{M_{DKM}(s,t,u)}{\tilde{M}_{GL}(s,t,u)}$$

with M_{DKM} : amplitude at one loop with $\mathcal{O}(e^2 m)$ effects

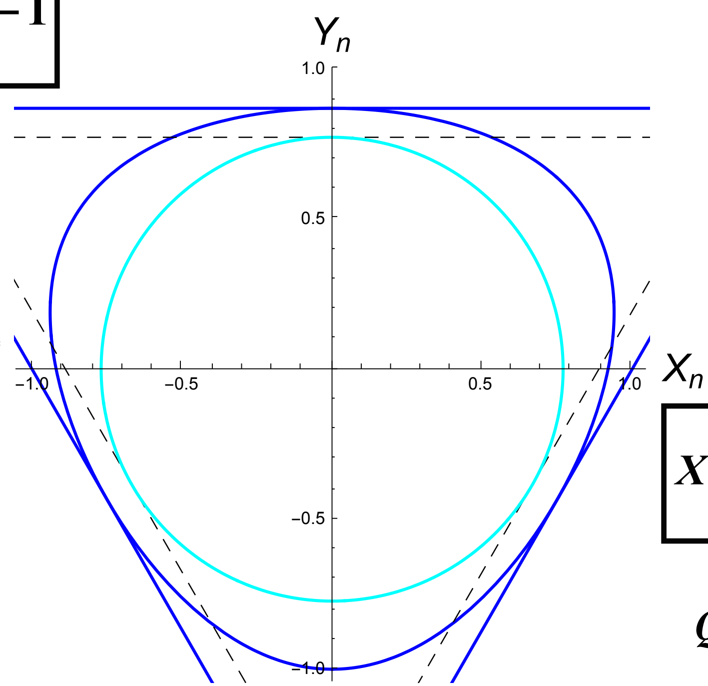
Ditsche, Kubis, Meissner'09

M_{GL}: amplitude at one loop in the isospin limit

Gasser & Leutwyler'85

$$Y_n = \frac{3T_3}{Q_n} - 1$$

Neutral channel



$$X_n = \sqrt{3}\,\frac{T_2 - T_1}{Q_n}$$

$$Q_n \equiv M_\eta - 3M_{\pi^0}$$

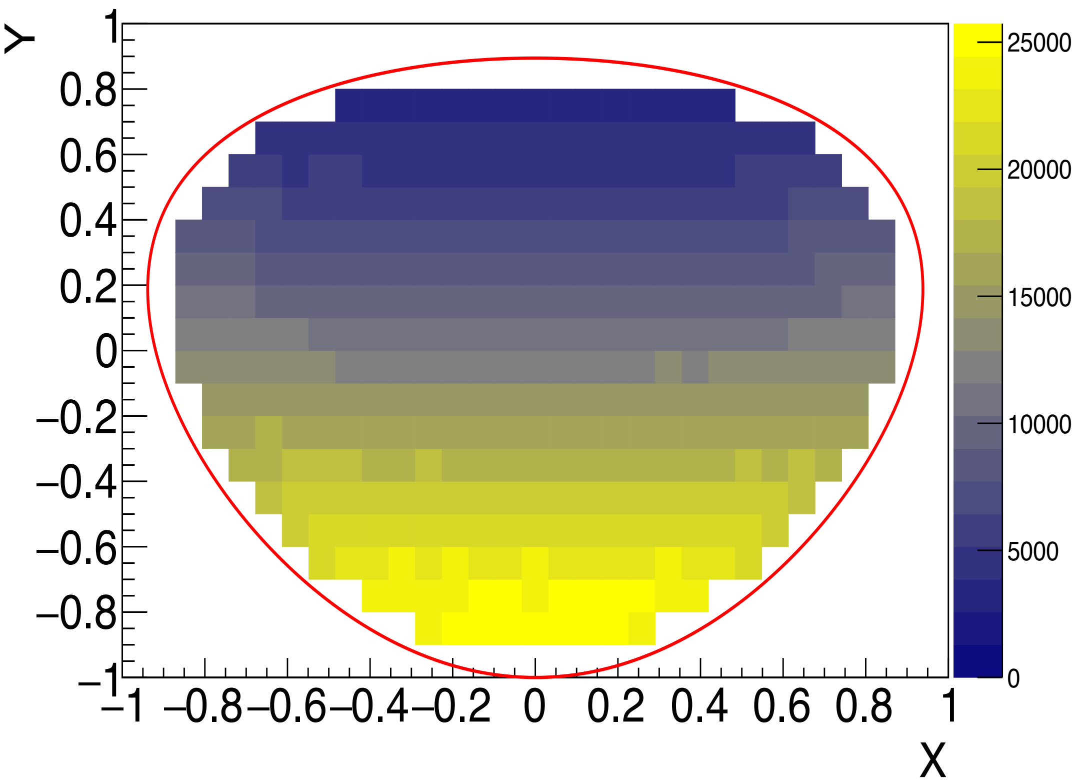
Kinematic map:
isospin symmetric boundaries

⇨ physical boundaries

$$M_{GL} \rightarrow \tilde{M}_{GL}$$

4.2 $\eta \to 3\pi$ Dalitz plot

- In the charged channel: experimental data from *WASA, KLOE, BESIII*



KLOE'16

$$\left|A(s,t,u)\right|^2 = N\begin{pmatrix} 1 + aY + bY^2 \\ + dX^2 + fY^3 + \ldots \end{pmatrix}$$

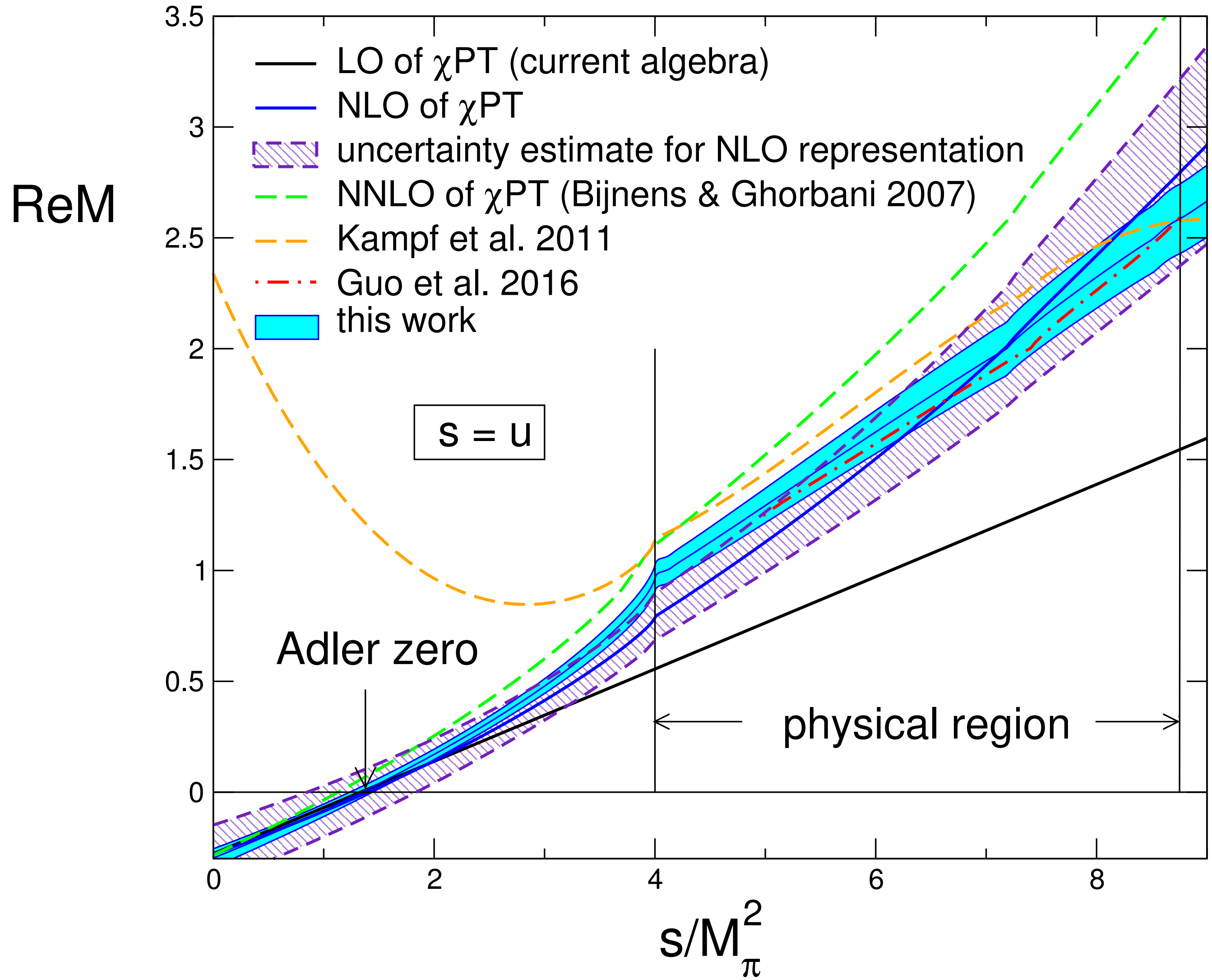
$$X = \sqrt{3}\,\frac{T_+ - T_-}{Q_c} = \frac{\sqrt{3}}{2M_\eta Q_c}(u-t)$$

$$Y = \frac{3T_0}{Q_c} - 1 = \frac{3}{2M_\eta Q_c}\left(\left(M_\eta - M_{\pi^0}\right)^2 - s\right) - 1$$

- New data expected from *CLAS* and *GlueX* with very different systematics

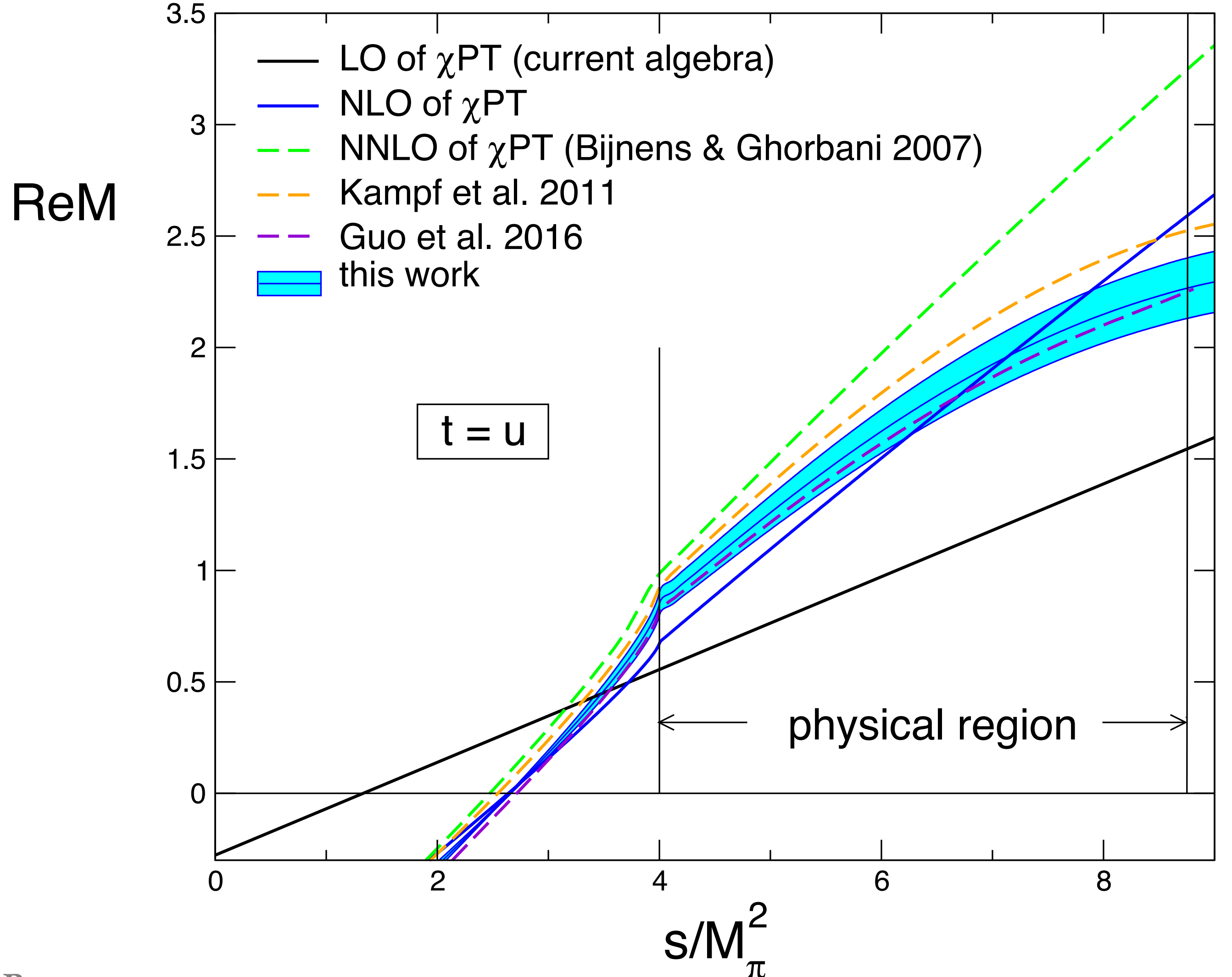
4.3 Results: Amplitude for $\eta \rightarrow \pi^+ \pi^- \pi^0$ decays

- The amplitude along the line s = u :



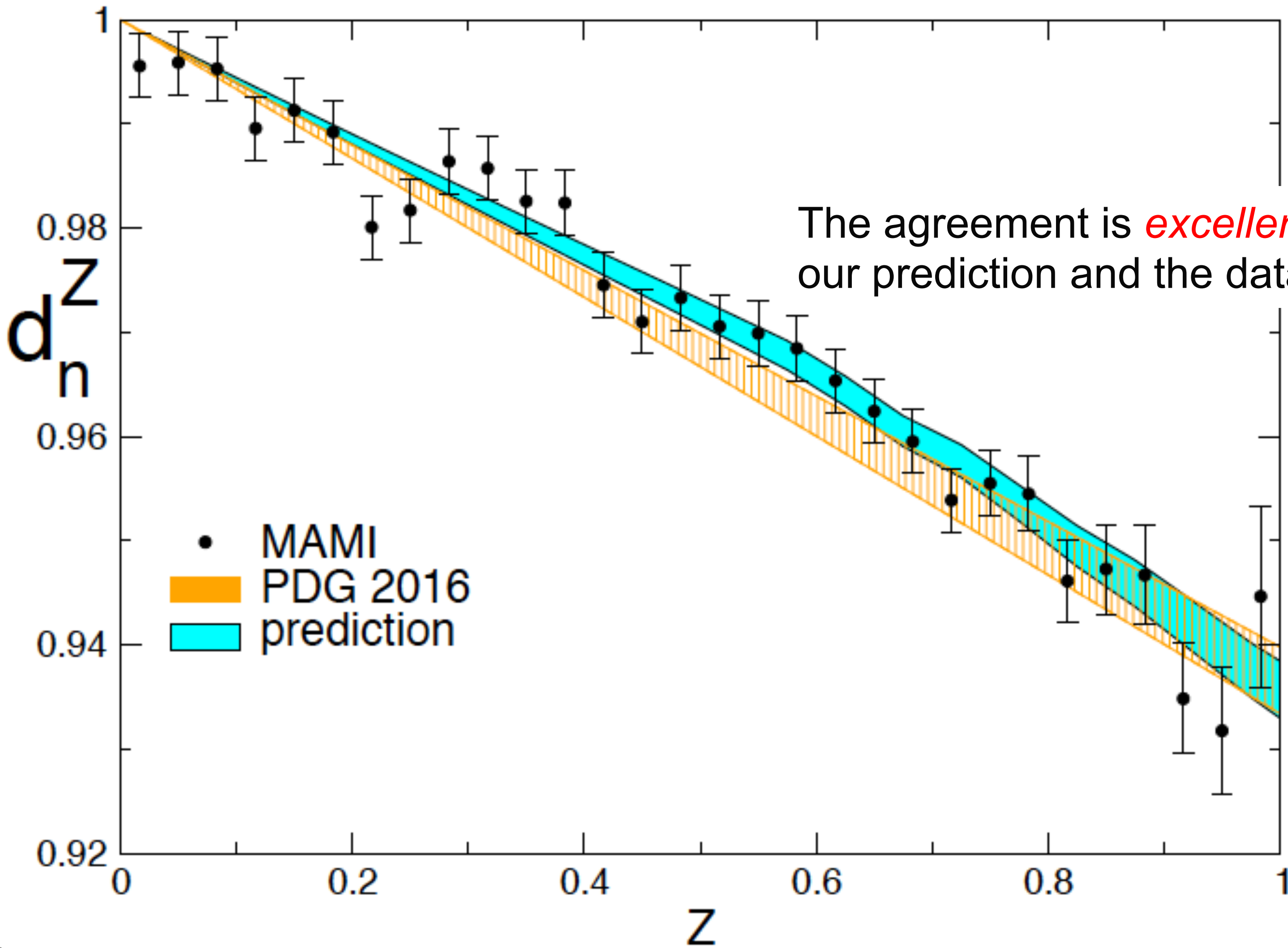
4.3 Results: Amplitude for $\eta \to \pi^+ \pi^- \pi^0$ decays

- The amplitude along the line t = u :



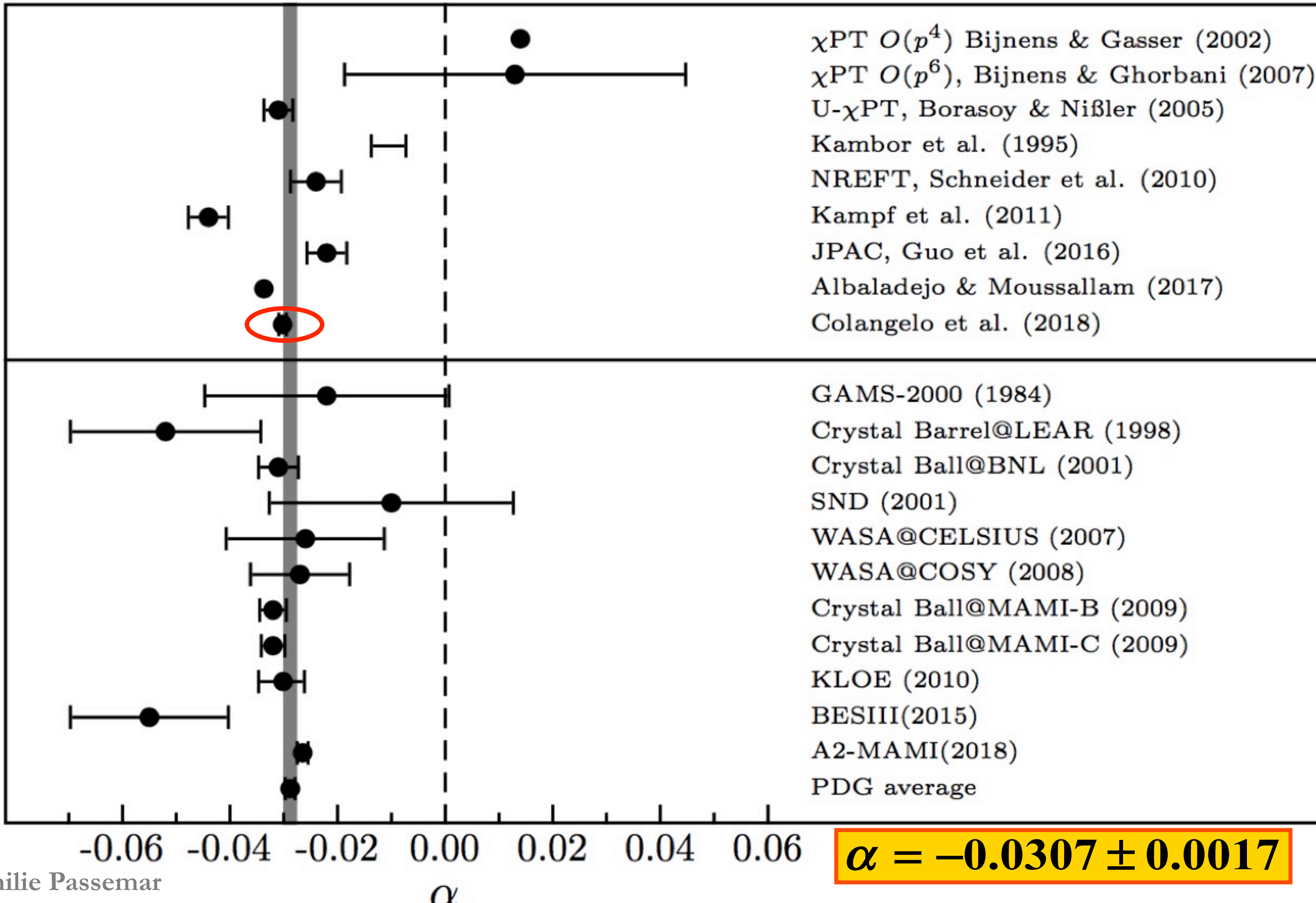
4.4 Z distribution for $\eta \rightarrow \pi^0 \pi^0 \pi^0$ decays

- The amplitude squared in the neutral channel is



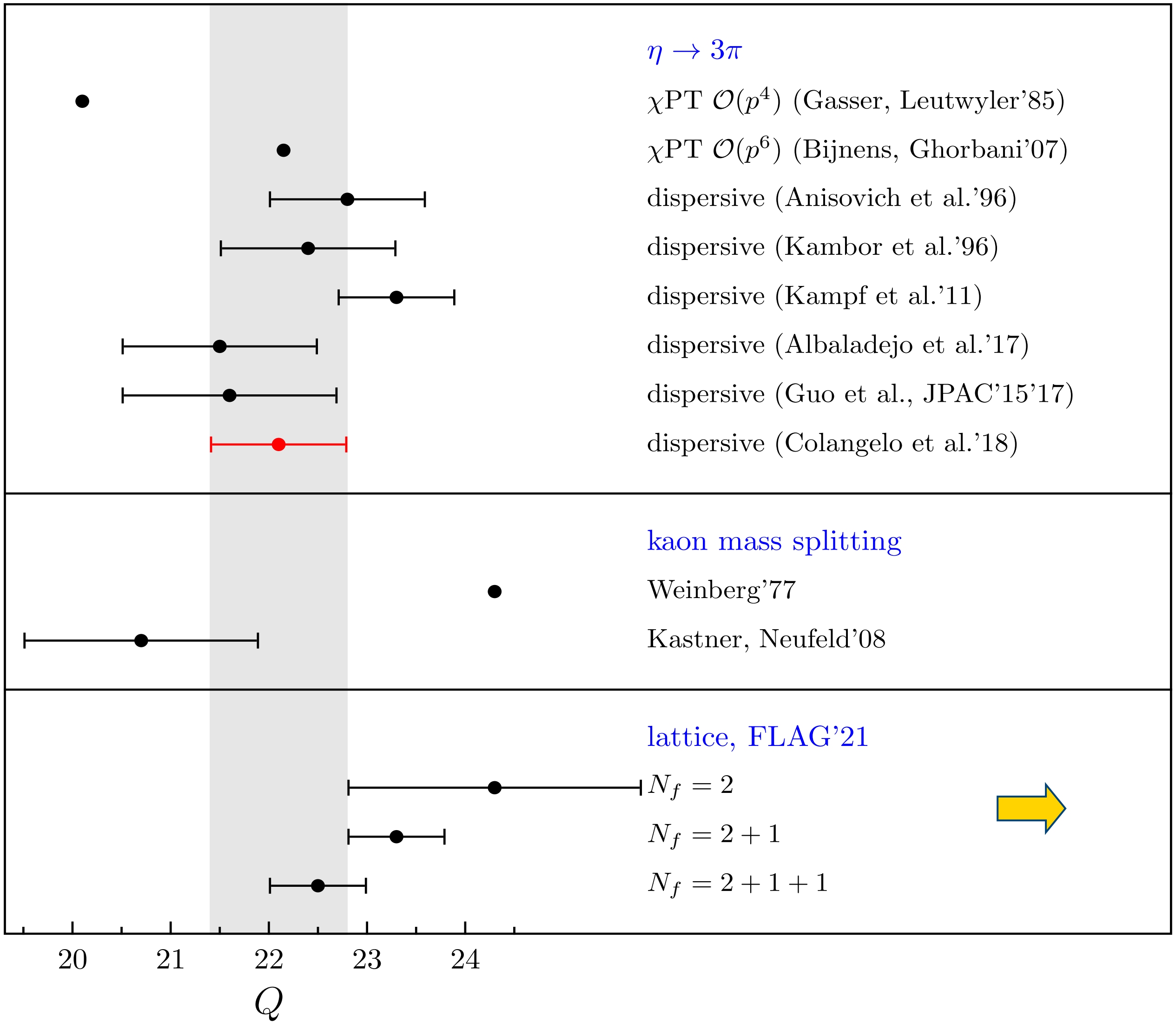
The agreement is *excellent* between our prediction and the data!

Comparison of results for α



$\alpha = -0.0307 \pm 0.0017$

4.5 Quark mass ratio



$$Q = 22.1 \pm 0.7$$

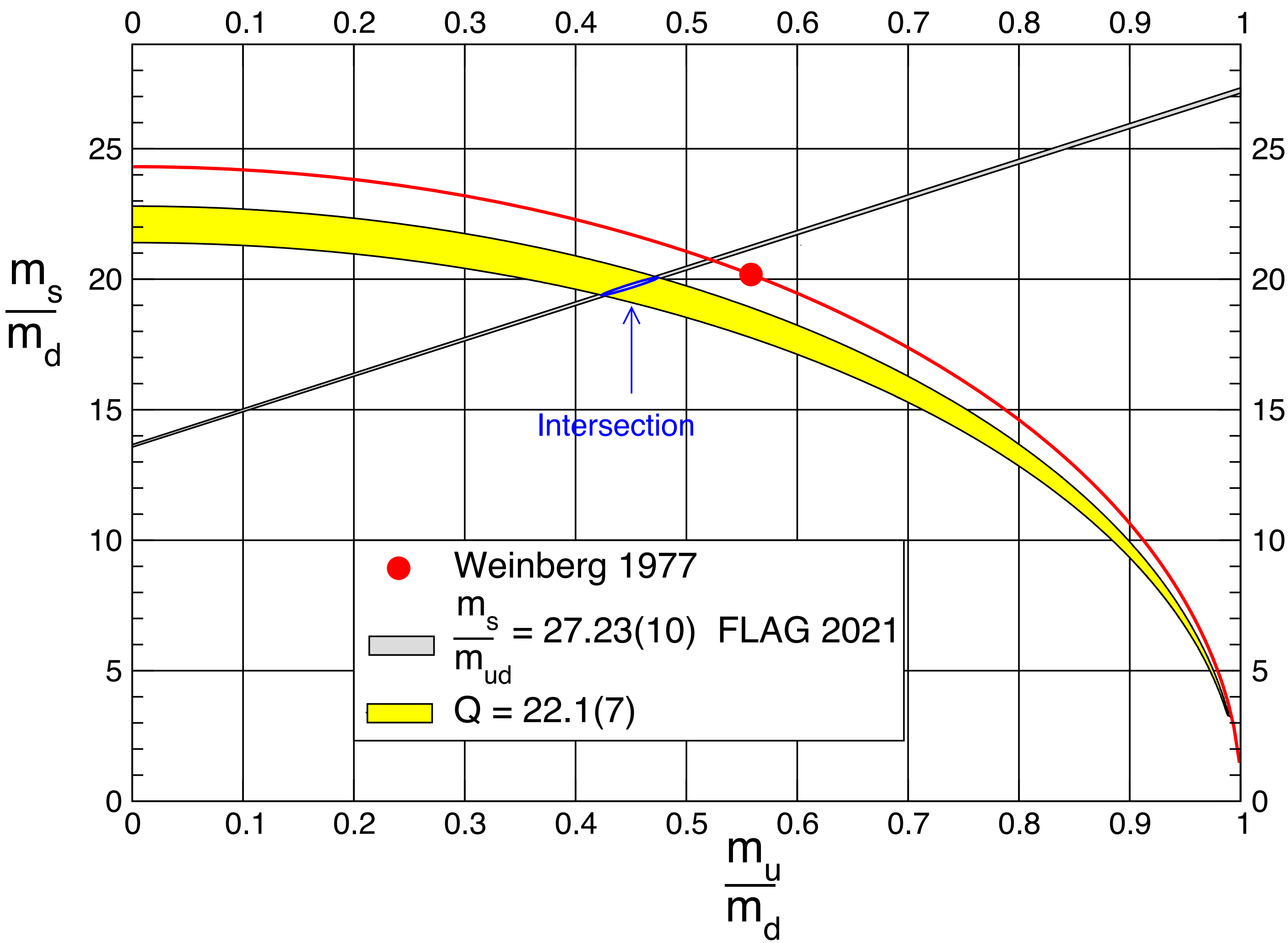
New lattice results

Shift of Q towards smaller values

Better *agreement* with $\eta \to 3\pi$ result

- Experimental systematics needs to be taken into account

4.5 Light quark masses

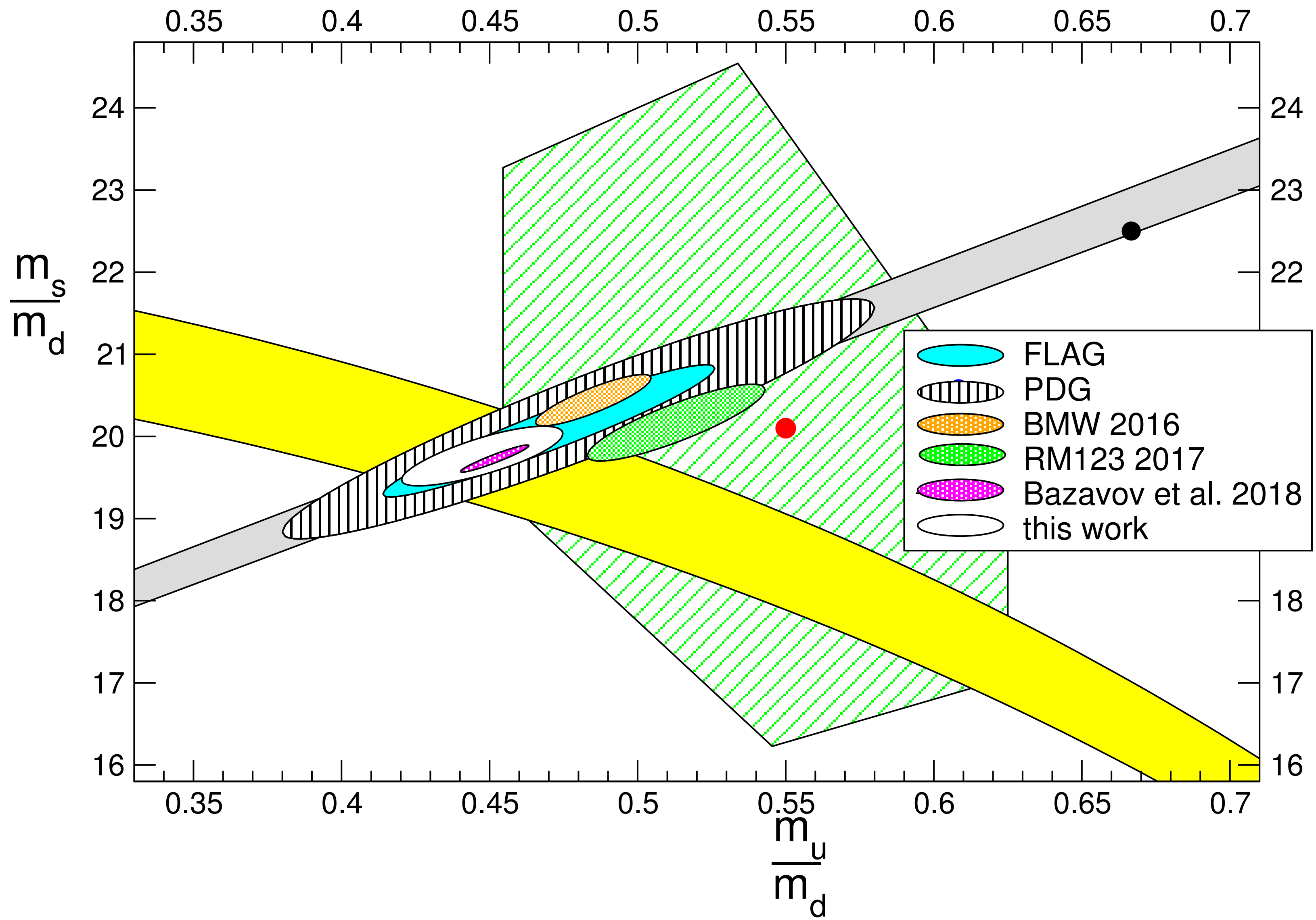


$Q = 22.1 \pm 0.7$

$\frac{m_u}{m_d} = 0.44 \pm 0.03$

- Smaller values for Q ⇨ smaller values for m_s/m_d and m_u/m_d than LO ChPT

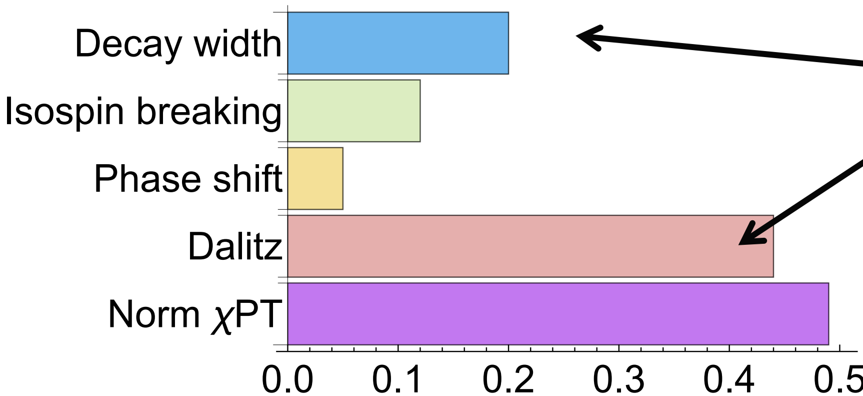
4.5 Light quark masses



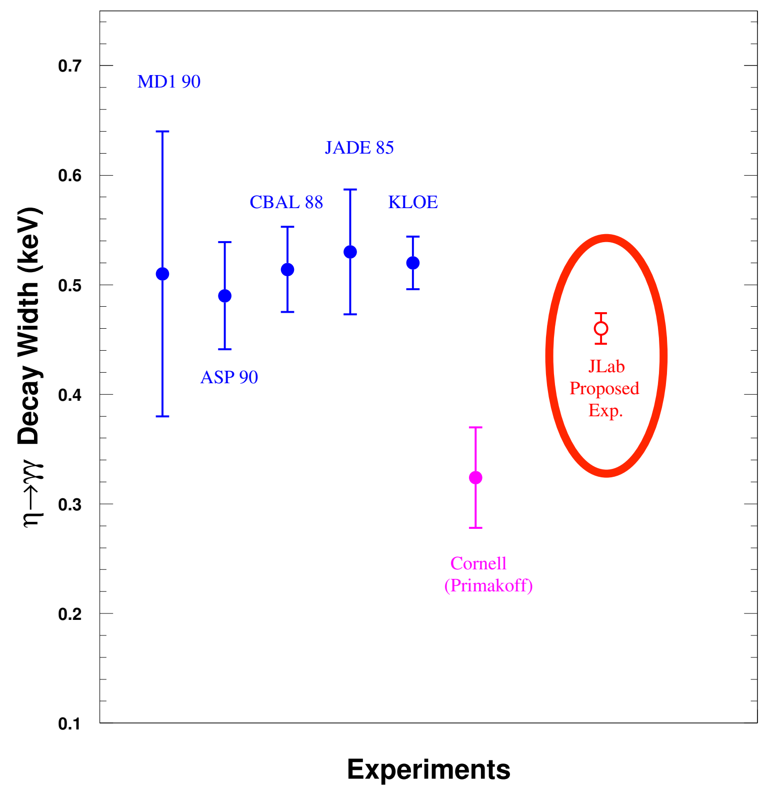
4.6 Prospects

Gan, Kubis, E. P., Tulin'22

- Uncertainties in the quark mass ratio

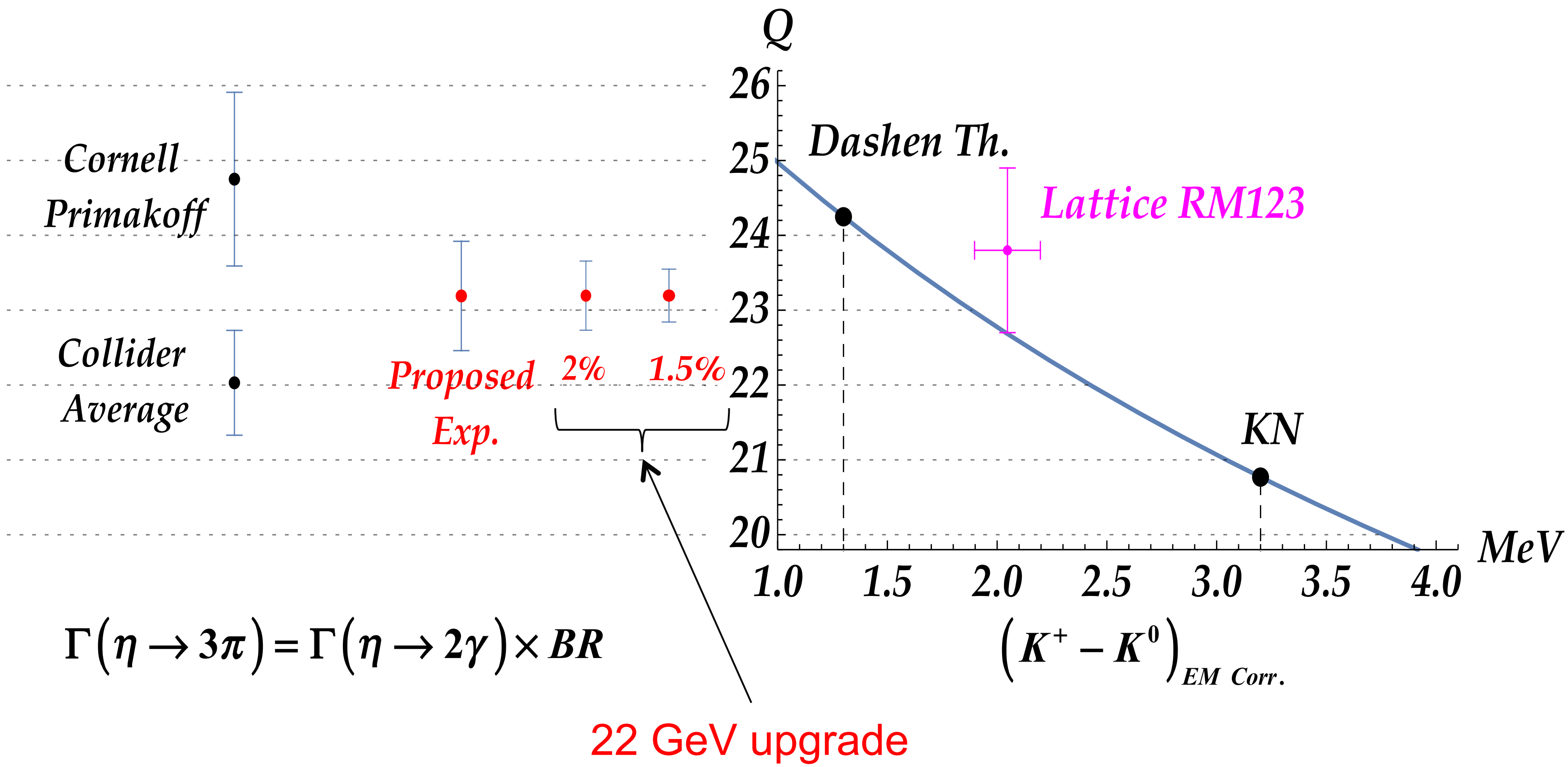


Can be investigated and reduced at *future facilities*



$$\Gamma_{\eta\to\pi^+\pi^-\pi^0} = \frac{1}{Q^4}\frac{M_K^4}{M_\pi^4}\frac{\left(M_K^2-M_\pi^2\right)^2}{6912\pi^3 F_\pi^4 M_\eta^3}\int_{s_{\min}}^{s_{\max}} ds \int_{u_-(s)}^{u_+(s)} du \, \left|M(s,t,u)\right|^2$$

4.7 Expected Impact of JLab 22 GeV program



$$\Gamma(\eta \to 3\pi) = \Gamma(\eta \to 2\gamma) \times BR$$

→ *JLab White Paper arXiv:2306.09360 [nucl-ex]*

Global Experimental Efforts in η Decays

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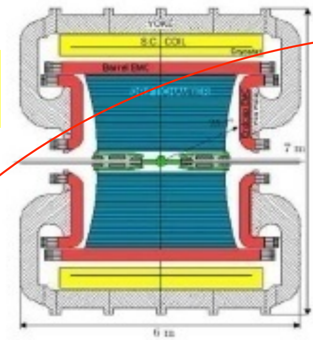
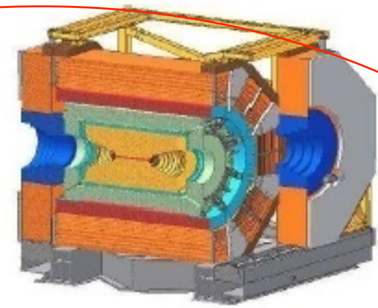
KLOE-2 at DAΦNE

BESIII at BEPCII

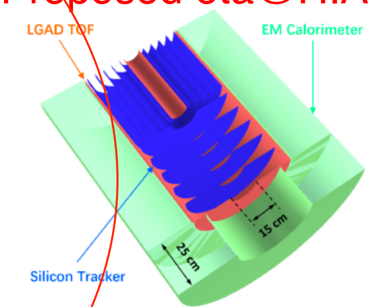
Low-energy

e⁺e⁻ Collider





Proposed eta@HIAF

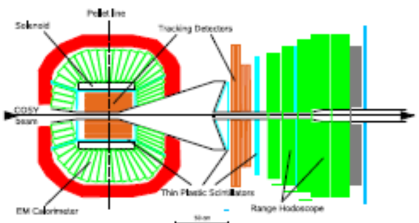


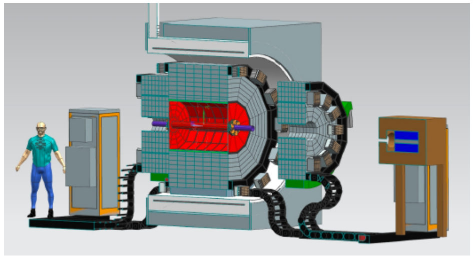
WASA at COSY

Proposed REDTOP

Fixed-target

hadroproduction





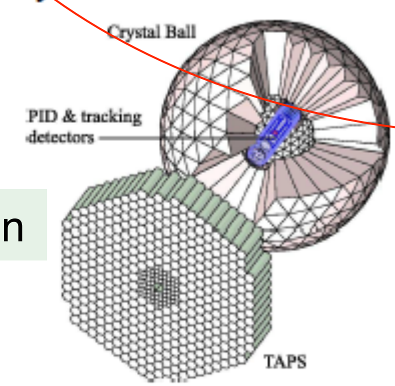
High-energy

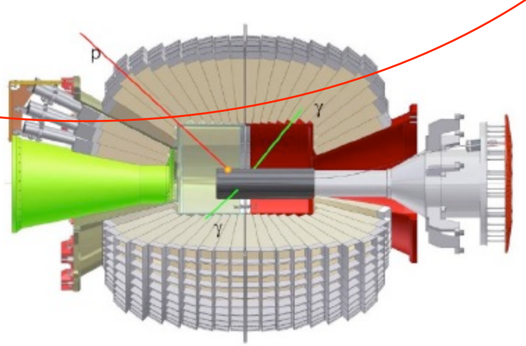
Crystall Ball at MAMI

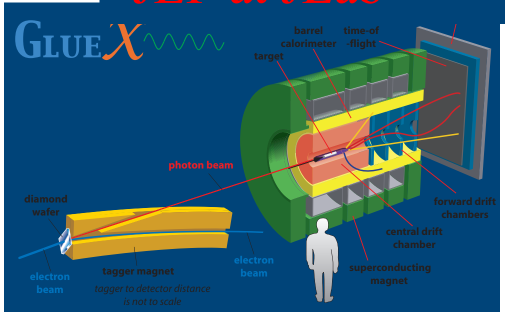
CBELSA/TAPS at ELSA

JEF at JLab

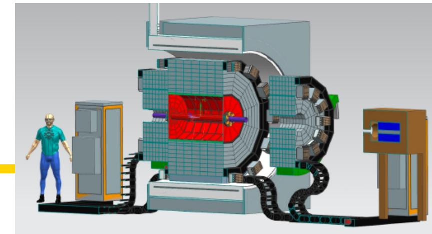
Photoproduction







A New Proposal: REDTOP



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Baseline option – medium-energy CW proton beam

- proton beam on thin Li/Be target : ~1.8 GeV - 30 W (10^{11} POT/sec)
- Low-cost, readily available (BNL, ESS, FNAL, GSI, HIAF)
- η : inelastic background = 1:200
- Untagged η production

vs LHCb@40 MHz

Inelastic interaction rate: ~ 0.7 GHz
Average event multiplicity ≈ 4 charged + 4 neutral
η/η' production rate: ~ 2.3 MHz

Preferred option – low-energy pion beam

- π^+ on Li/Be or π^- on LH: ~750 MeV - 2.5x10^9 πOT/sec
- More expensive but lower background (ESS, FNAL(?), FAIR, HIAF, ORNL)
- η : inelastic background = 1:50 → sensitivity to BSM increased by > 2x
- Semi-tagged η production

Inelastic interaction rate: ~ 0.1GHz
η/η' production rate: ~ 2.3 MHz

Ultimate option: Tagged 10^{13} η mesons

- high intensity proton beam on De target: ~0.9 GeV ; 0.1-1 MW
- Less readily available: (ESS, FAIR, CSNS, ORNL, PIP-II)
- Required fwd tagging detector for He_3^{++}
- Fully tagged production from nuclear reaction: $p+De \rightarrow \eta + He_3^{++}$

Inel. interaction rate: ~ 13 - 130 GHz
η/η' production rate: ~ 0.1 - 1 MHz

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arXiv:2203.07651

up to ~5x10^{13} η per year!

Another New Proposal: eta-Factory at HIAF

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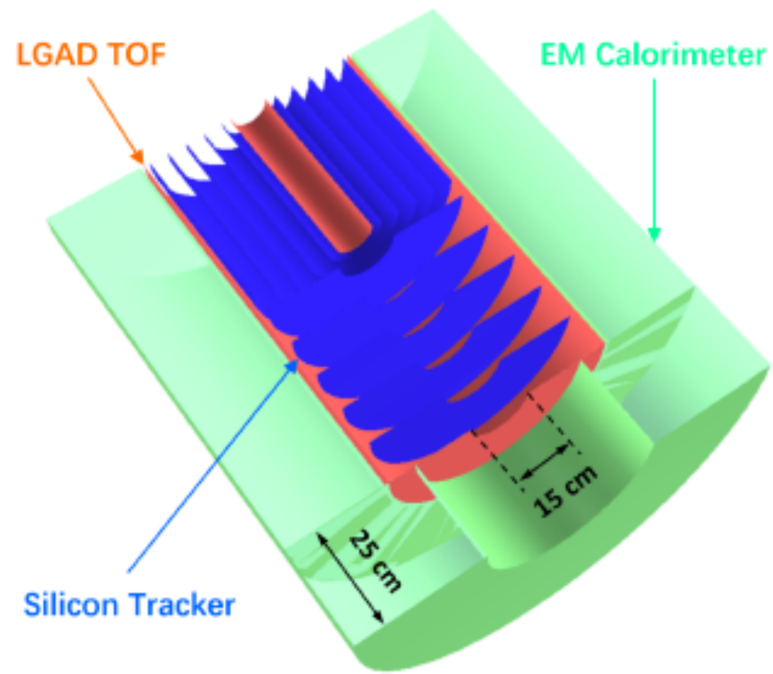
HIAF, Huizhou, China



arXiv:2407.00874v1

up to ~10^{13} η per year

5. Conclusion and Outlook

5.1 Conclusion

- η and η' allows to study the fundamental properties of QCD and test the SM
  - Extraction of fundamental parameters of the SM,
    ⇨ e.g. light quark masses
  - Study of chiral dynamics
  - Study of CP violation

- To studies η and η' with the best precision: Development of amplitude analysis techniques consistent with analyticity, unitarity, crossing symmetry dispersion relations allow to take into account *all rescattering effects* being as model independent as possible combined with ChPT ⇨ Provide parametrization for experimental studies

- In this talk, illustration with $\eta \to 3\pi$ and extraction of the light quark masses

- Many more topics could be explored with η and η'

Gan, Kubis, E. P., Tulin'22

5.2 Outlook

- New η and η' programs *JEF*, *REDTOP and HIEPA* *Gan, Kubis, E. P., Tulin'22*
- In our opinion the most promising channels to study:

Decay channel	Standard Model	Discrete symmetries	Light BSM particles
$\eta \to \pi^+\pi^-\pi^0$	light quark masses	C/CP violation	scalar bosons (also η')
$\eta^{(\prime)} \to \gamma\gamma$	η–η' mixing, precision partial widths		
$\eta^{(\prime)} \to \ell^+\ell^-\gamma$	$(g-2)_\mu$		Z' bosons, dark photon
$\eta \to \pi^0\gamma\gamma$	higher-order χPT, scalar dynamics		$U(1)_B$ boson, scalar bosons
$\eta^{(\prime)} \to \mu^+\mu^-$	$(g-2)_\mu$, precision tests	CP violation	
$\eta \to \pi^0\ell^+\ell^-$		C violation	scalar bosons
$\eta^{(\prime)} \to \pi^+\pi^-\ell^+\ell^-$	$(g-2)_\mu$		ALPs, dark photon
$\eta^{(\prime)} \to \pi^0\pi^0\ell^+\ell^-$		C violation	ALPs

- Synergies between different physics:
  - Standard Model precision analyses
  - Discrete symmetry tests
  - Search for light BSM particles

6. Back-up

Studying C & CP violation with $\eta \to 3\pi$ asymetries

Gardner & Shi'19
Akdag, Isken, Kubis'21
Akdag, Kubis, Wirzba'22

- $\eta(I^G = 0^+) \to 3\pi(I^G = 1^-)$ breaks G parity
  - In the SM: C conserved, isospin broken
  - Now in BSM: C broken, isospin either conserved or broken

$$\mathcal{M}(s,t,u) = \mathcal{M}_1^{C}(s,t,u) + \mathcal{M}_0^{\not{C}}(s,t,u) + \mathcal{M}_2^{\not{C}}(s,t,u)$$

- 2 additional amplitudes which are C violating:
  interference: $\pi^+ \leftrightarrow \pi^-$ asymmetries linear in BSM couplings

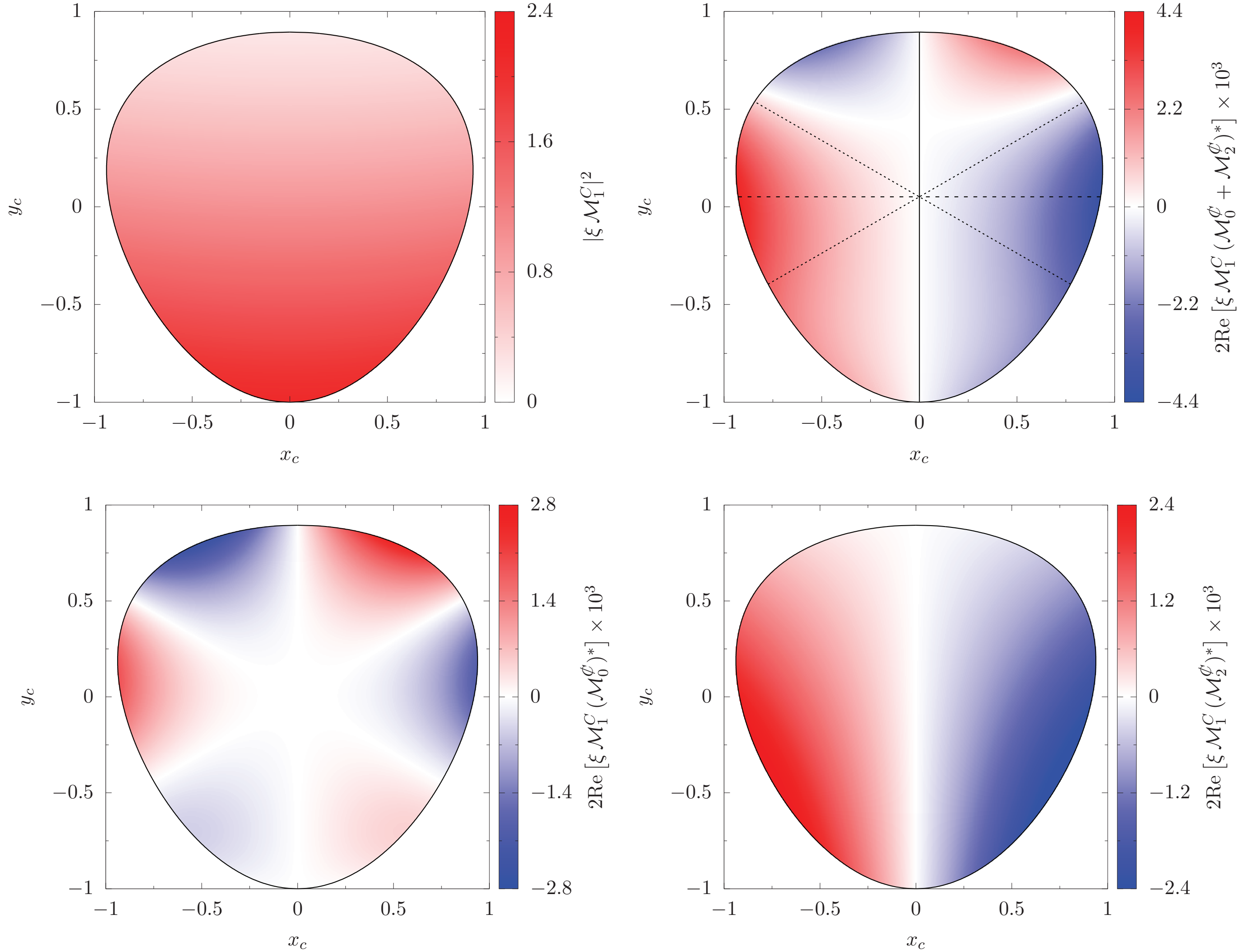
- Use KT approach to determine the hadronic amplitudes

$$|\mathcal{M}_c|^2 \approx |\mathcal{M}_1^{C}|^2 + 2\mathrm{Re}\left[\mathcal{M}_1^{C}\left(\mathcal{M}_0^{\not{C}}\right)^*\right] + 2\mathrm{Re}\left[\mathcal{M}_1^{C}\left(\mathcal{M}_2^{\not{C}}\right)^*\right]$$

- $\mathcal{M}_0^{\not{C}}$ and $\mathcal{M}_2^{\not{C}}$ lead to different interference patterns

Studying C & CP violation with $\eta \to 3\pi$ asymetries

Akdag, Isken, Kubis'21



- Asymmetries constrained to the *permille* level

Measurement of $\eta \rightarrow 3\pi$

- More information in the charged compared to the neutral channel

  ⇨ neutral channel sum over isospin:

$$\overline{A}(s,t,u) = A(s,t,u) + A(t,u,s) + A(u,s,t)$$

Only one Dalitz plot parameter determined α ⇨ $\left|A_n(s,t,u)\right|^2 = N\left(1 + 2\alpha Z\right)$

4.4 Dispersion Relations for the $M_I(s)$

- $$M_0(s) = \Omega_0(s)\left(\alpha_0 + \beta_0 s + \gamma_0 s^2 + \frac{s^3}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^3}\frac{\sin\delta_0^0(s')\,\hat{M}_0(s')}{|\Omega_0(s')|\left(s'-s-i\varepsilon\right)}\right)$$

  Omnès function

  Similarly for M_1 and M_2

- Four subtraction constants to be determined: α_0, β_0, γ_0 and one more in M_1 (β_1)

- Inputs needed for these and for the ππ phase shifts δ_l^I
  – M_0: ππ scattering, ℓ=0, I=0
  – M_1: ππ scattering, ℓ=1, I=1
  – M_2: ππ scattering, ℓ=0, I=2

- Solve dispersion relations numerically by an iterative procedure

Corrections to Dashen's theorem

- Dashen's Theorem

$$\left(M_{K^+}^2 - M_{K^0}^2\right)_{\rm em} = \left(M_{\pi^+}^2 - M_{\pi^0}^2\right)_{\rm em} \Rightarrow \boxed{\left(M_{K^+} - M_{K^0}\right)_{\rm em} = 1.3\ {\rm MeV}}$$

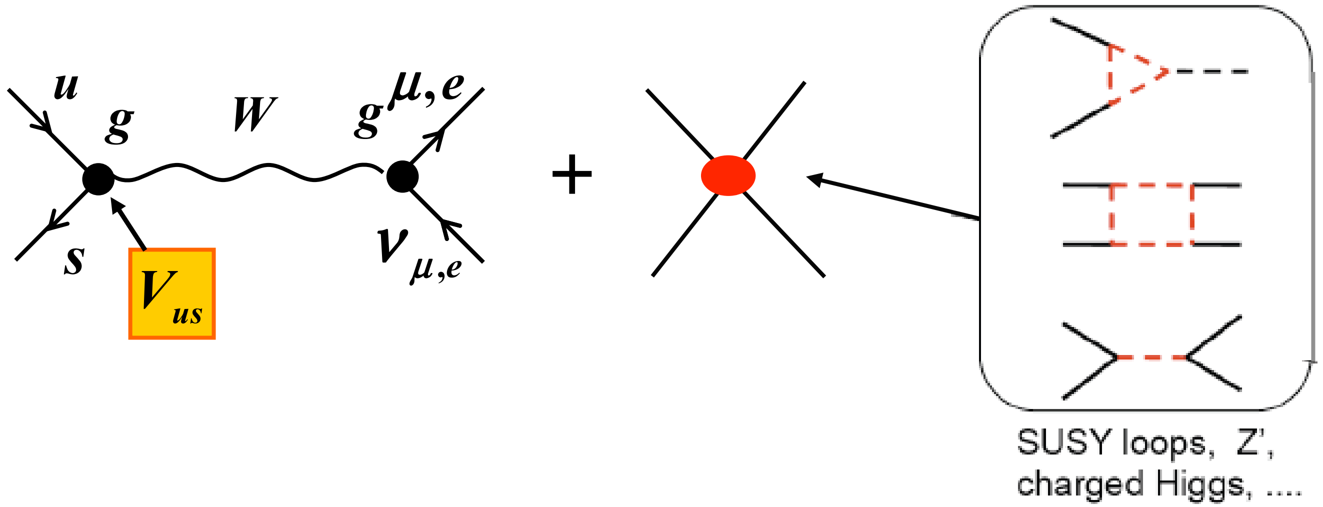
- With higher order corrections
  - Lattice : $\left(M_{K^+} - M_{K^0}\right)_{\rm em} = 1.9\ {\rm MeV},\ Q = 22.8$ *Ducan et al.'96*
  - ENJL model: $\left(M_{K^+} - M_{K^0}\right)_{\rm em} = 2.3\ {\rm MeV},\ Q = 22$ *Bijnens & Prades'97*
  - VMD: $\left(M_{K^+} - M_{K^0}\right)_{\rm em} = 2.6\ {\rm MeV},\ Q = 21.5$ *Donoghue & Perez'97*
  - Sum Rules: $\left(M_{K^+} - M_{K^0}\right)_{\rm em} = 3.2\ {\rm MeV},\ Q = 20.7$ *Anant & Moussallam'04*

Update $\Rightarrow$ $Q = 20.7 \pm 1.2$ *Kastner & Neufeld'07*

1.1 Light quark masses

- Fundamental unknowns of the the QCD Lagrangian
  In the following, consider the 3 light flavours u,d,s
- High precision physics at low energy as a key of new physics?
  $m_d - m_u$: small isospin breaking corrections but to be taken into account for high precision physics

  Ex: V_{us} from $K^{\pm}_{l3}\left(K^{\pm}\to\pi^0 l^{\pm}\nu_l\right)$ decays   *NA62, KLOE-2*



- No direct access to the quarks due to confinement!

3.8 Subtraction constants

- Extension of the numbers of parameters compared to *Anisovich & Leutwyler'96*

$$P_0(s) = \alpha_0 + \beta_0 s + \gamma_0 s^2 + \delta_0 s^3$$
$$P_1(s) = \alpha_1 + \beta_1 s + \gamma_1 s^2$$
$$P_2(s) = \alpha_2 + \beta_2 s + \gamma_2 s^2$$

- In the work of *Anisovich & Leutwyler'96* matching to one loop ChPT
  Use of the SU(2) x SU(2) chiral theorem
  ⇒ The amplitude has an *Adler zero* along the line s=u

- Now data on the Dalitz plot exist from KLOE, WASA, MAMI and BES III

  ⇒ Use the data to directly fit the subtraction constants

- However normalization to be fixed to ChPT!

3.8 Subtraction constants

- The subtraction constants are

$$P_0(s) = \alpha_0 + \beta_0 s + \gamma_0 s^2 + \delta_0 s^3$$

$$P_1(s) = \alpha_1 + \beta_1 s + \gamma_1 s^2$$

$$P_2(s) = \alpha_2 + \beta_2 s + \gamma_2 s^2 + \delta_0 s^3$$

Only *6 coefficients* are of physical relevance

- They are determined from combining ChPT with a fit to KLOE Dalitz plot

- Taylor expand the dispersive M_I
Subtraction constants ⟺ Taylor coefficients

$$M_0(s) = A_0 + B_0 s + C_0 s^2 + D_0 s^3 + \ldots$$

$$M_1(s) = A_1 + B_1 s + C_1 s^2 + \ldots$$

$$M_2(s) = A_2 + B_2 s + C_2 s^2 + D_2 s^3 +$$

- Gauge freedom in the decomposition of M(s,t,u)

3.8 Subtraction constants

- Build some gauge independent combinations of Taylor coefficients

$$H_0 = A_0 + \frac{4}{3}A_2 + s_0\left(B_0 + \frac{4}{3}B_2\right)$$

$$H_1 = A_1 + \frac{1}{9}(3B_0 - 5B_2) - 3C_2 s_0$$

$$H_2 = C_0 + \frac{4}{3}C_2, \qquad H_3 = B_1 + C_2$$

$$H_4 = D_0 + \frac{4}{3}D_2, \qquad H_5 = C_1 - 3D_2$$

⇒

$$H_0^{ChPT} = 1 + 0.176 + O\left(p^4\right)$$

$$h_1^{ChPT} = \frac{1}{\Delta_{\eta\pi}}\left(1 - 0.21 + O\left(p^4\right)\right)$$

$$h_2^{ChPT} = \frac{1}{\Delta_{\eta\pi}^2}\left(4.9 + O\left(p^4\right)\right)$$

$$h_3^{ChPT} = \frac{1}{\Delta_{\eta\pi}^2}\left(1.3 + O\left(p^4\right)\right)$$

$$\left[h_i \equiv \frac{H_i}{H_0}\right]$$

⇒

$$\chi^2_{theo} = \sum_{i=1}^{3}\left(\frac{h_i - h_i^{ChPT}}{\sigma_{h_i^{ChPT}}}\right)^2$$

$$\sigma_{h_i^{ChPT}} = 0.3\left|h_i^{NLO} - h_i^{LO}\right|$$

Hat functions

- Discontinuity of M_I: by definition $disc\left[M_I(s)\right] \equiv disc\left[f_\ell^I(s)\right]$

  ⇨ $f_1^I(s) = M_I(s) + \hat{M}_I(s)$

  with $\hat{M}_I(s)$ real on the right-hand cut

- The left-hand cut is contained in $\hat{M}_I(s)$

- Determination of $\hat{M}_I(s)$:
  subtract M_I from the partial wave projection of $M(s,t,u)$

  $$M(s,t,u) = M_0(s) + (s-u)M_1(t) + \ldots$$

- $\hat{M}_I(s)$ singularities in the t and u channels, depend on the other M_I
  Angular averages of the other functions ⇨ Coupled equations

Hat functions

- Ex: $$\hat{M}_0(s) = \frac{2}{3}\langle M_0 \rangle + 2(s - s_0)\langle M_1 \rangle + \frac{20}{9}\langle M_2 \rangle + \frac{2}{3}\kappa(s)\langle z M_1 \rangle$$

where $\langle z^n M_I \rangle(s) = \frac{1}{2}\int_{-1}^{1} dz\, z^n M_I(t(s,z)),$

$z = \cos\theta$ scattering angle

Non trivial angular averages ⇨ need to deform the integration path to avoid crossing cuts

Anisovich & Anselm'66

2.3 Computation of the amplitude

- What do we know?
- The amplitude has an Adler zero: soft pion theorem *Adler'85*

➡ Amplitude has a zero for :

$$p_{\pi^+} \to 0 \Rightarrow s = u = 0,\ t = M_\eta^2$$

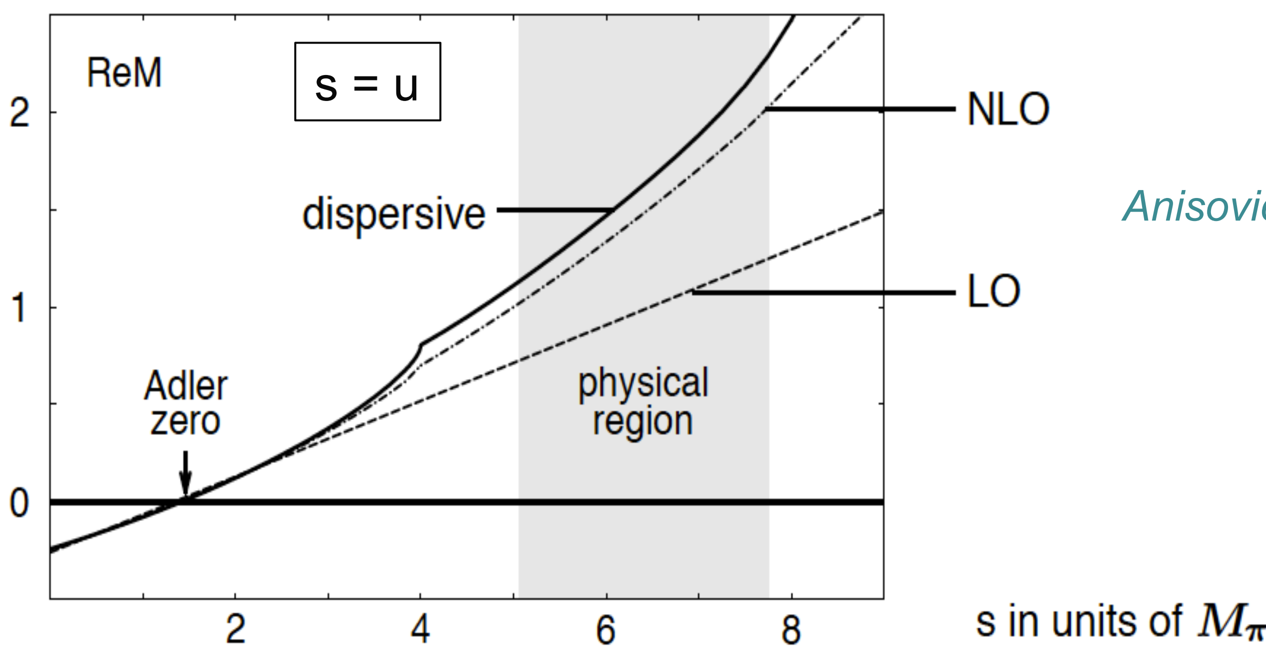
$$p_{\pi^-} \to 0 \Rightarrow s = t = 0,\ u = M_\eta^2$$

$M_\pi \neq 0$ ➡

$$s = u = \frac{4}{3} M_\pi^2,\ t = M_\eta^2 + \frac{M_\pi^2}{3}$$

$$s = t = \frac{4}{3} M_\pi^2,\ u = M_\eta^2 + \frac{M_\pi^2}{3}$$

SU(2) corrections



Anisovich & Leutwyler'96

2.4 Neutral channel : $\eta \to \pi^0 \pi^0 \pi^0$

- What do we know?

- We can relate charged and neutral channels

$$\overline{A}(s,t,u) = A(s,t,u) + A(t,u,s) + A(u,s,t)$$

⇨ *Correct formalism should be able to reproduce both charged and neutral channels*

- Ratio of decay width precisely measured

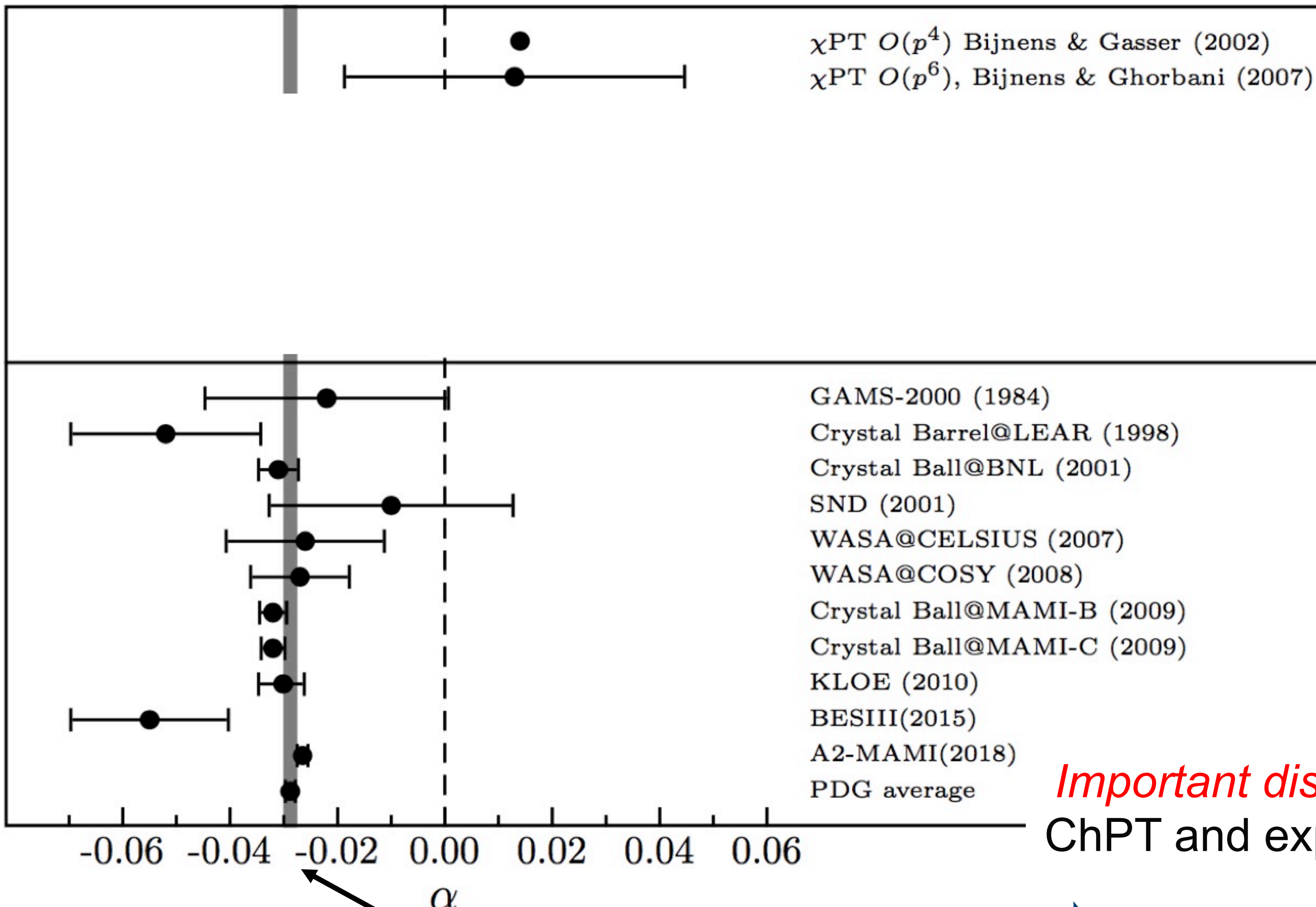
$$r = \frac{\Gamma\left(\eta \to \pi^0\pi^0\pi^0\right)}{\Gamma\left(\eta \to \pi^+\pi^-\pi^0\right)} = 1.426 \pm 0.026$$

PDG'19

2.4 Neutral Channel : $\eta \to \pi^0 \pi^0 \pi^0$

$$Q_n \equiv M_\eta - 3M_{\pi^0}$$

- Decay amplitude $\Gamma_{\eta\to 3\pi} \propto |\bar{A}|^2 \propto 1 + 2\alpha Z$ with $Z = \frac{2}{3}\sum_{i=1}^{3}\left(\frac{3T_i}{Q_n} - 1\right)^2$



$\alpha \sim 0.015$

$\alpha = -0.0288 \pm 0.0012$

Important discrepancy between ChPT and experiment!

⇨ Help of a dispersive treatment?

3.3 Neutral Channel : $\eta \to \pi^0 \pi^0 \pi^0$

- Decay amplitude $\Gamma_{\eta\to 3\pi} \propto \left|\bar{A}\right|^2 \propto 1 + 2\alpha Z$ with $Z = \frac{2}{3}\sum_{i=1}^{3}\left(\frac{3T_i}{Q_n} - 1\right)^2$

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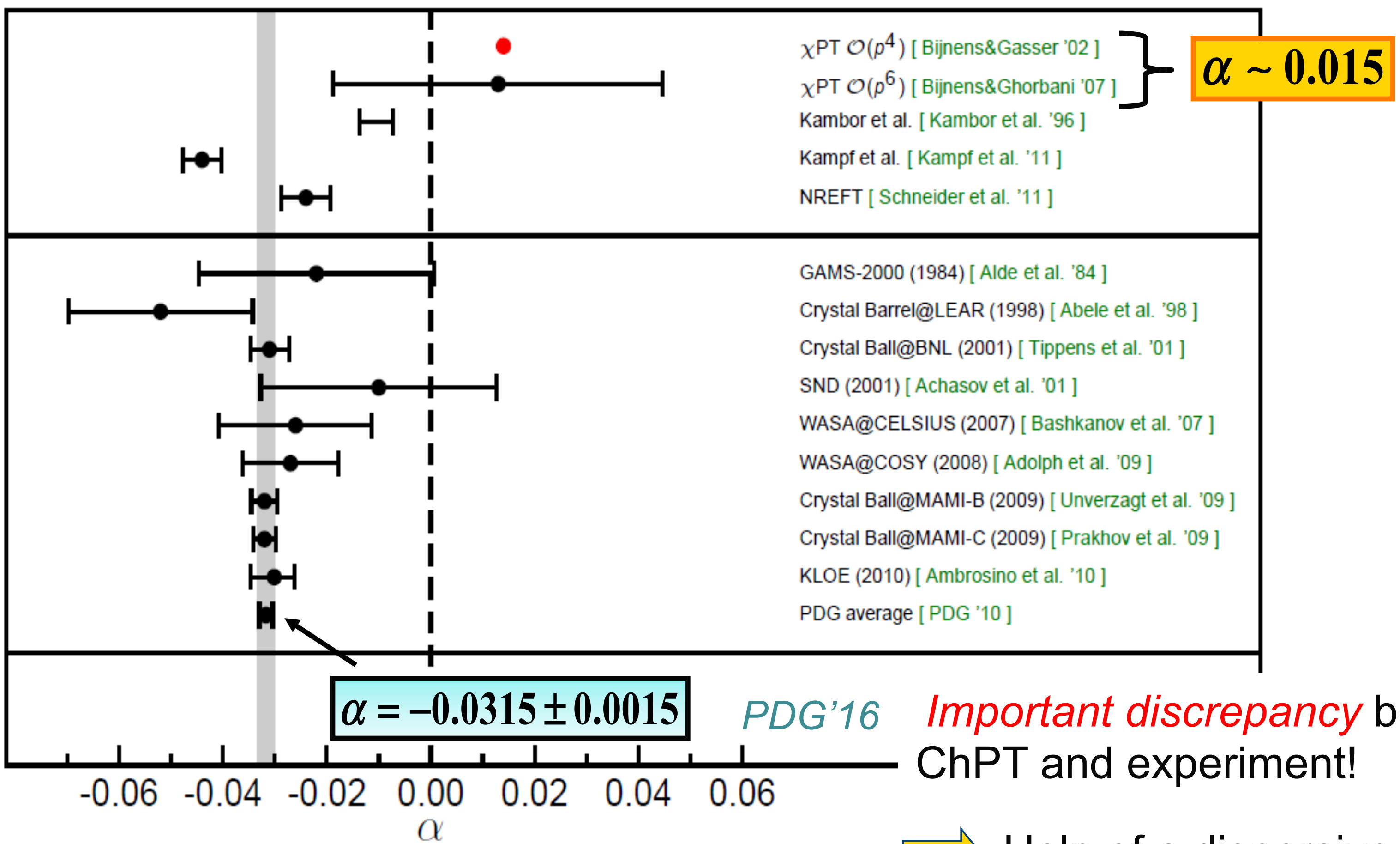


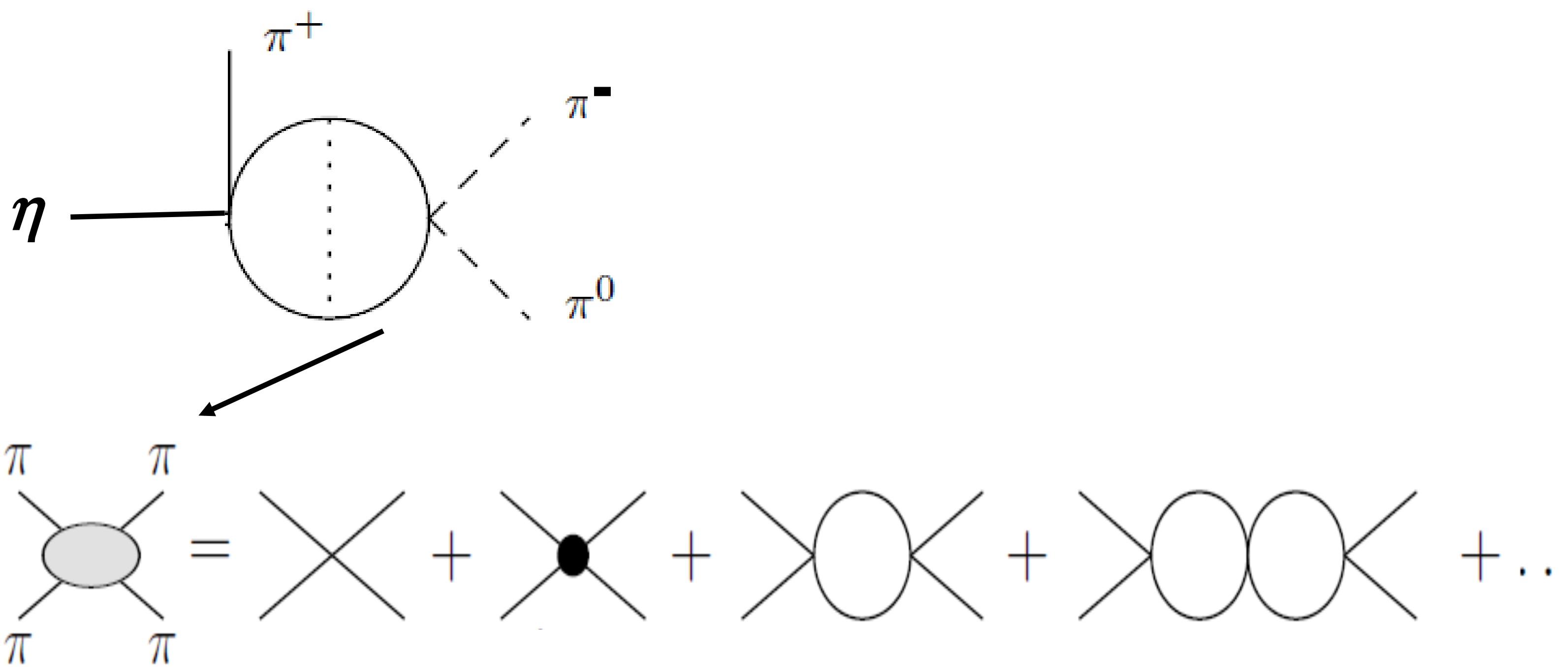
Important discrepancy between ChPT and experiment!

⇒ Help of a dispersive treatment?

2.5 Dispersive treatment

- The Chiral series has convergence problems

➡ Large *ππ final state interactions* *Roiesnel & Truong'81*



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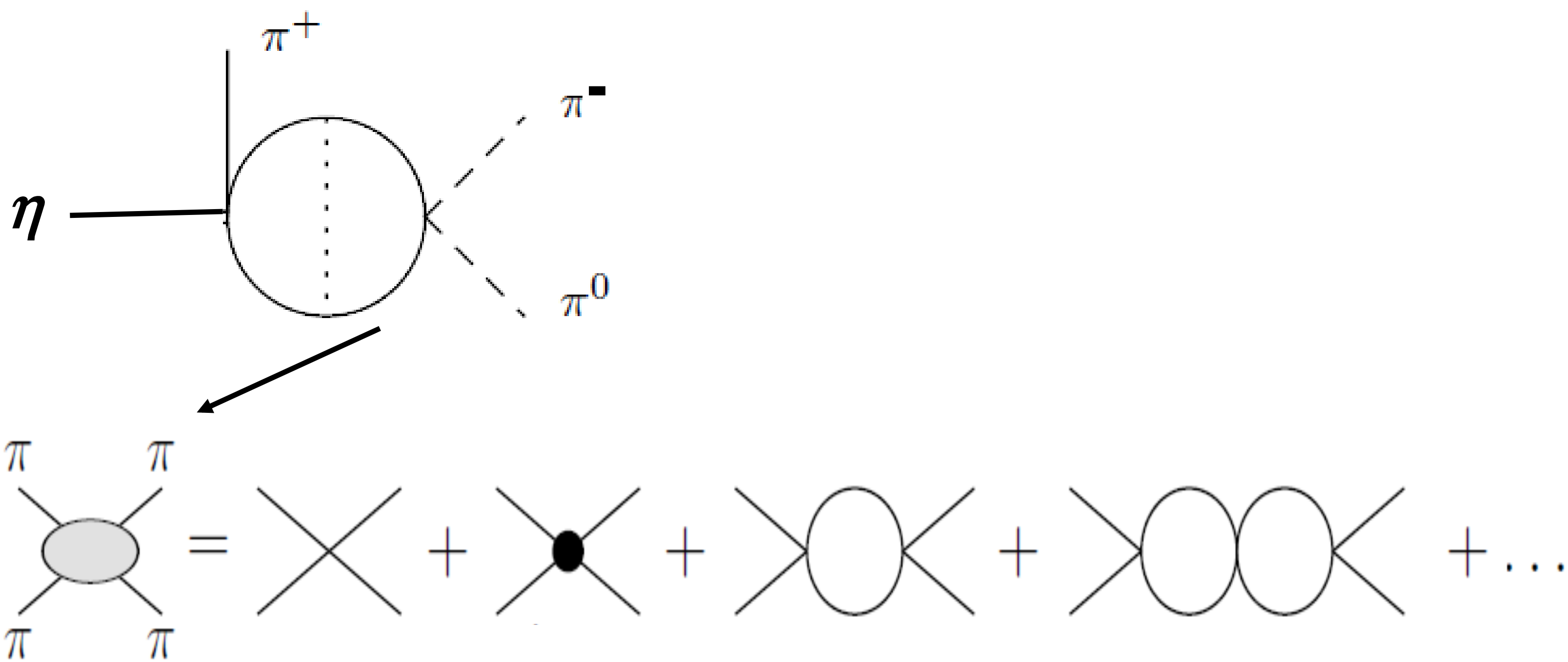


- *Dispersive treatment :*
  – analyticity, unitarity and crossing symmetry
  – Take into account all the rescattering effects

2.6 Why a new dispersive analysis?

- Several new ingredients:
  - New inputs available: extraction $\pi\pi$ phase shifts has improved

    Ananthanarayan et al'01, Colangelo et al'01

    Descotes-Genon et al'01

    Kaminsky et al'01, Garcia-Martin et al'09

  - New experimental programs, precise Dalitz plot measurements

    TAPS/CBall-MAMI (Mainz), WASA-Celsius (Uppsala), WASA-Cosy (Juelich)

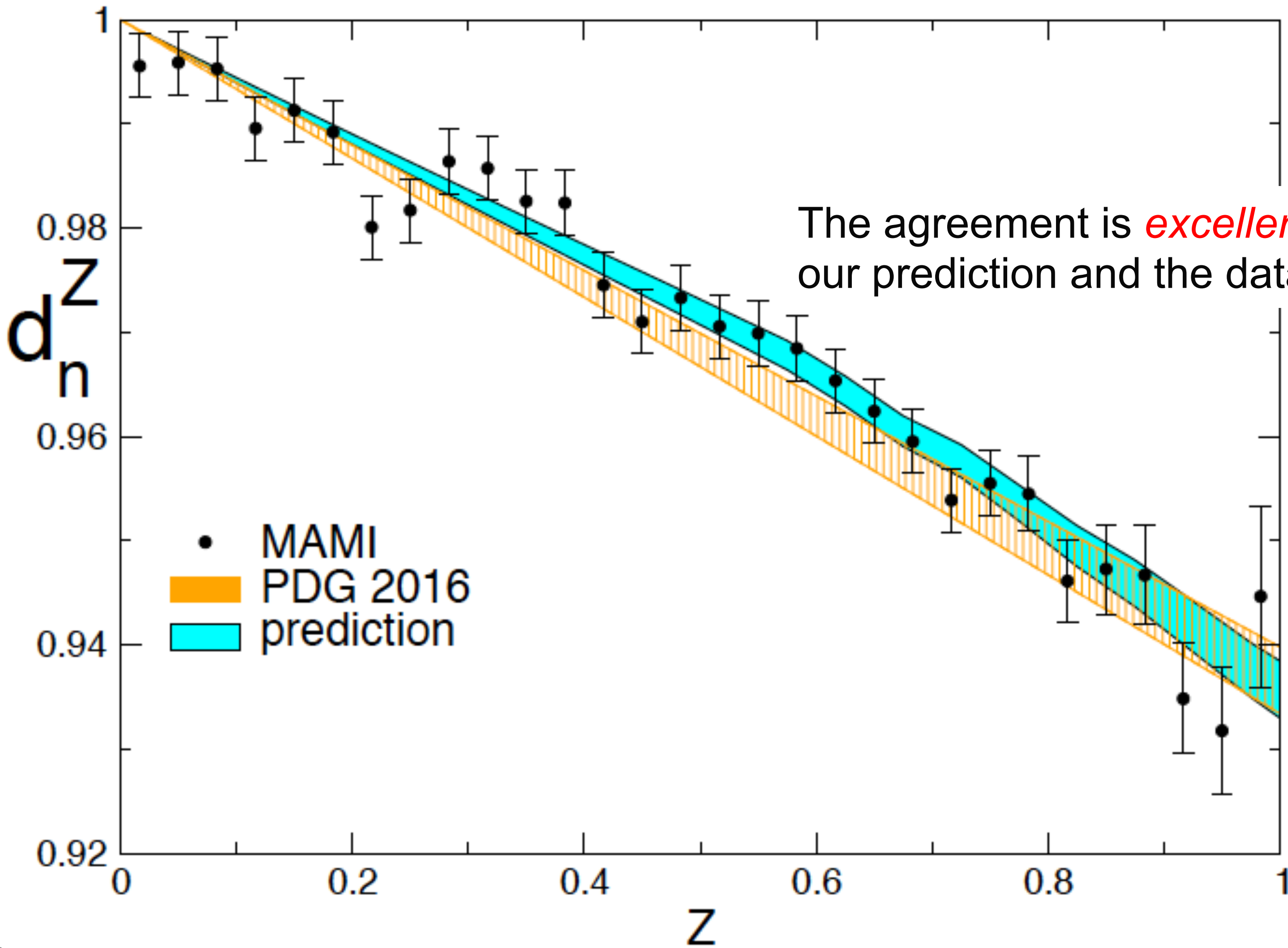
    CBall-Brookhaven, CLAS, GlueX (JLab), KLOE I-II (Frascati)

    BES III (Beijing)

  - Many improvements needed in view of very precise data: inclusion of
    - Electromagnetic effects ($\mathcal{O}(e^2m)$) *Ditsche, Kubis, Meissner'09*
    - Isospin breaking effects *Gullstrom, Kupsc, Rusetsky'09, Schneider, Kubis, Ditsche'11*
    - Inelasticities *Albaladejo & Moussallam'15*

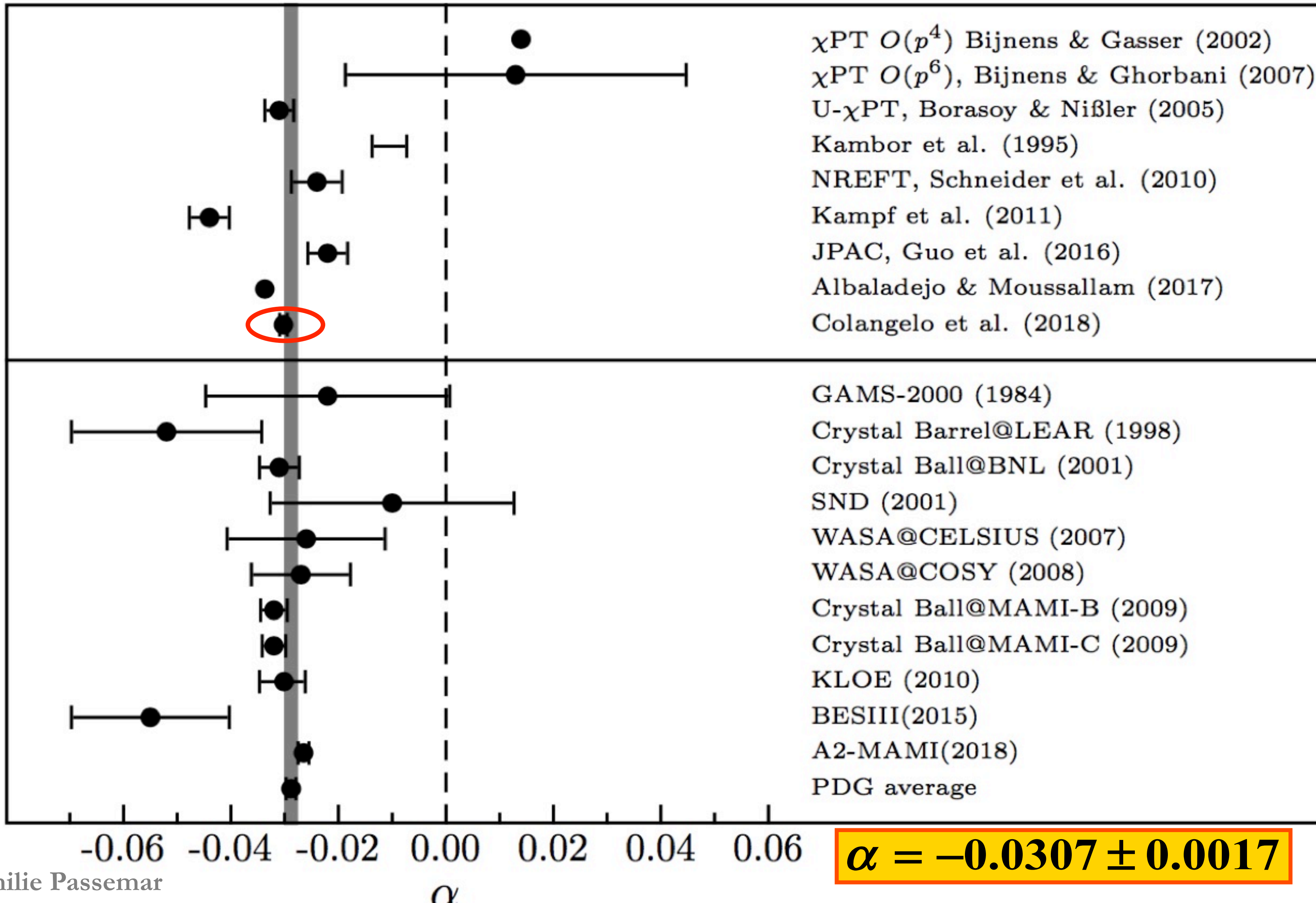
2.11 Z distribution for $\eta \to \pi^0 \pi^0 \pi^0$ decays

- The amplitude squared in the neutral channel is



The agreement is *excellent* between our prediction and the data!

2.12 Comparison of results for α



$$\alpha = -0.0307 \pm 0.0017$$

Experimental Facilities and Role of JLab 12

M. J. Amaryan et al.
CLAS Analysis Proposal, (2014)

π	$e^+e^-\gamma$			
η	$e^+e^-\gamma$	$\pi^+\pi^-\gamma$	$\pi^+\pi^-\pi^0$, $\pi^+\pi^-$	$\pi^+\pi^-e^+e^-$
η'	$e^+e^-\gamma$	$\pi^+\pi^-\gamma$	$\pi^+\pi^-\pi^0$, $\pi^+\pi^-$	$\pi^+\pi^-\eta$, $\pi^+\pi^-e^+e^-$
ρ		$\pi^+\pi^-\gamma$		
ω	$e^+e^-\pi^0$	$\pi^+\pi^-\gamma$	$\pi^+\pi^-\pi^0$	
φ			$\pi^+\pi^-\pi^0$	$\pi^+\pi^-\eta$

2.3 Computation of the amplitude

- What do we know?

- Compute the amplitude using ChPT : the effective theory that describe dynamics of the Goldstone bosons (kaons, pions, eta) at low energy

- Goldstone bosons interact weakly at low energy and $m_u, m_d \ll m_s < \Lambda_{QCD}$
  Expansion organized in external momenta and quark masses

Weinberg's power counting rule

$$\mathcal{L}_{eff} = \sum_{d\geq 2} \mathcal{L}_d \, , \mathcal{L}_d = \mathcal{O}\left(p^d\right) \, , p \equiv \left\{q, m_q\right\}$$

$$\mathbf{p} << \Lambda_H = 4\pi F_\pi \sim 1 \text{ GeV}$$

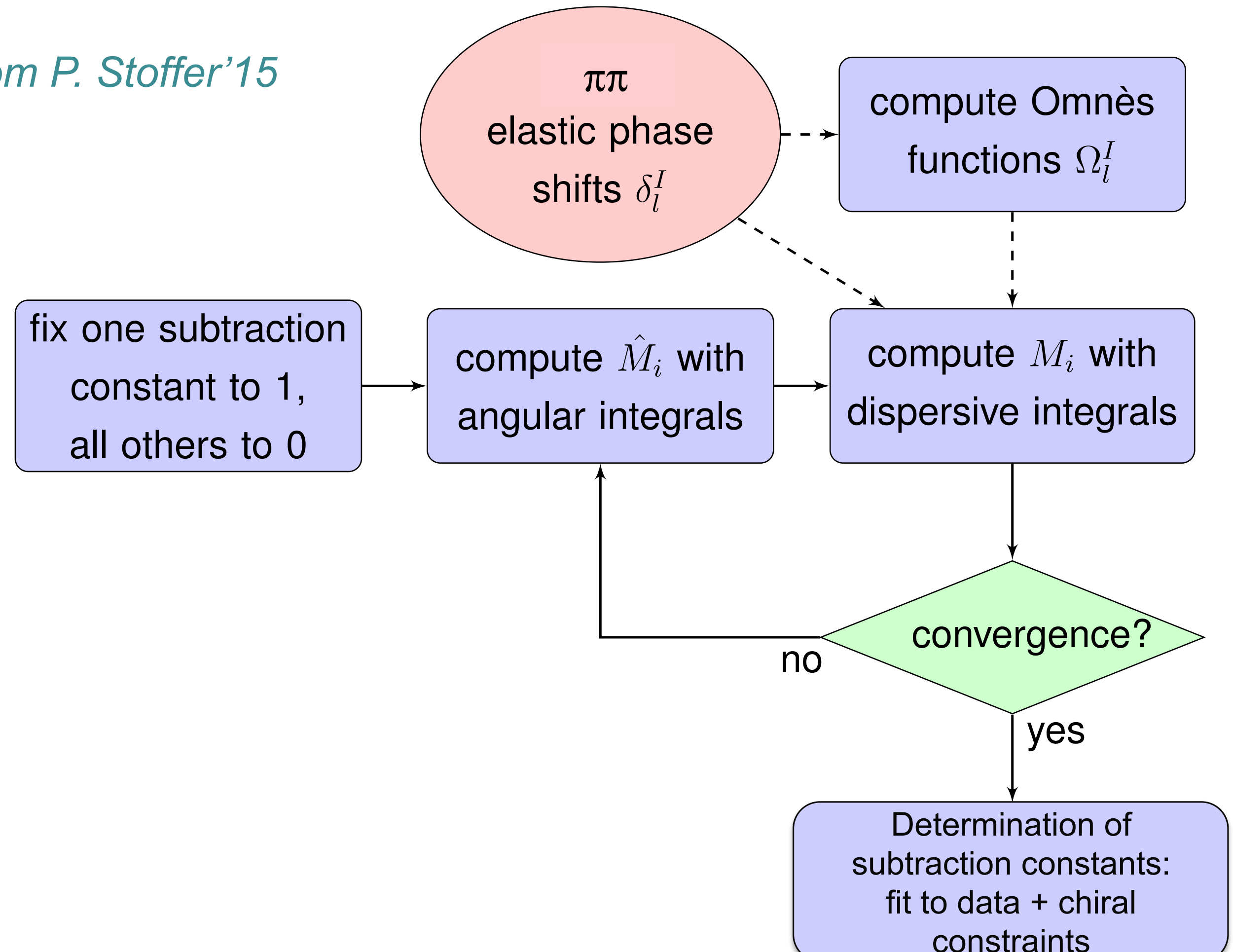
2.5 Iterative Procedure

- Solution *linear* in the *subtraction constants* *Anisovich & Leutwyler'96*

$$M(s,t,u) = \alpha_0 M_{\alpha_0}(s,t,u) + \beta_0 M_{\beta_0}(s,t,u) + \ldots$$

⇨ makes the fit much easier

Adapted from P. Stoffer'15



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$$P_0(s) = \alpha_0 + \beta_0 s + \gamma_0 s^2 + \delta_0 s^3$$
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- Gauge freedom in the decomposition of M(s,t,u)

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- Build some gauge independent combinations of Taylor coefficients

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$$\left[h_i \equiv \frac{H_i}{H_0} \right]$$

$\Rightarrow$

$$\chi^2_{theo} = \sum_{i=1}^{3} \left(\frac{h_i - h_i^{ChPT}}{\sigma_{h_i^{ChPT}}} \right)^2$$

$$\sigma_{h_i^{ChPT}} = 0.3 \left| h_i^{NLO} - h_i^{LO} \right|$$

Isospin breaking corrections

- Dispersive calculations in the isospin limit ⇨ to fit to data one has to include isospin breaking corrections

- $$M_{c/n}(s,t,u) = M_{disp}(s,t,u)\frac{M_{DKM}(s,t,u)}{\tilde{M}_{GL}(s,t,u)}$$

with M_{DKM} : amplitude at one loop with $\mathcal{O}(e^2 m)$ effects

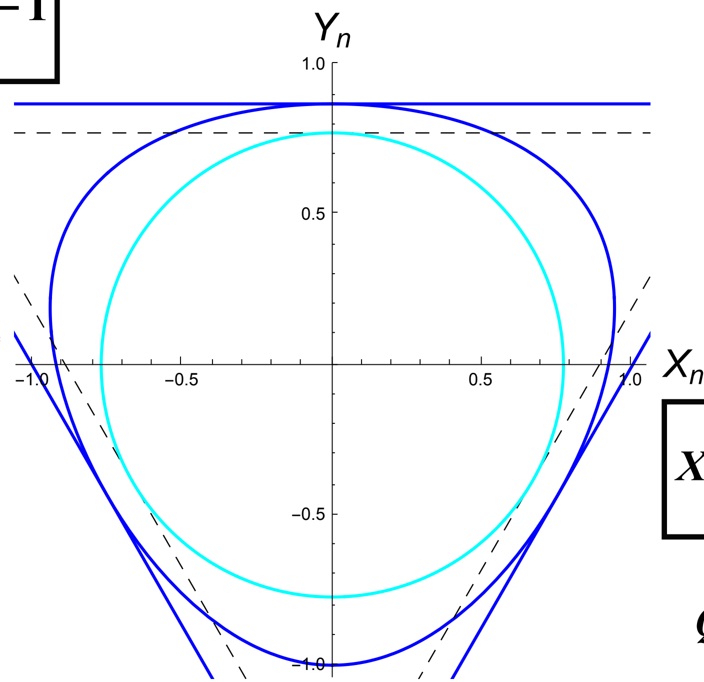
Ditsche, Kubis, Meissner'09

M_{GL}: amplitude at one loop in the isospin limit

Gasser & Leutwyler'85

$$Y_n = \frac{3T_3}{Q_n} - 1$$

Neutral channel



$$X_n = \sqrt{3}\,\frac{T_2 - T_1}{Q_n}$$

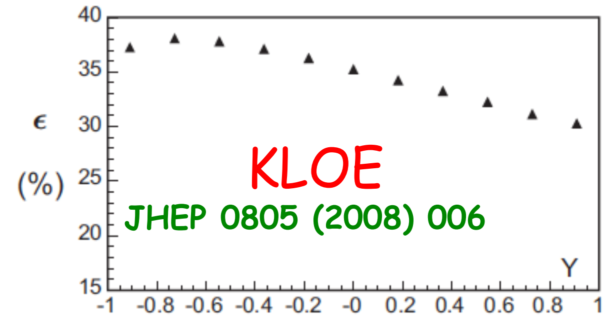
$$Q_n \equiv M_\eta - 3M_{\pi^0}$$

Kinematic map: isospin symmetric boundaries ⇨ physical boundaries

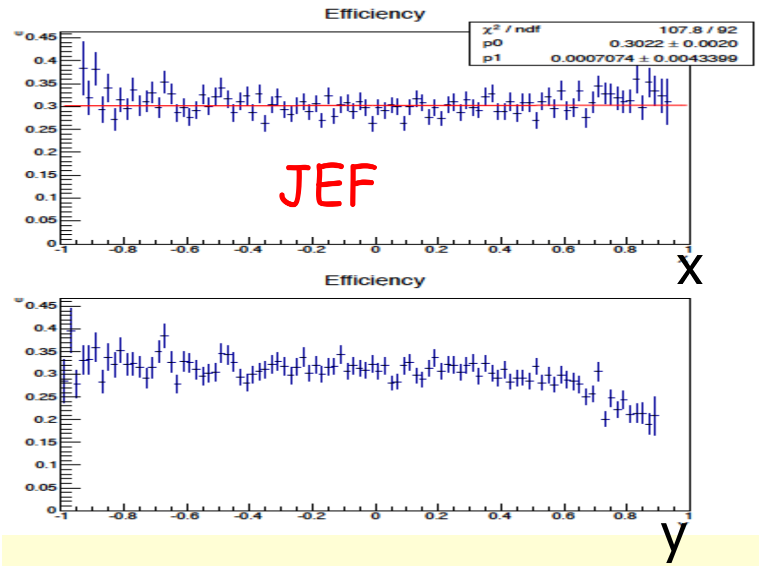
$$M_{GL} \rightarrow \tilde{M}_{GL}$$

2.15 Prospects

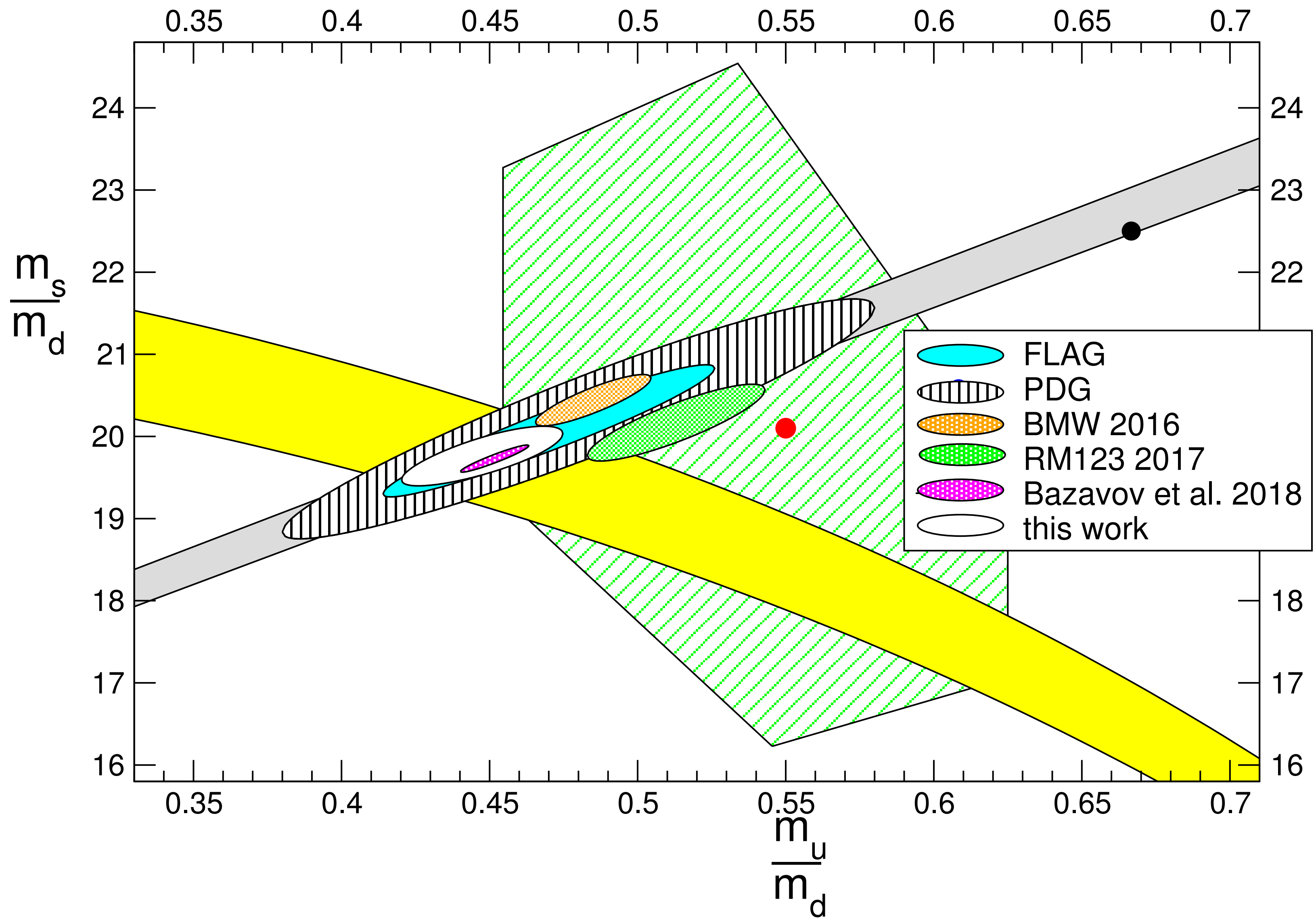
Exp.	$3\pi^0$ Events (10^6)	$\pi^+ \pi^- \pi^0$ Events (10^6)
Total world data (include prel. WASA and prel. KLOE)	6.5	6.0
GlueX+PrimEx-η +JEF	20	19.6



- Existing data from the low energy facilities are sensitive to the detection threshold effects
- JEF at high energy has uniform detection efficiency over Dalitz phase space
- JEF will offer large statistics and different systematics



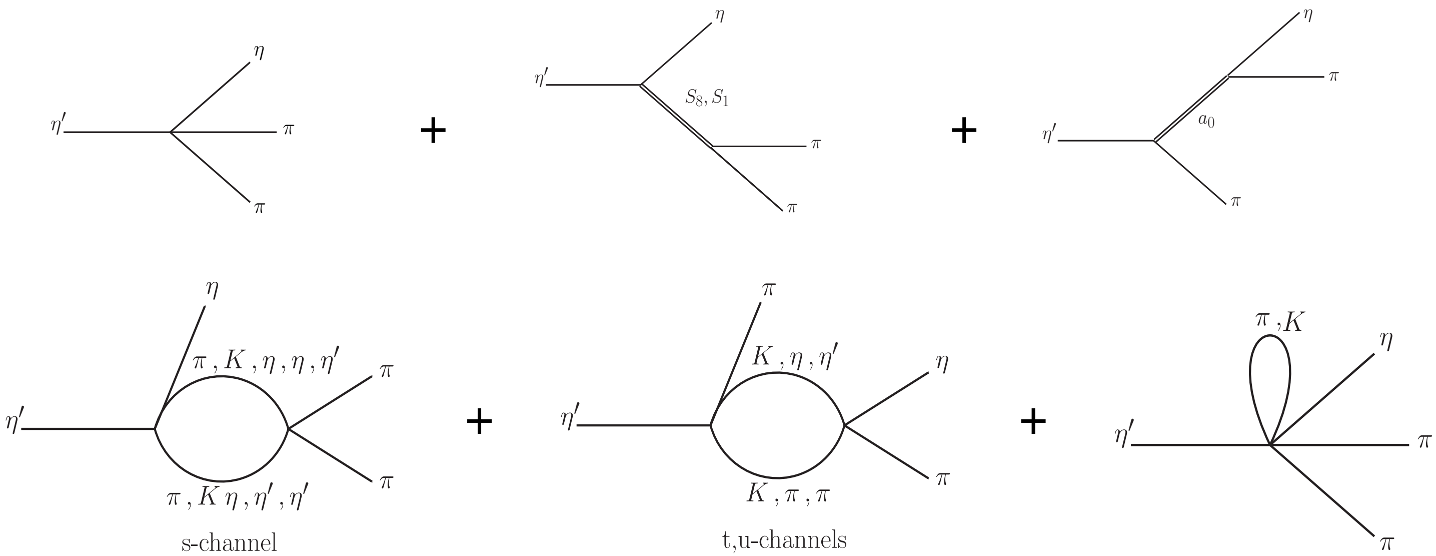
2.14 Comparison with Lattice

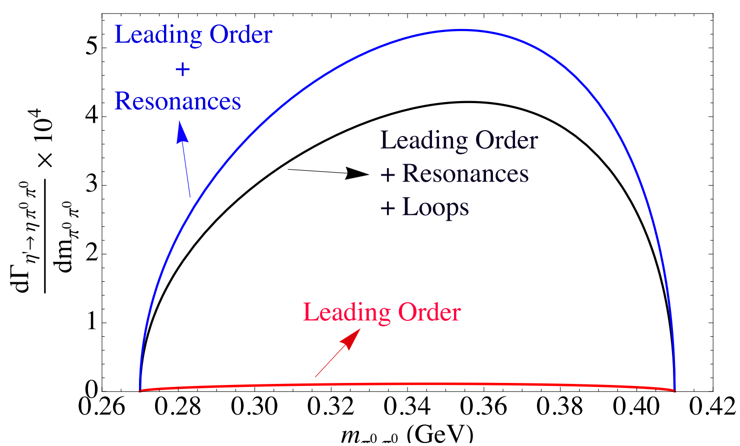


3.2 Theoretical Framework

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \eta_8 \\ \eta_1 \end{pmatrix}$$

- U(3) ChPT with resonances at one-loop





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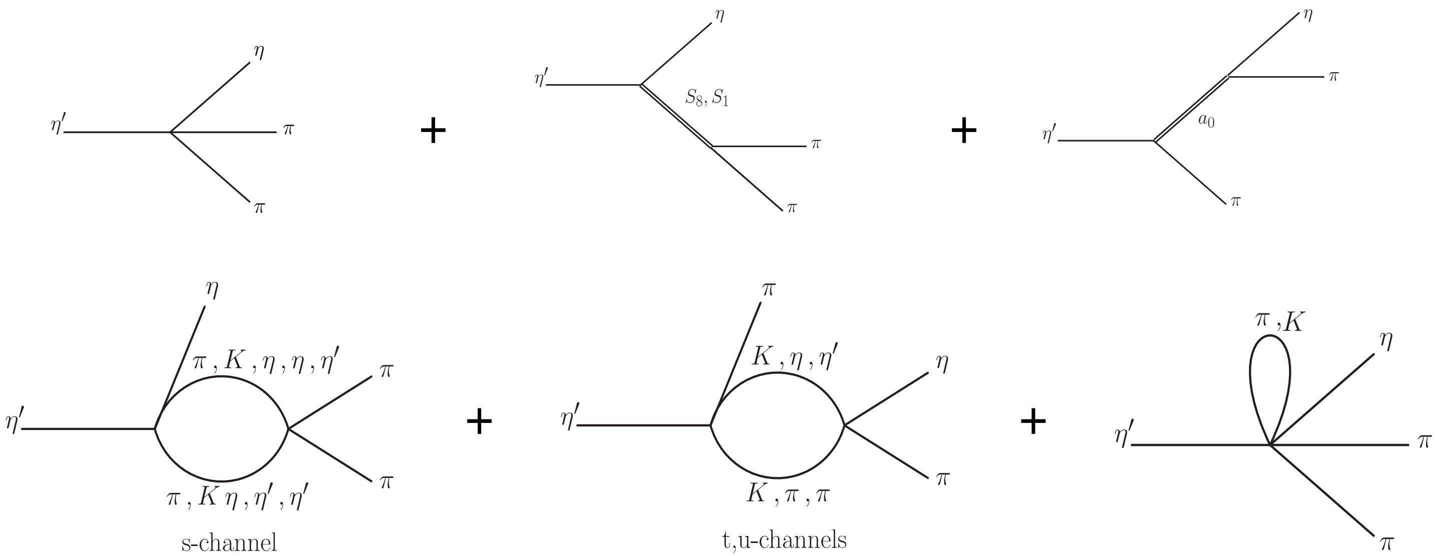
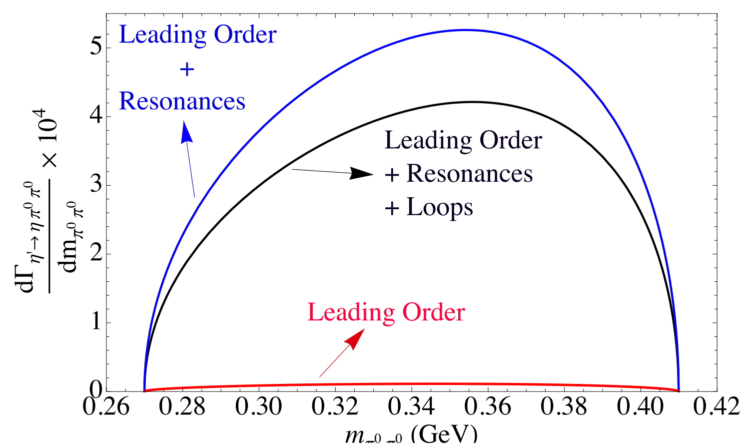
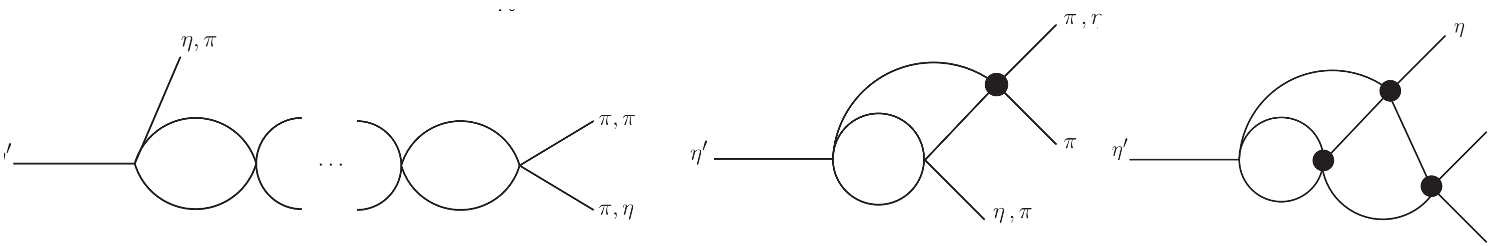




Final-state interaction through the N/D unitarization method

- Unitarity relations

$$\mathrm{Im}\mathcal{M}_{\eta'\to\eta\pi\pi} = \frac{1}{2}\sum_n (2\pi)^4\,\delta^4\left(p_\eta + p_1 + p_2 - p_n\right)\mathcal{T}^*_{n\to\eta\pi\pi}\mathcal{M}_{\eta'\to n}$$



not included

- A dispersive analysis also exists by *Isken et al.'17* but here we include D waves as well as kaon loops