Helicity Formalism Part II

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Outline

Following SU Chung, Spin formalisms, https://suchung.web.cern.ch/spinfm1.pdf

See also Weinberg, the quantum theory of fields, vol I

Helicity vs Canonical States

Helicity states

$$|\vec{p}, \lambda\rangle_h = R(\Omega)L_z(p)\,|\vec{0}, \lambda\rangle$$

$$|\vec{p}, \lambda\rangle_h = R(\Omega)L_z(p)\; R^{-1}(\Omega)R(\Omega)\,|\vec{0}, \lambda\rangle$$

$$= R(\Omega)L_z(p)\; R^{-1}(\Omega) \sum_m D^s_{m,\lambda}(\Omega)\,|\vec{0}, m\rangle$$

Canonical states

$$|\vec{p}, m\rangle_z = R(\Omega)L_z(p)R^{-1}(\Omega)\,|\vec{0}, m\rangle$$

Relation between the two quantization methods

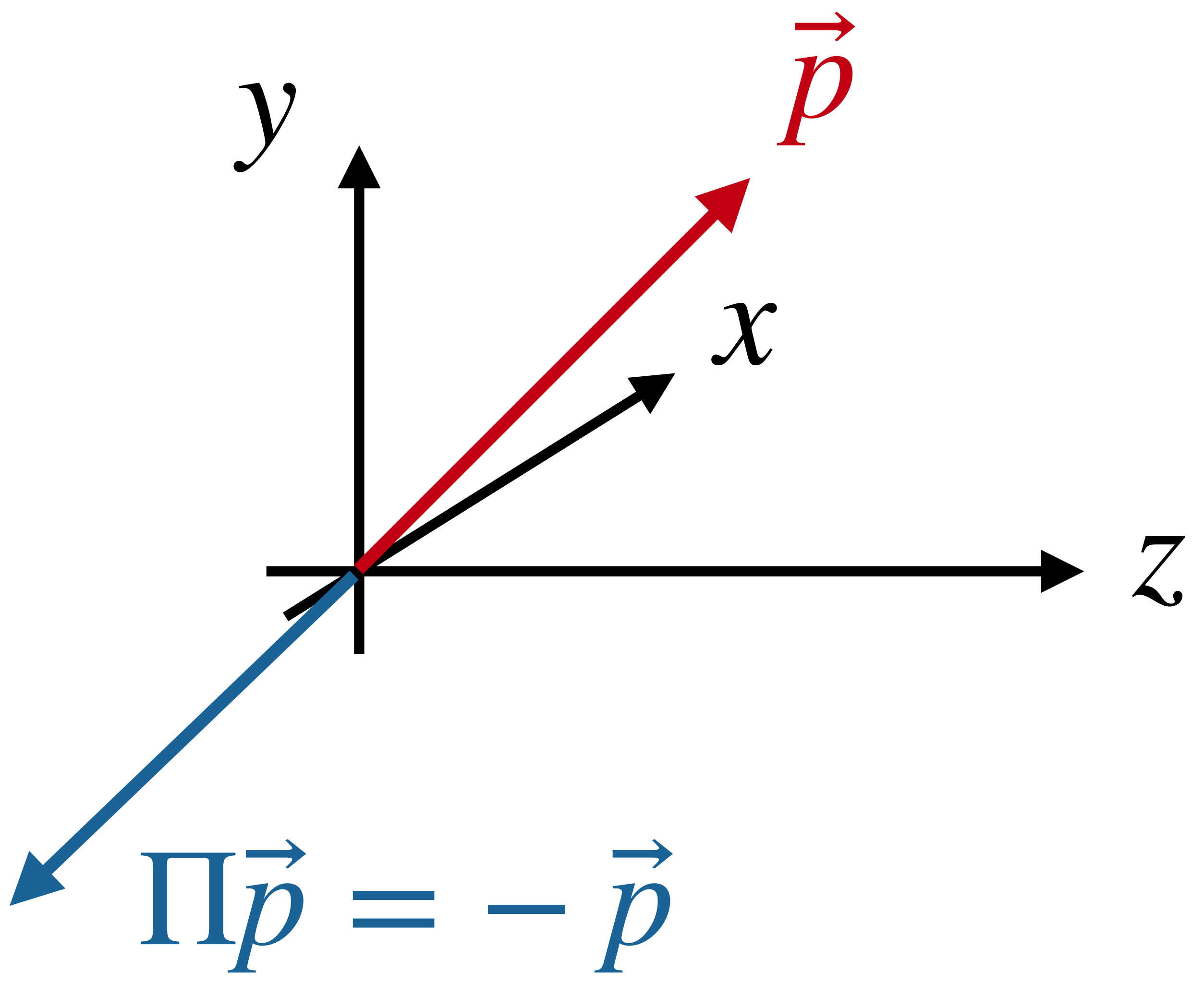
$$|\vec{p}, \lambda\rangle_h = \sum_m D^s_{m,\lambda}(\Omega)\,|\vec{p}, m\rangle_z$$

Intrinsic Parity

Every particle has an intrinsic parity η

$$\Pi\,|\vec{0}, m\rangle = \eta\,|\vec{0}, m\rangle$$

The parity operator changes the sign of all momenta



$$\Pi\,|\vec{p}\rangle = -\,|\vec{p}\rangle$$

Parity commutes with rotations

$$\Pi R(\Omega) = R(\Omega)\Pi$$

But it reverses the boost along z

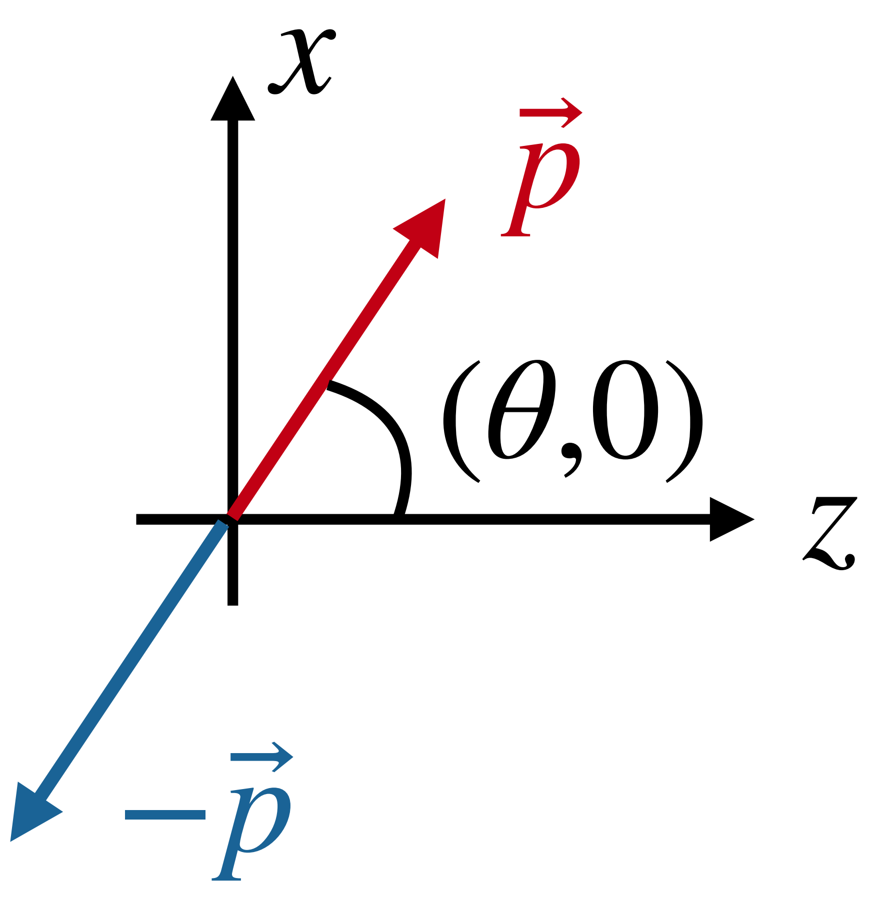
$$\Pi L_z(p) = L_z(-p)\Pi \qquad p \equiv |\vec{p}\,|$$

Relations between angles

$$\Omega_{\vec{p}} = (\theta, \phi)$$

$$\Omega_{-\vec{p}} = (\pi - \theta, \pi \pm \phi)$$

Sign such that $\pi \pm \phi \in [0,2\pi[$



$\Omega_{-\vec{p}} \neq (\pi + \theta, 0)$

$\Omega_{-\vec{p}} = (\pi - \theta, \pi)$

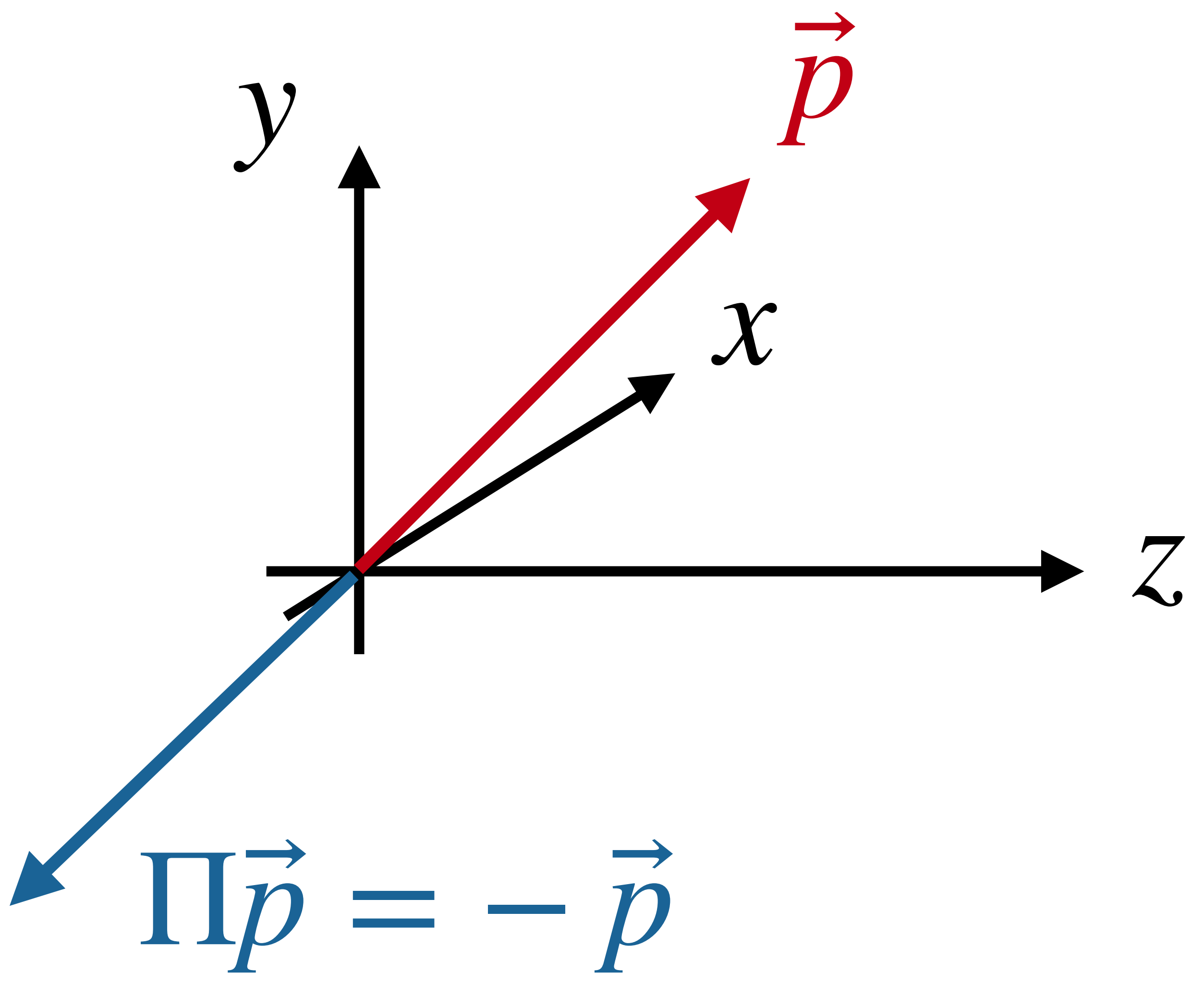
Parity Operator on States

Every particle has an intrinsic parity η

$$\Pi\,|\vec{0},m\rangle = \eta\,|\vec{0},m\rangle$$

$$\Pi R(\Omega) = R(\Omega)\Pi$$

$$\Pi L_z(p) = L_z(-p)\Pi \qquad p \equiv |\vec{p}\,|$$



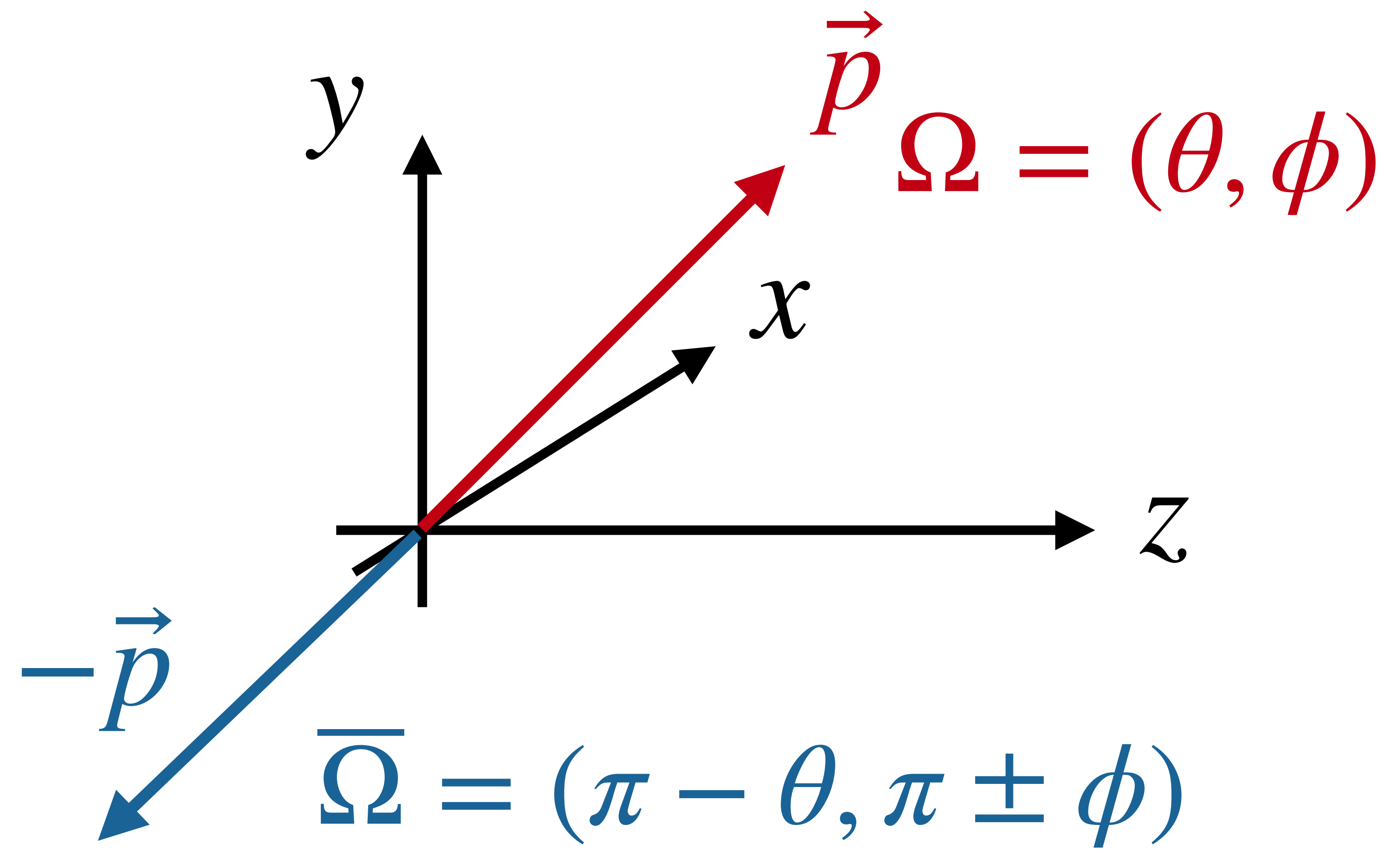
Apply parity to canonical states

$$\Pi\,|\vec{p},m\rangle_z = \Pi R(\Omega)L_z(p)R^{-1}(\Omega)\,|\vec{0},m\rangle_z$$

$$= R(\Omega)L_z(-p)R^{-1}(\Omega)\Pi\,|\vec{0},m\rangle_z$$

$$\Pi\,|\vec{p},m\rangle_z = \eta\,|-\vec{p},m\rangle_z$$

Intrinsic Parity



Parity acting on canonical states

$$\Pi\,|\vec{p}, m\rangle_z = \eta\,|-\vec{p}, m\rangle_z$$

Relation with helicity state

$$|\vec{p}, \lambda\rangle_h = \sum_m D^s_{m,\lambda}(\Omega)\,|\vec{p}, m\rangle_z$$

$$D^s_{m,\lambda}(\Omega) = (-1)^s D^s_{m,-\lambda}(\overline{\Omega})$$

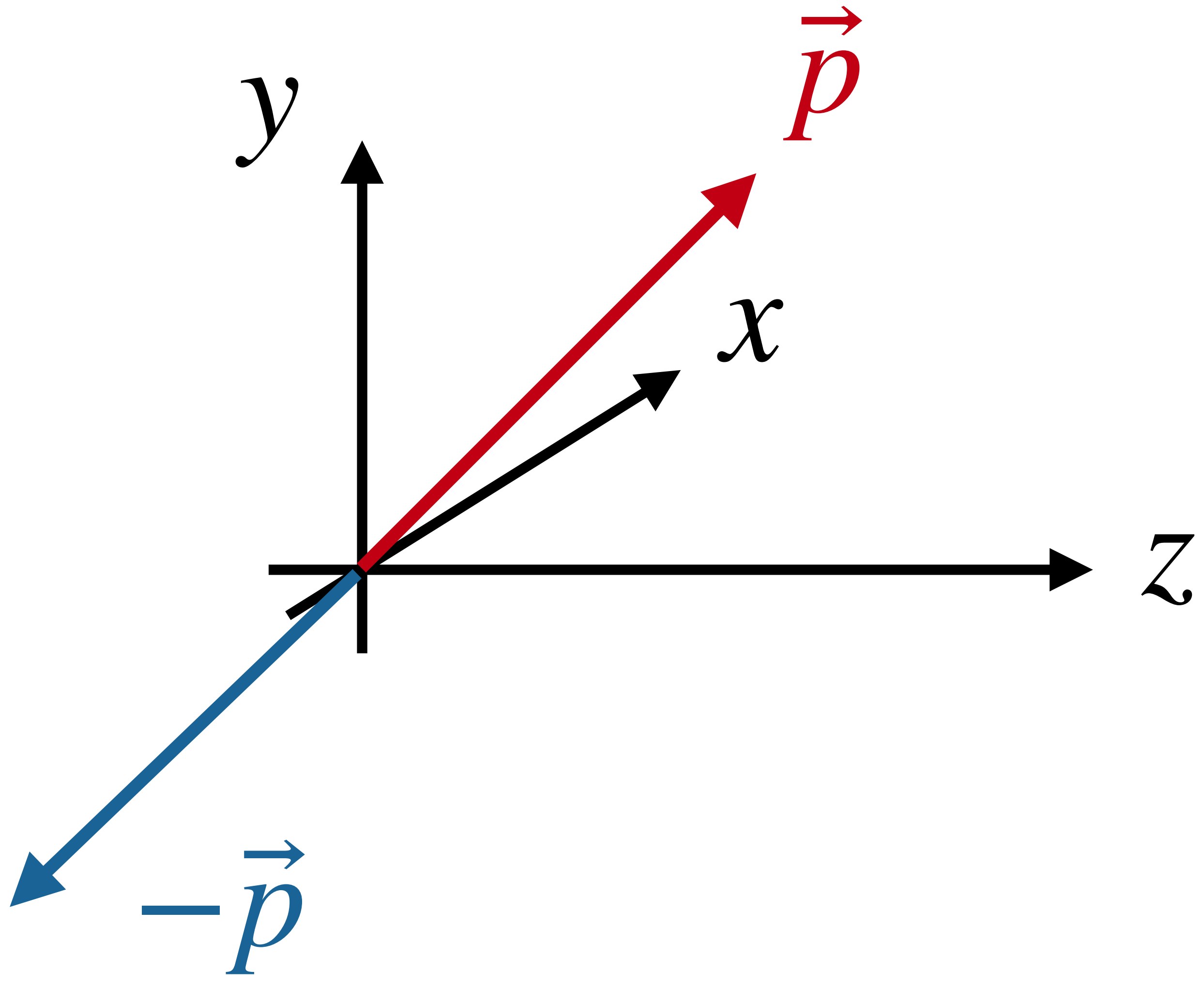
(Only for boson, $s \in N$)

$$\Pi\,|\vec{p}, \lambda\rangle_h = \eta \sum_m D^s_{m,\lambda}(\Omega)\,|-\vec{p}, m\rangle_z$$

$$= \eta(-1)^s \sum_m D^s_{m,-\lambda}(\bar{\Omega})\,|-\vec{p}, m\rangle_z$$

$$\Pi\,|\vec{p}, \lambda\rangle_h = \eta(-1)^s\,|-\vec{p}, -\lambda\rangle_h$$

Two-Particle States (Canonical)



We want a 2-particle state with good spin

First boost the 2 particles back-to-back

$$|\vec{p}, m_1, m_2\rangle_z = |\vec{p}, m_1\rangle_z \otimes |-\vec{p}, m_2\rangle_z$$

$$= B_c(\vec{p} \leftarrow \vec{0})\,|\vec{0}, m_1\rangle_z \otimes B_c(-\vec{p} \leftarrow \vec{0})\,|\vec{0}, m_2\rangle_z$$

$$B_c(\vec{p} \leftarrow \vec{0}) = R(\Omega)L_z(p)R^{-1}(\Omega)$$

$$B_c(-\vec{p} \leftarrow \vec{0}) = R(\Omega)L_z(-p)R^{-1}(\Omega)$$

Couple the spins and add angular momentum

$$|\vec{p}, Sm_s\rangle = \sum_{m_1,m_2} C^{Sm_s}_{s_1m_1;s_2m_2}\,|\vec{p}, m_1, m_2\rangle_z$$

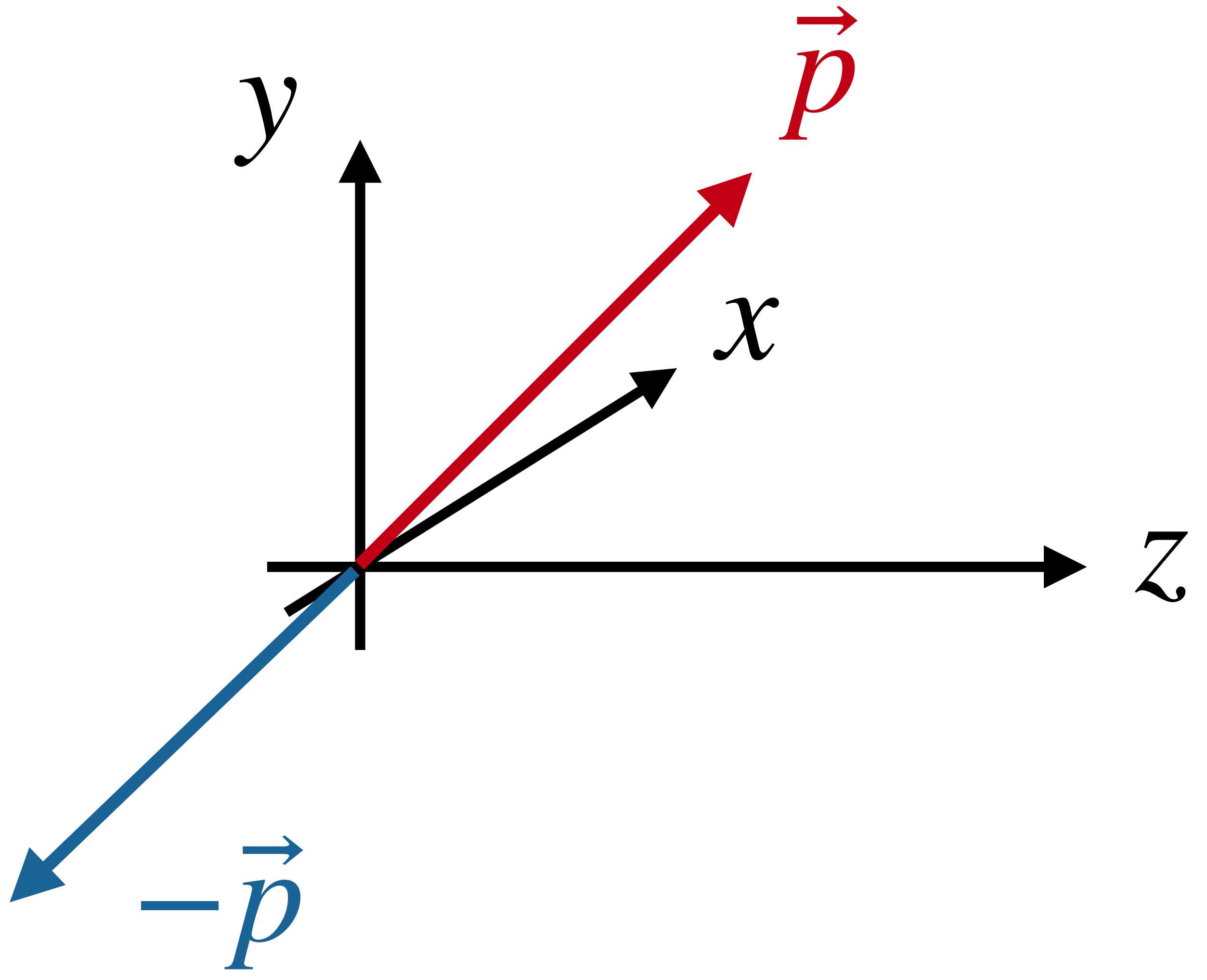
$$|Lm_L, Sm_s\rangle = \int d\Omega\; Y^L_{m_L}(\Omega)\,|\vec{p}, S, m_S\rangle$$

Coupling L and S

$$|JMLS\rangle = \sum_{m_L,m_S} C^{JM}_{Lm_L;Sm_S}\,|Lm_L; Sm_S\rangle$$

$$\Pi\,|JMLS\rangle = \eta_1\eta_2(-1)^L\,|JMLS\rangle$$

Two-Particle States (Helicity)



We want a 2-particle state with good spin

First boost the 2 particles back-to-back

$$|\vec{p}, \lambda_1, \lambda_2\rangle_h = |\vec{p}, \lambda_1\rangle_h \otimes |-\vec{p}, \lambda_2\rangle_h$$

$$= B_h(\vec{p} \leftarrow \vec{0})\,|\vec{0}, \lambda_1\rangle_z \otimes B_h(-\vec{p} \leftarrow \vec{0})\,|\vec{0}, \lambda_2\rangle_h$$

$$B_h(\vec{p} \leftarrow \vec{0}) = R(\Omega) L_z(p)$$

$$B_h(-\vec{p} \leftarrow \vec{0}) = (-1)^{s-\lambda_2} R(\Omega) L_z(-p)$$

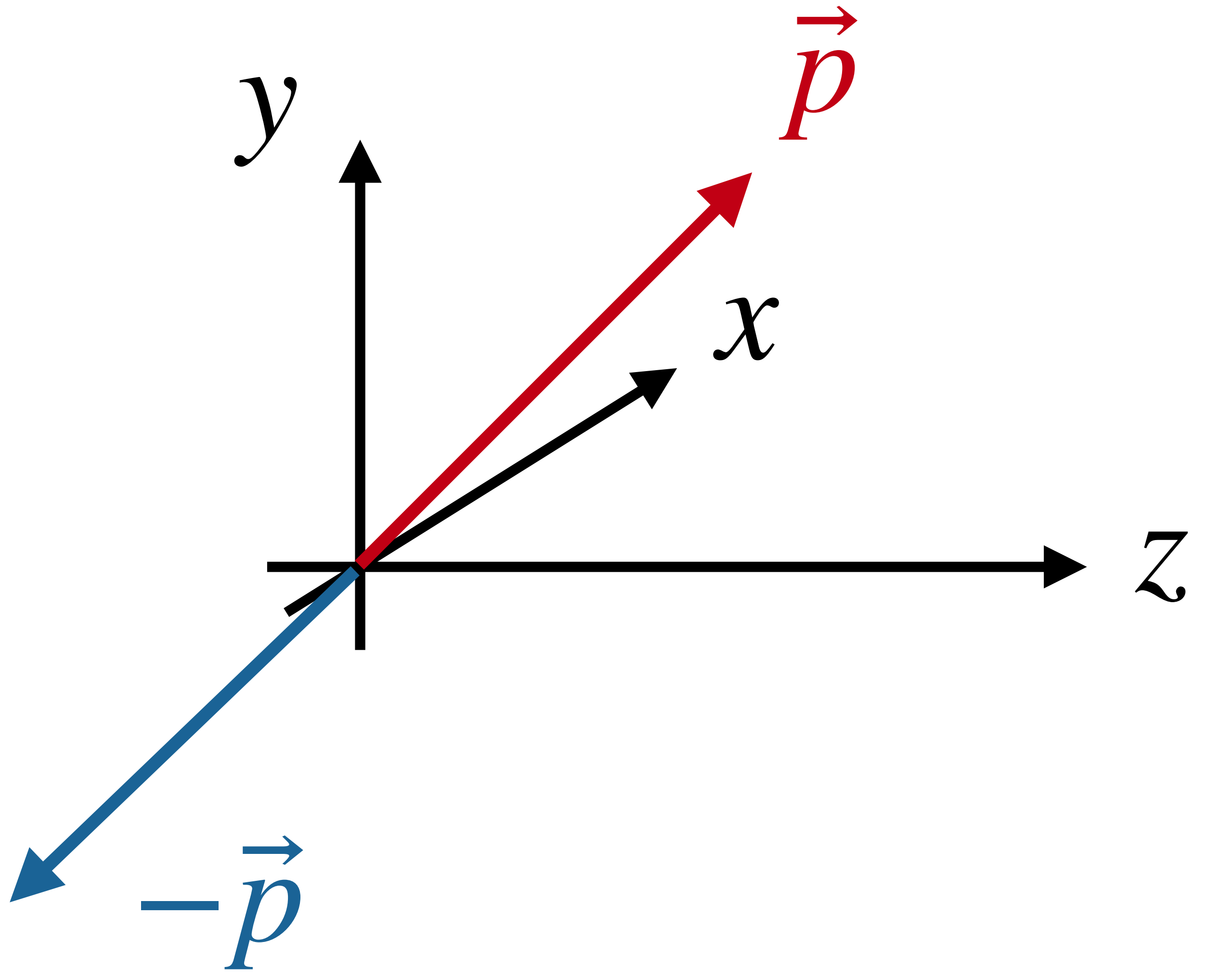
Good total spin state

$$|JM\lambda_1\lambda_2\rangle = \left(\frac{2J+1}{4\pi}\right)^{\frac{1}{2}} \int d\Omega\ D^{J*}_{M,\lambda_1-\lambda_2}(\Omega)\ |\vec{p}\lambda_1, \lambda_2\rangle$$

It transforms like a irreps of spin J

$$R(\Omega)\,|JM\lambda_1\lambda_2\rangle = \sum_{M'} D^{J*}_{M',M}(\Omega)\,|JM'\lambda_1\lambda_2\rangle$$

Parity on Two-Particle States



Good total spin state

$$|JM\lambda_1\lambda_2\rangle = \left(\frac{2J+1}{4\pi}\right)^{\frac{1}{2}} \int d\Omega\ D^{J*}_{M,\lambda_1-\lambda_2}(\Omega)\ |\vec{p}\lambda_1,\lambda_2\rangle$$

Parity on a single state

$$\Pi\,|\vec{p},\lambda\rangle_h = \eta(-1)^s\,|-\vec{p},-\lambda\rangle_h$$

Parity on a two-body state with good spin

$$\Pi\,|JM\lambda_1\lambda_2\rangle = \eta_1\eta_2(-1)^{J+s_1+s_2}\,|JM-\lambda_1-\lambda_2\rangle$$

Parity on a two-body state

$$\Pi\,|\vec{p},\lambda_1,\lambda_2\rangle_h = \eta_1\eta_2(-1)^{s_1+s_2}\,|-\vec{p},-\lambda_1,-\lambda_2\rangle_h$$

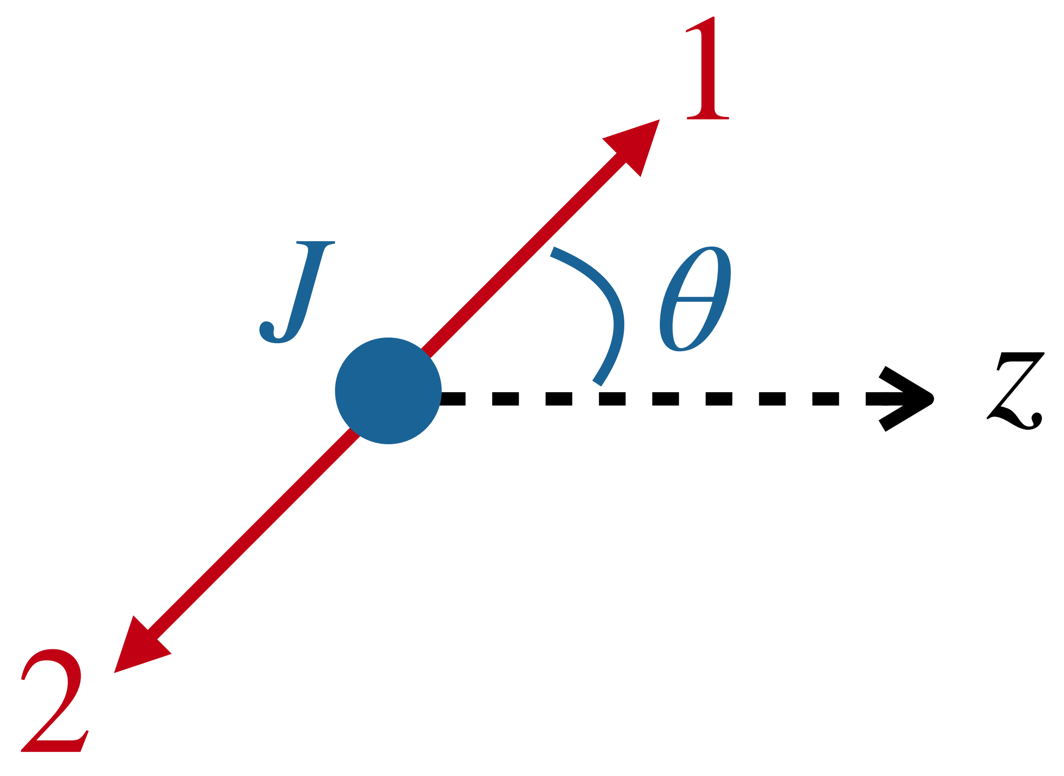
Parity on a Wigner function $D^J_{m,\lambda}(\Omega) = (-1)^J D^J_{m,-\lambda}(\overline{\Omega})$

Applications

Parity on a two-body state with good spin

$$\Pi | JM\lambda_1\lambda_2\rangle = \eta_1\eta_2(-1)^{J+s_1+s_2} | JM -\lambda_1 -\lambda_2\rangle$$

Two pseudoscalar only couples to natural mesons



$$\Pi | JM00\rangle = (-1)(-1)(-1)^{J+0+0} | JM00\rangle$$
$$= (-1)^J | JM00\rangle$$

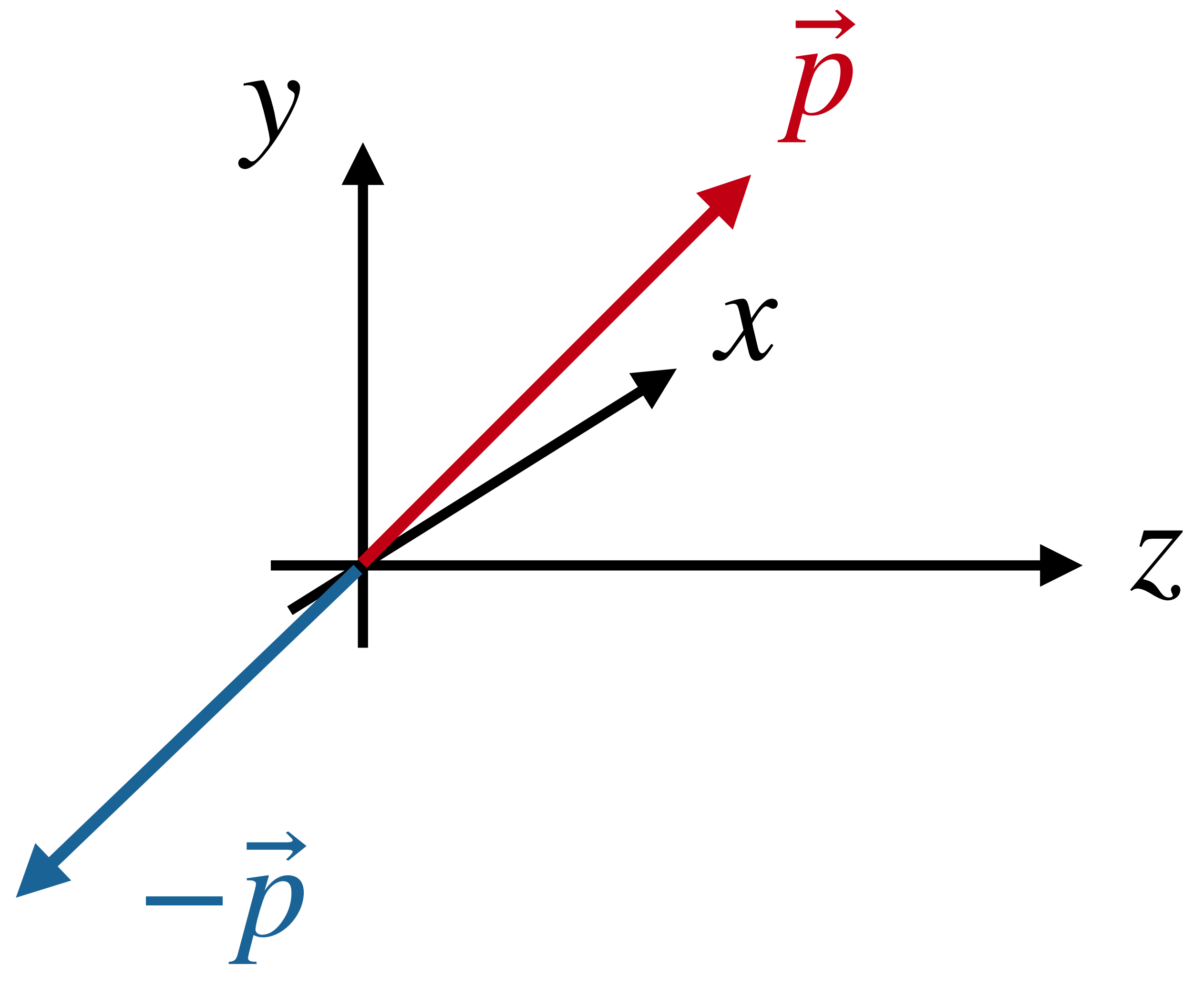
Natural mesons have $P(-1)^J = +1$ $\quad 0^+, 1^-, 2^+, 3^-, \ldots$

Unnatural mesons have $P(-1)^J = -1$ $\quad 0^-, 1^+, 2^-, 3^+, \ldots$

Helicity vs Canonical States

Relation between helicity and canonical states

$$|\vec{p}, \lambda\rangle_h = \sum_m D^s_{m,\lambda}(\Omega)\, |\vec{p}, m\rangle_z$$



Relation between the two quantization methods

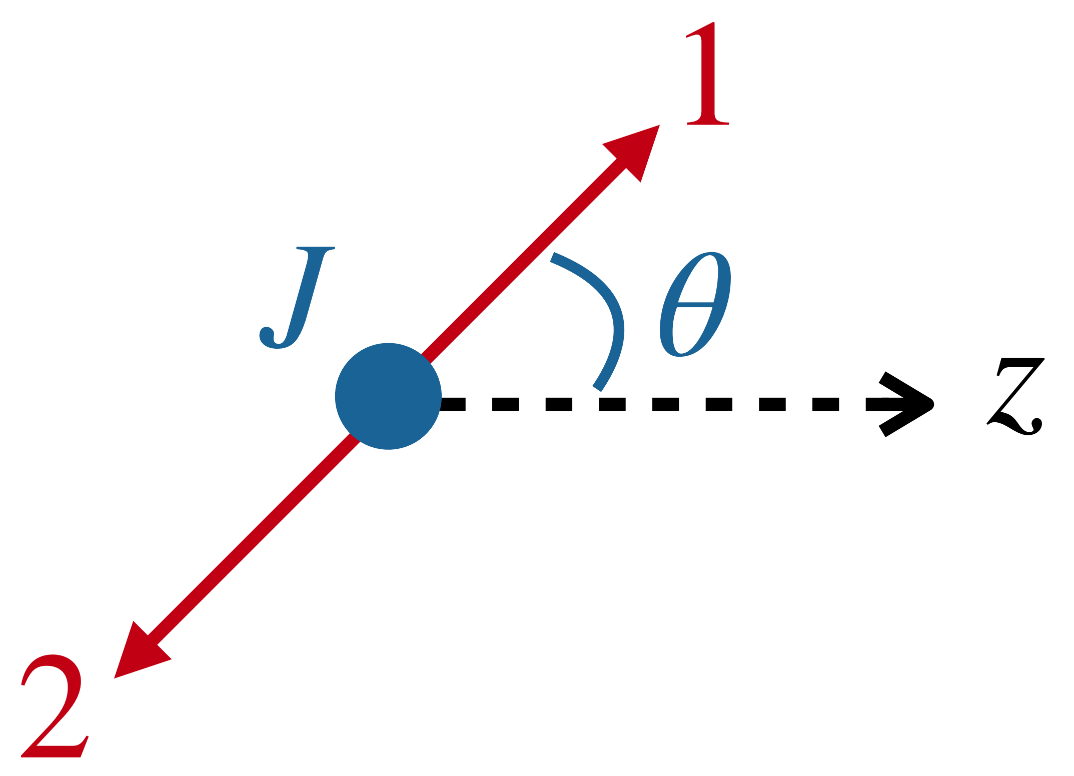
$$|JM\lambda_1\lambda_2\rangle = \sum_{\ell s} \left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} C^{J\lambda}_{\ell 0; s\lambda} C^{s\lambda}_{s_1\lambda_1; s_2 -\lambda_2} |JM\ell s\rangle$$

$$|JM\ell s\rangle = \sum_{\lambda_1,\lambda_2} \left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} C^{J\lambda}_{\ell 0; s\lambda} C^{s\lambda}_{s_1\lambda_1; s_2 -\lambda_2} |JM\lambda_1\lambda_2\rangle$$

Valid only with
the second particle convention

$$B_h(-\vec{p} \leftarrow \vec{0}) = (-1)^{s-\lambda_2} R(\Omega) L_z(-p)$$

Two-Body Decay



M is quantized along the z axis

$$A_{M;\lambda_1,\lambda_2} = \langle \vec{p}\lambda_1, \lambda_2 | A | JM \rangle$$

$$= \langle \vec{p}\lambda_1, \lambda_2 | JM\lambda_1\lambda_2 \rangle \langle JM\lambda_1\lambda_2 | A | JM \rangle$$

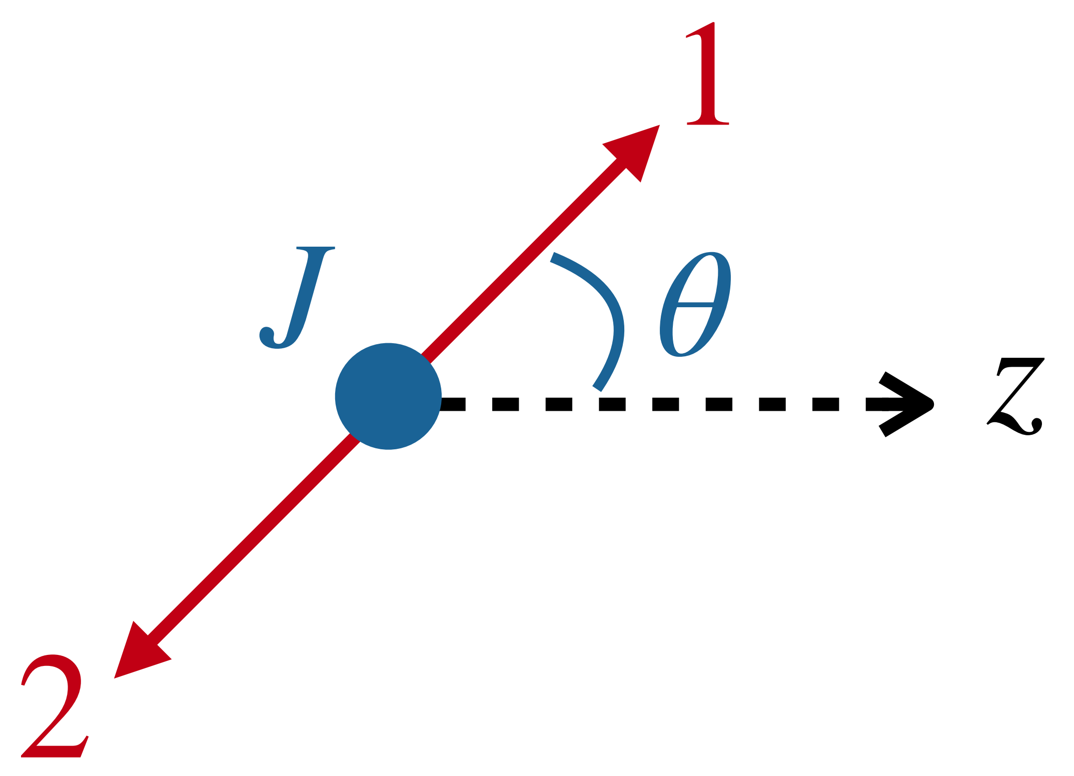
$$= a^J_{\lambda_1,\lambda_2} D^{J*}_{M,\lambda_1-\lambda_2}(\Omega)$$

Ω are the angles of particle 1

The width is given by

$$\Gamma = \frac{1}{2J+1} \frac{1}{8\pi} \frac{p_1}{M^2} \sum_{M;\lambda_1,\lambda_2} \int |A_{M;\lambda_1,\lambda_2}|^2 \times \frac{d\Omega}{4\pi}$$

Two-Body Decay



M is quantized along the z axis

$$A_{M;\lambda_1,\lambda_2} = a^J_{\lambda_1,\lambda_2} D^{J*}_{M,\lambda_1-\lambda_2}(\Omega)$$

The width is given by

$$\Gamma = \frac{1}{2J+1}\frac{1}{8\pi}\frac{p}{M^2}\sum_{M,\lambda_1,\lambda_2}\int |A_{M;\lambda_1,\lambda_2}|^2 \times \frac{d\Omega}{4\pi}$$

Wigner function normalization

$$\int |D^J_{M,\lambda_1,\lambda_2}(\Omega)|^2 d\Omega = \frac{4\pi}{2J+1}$$

We first compute

$$\int \sum_{M;\lambda_1,\lambda_2} |A_{M,\lambda_1,\lambda_2}|^2 d\Omega = 4\pi \sum_{\lambda_1,\lambda_2} |a^J_{\lambda_1,\lambda_2}|^2$$

The width becomes

$$\Gamma = \frac{1}{2J+1}\frac{1}{8\pi^2}\frac{p}{M^2}\sum_{\lambda_1,\lambda_2}|a_{\lambda_1,\lambda_2}|^2$$

Two-Particle States (Helicity)

$$N_J^2 = \frac{2J+1}{4\pi}$$

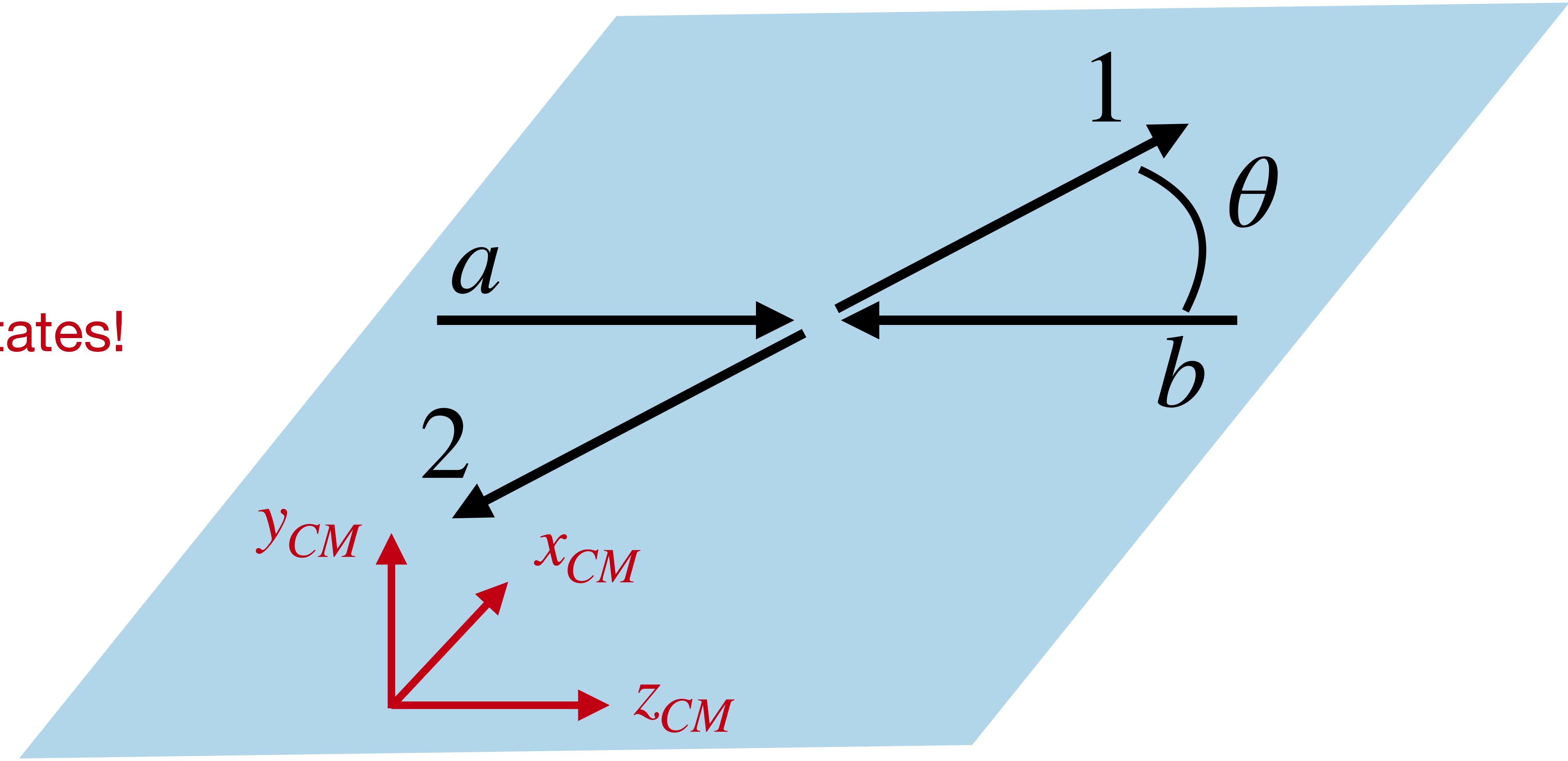
Consider the matrix element

$$\langle \Omega \lambda_1 \lambda_2 | T(s) | 00 \lambda_a \lambda_b \rangle = \sum_{JM} \underbrace{\langle \Omega \lambda_1 \lambda_2 | JM \lambda_1 \lambda_2 \rangle}_{N_J D^{J*}_{M,\lambda_1\lambda_2}(\Omega)} \underbrace{\langle JM \lambda_1 \lambda_2 | T(s) | JM \lambda_a \lambda_b \rangle}_{t^J_{\lambda_1,\lambda_2,\lambda_a,\lambda_b}(s)} \underbrace{\langle JM \lambda_a \lambda_b | 00 \lambda_a \lambda_b \rangle}_{N_J D^{J*}_{M,\lambda_a\lambda_b}(0,0,0)}$$

We obtain the partial wave expansion

$$A_{\lambda_1\lambda_2,\lambda_a,\lambda_b}(s,\Omega) = \sum_J \frac{2J+1}{4\pi} t^J_{\lambda_1,\lambda_2,\lambda_a,\lambda_b}(s)\, D^{J*}_{\lambda_a-\lambda_b,\lambda_1-\lambda_2}(\Omega)$$

It includes both parity states!



Normalization

$$\int |A_{\lambda_1\lambda_2,\lambda_a,\lambda_b}(s,\Omega)|^2 d\Omega = \sum_J |t^J_{\lambda_1,\lambda_2,\lambda_a,\lambda_b}(s)|^2$$

Two-Particle States (Helicity)

We obtain the partial wave expansion

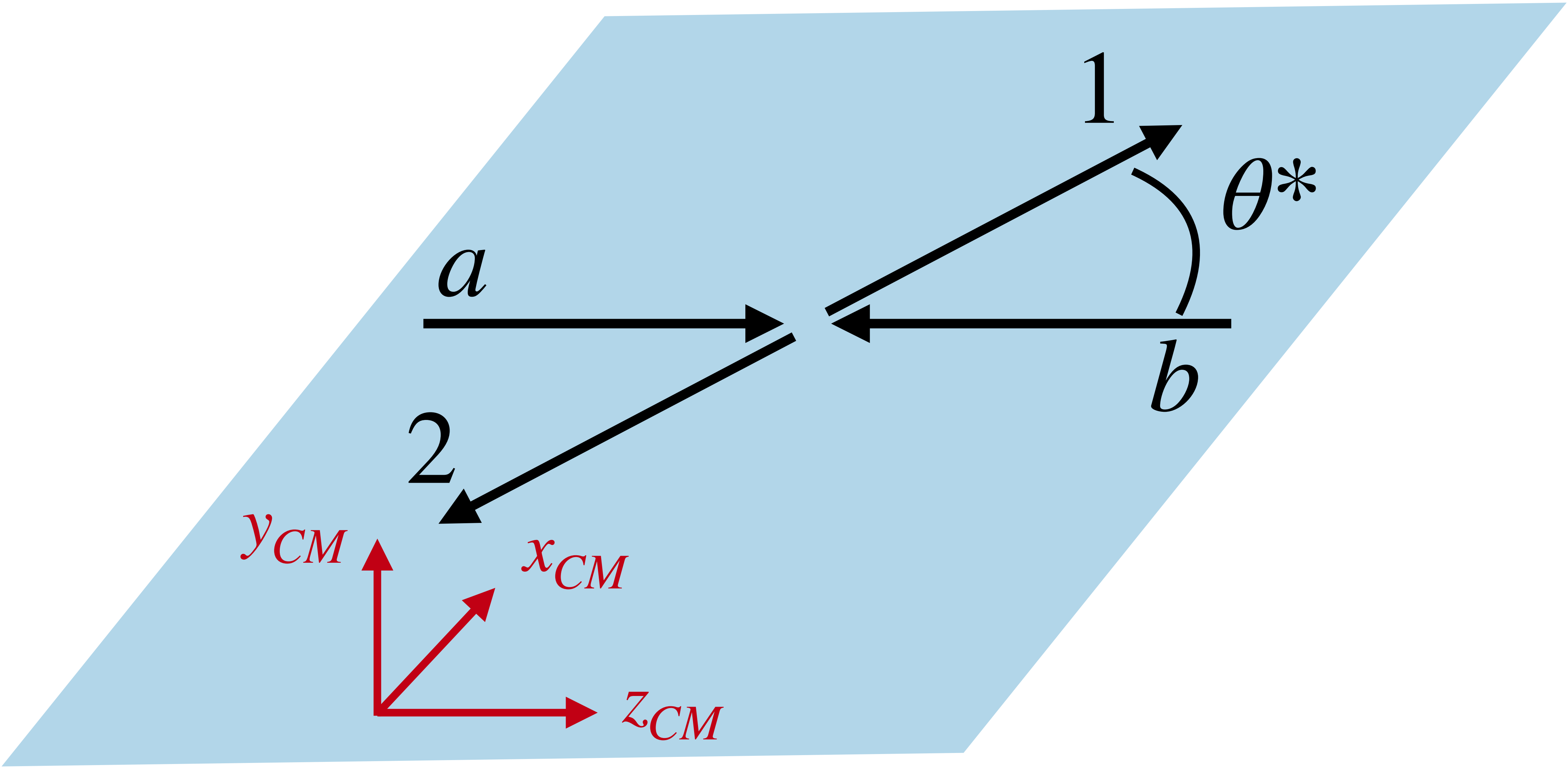
$$A_{\lambda_1\lambda_2,\lambda_a,\lambda_b}(s,\Omega) = \sum_J \frac{2J+1}{4\pi}\ t^J_{\lambda_1,\lambda_2,\lambda_a,\lambda_b}(s)\ D^{J*}_{\lambda_a-\lambda_b,\lambda_1-\lambda_2}(\Omega)$$

For (pseudo-)scalar scattering

$$A(s,\cos\theta) = \frac{1}{4\pi}\sum (2J+1)\ t^J(s)\ P_J(\cos\theta)$$

Normalization

$$\int |A(s,\cos\theta^*)|^2 d\Omega = \sum_J |t^J(s)|^2$$

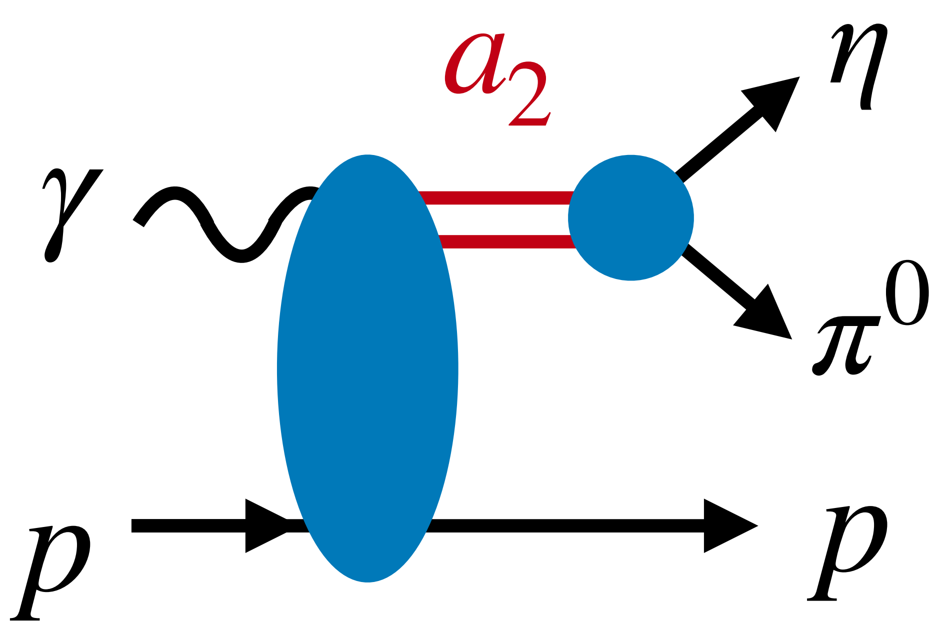


Examples

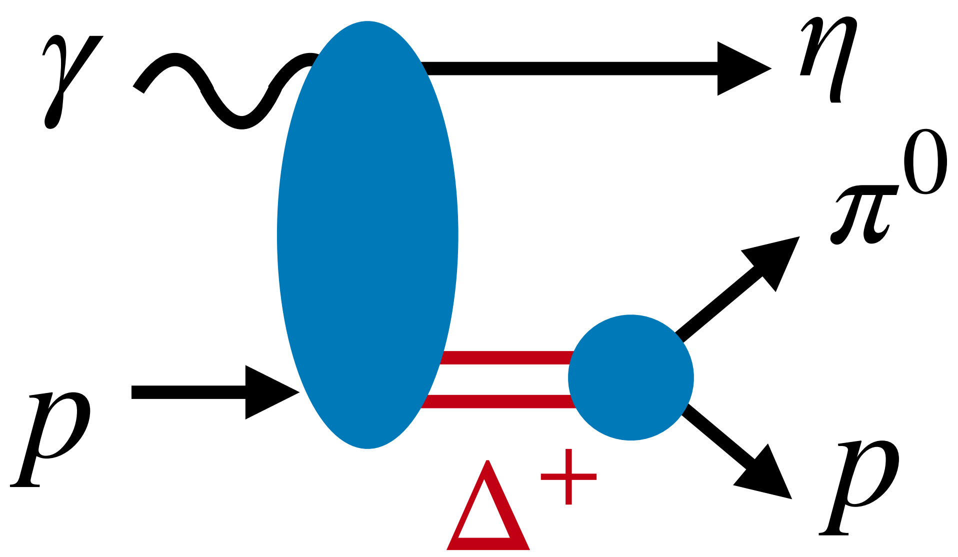
The model

$$I = |A|^2 \qquad A = A^{12} + A^{23} + A^{31} \qquad 1 \equiv \eta; 2 \equiv \pi^0; 3 \equiv p$$

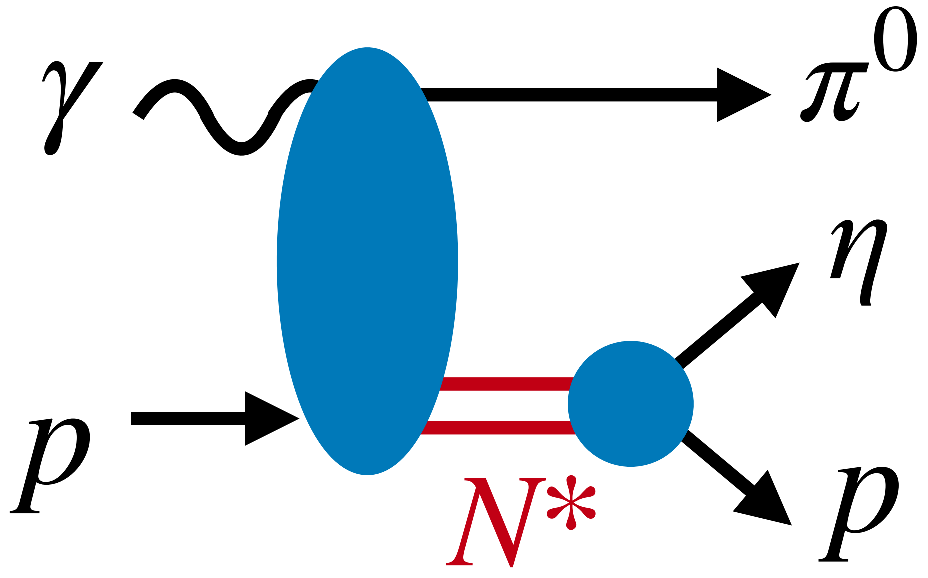
$$A^{12} = \frac{\sum a_m Y_2^m(\Omega_1)}{s - m_{a_2}^2 + i m_{a_2}\Gamma_{a_2}} \times s^{0.5+0.9u_3}$$



$$A^{23} = \frac{\sum b_m Y_1^m(\Omega_2)}{s - m_{\Delta}^2 + i m_{\Delta}\Gamma_{\Delta}} \times s^{0.5+0.9t_1}$$



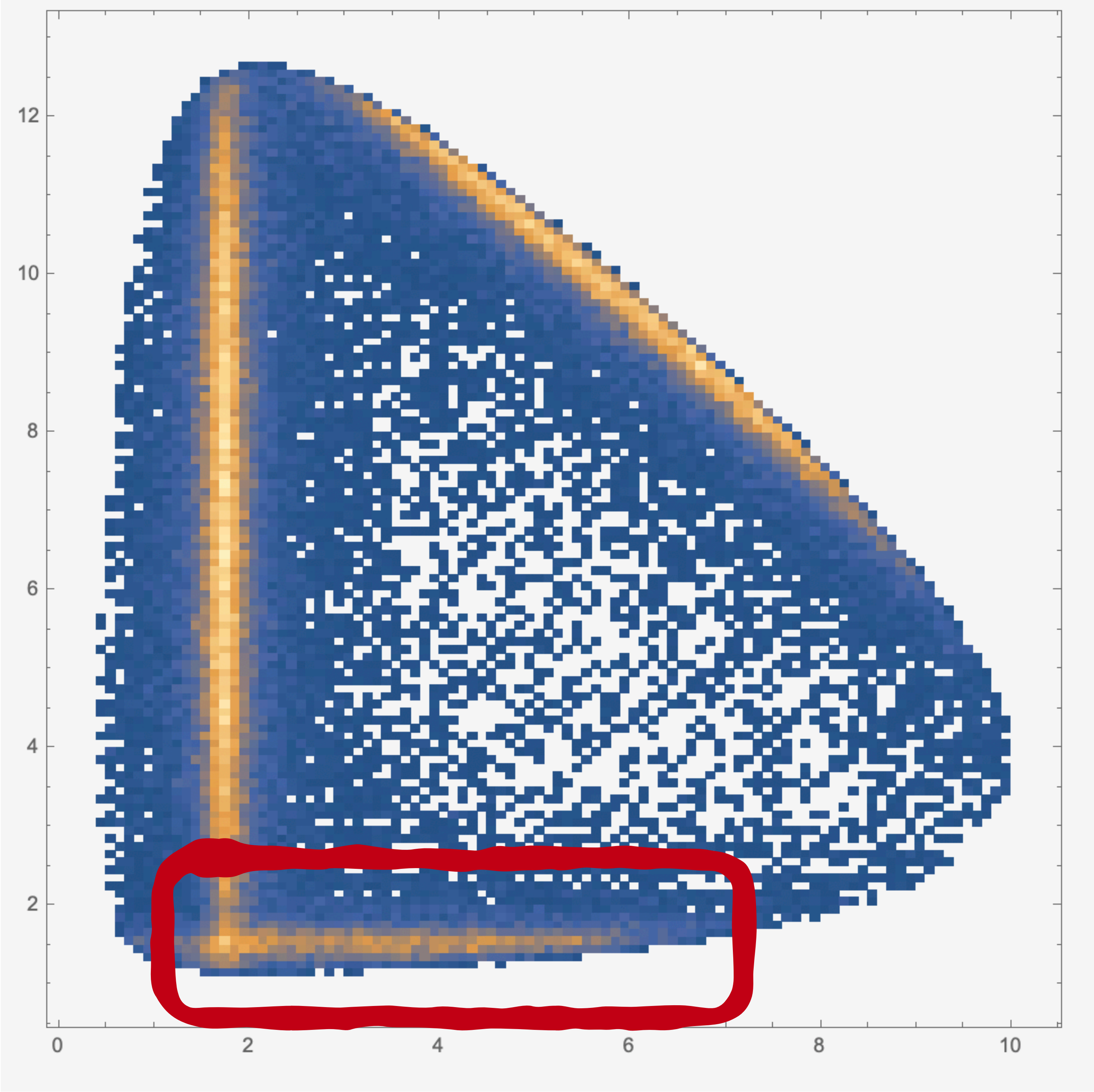
$$A^{31} = \frac{c_0}{s - m_{N*}^2 + i m_{N*}\Gamma_{N*}} \times s^{1.08+0.2t_2}$$



The model

$$I = |A|^2 \qquad A = A^{12} + A^{23} + A^{31}$$

$$A^{23} = \frac{\sum b_m Y_1^m(\Omega_2)}{s - m_\Delta^2 + i m_\Delta \Gamma_\Delta} \times s^{0.5+0.9t_1} \simeq \sum_m b_m Y_1^m(\Omega_2)$$



$$|A|^2 \simeq |\sum_m b_m Y_1^m(\Omega_2)|^2 = \sum_{m,m'} \rho_{m,m'} Y_1^m(\Omega_2) Y_1^{m'*}(\Omega_2)$$

$$= \frac{3}{8\pi}\left[2a_0^2 \cos^2\theta + \sqrt{2}a_0(a_{-1} - a_1)\sin 2\theta \cos\phi + (a_1 + a_{-1})\sin^2\theta - 2a_1a_{-1}\sin^2\theta\cos^2\phi\right]$$

The model

$$I = |A|^2 \qquad A = A^{12} + A^{23} + A^{31}$$

$$A^{12} = \frac{\sum a_m Y_2^m(\Omega_1)}{s - m_{a_2}^2 + i m_{a_2}\Gamma_{a_2}} \times s^{0.5+0.9u_3} \simeq \sum a_m Y_2^m(\Omega_1)$$

$$|A|^2 \simeq |\sum a_m Y_2^m(\Omega_1)|^2 = |\sum_{m,m'} \rho_{m,m'} Y_2^m(\Omega_1) Y_2^{m'*}(\Omega_1)$$