Regge Theory

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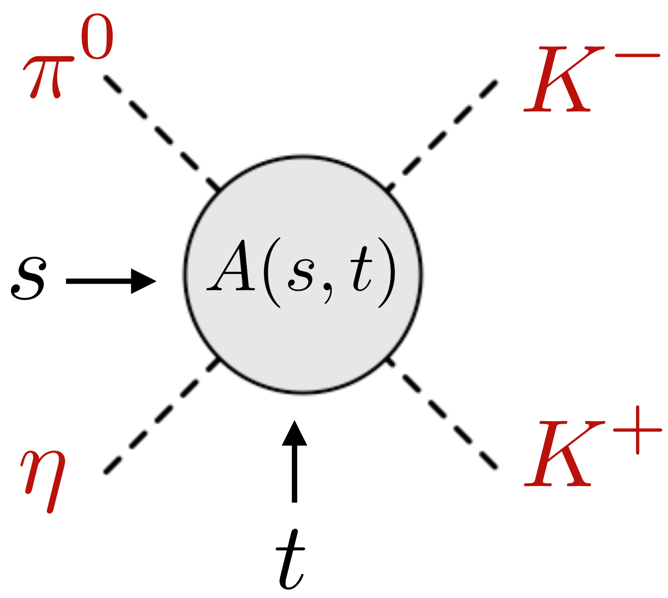
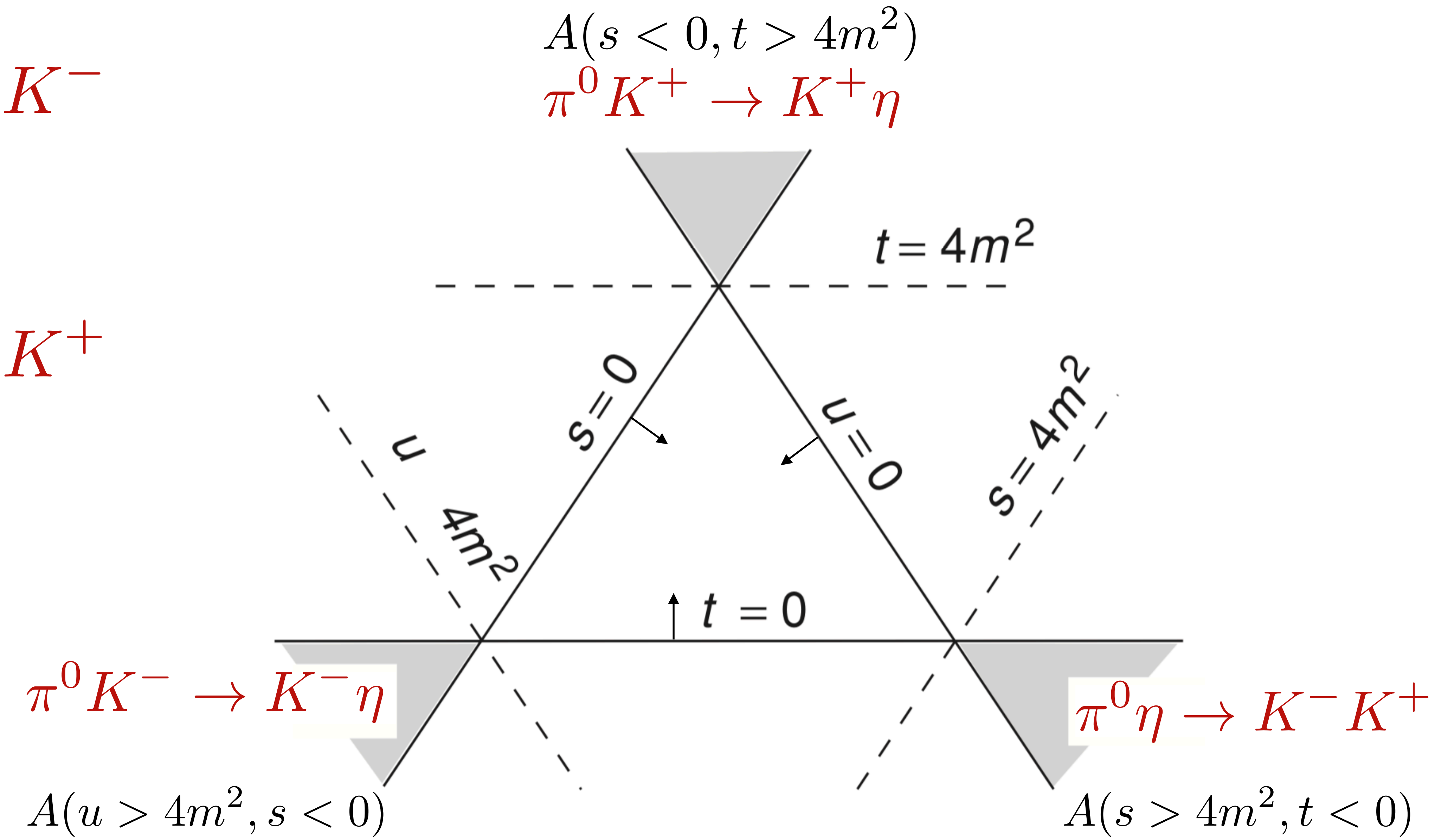
Crossing Symmetry

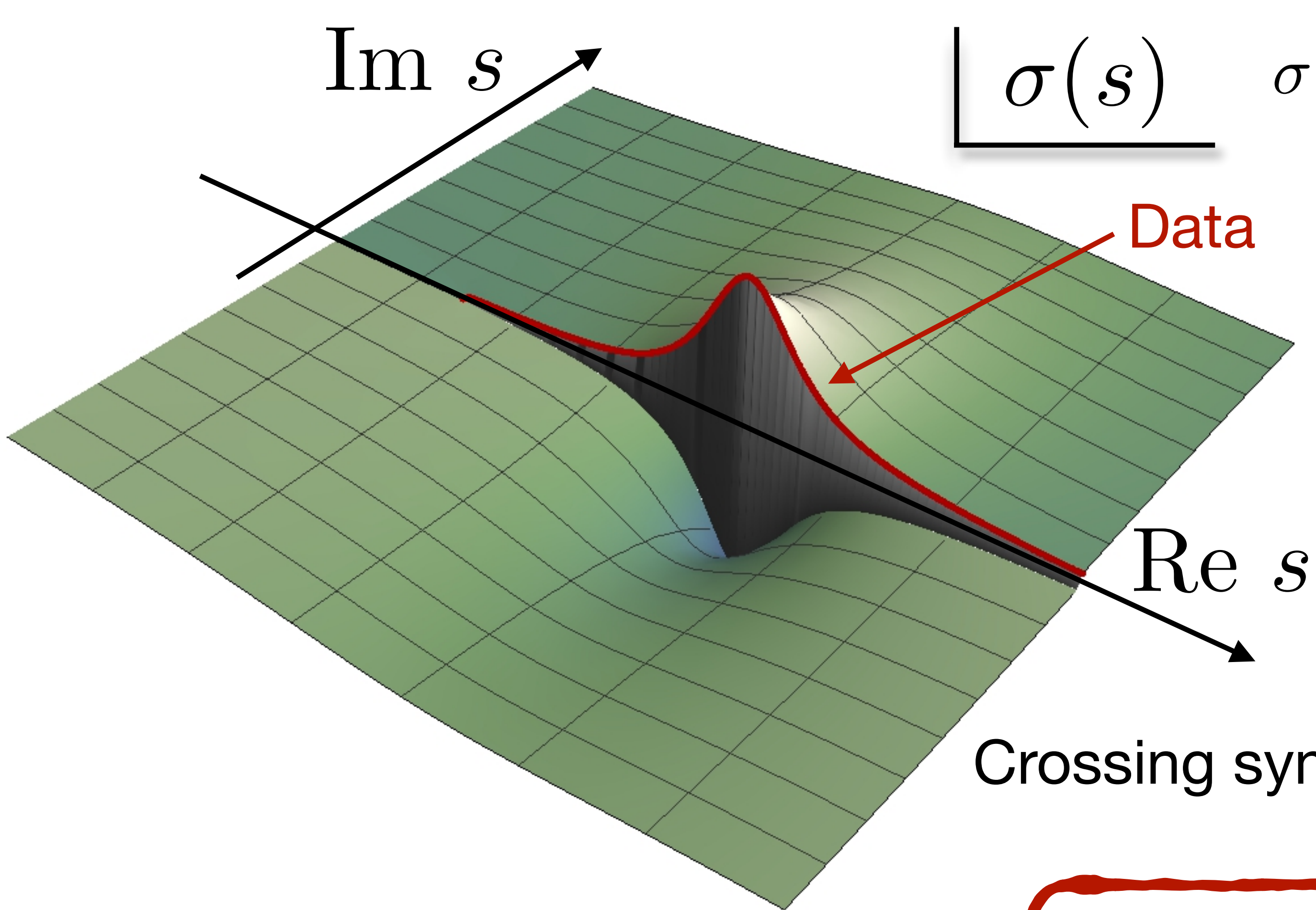




$$s + t + u = 4m^2$$

All mesons having mass m

Analyticity



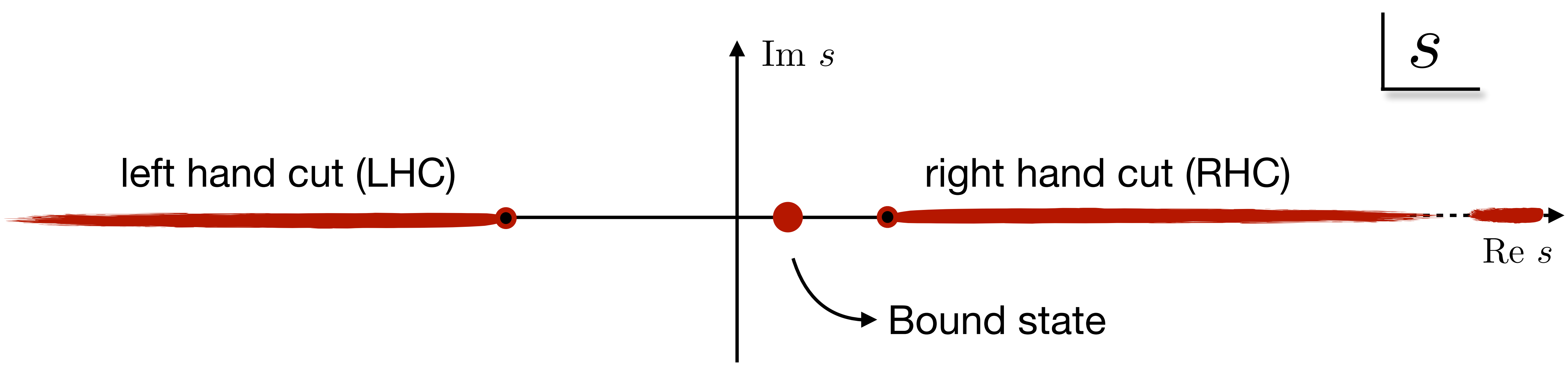
$$\sigma(E_{\text{lab}}) = \text{Im} \int d^3r \int_{\sqrt{r^2}}^{\infty} dt e^{iE_{\text{lab}}(t-vr)} f(r)$$

causality: $t - vr > 0$

Analyticity in upper half-plane $\text{Im}\ E_{\text{lab}} > 0$

Crossing symmetry implies analyticity in lower half-plane

Only allowed singularities are on the real axis



Unitarity

Conservation of probability

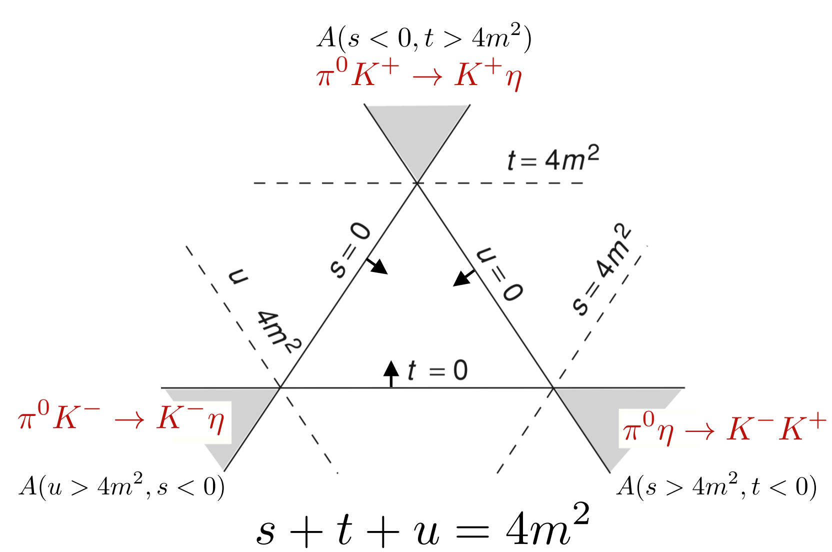
$$SS^\dagger = 1$$
$$S = 1 + iT$$
$$T - T^\dagger = iTT^\dagger$$

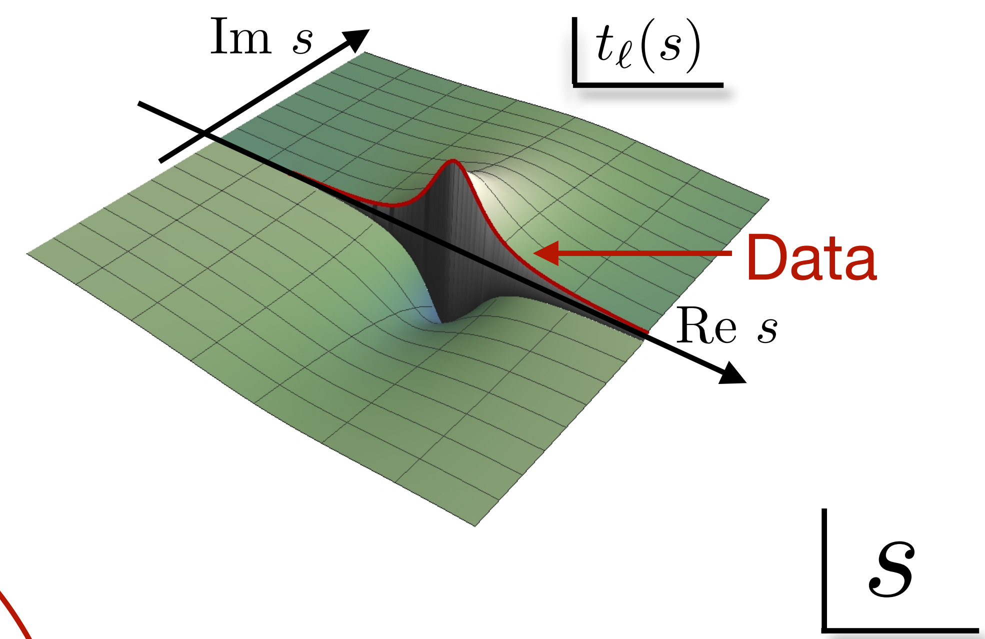
PW diagonalizes unitarity

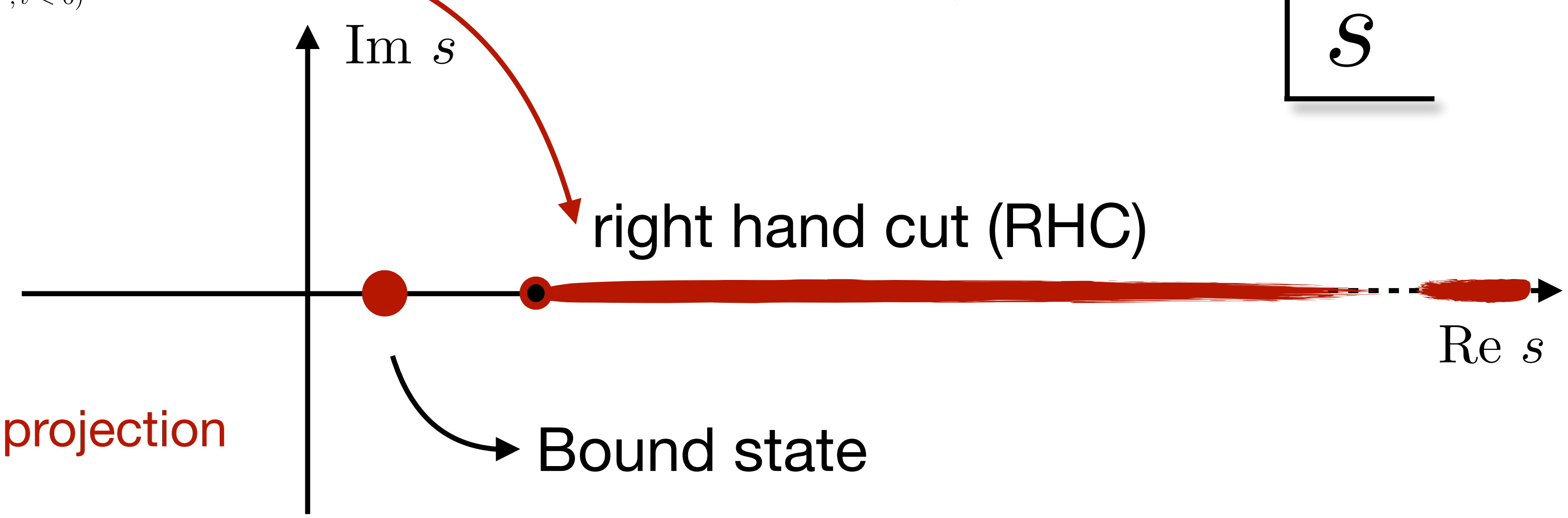
$$t_\ell(s+i\epsilon) - t_\ell(s-i\epsilon) = i\rho(s)t_\ell(s+i\epsilon)t_\ell(s-i\epsilon) \quad \text{(for real s)}$$

Phase space (for real s)

$$\rho(s) \propto \sqrt{1-4m^2/s}\;\theta(s-4m^2)$$







LHC from u-channel unitarity and PW projection

Unitarity

Conservation of probability

$$SS^\dagger = 1$$
$$S = 1 + iT$$

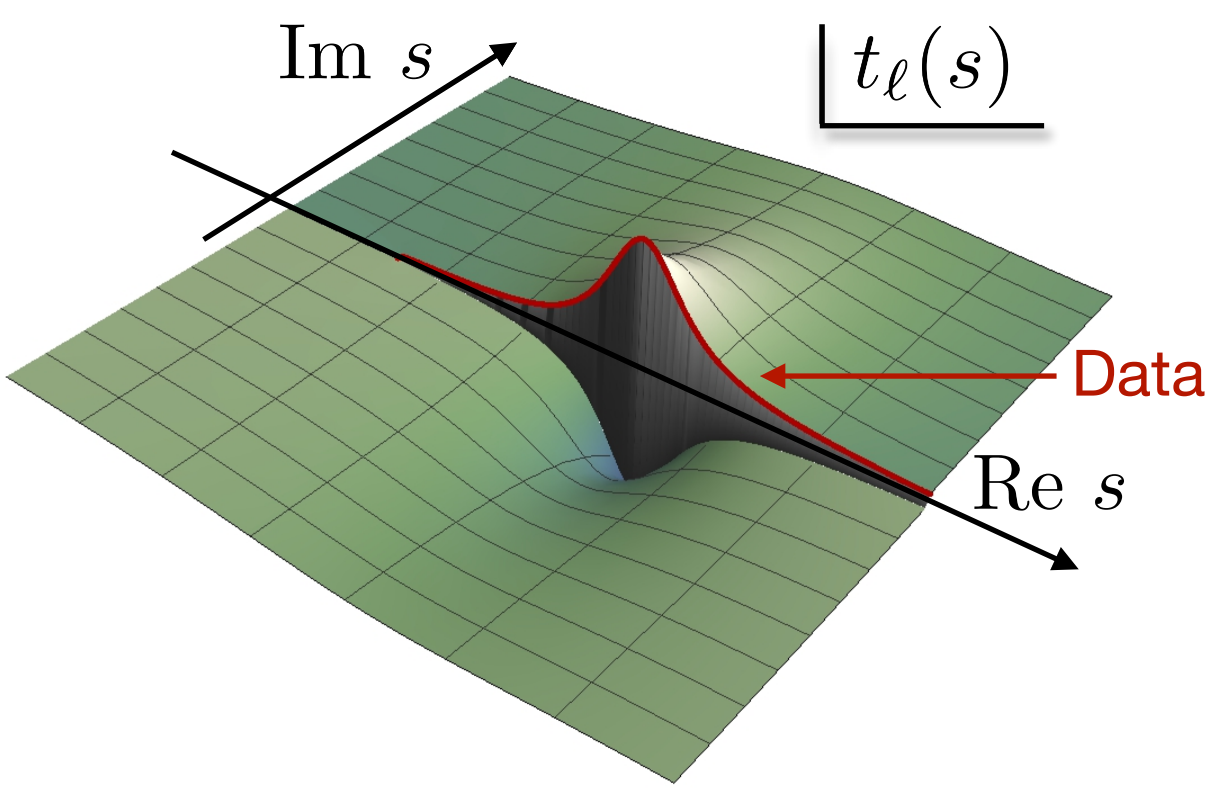
$$T - T^\dagger = iTT^\dagger$$

PW diagonalizes unitarity

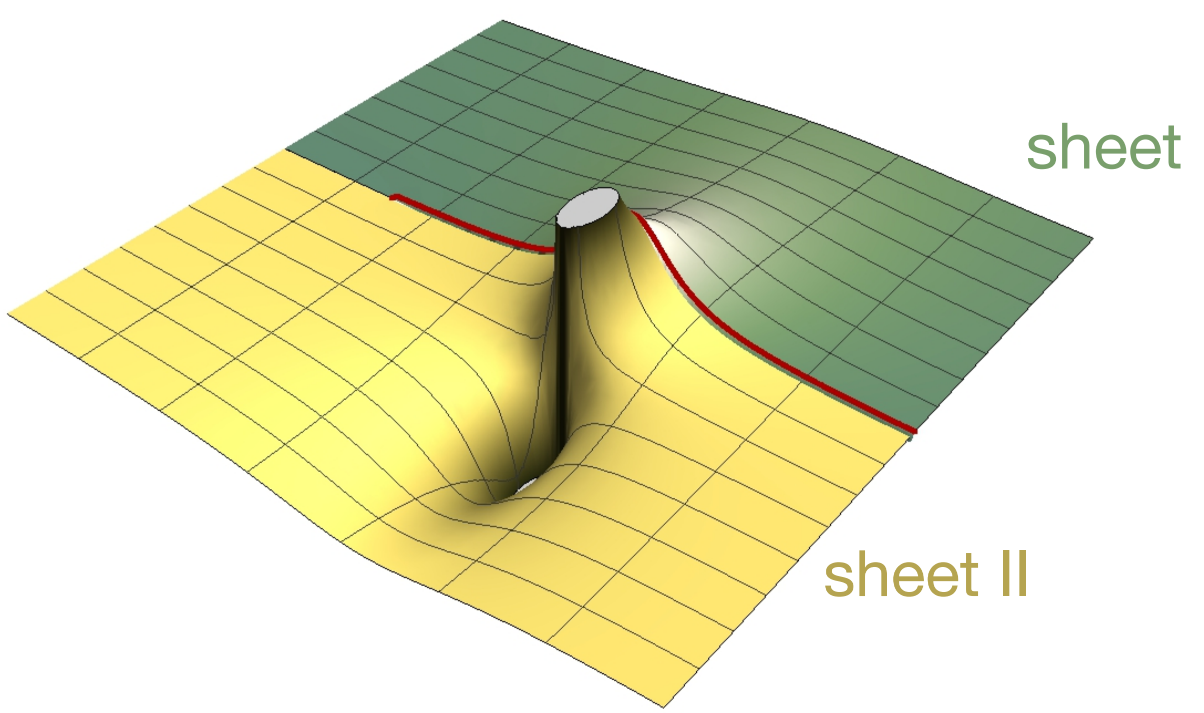
$$t_\ell(s+i\epsilon) - t_\ell(s-i\epsilon) = i\rho(s)t_\ell(s+i\epsilon)t_\ell(s-i\epsilon)$$ (for real s)

Exploring sheet II

$$t^I(s+i\epsilon) = t^{II}(s-i\epsilon) = \frac{t_\ell(s-i\epsilon)}{1-i\rho(s)t_\ell(s-i\epsilon)} = \frac{1}{t_\ell^{-1}(s-i\epsilon)-i\rho(s)}$$



There will be pole(s) on sheet II



Unitarity

PW diagonalizes unitarity $t_\ell(s+i\epsilon) - t_\ell(s-i\epsilon) = i\rho(s) t_\ell(s+i\epsilon) t_\ell(s-i\epsilon)$ (for real s)

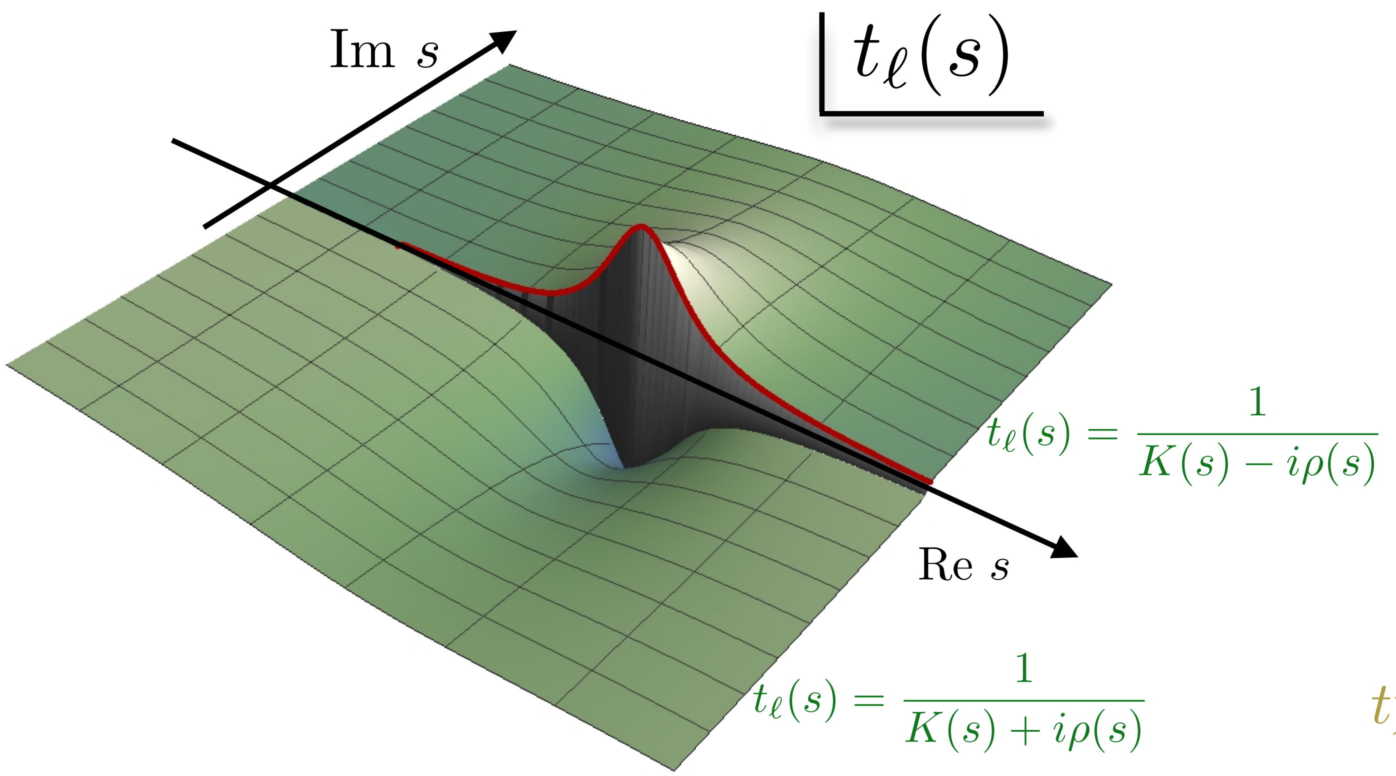
On the real axis: $\Delta t_\ell(s) = 2i\rho(s)|t_\ell(s)|^2$ $\quad \mathrm{Im}\, t_\ell^{-1}(s) = -\rho(s)$

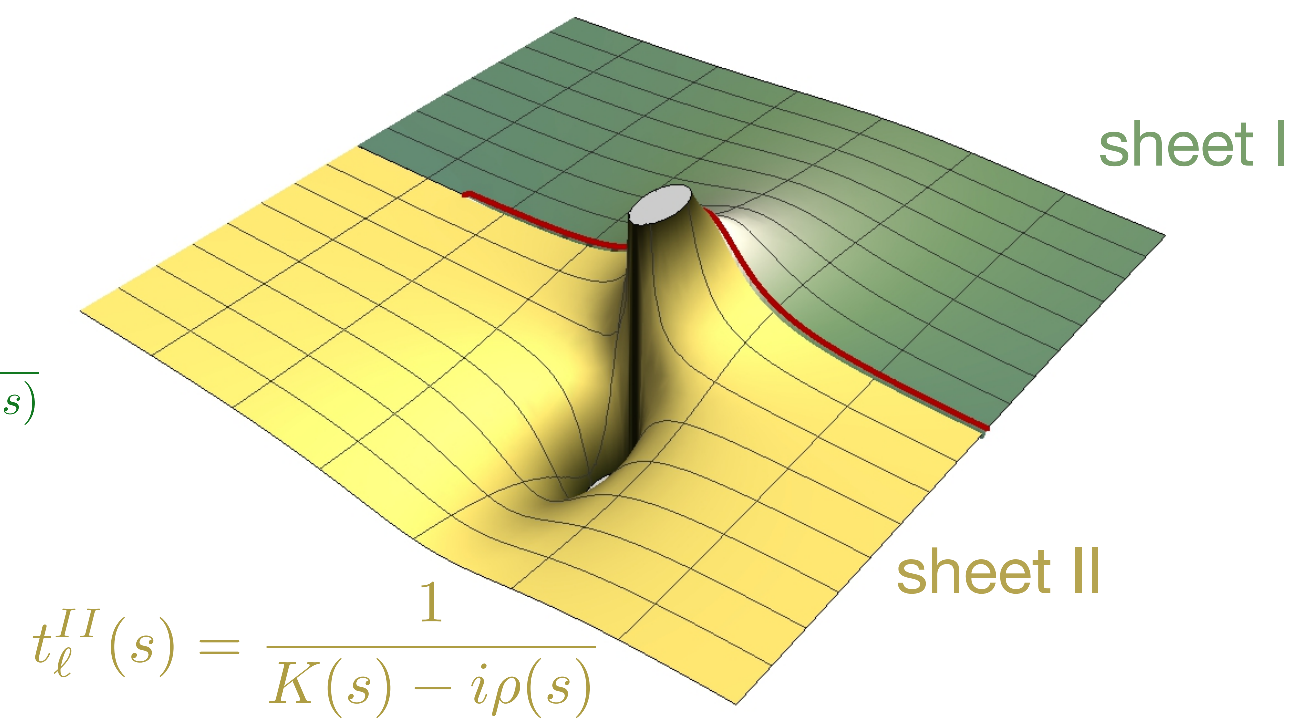
$$t_\ell(s \pm i\epsilon) = \frac{1}{K(s) \mp i\rho(s)}$$

real → $K(s)$

example ok with analyticity: $K(s) = \frac{m^2 - s}{m\Gamma}$

on sheet II: $t^{II}(s) = \frac{m\Gamma}{m^2 - s - i\rho(s) m\Gamma}$





There is no left hand cut

Unitarity

(inelastic) unitarity

$$\operatorname{Im} t_\ell(s) = \rho(s)\,|\,t_\ell(s)\,|^2 + \Delta_\ell \qquad \rho(s) = \frac{1}{16\pi}\frac{2q}{\sqrt{s}} \simeq \frac{1}{16\pi}$$

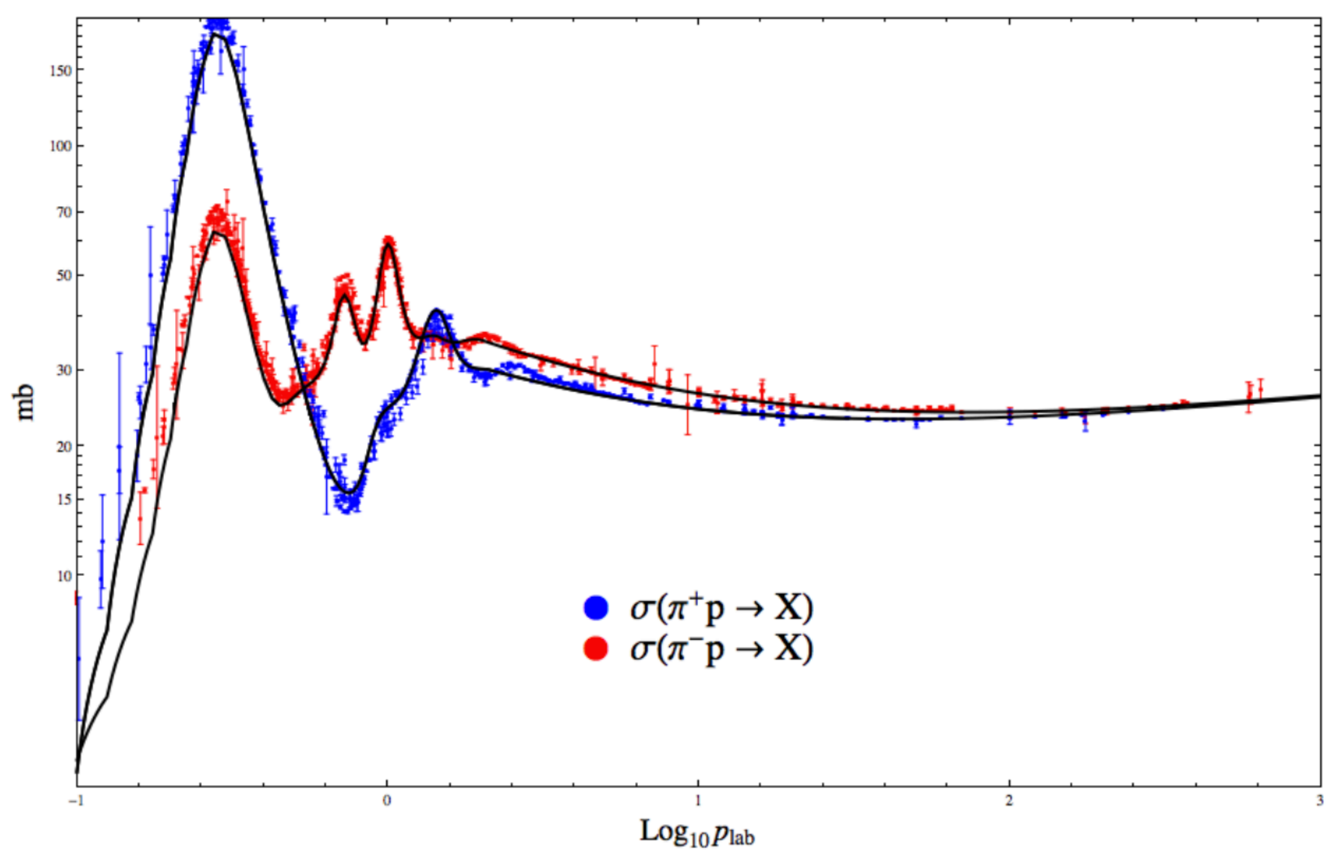
$$\operatorname{Im} t_\ell(s) \geq \frac{1}{16\pi}\,|\,t_\ell(s)\,|^2 \geq \frac{1}{16\pi}\,|\operatorname{Im} t_\ell(s)\,|^2$$

Bound on partial waves

$$\operatorname{Im} t_\ell(s) \leq 16\pi$$

Optical theorem:

$$\begin{aligned}\sigma_{\text{tot}}(s) &\simeq \frac{1}{s}\operatorname{Im} A(s, t=0)\\ &\simeq \frac{1}{s}\sum_\ell (2\ell+1)\operatorname{Im} t_\ell(s) P_\ell(1)\\ &\leq \frac{8\pi}{s}\sum_\ell (2\ell+1) \simeq \frac{16\pi}{s}\ell^2_{\text{max}}\end{aligned}$$

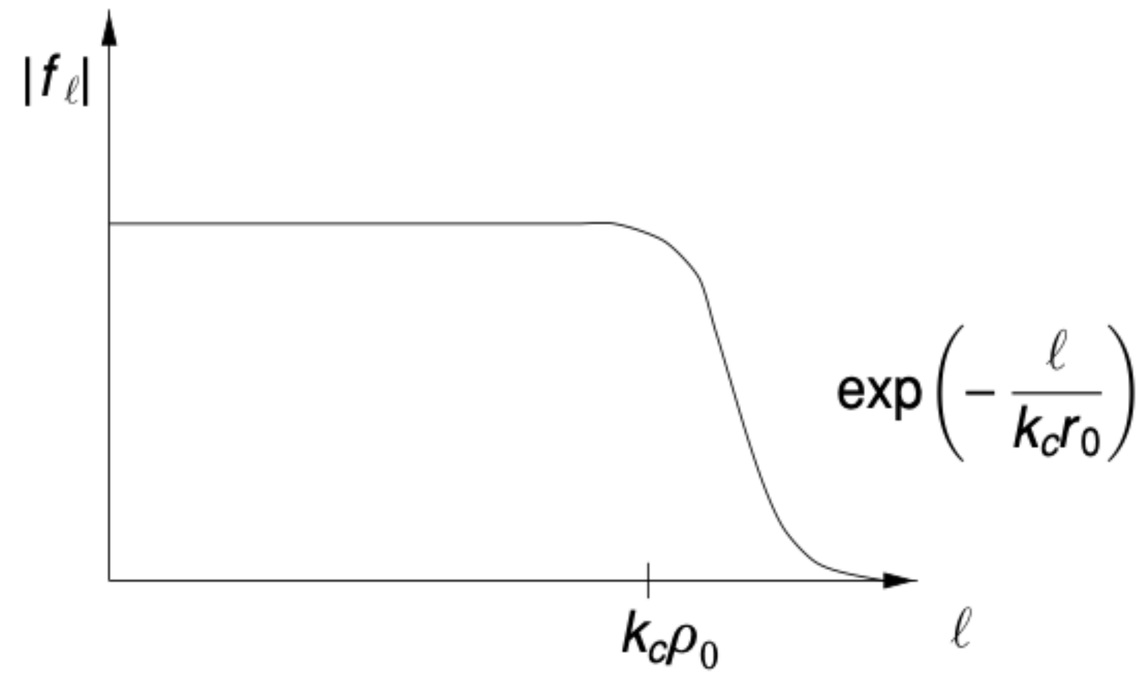


Radius of interaction

Classical picture:
cross section means transverse area

$$\sigma(s) \simeq \frac{16\pi}{s}\ell_{\max}^2 \equiv \pi\rho_0^2$$

Radius of interaction $\rho_0(s) \simeq \frac{2\ell_{\max}}{\sqrt{s}} = \frac{\ell_{\max}}{k_c}$



Partial wave expansion converges
up to the t-channel singularity

$$A(s,z) = \sum_\ell (2\ell+1)a_\ell(s)P_\ell(z)$$

That is up to $z_0 = 1 + \frac{2t_0}{s-4m^2} > 1$ For which $P_\ell(z_0) \sim e^{\ell\frac{2\sqrt{t_0}}{s}} = e^{\frac{\ell}{r_0 k_c}}$

So the partial waves must behave as $a_\ell(s) \sim e^{-\frac{\ell}{r_0 k_c}}$

Froissart-Martin bound

Assume that at high energy $|A(s,t)| \leq s^{N(t)}$

Unitary tells us that $$|A(s,t)| \leq 16\pi \sum_{\ell=0}^{\ell_{\max}} (2\ell+1)\,|P_\ell(z)|$$

Makes sense up to $t = t_0$ $$|A(s,t)| \leq \ell_{\max} e^{\frac{\ell_{\max}}{r_0 k_c}} \simeq s^{N_1} \qquad N_1 = N(t=t_0)$$

$$\log(\ell_{\max}) + \frac{\ell_{\max}}{r_0 k_c} = N_1 \log s$$

$$\ell_{\max}(s) \simeq \frac{N_1}{r_0}\sqrt{s}\log s \qquad k_c = \sqrt{s}/2$$

So the cross section grows max as $$\sigma(s) \simeq \frac{16\pi}{s}\ell_{\max}^2 \propto \log^2(s)$$

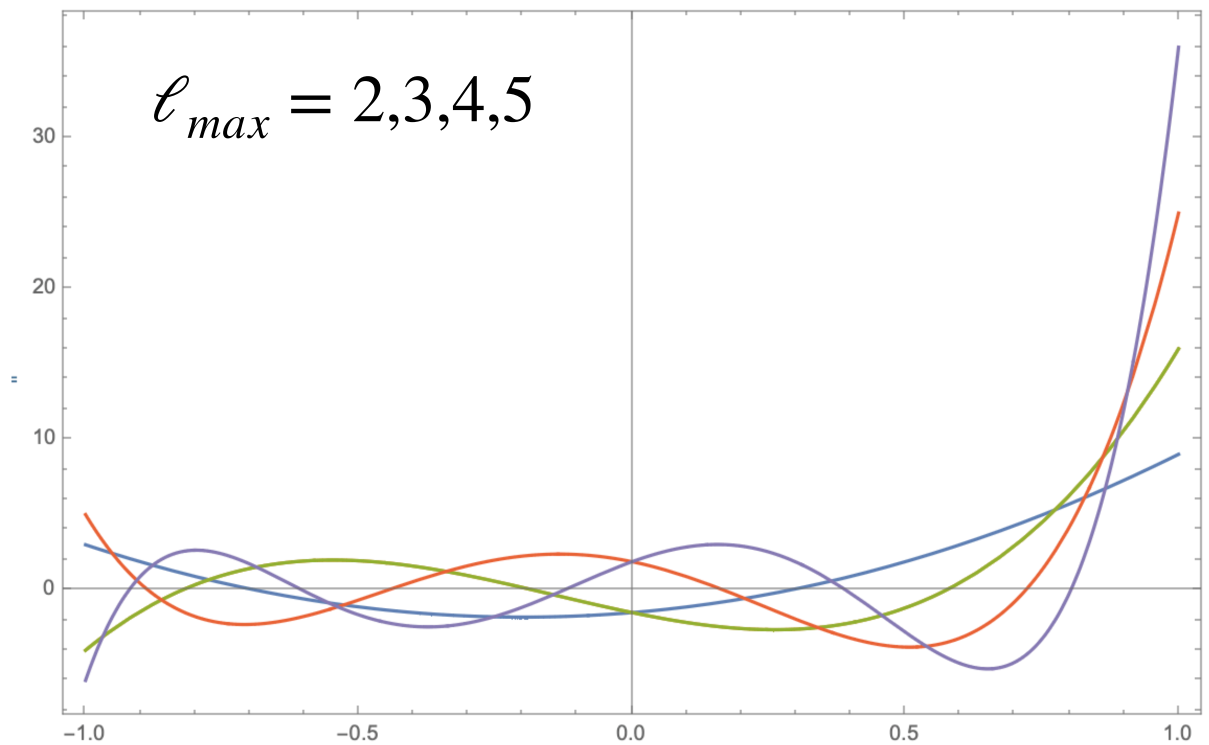
High Energy Physics

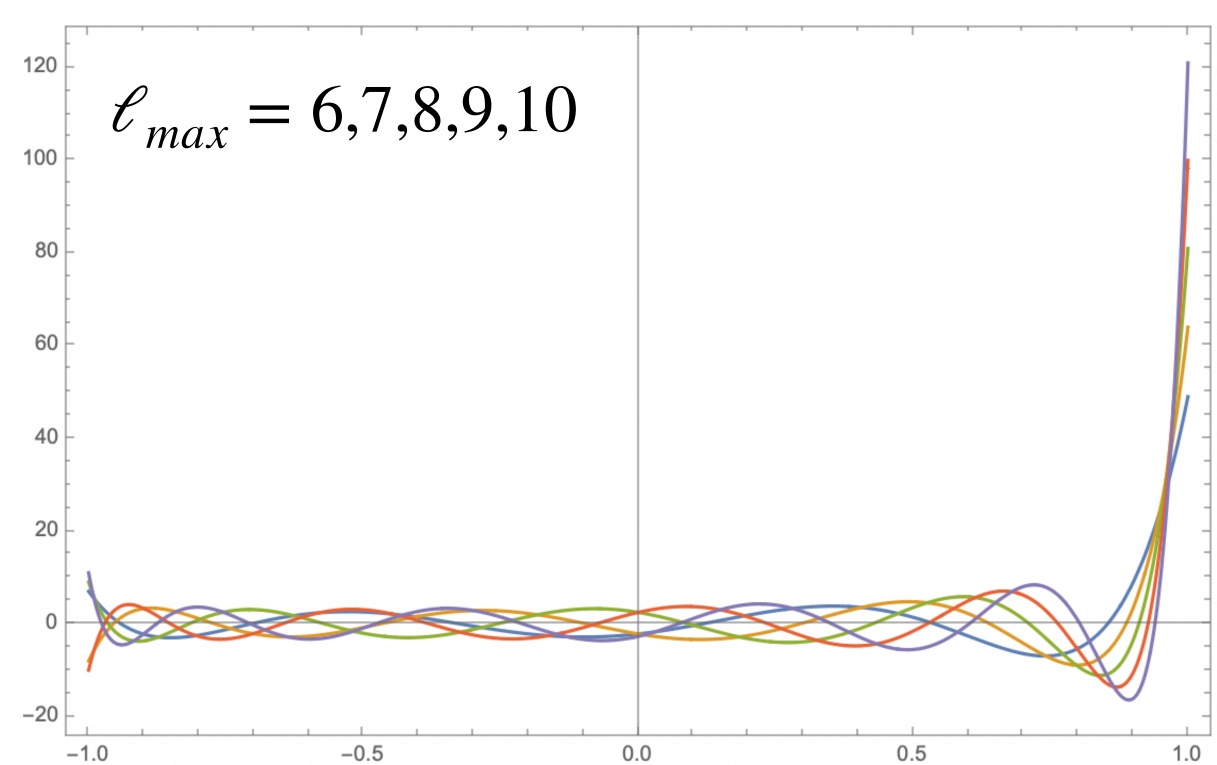
Assume constant partial waves

$$f(\ell_{max}, z) = \sum_{\ell=0}^{\ell_{max}} (2\ell + 1) P_\ell(z)$$

Legendre Polynomials interfere,
Except in the forward direction

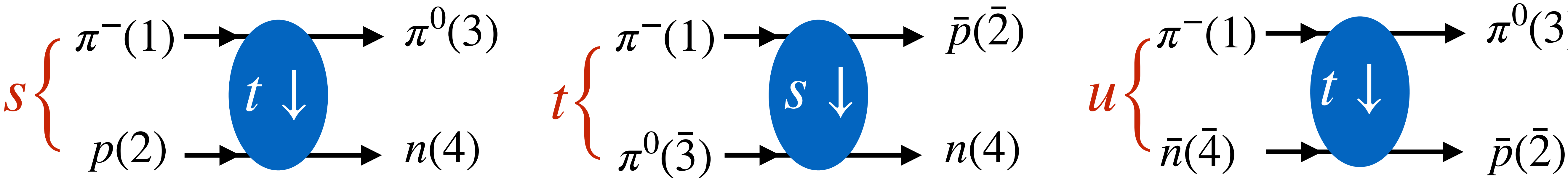
At high energy, most of the physics
is concentrated in the forward direction





Mandelstam plane and variables

All 3 reactions are described by the same complex function $A(s, t, u)$



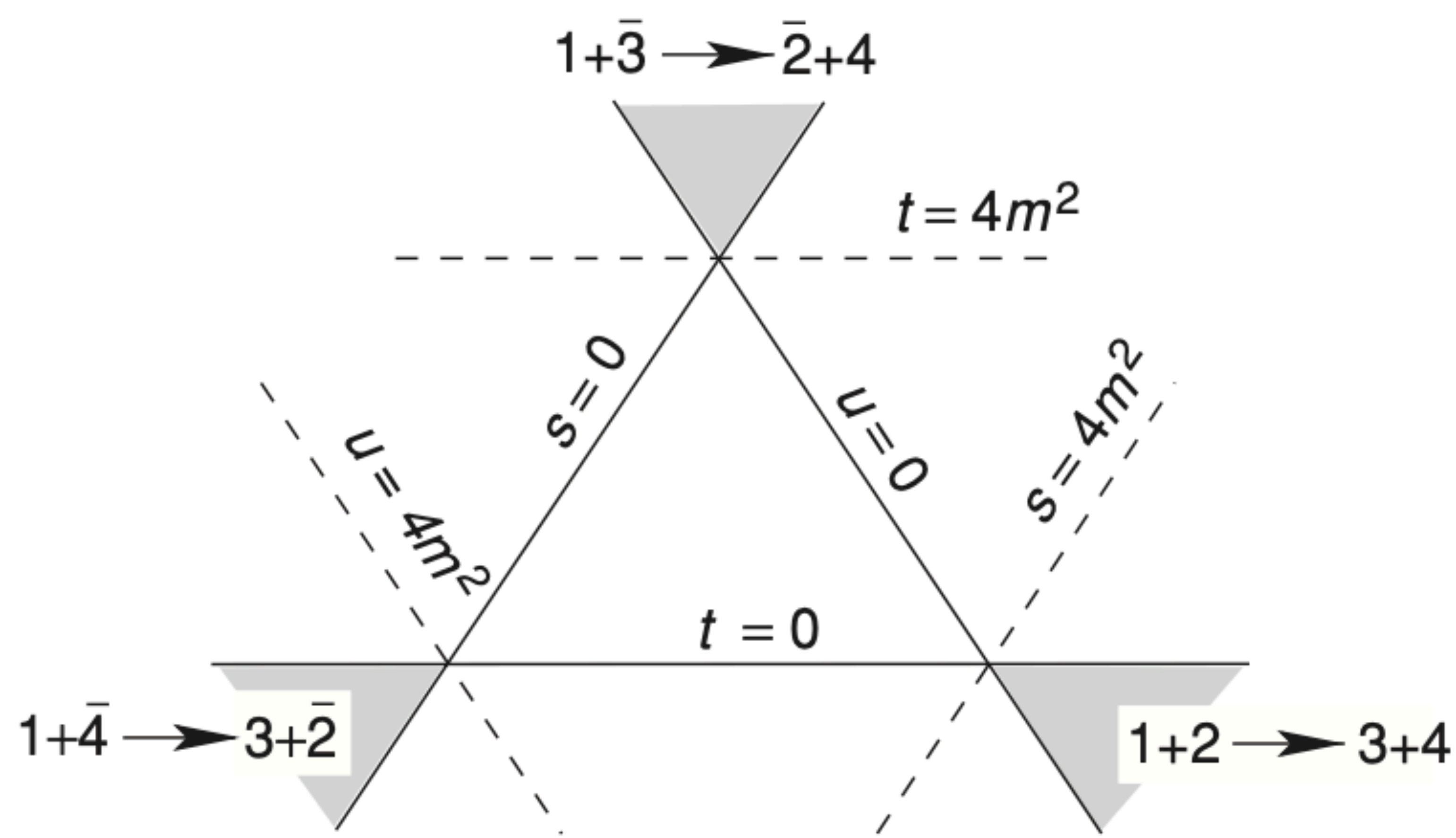
There are only 2 independent Mandelstam variables, $s + t + u = 4m^2$

All particles have same mass m

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2$$

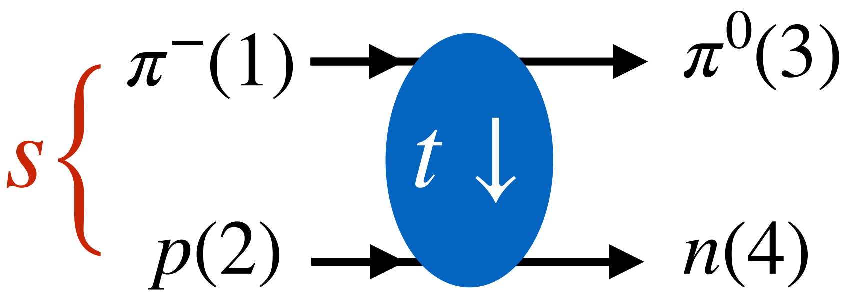
$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2$$

$$u = (p_1 - p_4)^2 = (p_3 - p_2)^2$$



Partial waves decomposition

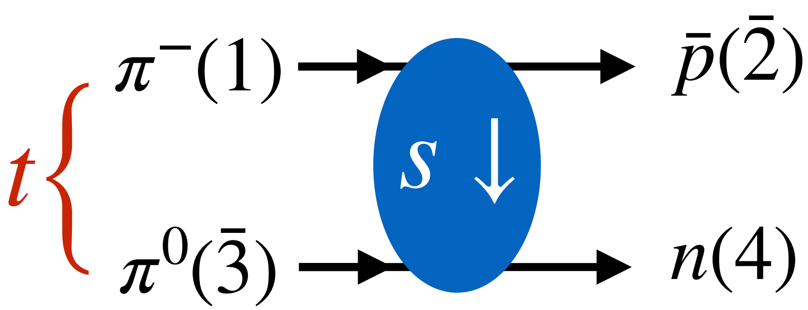
Consider the s-channel reaction



$$A(s, z_s) = \sum_{\ell=0}^{\infty} (2\ell + 1) a_\ell^s(s) P_\ell(z_s)$$

$$z_s = 1 + \frac{2t}{s - 4m^2} \qquad s \geq 4m^2 \qquad 4m^2 - s \leq t \leq 0$$

Consider the t-channel reaction

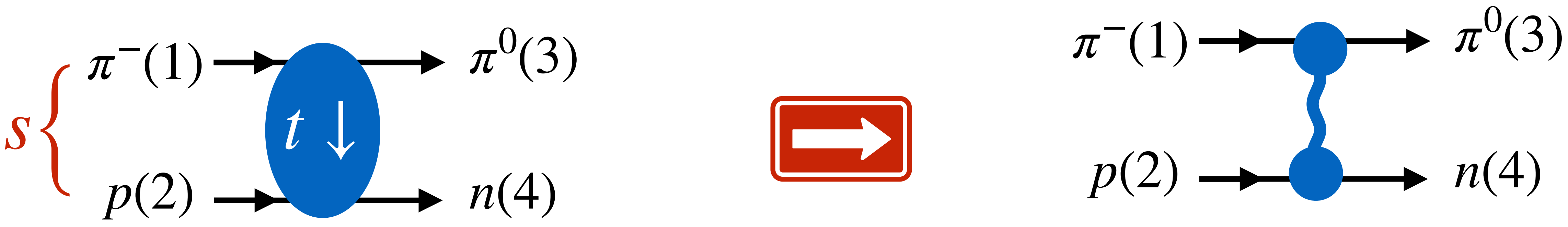


$$A(t, z_t) = \sum_{\ell=0}^{\infty} (2\ell + 1) a_\ell^t(t) P_\ell(z_t)$$

$$z_t = 1 + \frac{2s}{t - 4m^2} \qquad t \geq 4m^2 \qquad 4m^2 - t \leq s \leq 0$$

Partial waves decomposition

The s-channel reaction at high energies is dominated by the t-channel poles



The t-channel PW expansion doesn't converge in the s-channel physical region

$$A(t, z_t) = \sum_{\ell=0}^{\infty} (2\ell + 1) a_\ell^t(t) P_\ell(z_t)$$

$$z_t = 1 + \frac{2s}{t - 4m^2}$$

Need to analytically continue $A(t, z_t)$ for $t < 0, |z_t| > 1$

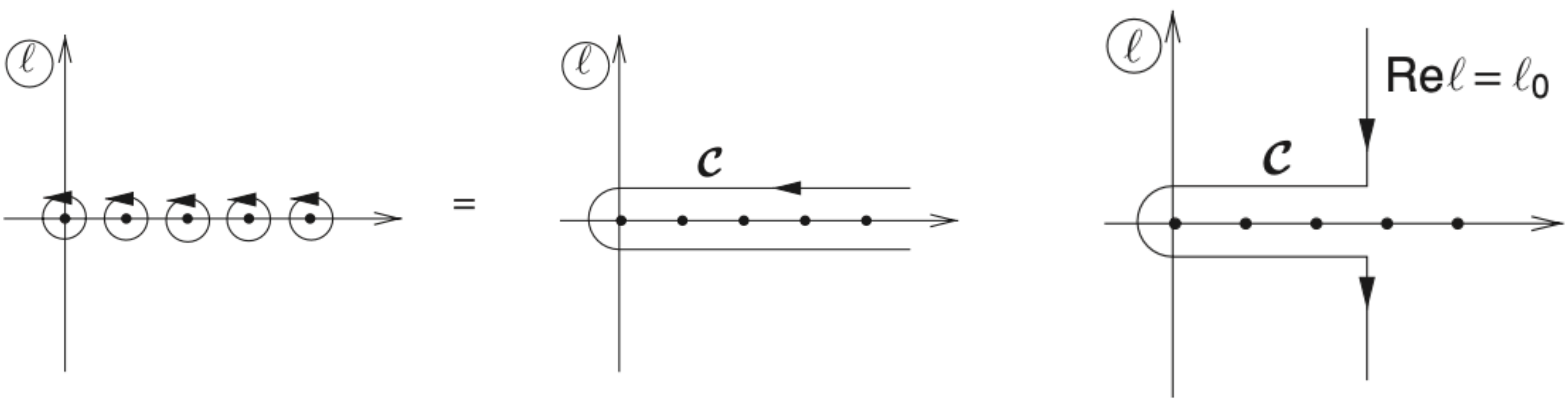
Sommerfeld-Watson transformation

Write the sum as a contour integration

$$A(t,z_t) = \sum_{\ell=0}^{\infty}(2\ell+1)a_\ell^t(t)P_\ell(z_t) = \frac{1}{2i}\oint\frac{d\ell}{\sin\pi\ell}(2\ell+1)a^t(\ell,t)P_\ell(-z_t)$$

We need to define $a^t(\ell,t)$ for complex values of ℓ

Then we will deform the contour



Froissard-Gribov Representation

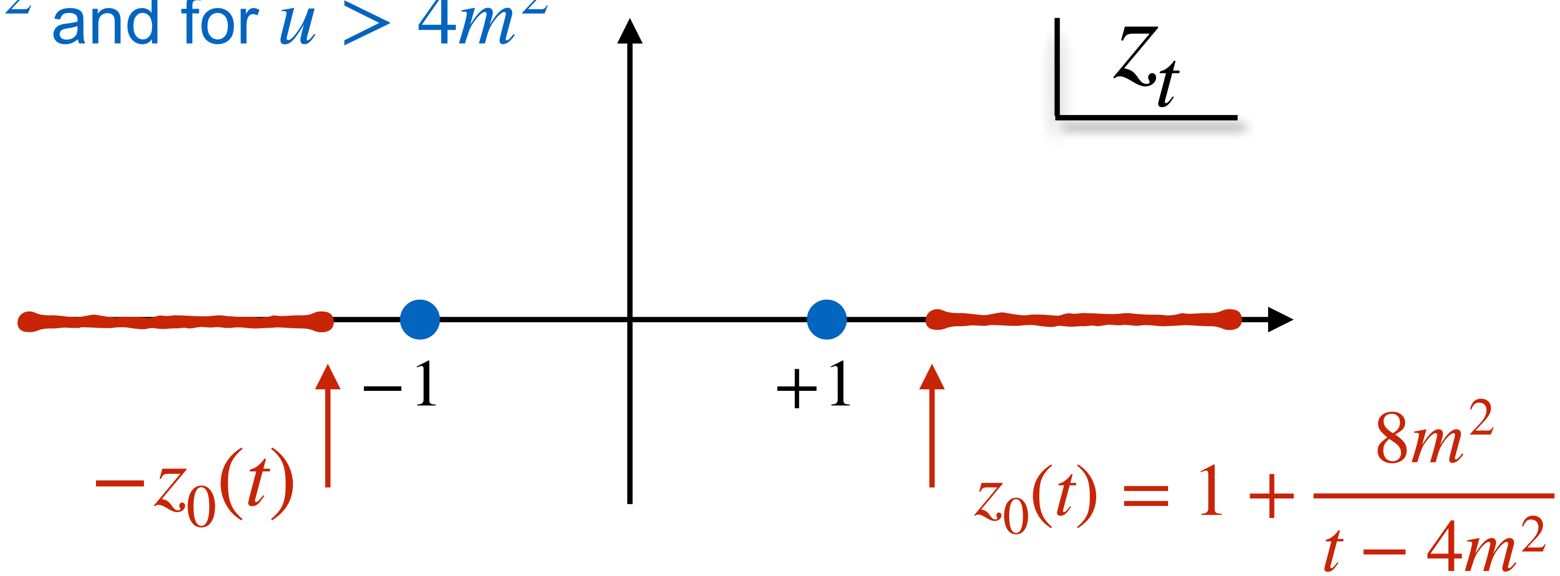
$$z_t = 1 + \frac{2s}{t - 4m^2} = -1 - \frac{2u}{t - 4m^2}$$

Naive definition of the partial wave

$$A(t, z_t) = \sum_{\ell=0}^{\infty} (2\ell + 1) a^t_\ell(t) P_\ell(z_t) \qquad a^t(\ell, t) = \frac{1}{2}\int_{-1}^{1} A(t, z_t) P_\ell(z_t) \mathrm{d}z_t$$

The series converges **for t fixed**, if $|z_t| < 1$

$A(t, z_t)$ will diverge for $s > 4m^2$ and for $u > 4m^2$



$$A(t, z_t) = \frac{1}{\pi}\int_{z_0(t)}^{\infty} \mathrm{d}z' \frac{\mathrm{D}_s A(t, z_t[s])}{z' - z_t} + \frac{1}{\pi}\int_{-\infty}^{-z_0(t)} \mathrm{d}z' \frac{\mathrm{D}_u A(t, z_t[u])}{z' - z_t}$$

Froissard-Gribov Representation

$$Q_\ell(z) = \frac{1}{2}\int_{-1}^{1} \frac{P_\ell(z')}{z'-z}\mathrm{d}z$$

Dispersive representation of the amplitude for fixed t

$$A(t,z_t) = \frac{1}{\pi}\int_{z_0(t)}^{\infty} \frac{\mathrm{D}_s A(t,z_t[s])}{z'-z_t}\mathrm{d}z' + \frac{1}{\pi}\int_{z_0(t)}^{\infty} \frac{\mathrm{D}_u A(t,-z_t[u])}{z'+z_t}\mathrm{d}z'$$

The discontinuities are

$$\mathrm{D}_s A(t,z_t[s]) = \frac{1}{2i}\left[A(t,z_t[s+i\epsilon]) - A(t,z_t[s-i\epsilon])\right]$$

Definition of the partial wave $\quad a^t(\ell,t) = \frac{1}{2}\int_{-1}^{1} A(t,z_t)P_\ell(z_t)\mathrm{d}z_t$

Obtenemos the Froissard-Gribov representation of the partial waves

$$a^t(\ell,t) = \frac{1}{\pi}\int_{z_0(t)}^{\infty} \mathrm{D}_s A(t,z_t[s])\; Q_\ell(z')\,\mathrm{d}z' - \frac{1}{\pi}\int_{z_0(t)}^{\infty} \mathrm{D}_u A(t,-z_t[u])\; Q_\ell(-z')\,\mathrm{d}z'$$

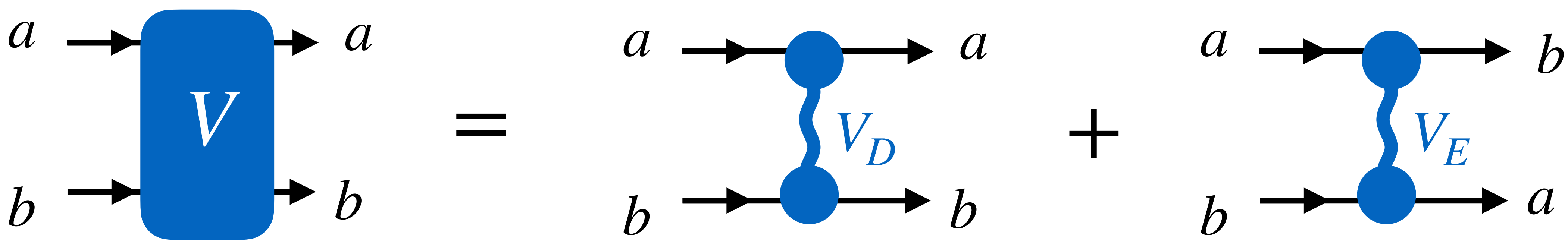
Signature

$$a^t(\ell,t) = \frac{1}{\pi}\int_{z_0(t)}^{\infty} \mathrm{D}_sA(t,z_t[s])\; Q_\ell(z')\,\mathrm{d}z' - \frac{1}{\pi}\int_{z_0(t)}^{\infty} \mathrm{D}_uA(t,-z_t[u])\; Q_\ell(-z')\,\mathrm{d}z'$$

The second term causes problem $Q_\ell(-z) = (-1)^{\ell+1} Q_\ell(z)$

We can only analytically continue the P.W. for either even or either odd ℓ

$$a^t_\pm(\ell,t) = \frac{1}{\pi}\int_{z_0(t)}^{\infty} \left[\mathrm{D}_sA(t,z_t[s]) \pm \mathrm{D}_uA(t,-z_t[u])\right]\; Q_\ell(z')\,\mathrm{d}z'$$



$$V = V_D + (-1)^\ell V_E$$

Trajectory quantum numbers (only mesons)

Isospin I

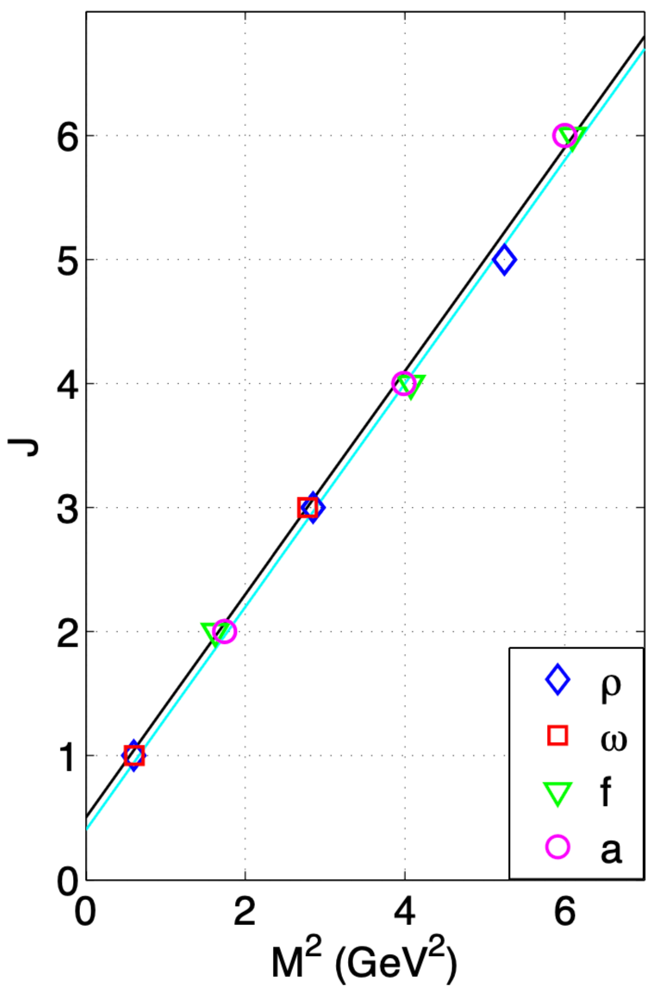
G-parity G, the parity of the pion number

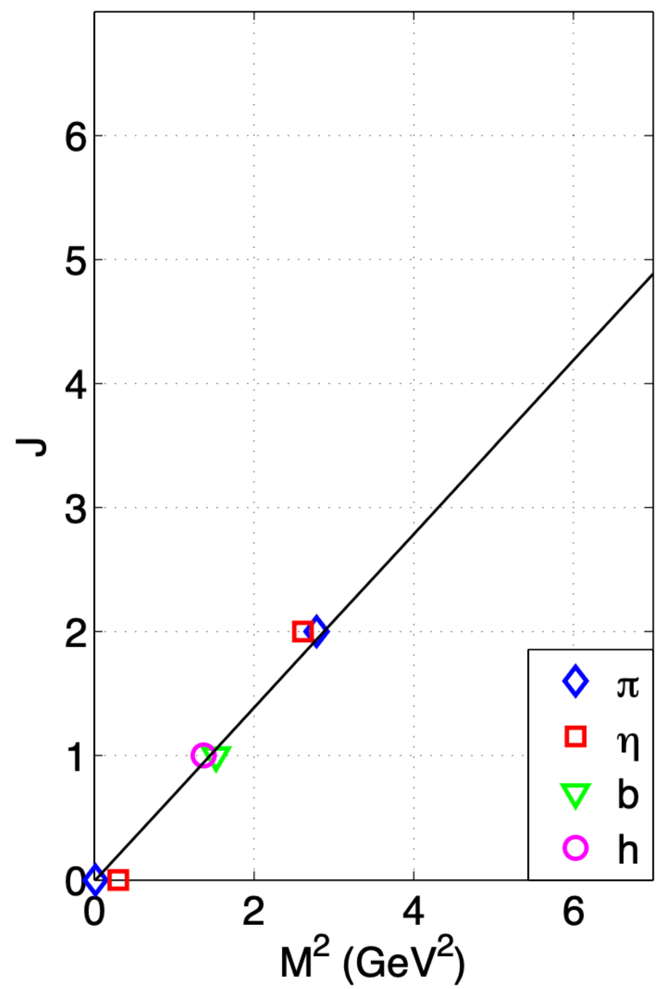
Naturality of the spin, $\eta = P(-1)^J$

Signature of the spin, $\tau = (-1)^J$

Table 1: Regge Trajectories

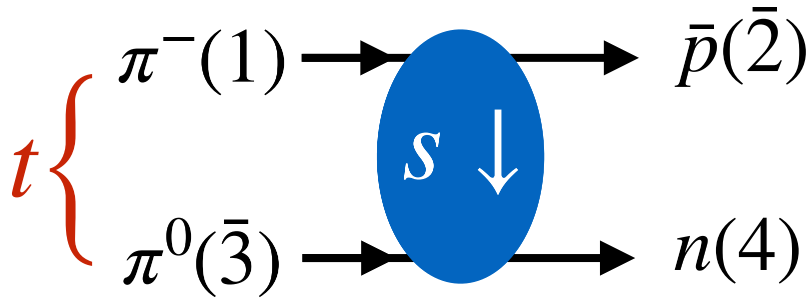
$I^{G\tau\eta}$			$I^{G\tau\eta}$	
0^{+++}	f	$0.5+0.9t$	0^{+--}	$\bar{f}$
0^{--+}	ω	$0.5+0.9t$	0^{-+-}	$\bar{\omega}$
1^{-++}	a	$0.4+0.9t$	1^{---}	$\bar{a}$
1^{+-+}	ρ	$0.5+0.9t$	1^{++-}	$\bar{\rho}$
0^{++-}	η	$0.7(t-m_\pi^2)$	0^{+-+}	$\bar{\eta}$
0^{---}	h	$0.7(t-m_\pi^2)$	0^{-++}	$\bar{h}$
1^{-+-}	π	$0.7(t-m_\pi^2)$	1^{--+}	$\bar{\pi}$
1^{+--}	b	$0.7(t-m_\pi^2)$	1^{+++}	$\bar{b}$





Partial waves decomposition

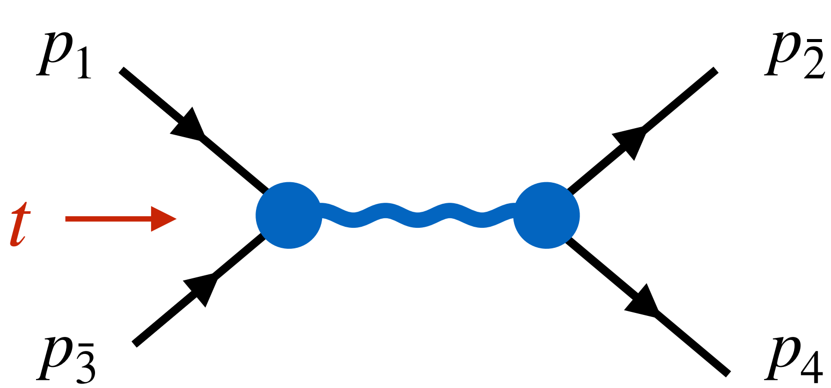
Consider the t-channel reaction



$$A(t, z_t) = \sum_{\ell=0}^{\infty} (2\ell + 1) a_\ell^t(t) P_\ell(z_t)$$

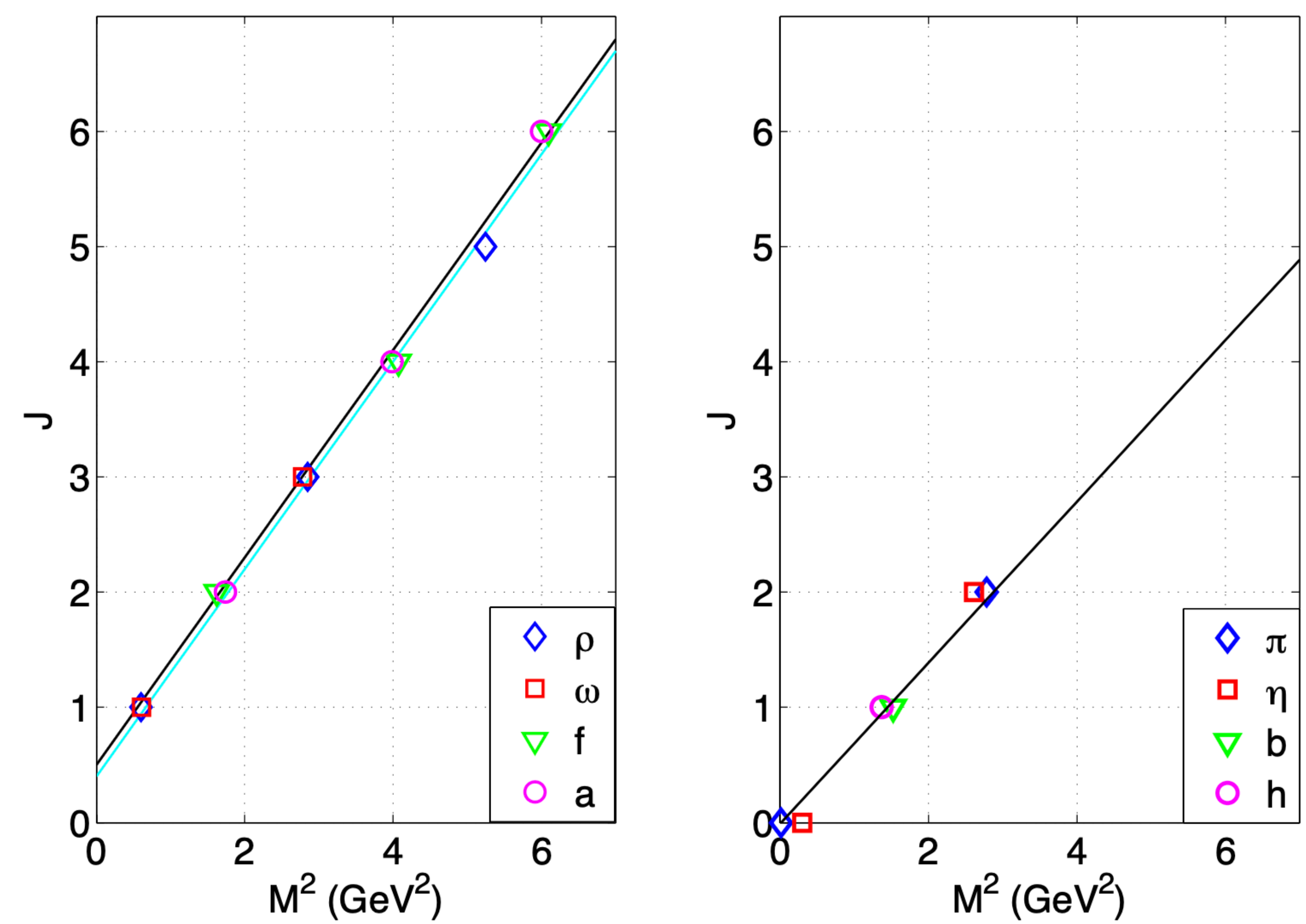
$$z_t = 1 + \frac{2s}{t - 4^2} \qquad t \geq 4m^2 \qquad 4m^2 - t \leq s \leq 0$$

Say there is a common pole in every wave



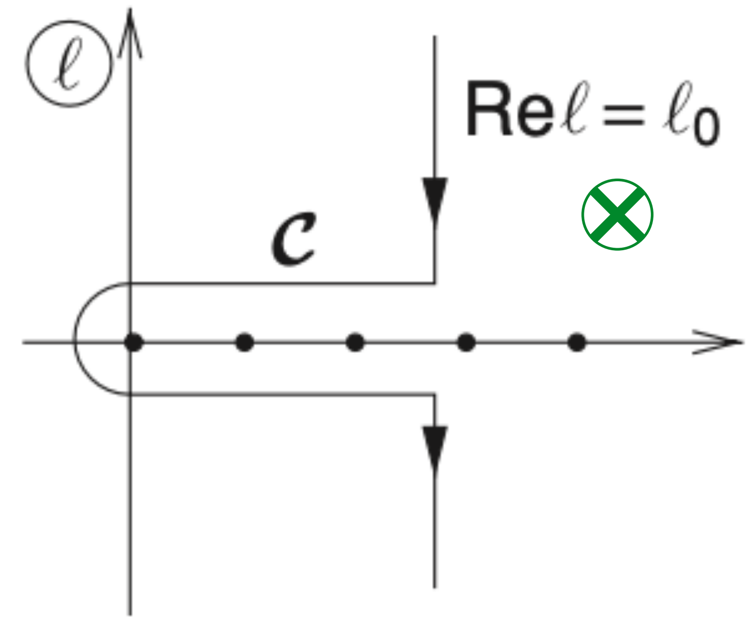
$$a_\ell^t(t) = \frac{g_\ell^2}{m_\ell^2 - t} \equiv \frac{\beta}{\ell - \alpha(t)}$$

$$\alpha(t) = \ell + \alpha'(t - m_\ell^2)$$



Regge Pole Formula

$$A(t, z_t) = \frac{1}{2i} \oint \frac{d\ell}{\sin \pi\ell} (2\ell + 1) a^t(\ell, t) P_\ell(-z_t)$$



$$a^t_\ell(t) = \frac{\beta}{\ell - \alpha(t)}$$

$$z_t = 1 + \frac{2s}{t - 4m^2}$$

Deforming the contour

$$A(t, z_t) = \pi[2\alpha(t) + 1]\beta(t) \frac{P_\ell(z_t)}{\sin \pi\alpha(t)} + \frac{1}{2i} \int_{\ell_0 - i\infty}^{\ell_0 + i\infty} \frac{(2\ell + 1) P_\ell(-z)}{\sin \pi\ell} a^t(\ell, t) d\ell$$

Large energy limit

$$A(t, z_t) = \tilde{\beta}(t) \frac{(s/s_0)^{\alpha(t)}}{\sin \pi\alpha(t)} + \mathcal{O}(s^{-1/2})$$

Restoring the signature factor

$$A(t, z_t) = \beta(t) \frac{\pm 1 + e^{-i\pi\alpha(t)}}{2 \sin \pi\alpha(t)} (s/s_0)^{\alpha(t)} + \mathcal{O}(s^{-1/2})$$

For the total cross section

$$\mathrm{Im} A(t, z_t) = \beta(t)(s/s_0)^{\alpha(t)}$$

Legendre Polynomial

Deforming the contour

$$A(t,z_t) = \pi[2\alpha(t)+1]\beta(t)\frac{P_\ell(z_t)}{\sin\pi\alpha(t)} + \frac{1}{2i}\int_{\ell_0-i\infty}^{\ell_0+i\infty}\frac{(2\ell+1)P_\ell(-z)}{\sin\pi\ell}a^t(\ell,t)\mathrm{d}\ell$$

Large energy limit

$$A(t,z_t) = \widetilde{\beta}(t)\frac{(s/s_0)^{\alpha(t)}}{\sin\pi\alpha(t)} + \mathcal{O}(s^{-1/2})$$

Defined by

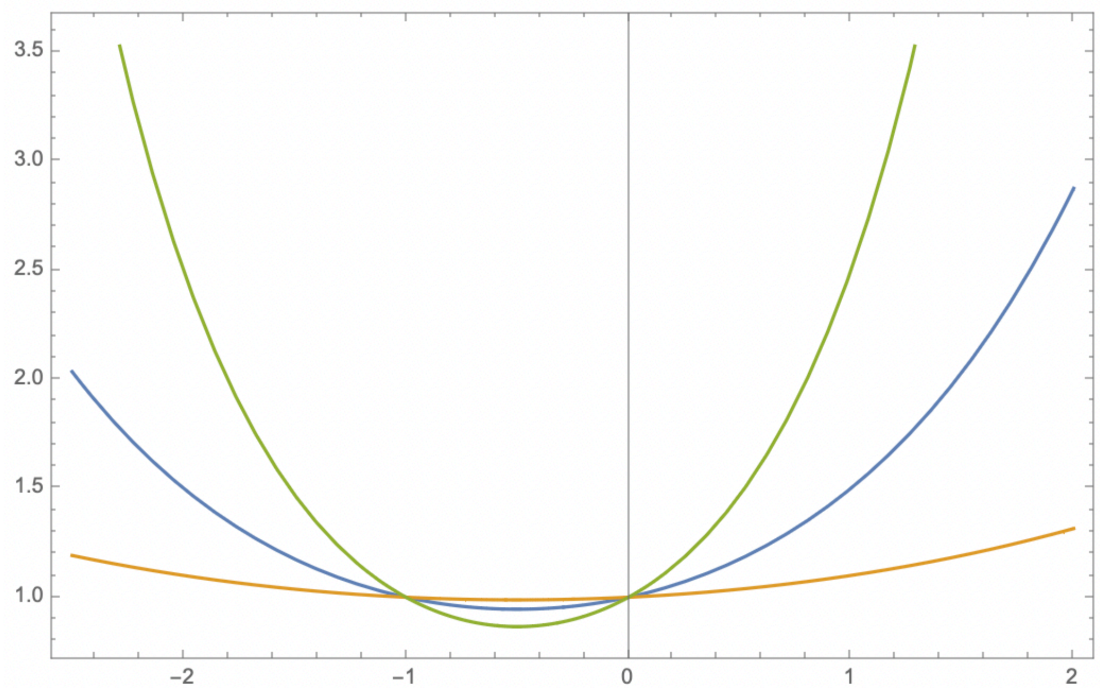
$$(1-z^2)P''_\ell(z) - 2zP'_\ell(z) + \ell(\ell+1)P_\ell(z) = 0$$

Symmetry

$$\ell \to -\ell - 1$$

$$\Rightarrow \quad P_{-\ell-1}(z) = P_\ell(z)$$

For $|z| > 1$, minimum for $\ell = -1/2$



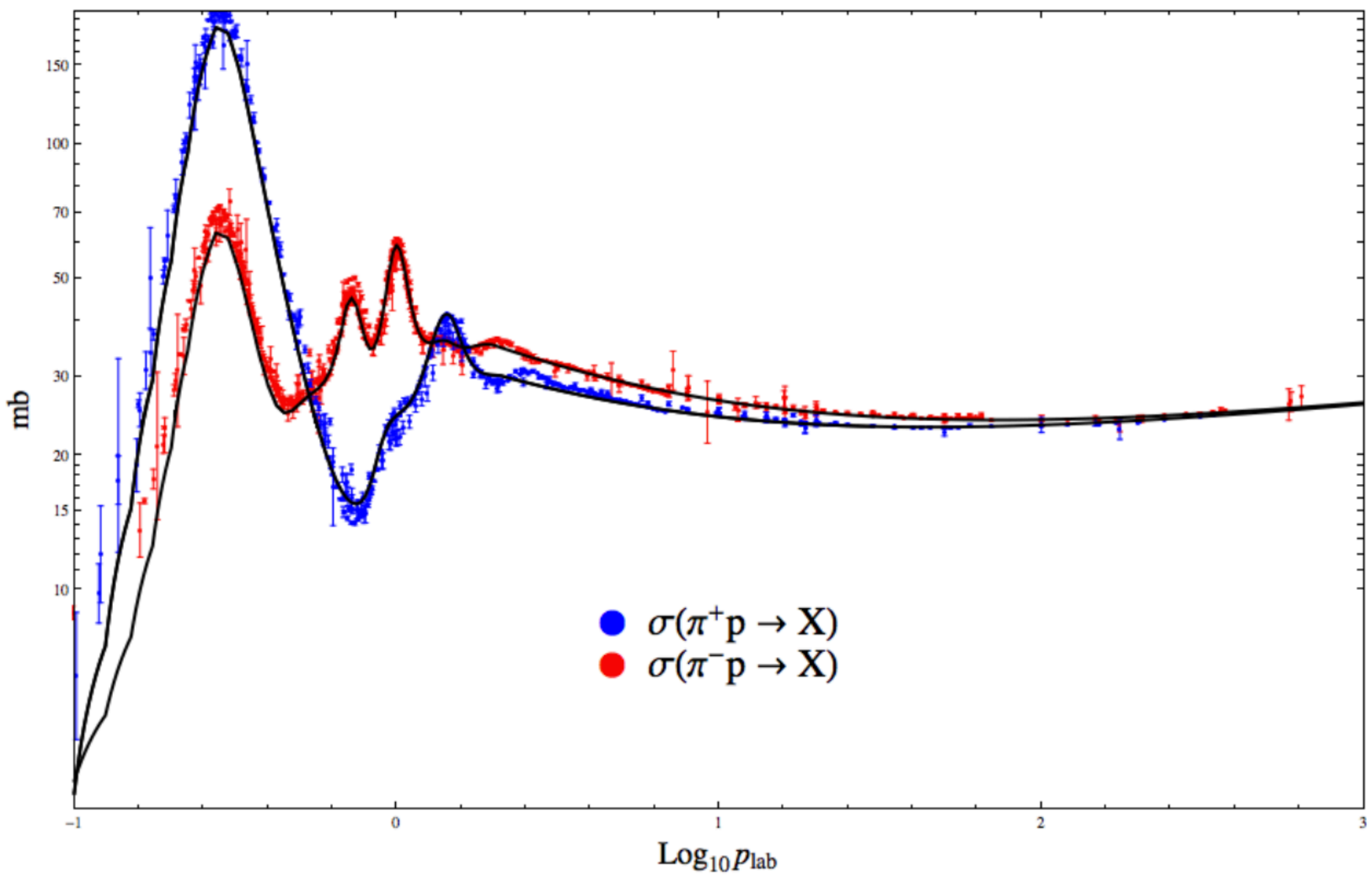
Regge Pole Formula

Restoring the signature factor

$$A(s,t) = \beta(t)\frac{\pm 1 + e^{-i\pi\alpha(t)}}{2\sin\pi\alpha(t)}(s/s_0)^{\alpha(t)} + \mathcal{O}(s^{-1/2})$$

For the total cross section

$$\mathrm{Im}A(s,t) = \beta(t)(s/s_0)^{\alpha(t)}$$



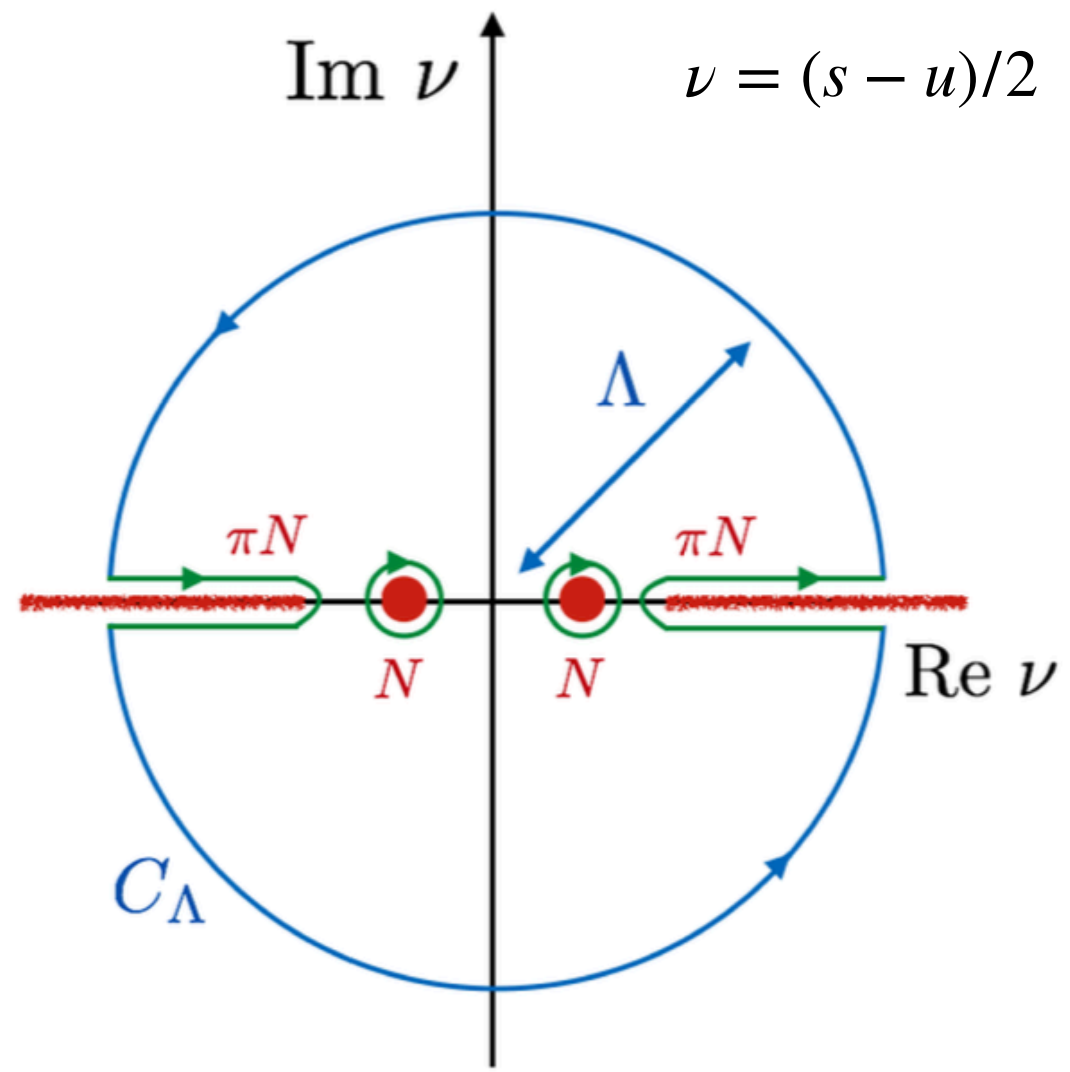
Finite Energy Sum Rules

How to determine $\beta(t)$?

$$A(t, z_t) = \beta(t)\frac{\pm 1 + e^{-i\pi\alpha(t)}}{2\sin\pi\alpha(t)}\left(\frac{s}{s_0}\right)^{\alpha(t)}$$

Write a contour integral with Λ large

$$\frac{1}{\Lambda^k}\int_0^\Lambda \mathrm{Im}A(\nu,t)\nu^k \mathrm{d}\nu = -\frac{1}{\Lambda^k}\oint_{C_\Lambda} A(s,t)\nu^k \frac{\mathrm{d}\nu}{2i}$$



Applied the Regge formula for the circle

$$\frac{1}{\Lambda^k}\int_0^\Lambda \mathrm{Im}A(\nu,t)\nu^k \mathrm{d}\nu = \frac{\beta(t)\Lambda^{\alpha(t)+1}}{\alpha(t)+k+1}$$

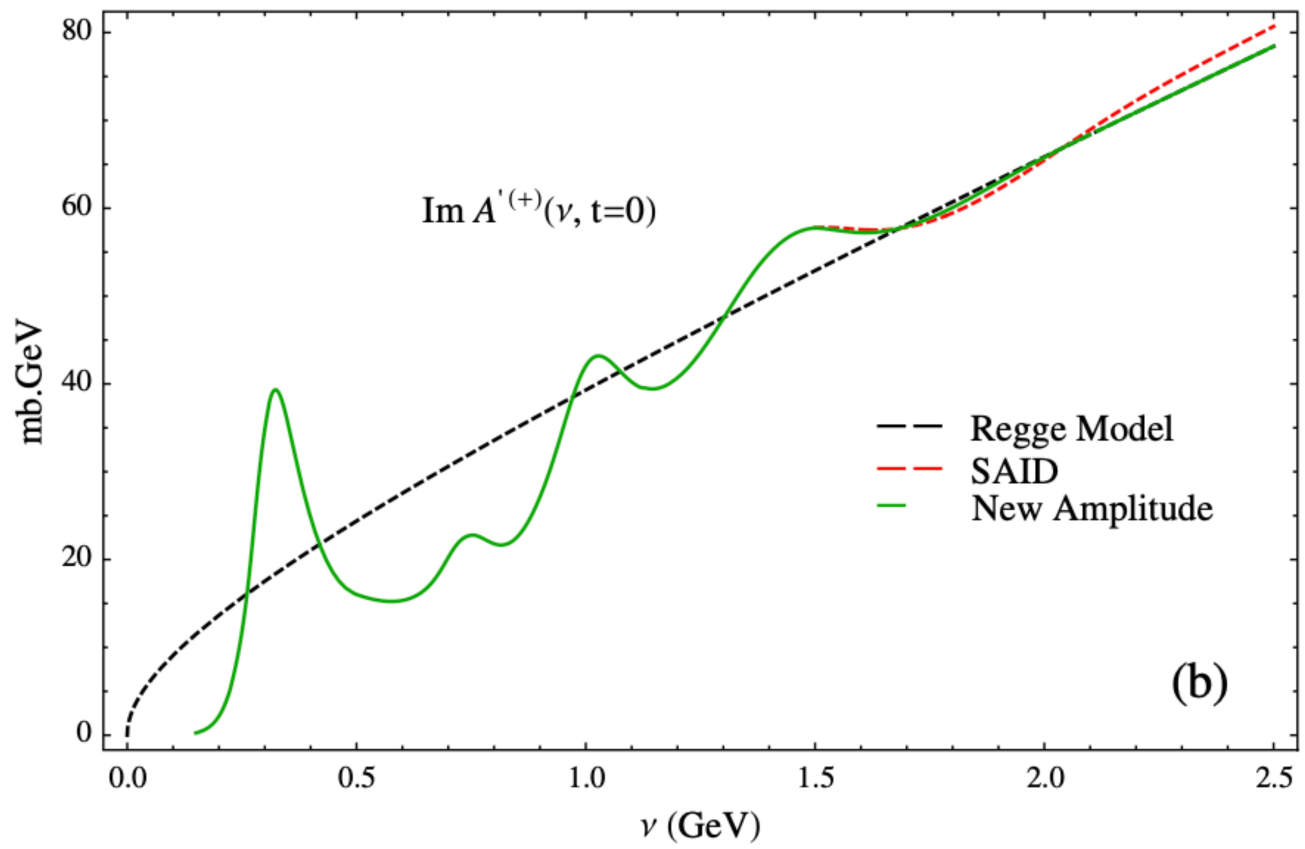
Finite Energy Sum Rules

PHYSICAL REVIEW D **92,** 074004 (2015)

Applied the Regge formula for the circle

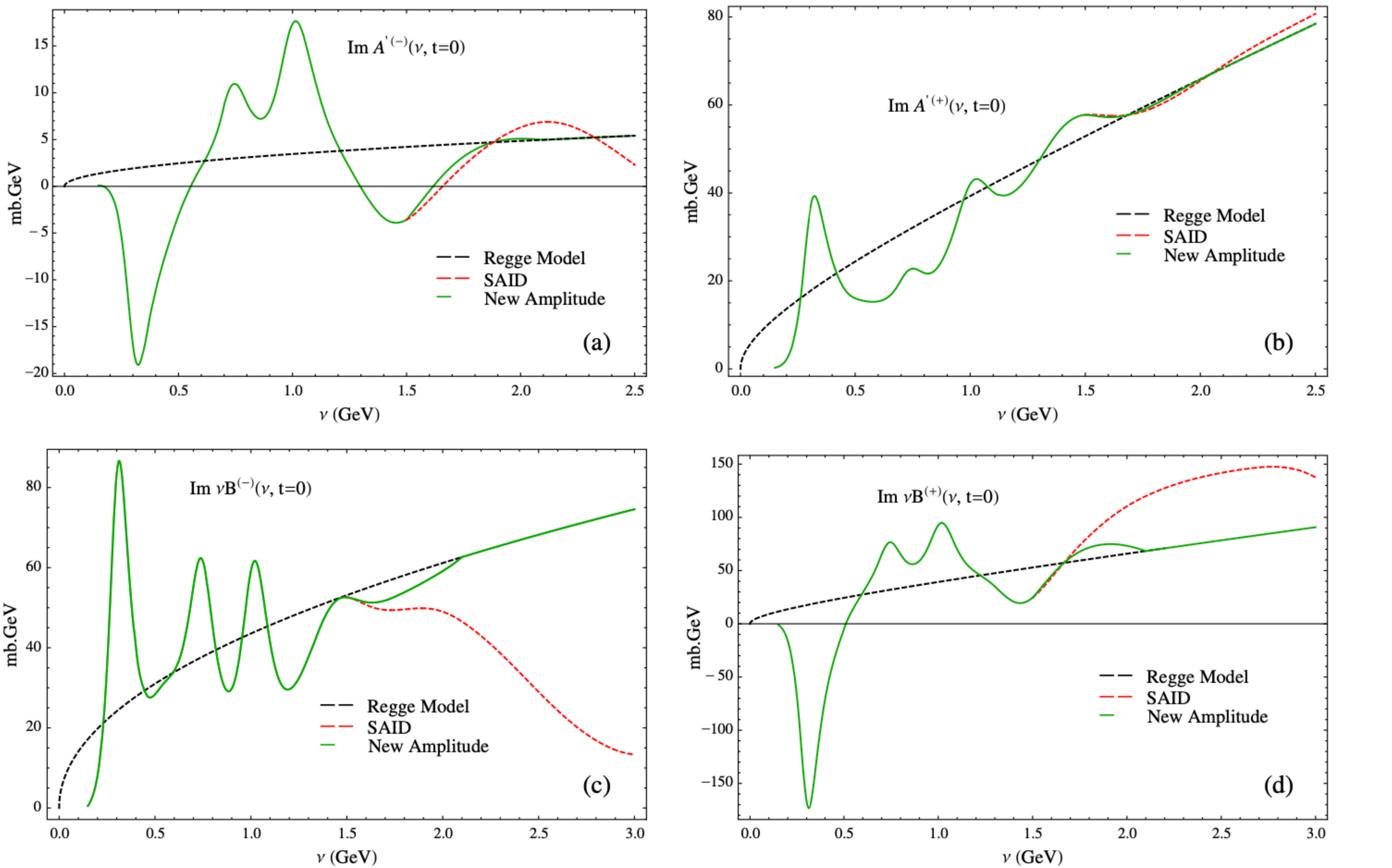
$$\int_0^\Lambda \mathrm{Im}A(\nu,t)\mathrm{d}\nu = \frac{\beta(t)\Lambda^{\alpha(t)+1}}{\alpha(t)+1}$$

$$k = 0$$



Finite Energy Sum Rules

PHYSICAL REVIEW D 92, 074004 (2015)



Application to photoproduction

EPL, **122** (2018) 41001

Determine $\beta(t)$ from the low energy solution

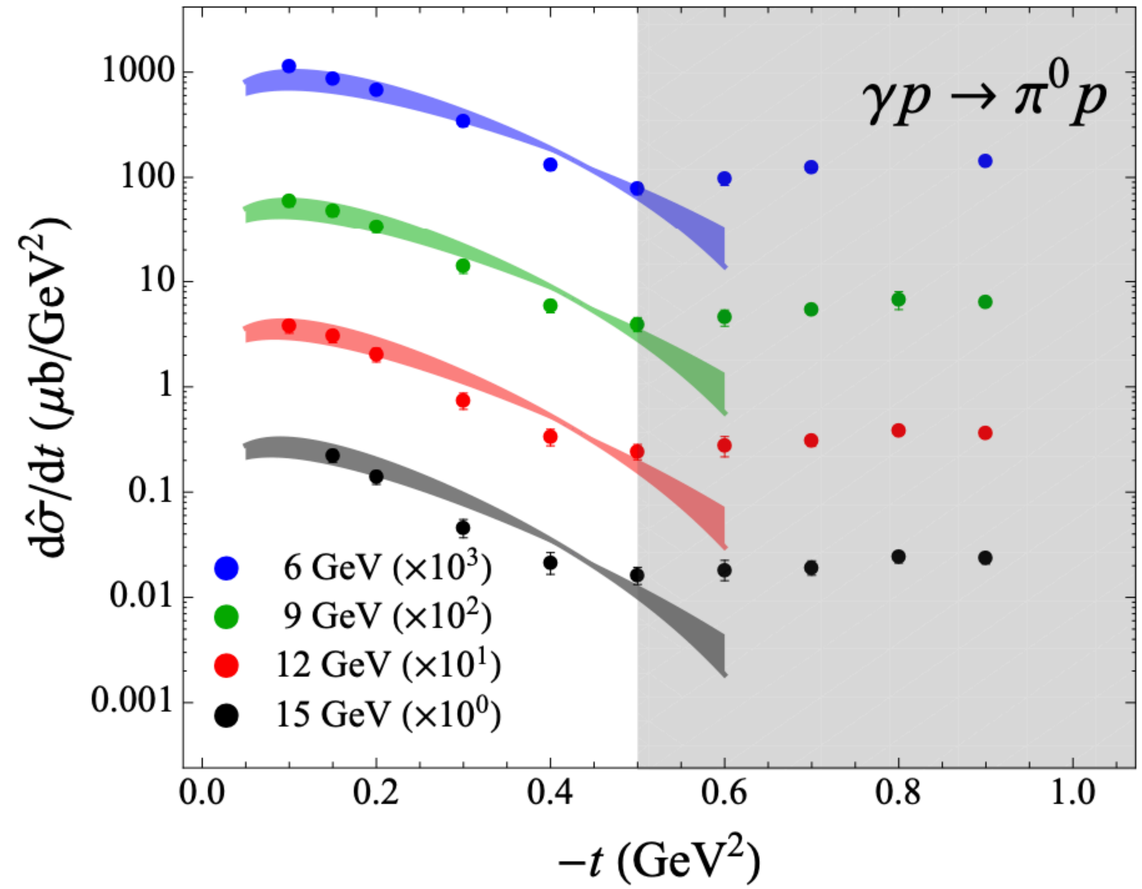
From known leading trajectory $\alpha(t) = 0.9t + 0.5$

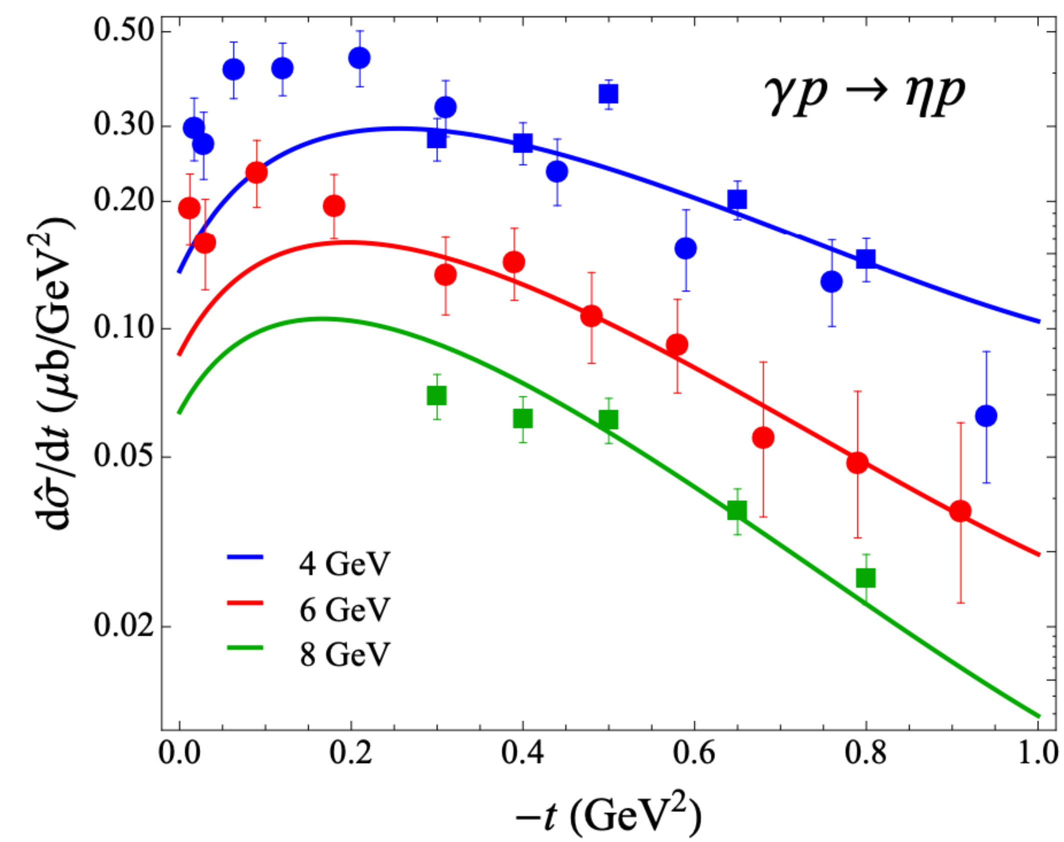
$$\widehat{\beta}_i(t) = \frac{\alpha(t)+k}{\Lambda^{\alpha(t)+k}} \int_0^{\Lambda} \mathrm{Im}\, A_i^{\mathrm{PWA}}(\nu,t)\,\nu^k\,\mathrm{d}\nu,$$

Make prediction for the high energy region

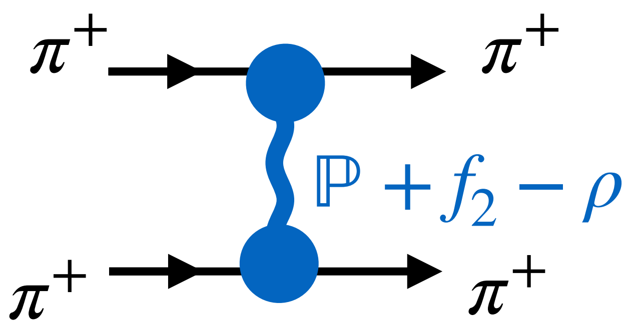
$$\widehat{A}_i(\nu,t) = \left[i + \tan\frac{\pi}{2}\alpha(t)\right]\widehat{\beta}_i(t)\,\nu^{\alpha(t)-1}.$$

$$\frac{\mathrm{d}\widehat{\sigma}}{\mathrm{d}t} \simeq \frac{1}{32\pi}\left[\left|\widehat{A}_1\right|^2 - t\left|\widehat{A}_4\right|^2\right]$$
$$= \frac{\nu^{2\alpha(t)-2}}{32\pi}\left[1+\tan^2\frac{\pi}{2}\alpha(t)\right]\left[\widehat{\beta}_1^2(t) - t\,\widehat{\beta}_4^2(t)\right].$$





Application to exotic reaction



No resonances in the s-channel

Average of low energy = average of high energy

= background $\qquad$ $= \mathrm{P} + f_2 - \rho$

Two component duality:

average of Pomeron = average of background

average of Regge = average of resonance

No resonance implies equality of Regge trajectories and residues

$$\beta_\rho^{\pi\pi}(t) s^{\alpha_\rho(t)} = \beta_{f_2}^{\pi\pi}(t) s^{\alpha_{f_2}(t)}$$

Valid for some s, for all t

$\beta_\rho^{\pi\pi}(t) = \beta_{f_2}^{\pi\pi}(t)$ And $\alpha_\rho(t) = \alpha_{f_2}(t)$

Repeat for kaons and get

$\beta_\rho^{KK}(t) = \beta_{f_2}^{KK}(t) = \beta_{a_2}^{KK}(t) = \beta_\omega^{KK}(t)$ And $\alpha_\rho(t) = \alpha_{f_2}(t) = \alpha_\omega(t) = \alpha_{a_2}(t)$

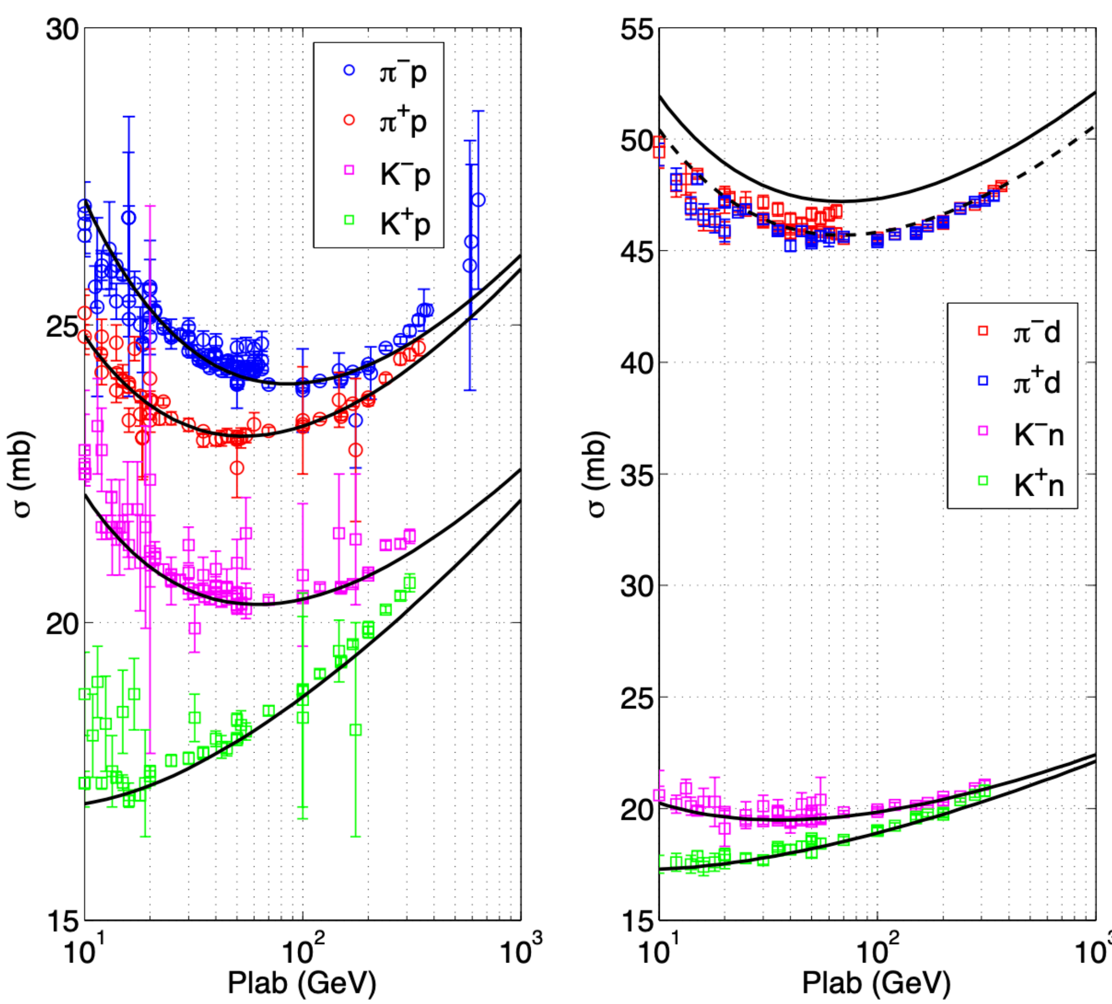
Total cross sections

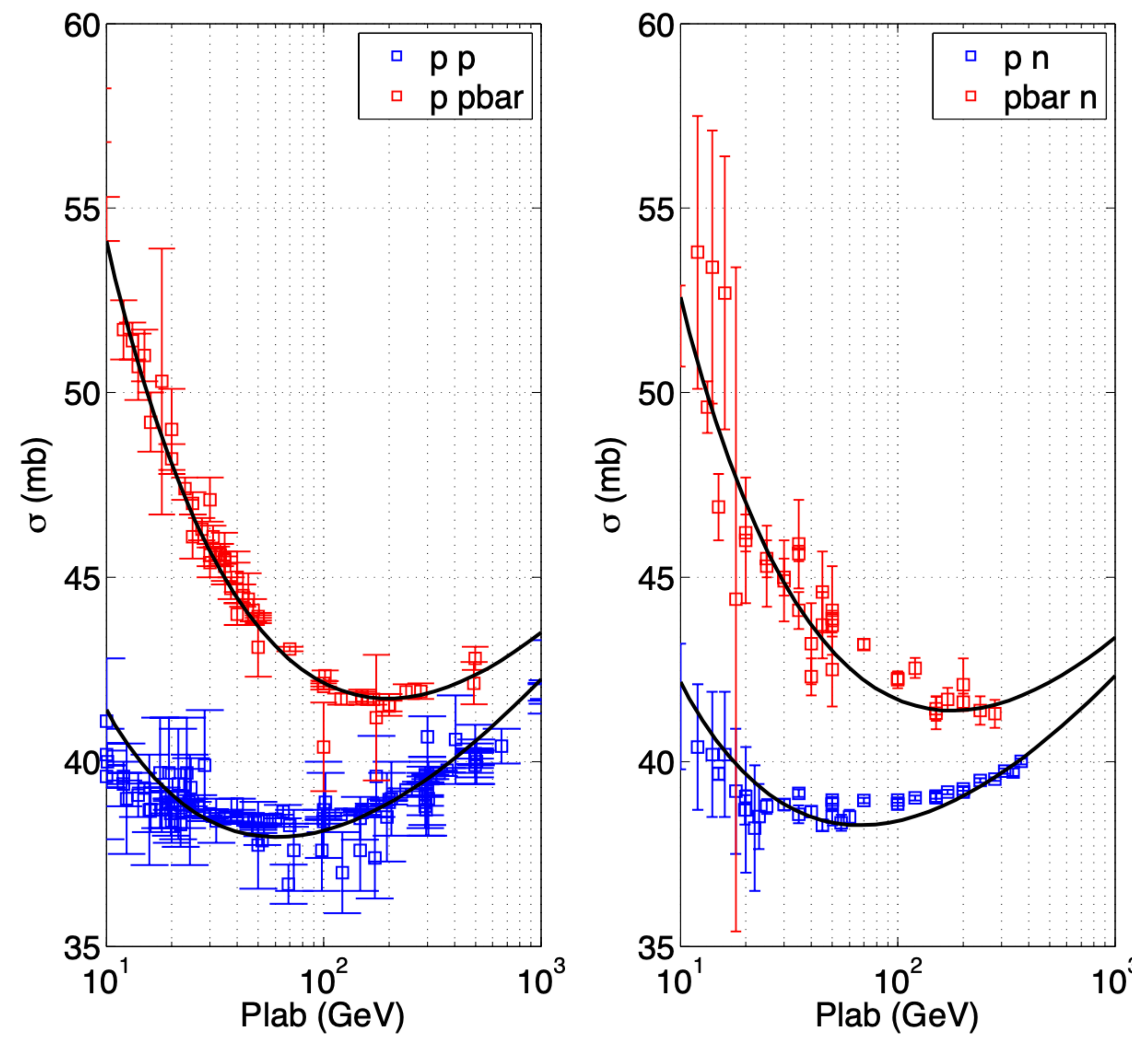
$$\sigma(s) = \frac{1}{s}\left[As^{1.08} + Bs^{0.5}\right]$$

Exch.	$\mathbb{P}$	f	ρ	ω	a
p_+p_+	5.93	21.88	2.67	8.51	2.60
$\pi^+\pi^-$	3.65	9.78	5.37	-	-
K^+K^-	3.17	4.58	2.62	3.01	4.34

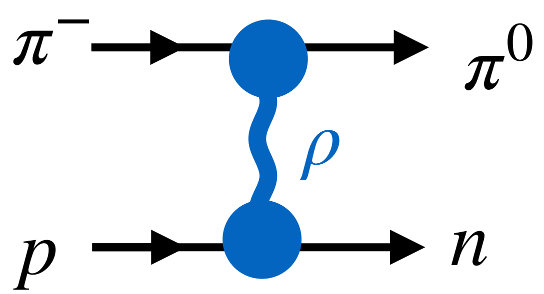
Table 2: Regge exchanges for elastic scatterings

$\pi^\pm p$	$\mathbb{P} + f \pm \rho$
$\pi^\pm n$	$\mathbb{P} + f \mp \rho$
$K^\pm p$	$\mathbb{P} + f \pm \rho + a \pm \omega$
$K^\pm n$	$\mathbb{P} + f \mp \rho - a \pm \omega$
pp	$\mathbb{P} + f + \rho + a + \omega$ (+ unnat)
$\bar{p}p$	$\mathbb{P} + f - \rho + a - \omega$ (+ unnat)
pn	$\mathbb{P} + f - \rho - a + \omega$ (+ unnat)
$\bar{p}n$	$\mathbb{P} + f + \rho - a - \omega$ (+ unnat)



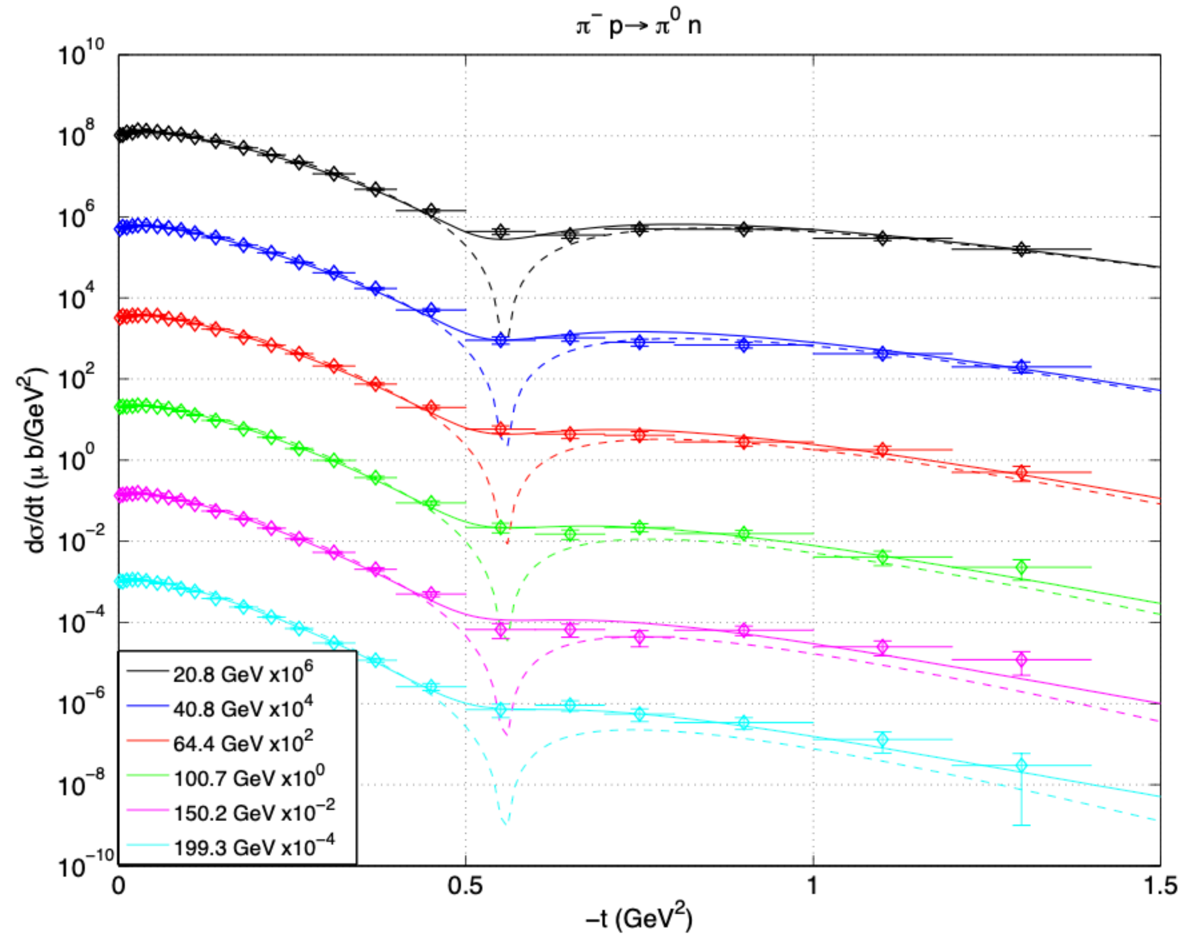


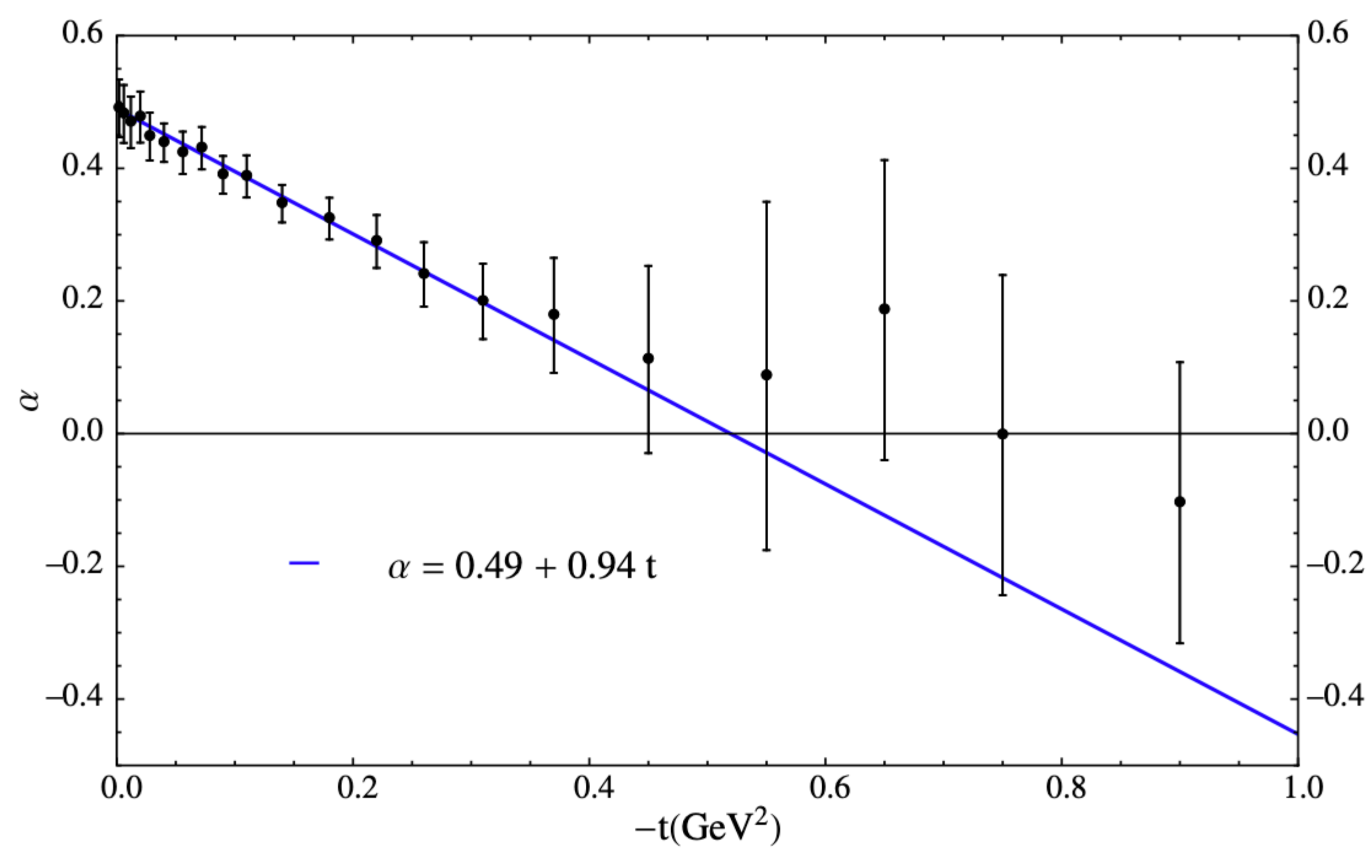
The ρ trajectory



$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{p_L^2} \, |\beta(t) s^{\alpha(t)}|^2$$

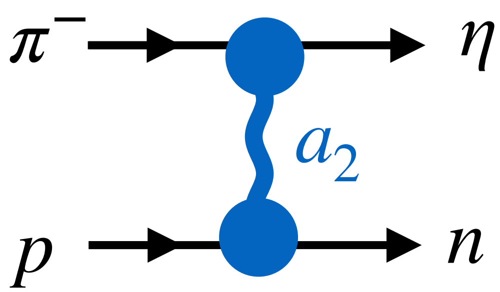
$$\alpha(t) = \frac{1}{2} \log \left(\frac{p_L^2 \frac{\mathrm{d}\sigma}{\mathrm{d}t}\big|_a}{p_L^2 \frac{\mathrm{d}\sigma}{\mathrm{d}t}\big|_b} \right) / \log \frac{s_a}{s_b}$$

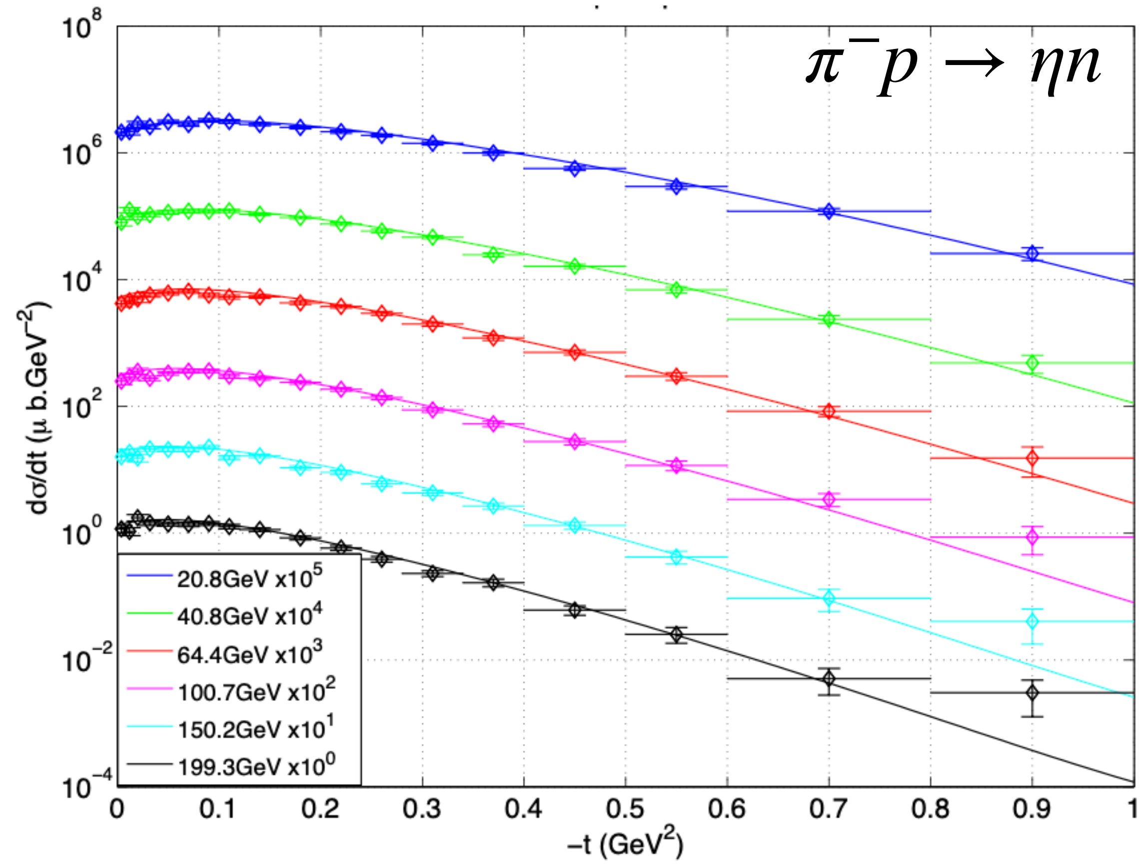


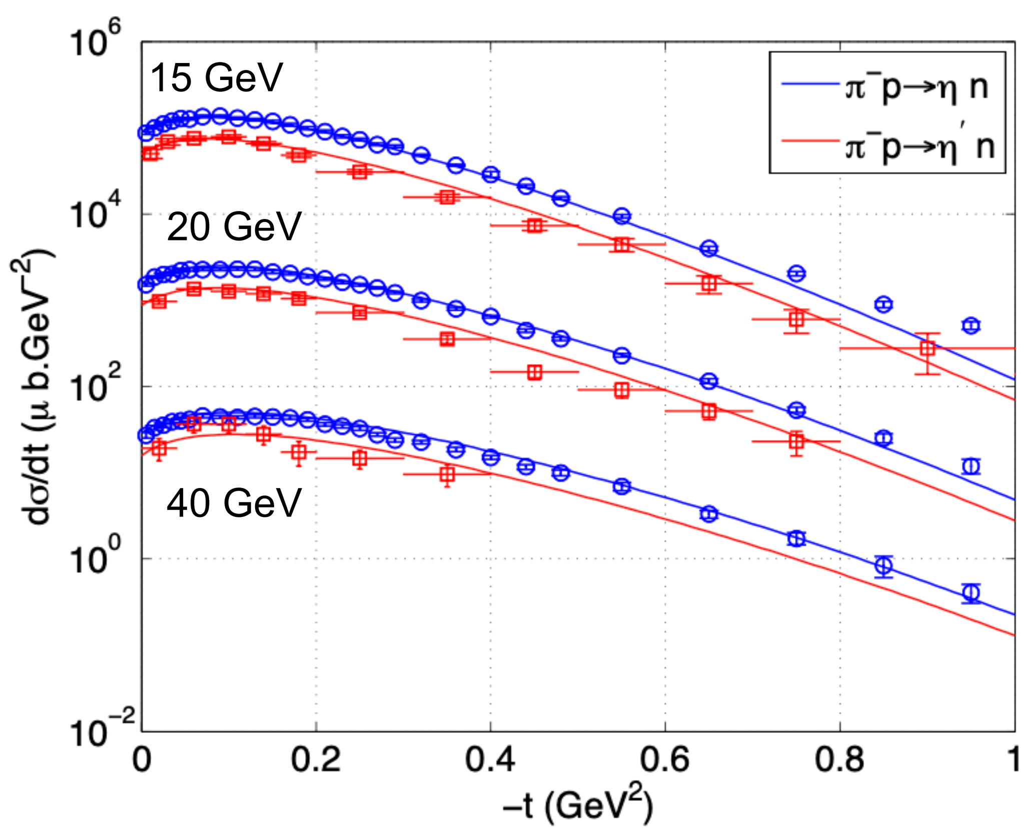


The a_2 trajectory

$$\frac{d\sigma}{dt} = \frac{1}{s} |\beta(t) s^{\alpha(t)}|^2$$





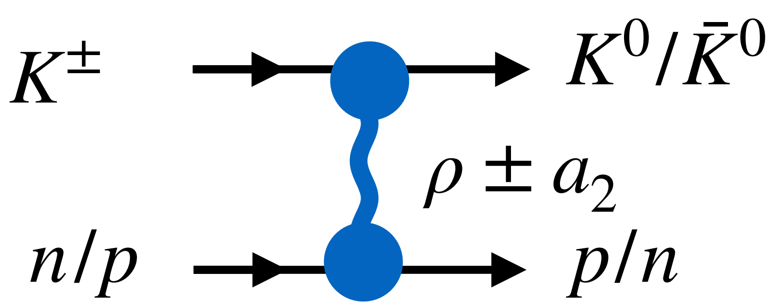


Kaon CEX

EXD implies $\alpha_\rho(t) = \alpha_{a_2}(t)$, $\beta_\rho^{KK}(t) = \beta_{a_2}^{KK}(t)$ and $\beta_\rho^{pn}(t) = \beta_{a_2}^{pn}(t)$

$$A^\rho = \beta(t)(1 - e^{-i\pi\alpha(t)})\left(s/s_0\right)^{\alpha(t)}$$

$$A^{a_2} = \beta(t)(1 + e^{-i\pi\alpha(t)})\left(s/s_0\right)^{\alpha(t)}$$



Sum and difference are

$$A^{(a)} = 2\beta(t)\left(s/s_0\right)^{\alpha(t)}$$

$$A^{(a)} = 2\beta(t)\left(s/s_0\right)^{\alpha(t)} e^{-i\pi\alpha(t)}$$