# Helicity Formalism Part I

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Exotic Hadron Topical Collaboration

Horizon2020 Summer School

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# Outline

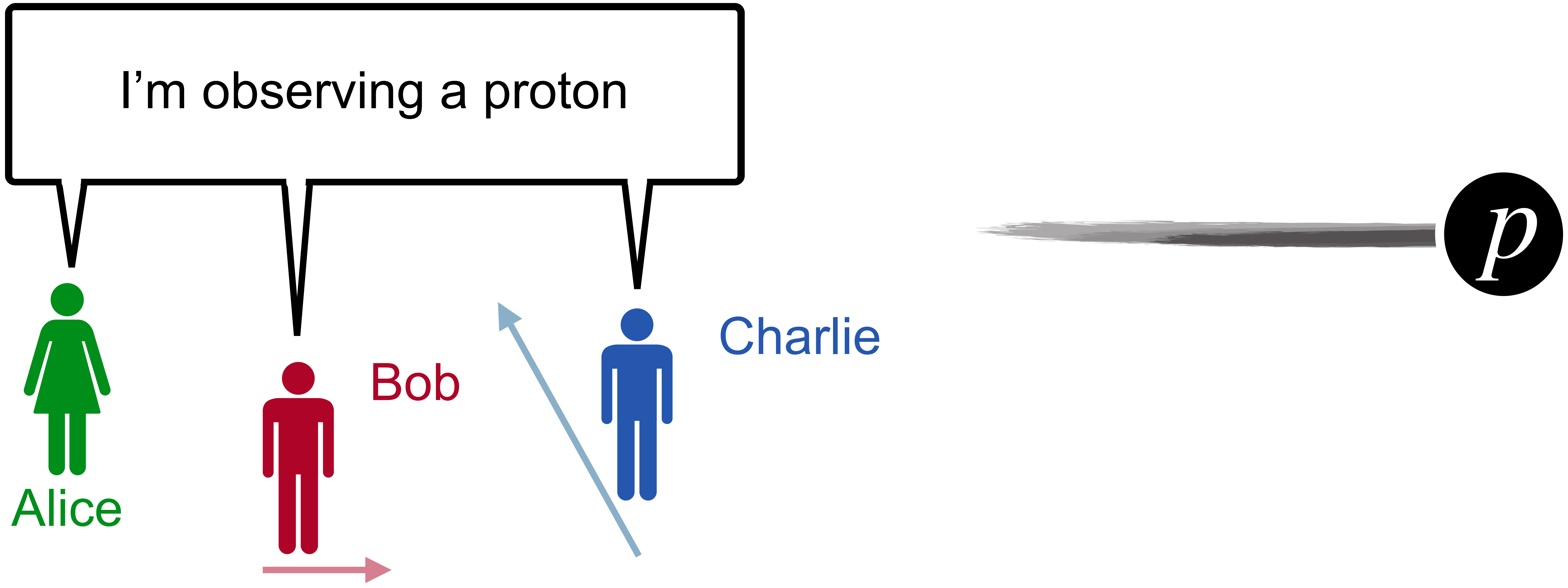
Following SU Chung, Spin formalisms, https://suchung.web.cern.ch/spinfm1.pdf

See also Weinberg, the quantum theory of fields, vol I

# What is a particle?

Alice, Bob and Charlie are different observers, in relative motion

The intrinsic properties of a particle must be identical for all observers




$M \simeq 1$ GeV

$S = \frac{1}{2}$

Group of symmetries of the space-time is the Poincaré group

It has two invariant quantities: mass and spin

# States

## Lorentz group include boosts and rotations

$$\begin{pmatrix} p'_x \\ p'_y \\ p'_z \end{pmatrix} = R(\alpha, \beta, \gamma) \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

3x3 matrix

Spelling out the matrix form

$$p'_{m'} = \sum_{m=-1}^{1} R_{m,m'}(\alpha, \beta, \gamma) p_m$$

## Lorentz group has other representations

$$|m'\rangle = \sum_{m=-J}^{J} D^J_{m,m'}(\alpha, \beta, \gamma) |m\rangle$$

A spin J has 2J+1 component $-J, \ldots, J$

## Explicit form

$$D^J_{m,m'}(\alpha, \beta, \gamma) = e^{-im\alpha} d^J_{m,m'}(\beta) e^{-im\gamma}$$

## Convention

$$d^1_{1,0}(\beta) = -\frac{\sin\beta}{\sqrt{2}}$$

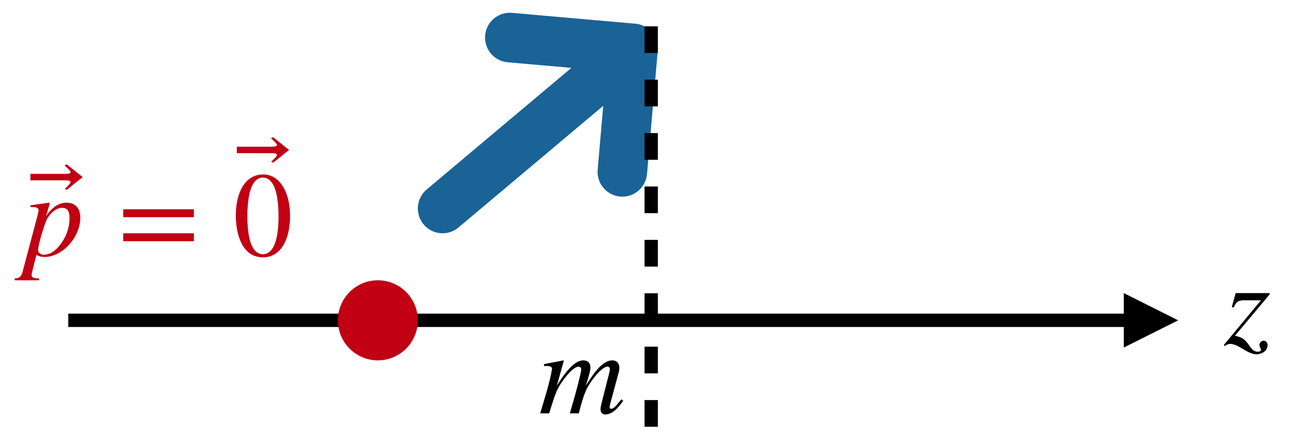
# States at Rest

States are tensorial product

$$|\vec{p}, m\rangle = |\vec{p}\rangle \otimes |m\rangle$$

Implicit dependence on $M, s$

Under a rotation,
the spin projection $m$ changes

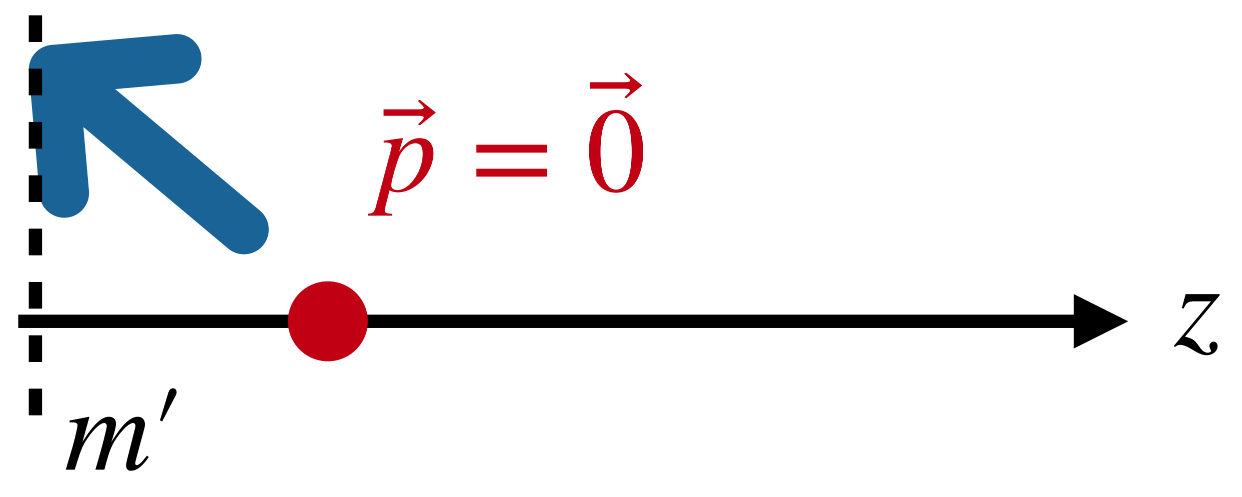



Let's first consider state at rest $\vec{p} = \vec{0}$

The spin projection $m$ is defined by $z$ axis

$$|m'\rangle = \sum_{m=-J}^{J} D^J_{m,m'}(\alpha, \beta, \gamma)\,|m\rangle$$

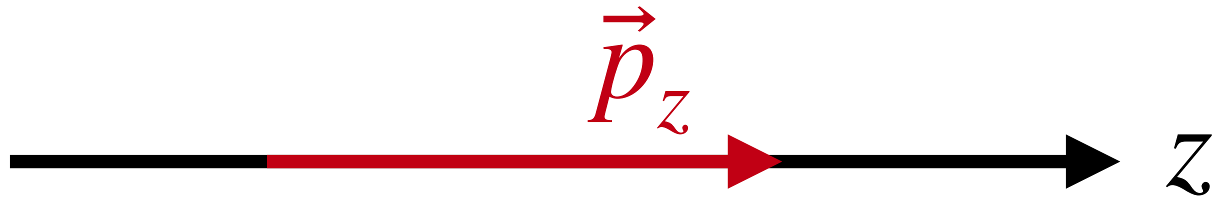
$$|m'\rangle = \sum_{m=-J}^{J} d^J_{m,m'}(\beta)\,|m\rangle$$



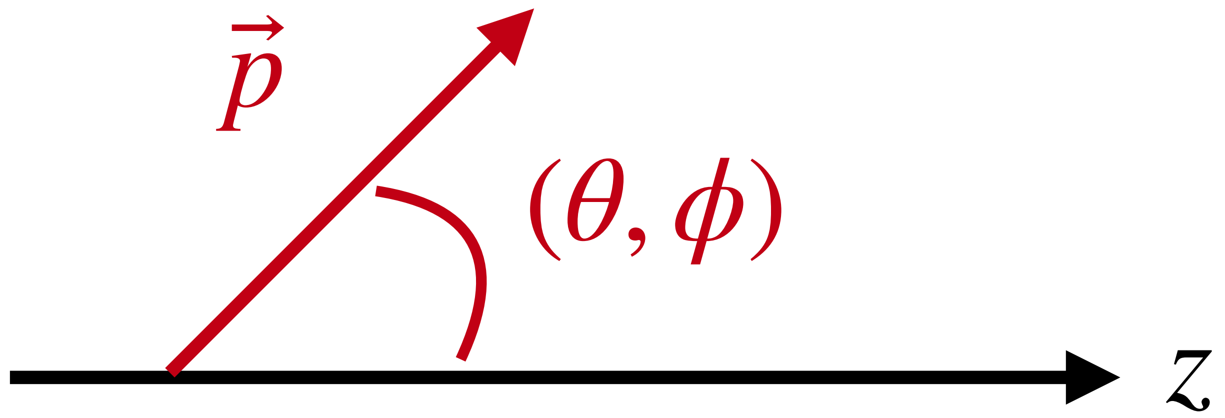
# Boosted States

How do we boost from $\vec{0}$ to $\vec{p}$?


$$|\vec{p}_z\rangle = L_z(p)\,|\vec{0}\rangle$$



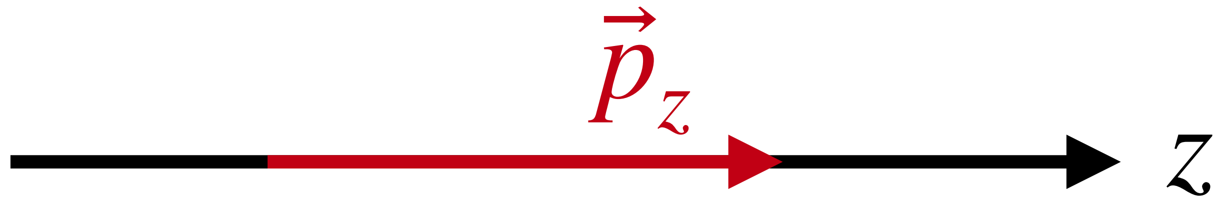

$$|\vec{p}\rangle = R(\phi,\theta,0)\,|\vec{p}_z\rangle$$



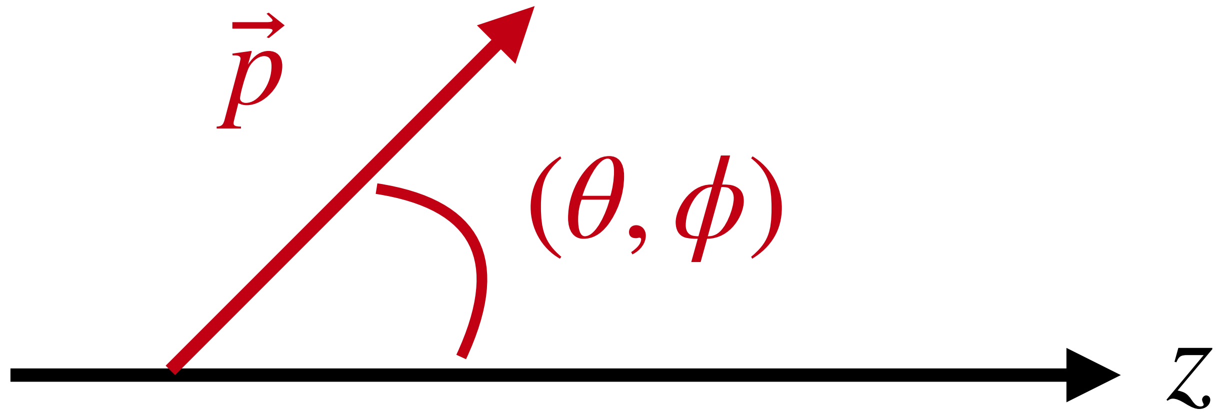

$$|\vec{p}\rangle = R(\phi,\theta,0)L_z(p)\,|\vec{0}\rangle$$


$$|\vec{0}\rangle = R^{-1}(\phi,\theta,0)\,|\vec{0}\rangle$$


$$|\vec{p}_z\rangle = L_z(p)\,|\vec{0}\rangle$$




$$|\vec{p}\rangle = R(\phi,\theta,0)\,|\vec{p}_z\rangle$$




$$|\vec{p}\rangle = R(\phi,\theta,0)L_z(p)R^{-1}(\phi,\theta,0)\,|\vec{0}\rangle$$

# Rotation of States

## Helicity states

$$|\vec{p},\lambda\rangle = R(\Omega)L_z(p)\,|\vec{0},\lambda\rangle$$

$$R(\Omega')\,|\vec{p},\lambda\rangle = R(\Omega')R(\Omega)L_z(p)\,|\vec{0},\lambda\rangle$$

$$= R(\Omega'\cdot\Omega)L_z(p)\,|\vec{0},\lambda\rangle$$

$$= |\vec{p}\,',\lambda\rangle$$

With $\vec{p}\,' = R(\Omega')\vec{p}$

Under a rotation, the helicity is conserved

Helicity is the spin projection on $\vec{p}$

## Canonical states

$$|\vec{p},m\rangle = R(\Omega)L_z(p)R^{-1}(\Omega)\,|\vec{0},m\rangle$$

$$R(\Omega')\,|\vec{p},m\rangle = R(\Omega')R(\Omega)L_z(p)R^{-1}(\Omega)\,|\vec{0},m\rangle$$

$$= R(\Omega'\cdot\Omega)L_z(p)R^{-1}(\Omega'\cdot\Omega)\;R(\Omega')\,|\vec{0},m\rangle$$

$$= \sum_{m'} D^{s}_{m',m}(\Omega')\,|\vec{p}\,',m'\rangle$$

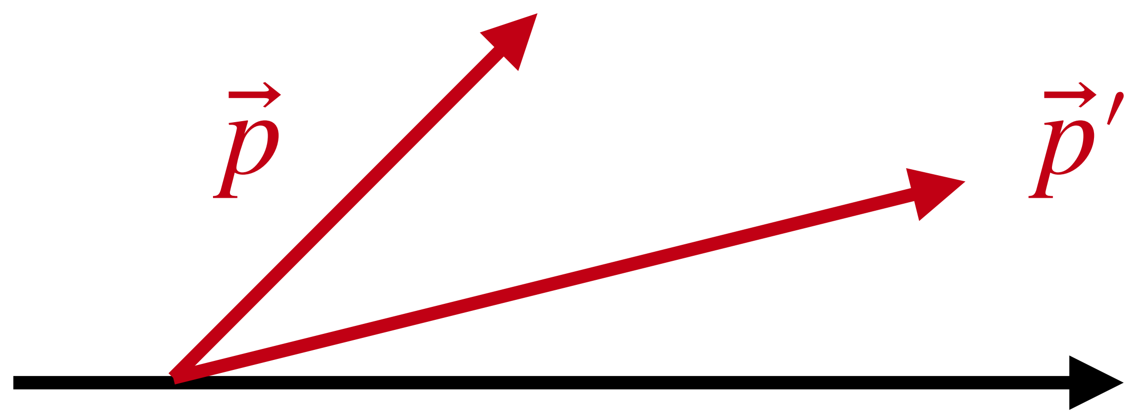
With $\vec{p}\,' = R(\Omega')\vec{p}$

Under a rotation, the spin projection changes

# Boosting States


Consider two different frames

$$|\vec{p}\rangle = B(\vec{p} \leftarrow \vec{0})\,|\vec{0}\rangle \qquad |\vec{p}\,'\rangle = B(\vec{p}\,' \leftarrow \vec{0})\,|\vec{0}\rangle$$




Related by a Lorentz transf. $\quad \Lambda(\vec{p} \leftarrow \vec{p}\,')\,|\vec{p}\,'\rangle = |\vec{p}\rangle$

Consider the cycle $\vec{0} \rightarrow \vec{p}\,' \rightarrow \vec{p} \rightarrow \vec{0}$

$$B(\vec{0} \leftarrow \vec{p})\Lambda(\vec{p} \leftarrow \vec{p}\,')B(\vec{p}\,' \leftarrow \vec{0}) = R(\Omega_W)$$

The rest states $|\vec{0}\rangle$ can differ by a rotation!

$$\Lambda(\vec{p} \leftarrow \vec{p}\,')B(\vec{p}\,' \leftarrow \vec{0}) = B(\vec{p} \leftarrow \vec{0})R(\Omega_W)$$

So the spin projection rotates as well!

$$\Lambda(\vec{p} \leftarrow \vec{p})\,|\vec{p}\,', \lambda\rangle = \sum_{\lambda'} D^s_{\lambda',\lambda}(\Omega_W)\,|\vec{p}, \lambda'\rangle$$

$\Omega_W$ is called the Wigner rotation

# Wigner Rotations

Under a Lorents boost, states undergo a Wigner rotation

$$\Lambda(\vec{p} \leftarrow \vec{p}\,')\,|\,\vec{p}\,', \lambda\rangle = \sum_{\lambda'} D^{s}_{\lambda',\lambda}(\Omega_W)\,|\,\vec{p}, \lambda'\rangle$$

The Wigner rotation is determined by the boosting chain

$$R(\Omega_W) = B(\vec{0} \leftarrow \vec{p})\Lambda(\vec{p} \leftarrow \vec{p}\,')B(\vec{p}\,' \leftarrow \vec{0})$$

$\Omega_W$ different for helicity and canonical

The boosts depend on the quantification!

Helicity states $$B(\vec{p} \leftarrow \vec{0}) = R(\Omega_p)L_z(p)$$

Canonical states $$B(\vec{p} \leftarrow \vec{0}) = R(\Omega_p)L_z(p)R^{-1}(\Omega_p)$$

# A Concrete Example

https://halldweb.jlab.org/DocDB/0048/004829/004/coordinates.pdf

## GlueX coordinates systems

# Transformations of States

## Helicity states

$$|\vec{p}, \lambda\rangle = R(\Omega)L_z(p)\,|\vec{0}, \lambda\rangle$$

$$R(\Omega')\,|\vec{p}, \lambda\rangle = |\vec{p}\,', \lambda\rangle$$

With $\vec{p}\,' = R(\Omega')\vec{p}$

Under a rotation, the helicity is conserved

$$\Lambda(\vec{p} \leftarrow \vec{p}\,')\,|\vec{p}\,', \lambda\rangle = \sum_{\lambda'} D^s_{\lambda',\lambda}(\Omega_W)\,|\vec{p}, \lambda'\rangle$$

If $\vec{p}\,'$ and $\vec{p}$ are parallel, the helicity is conserved

## Canonical states

$$|\vec{p}, m\rangle = R(\Omega)L_z(p)R^{-1}(\Omega)\,|\vec{0}, m\rangle$$

$$R(\Omega')\,|\vec{p}, m\rangle = \sum_{m'} D^s_{m',m}(\Omega')\,|\vec{p}\,', m'\rangle$$

Under a rotation, the spin projection changes

$$\Lambda(\vec{p} \leftarrow \vec{p}\,')\,|\vec{p}\,', m\rangle = \sum_{m'} D^s_{m',m}(\Omega^c_W)\,|\vec{p}, m'\rangle$$

In the NR limit, the helicity is conserved

# Transformations of Helicity States

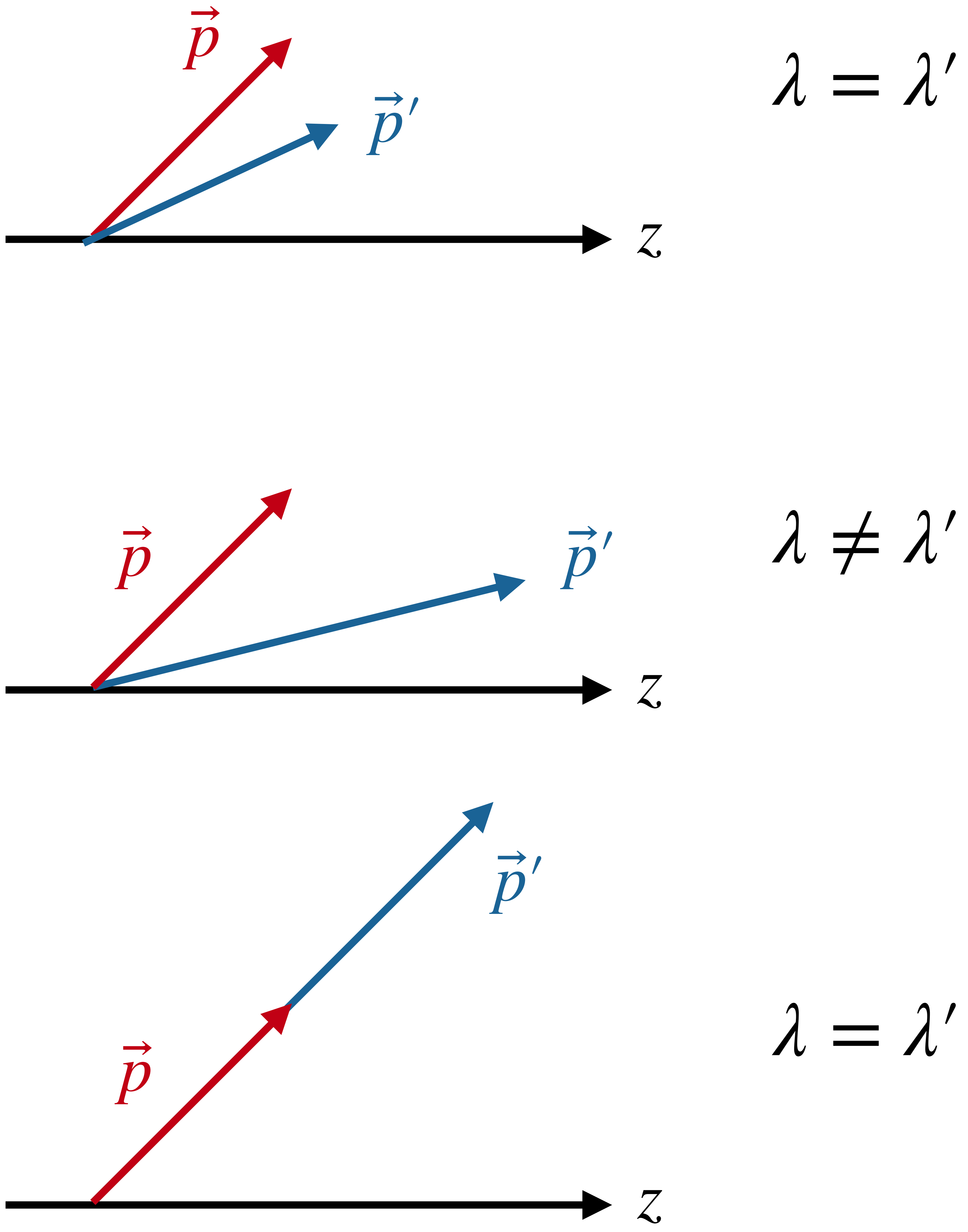
$$|\vec{p},\lambda\rangle = R(\Omega)L_z(p)\,|\vec{0},\lambda\rangle$$

$$R(\Omega')\,|\vec{p},\lambda\rangle = |\vec{p}\,',\lambda\rangle$$ With $\vec{p}\,' = R(\Omega')\vec{p}$

Under a rotation, the helicity is conserved

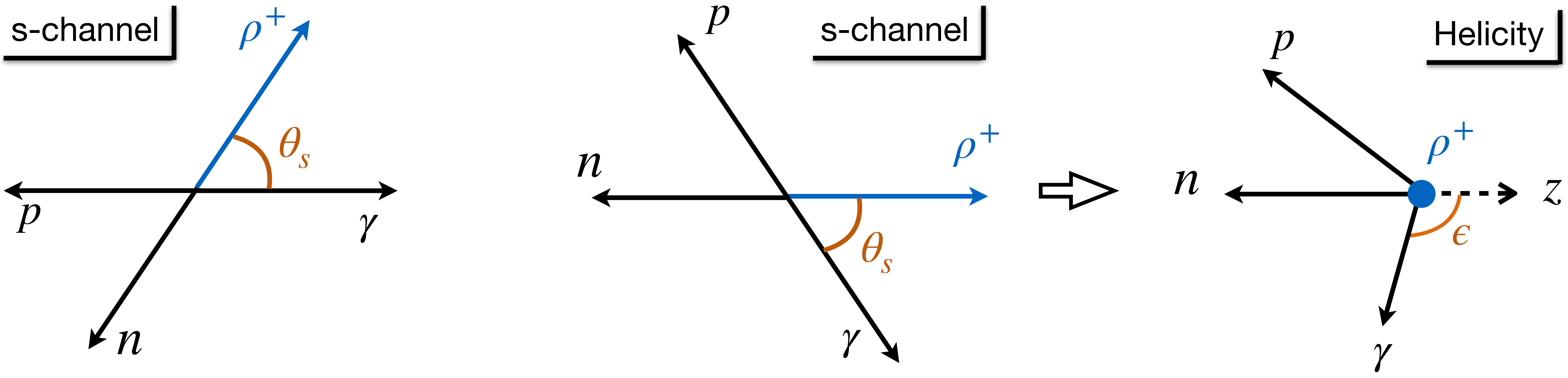
$$\Lambda(\vec{p} \leftarrow \vec{p}\,')\,|\vec{p}\,',\lambda\rangle = \sum_{\lambda'} D^s_{\lambda',\lambda}(\Omega_W)\,|\vec{p},\lambda'\rangle$$

If $\vec{p}\,'$ and $\vec{p}$ are parallel, the helicity is conserved



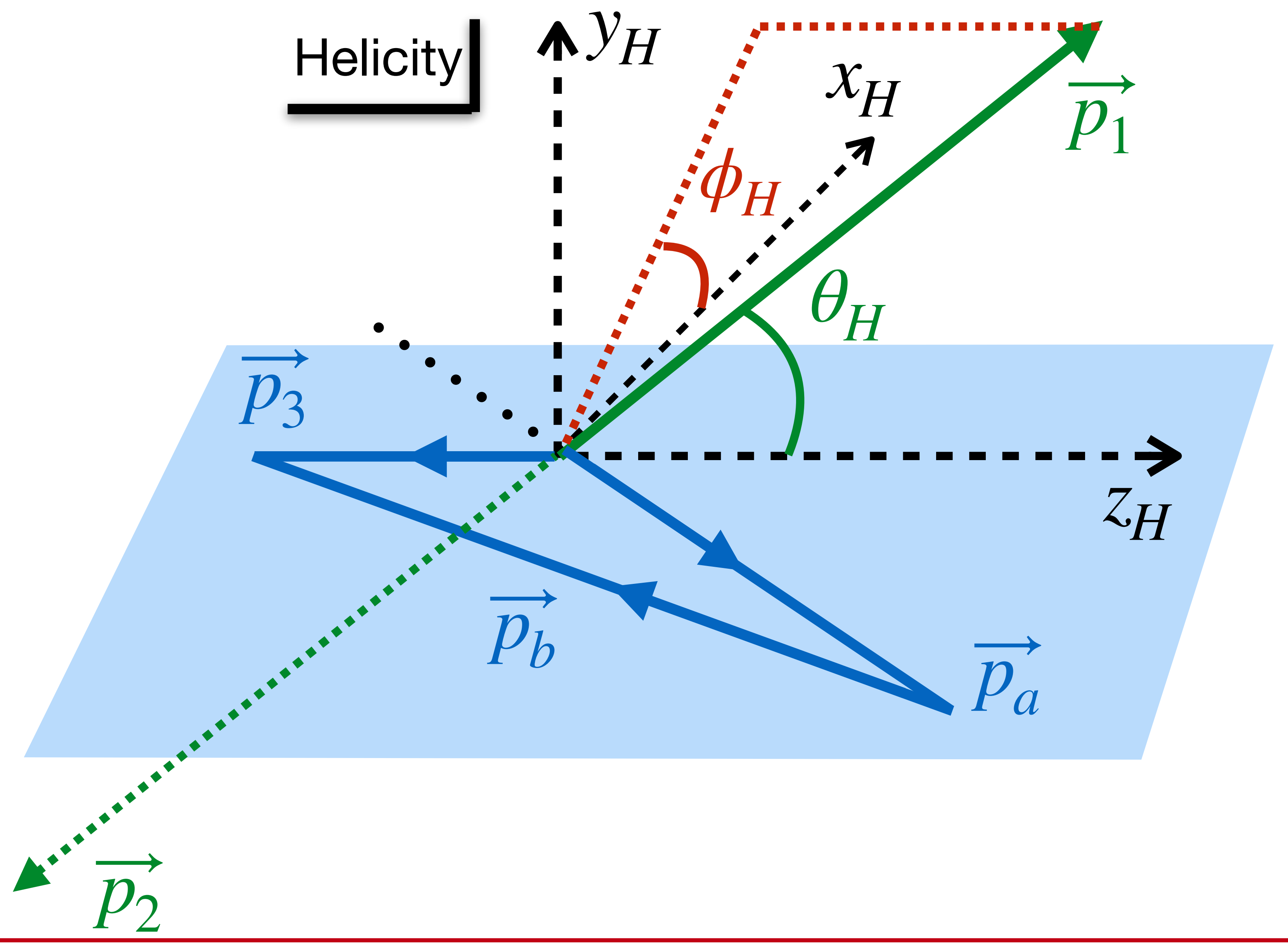
# Helicity Frame

CoM = s-channel




The scattering amplitude depends
on helicities (which are frame dependent)

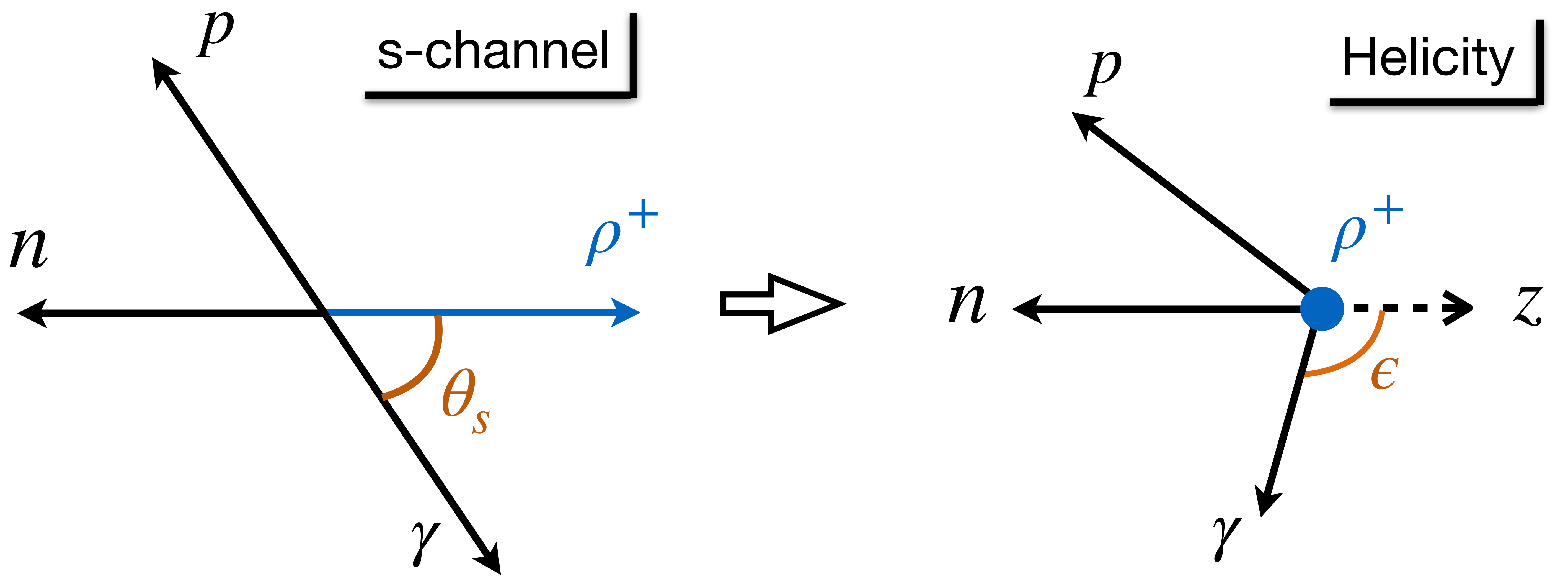
In the helicity frame, the resonance has the same helicity as in the CoM



# Helicity Frame



In the helicity frame, the resonance has the same helicity as in the CoM




The scattering amplitude depends on helicity (which are frame dependent)

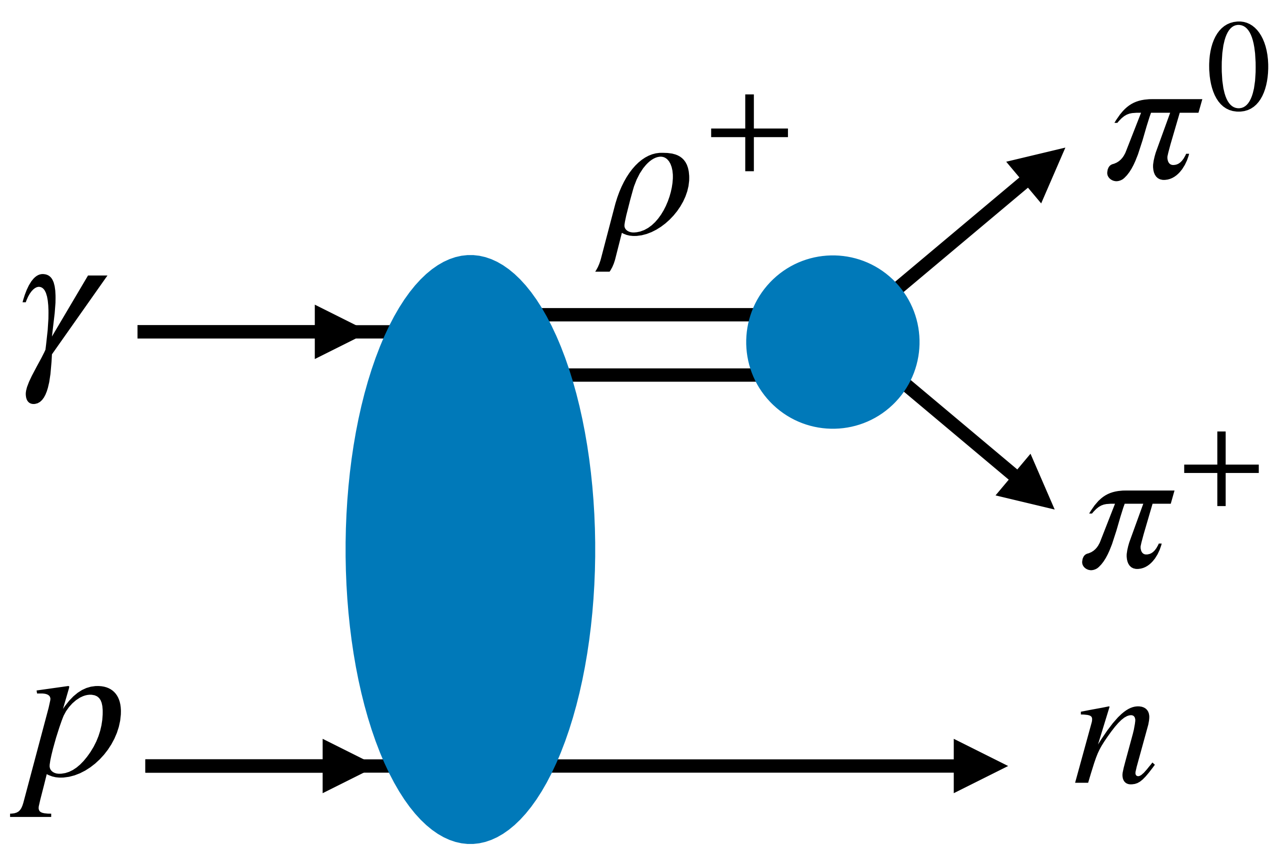
$$\sum_m A_{\lambda_\gamma,\lambda,\lambda',m} Y^1_m(\theta,\phi)$$

$$I(\theta,\phi) \propto \sum_{\lambda_\gamma,\lambda,\lambda} \left[ \sum_m A_{\lambda_\gamma,\lambda,\lambda',m} Y^1_m(\theta,\phi) \right] \left[ \sum_{m'} A_{\lambda_\gamma,\lambda,\lambda',m} Y^1_{m'}(\theta,\phi) \right]^* = \sum_{m,m'} \rho_{m,m'} Y^1_m(\theta,\phi) Y^{1*}_{m'}(\theta,\phi)$$

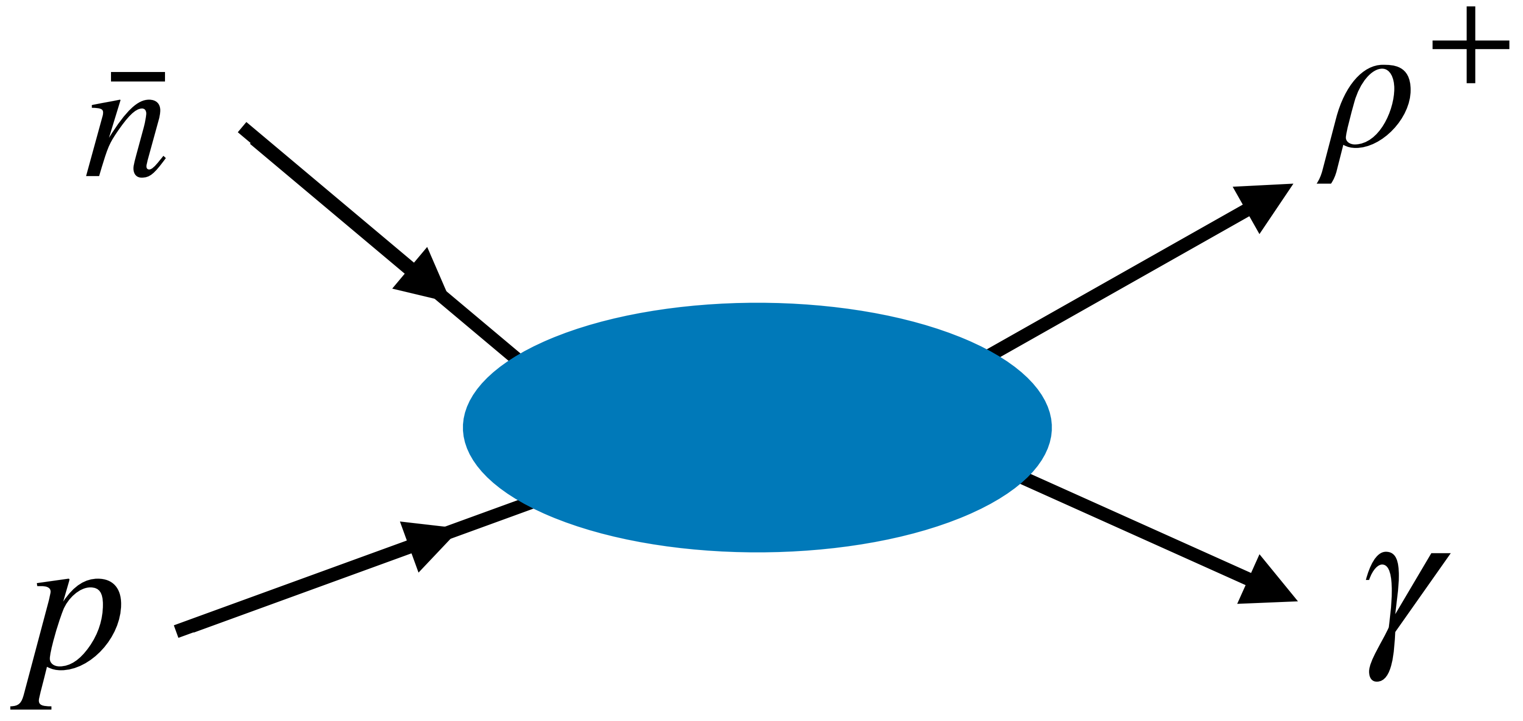
With $\rho_{m,m'} = \sum_{\lambda_\gamma,\lambda,\lambda} A_{\lambda_\gamma,\lambda,\lambda',m} (A_{\lambda_\gamma,\lambda,\lambda',m'})^*$ $\rho_{m,m'}$ is the same in the helicity frame and s-channel!

# Gottfried-Jackson frame

## Consider the reaction

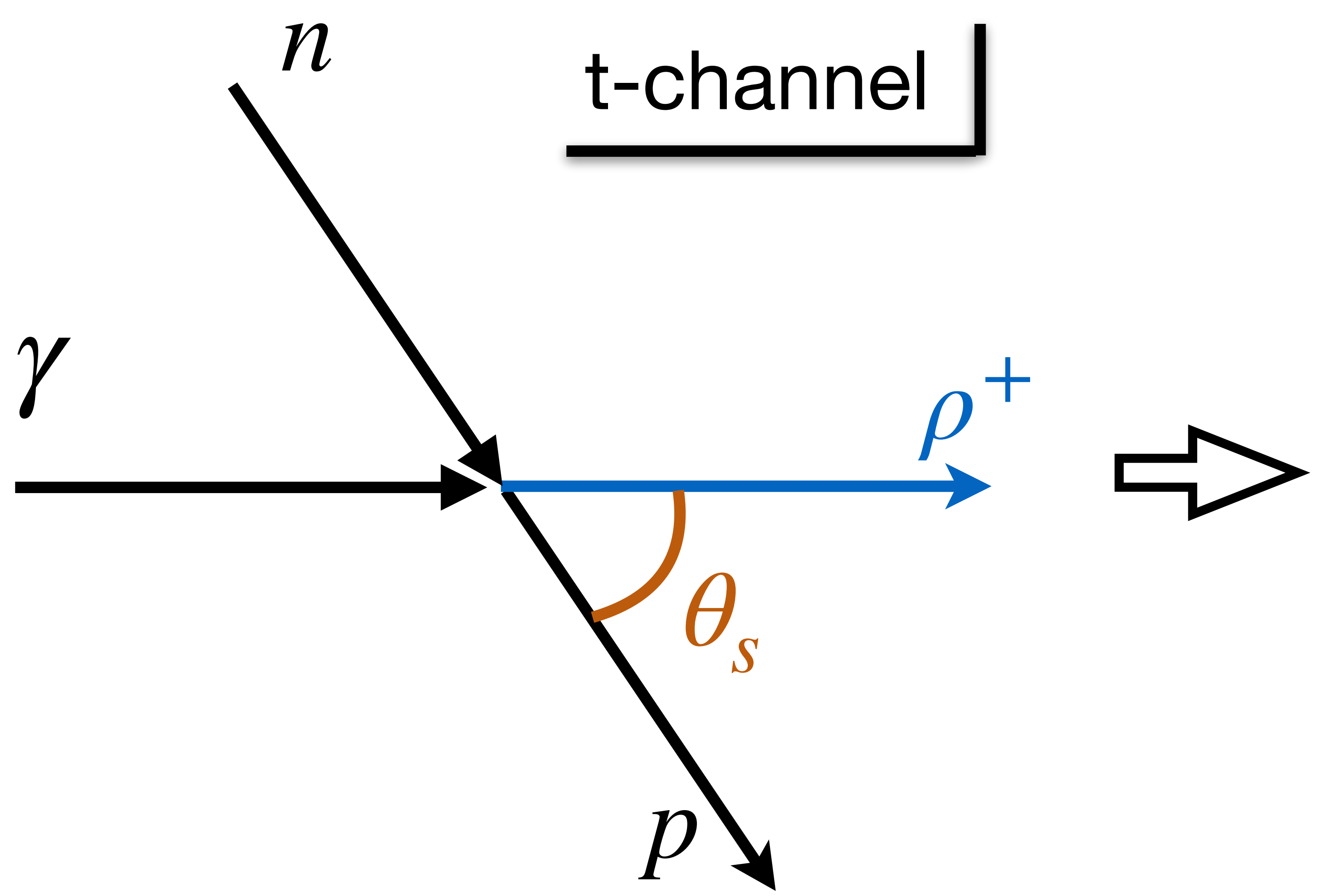


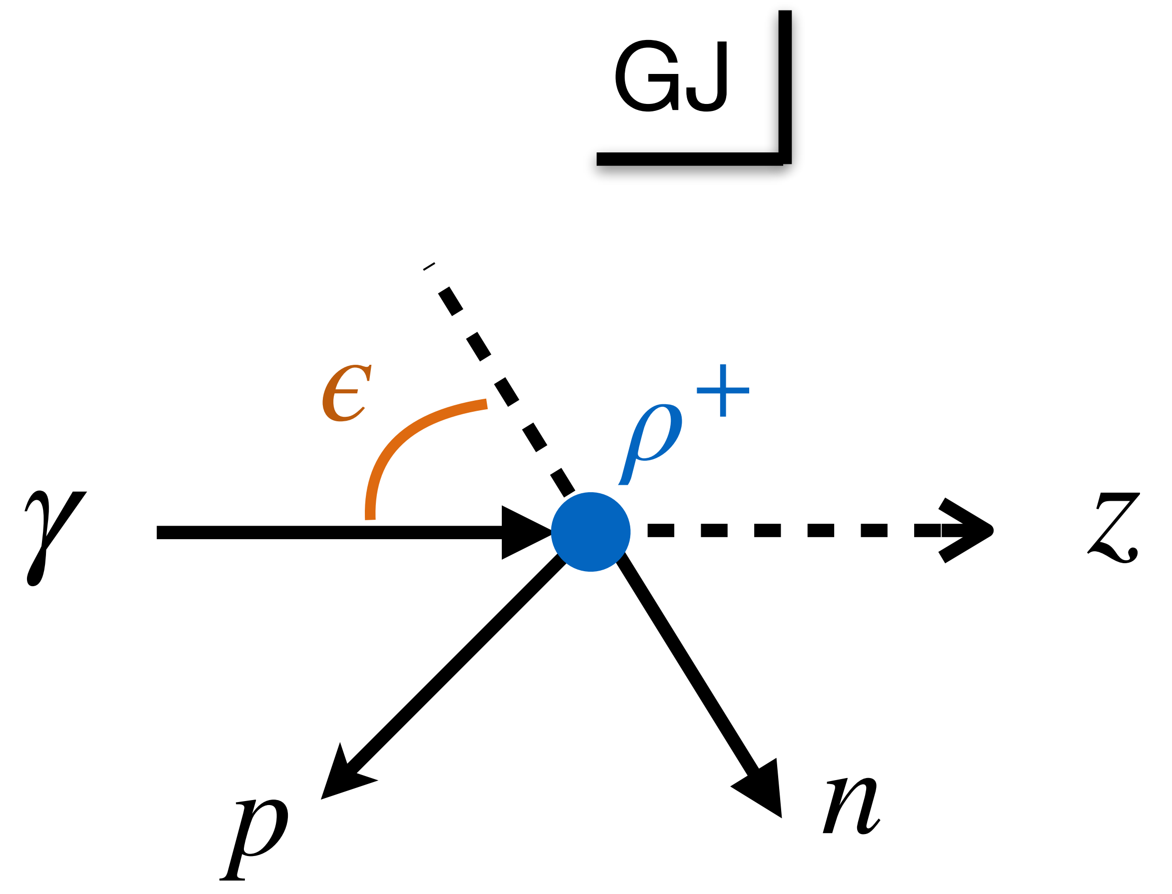

## The crossed reaction is




The t-channel is
the CoM of the crossed reaction

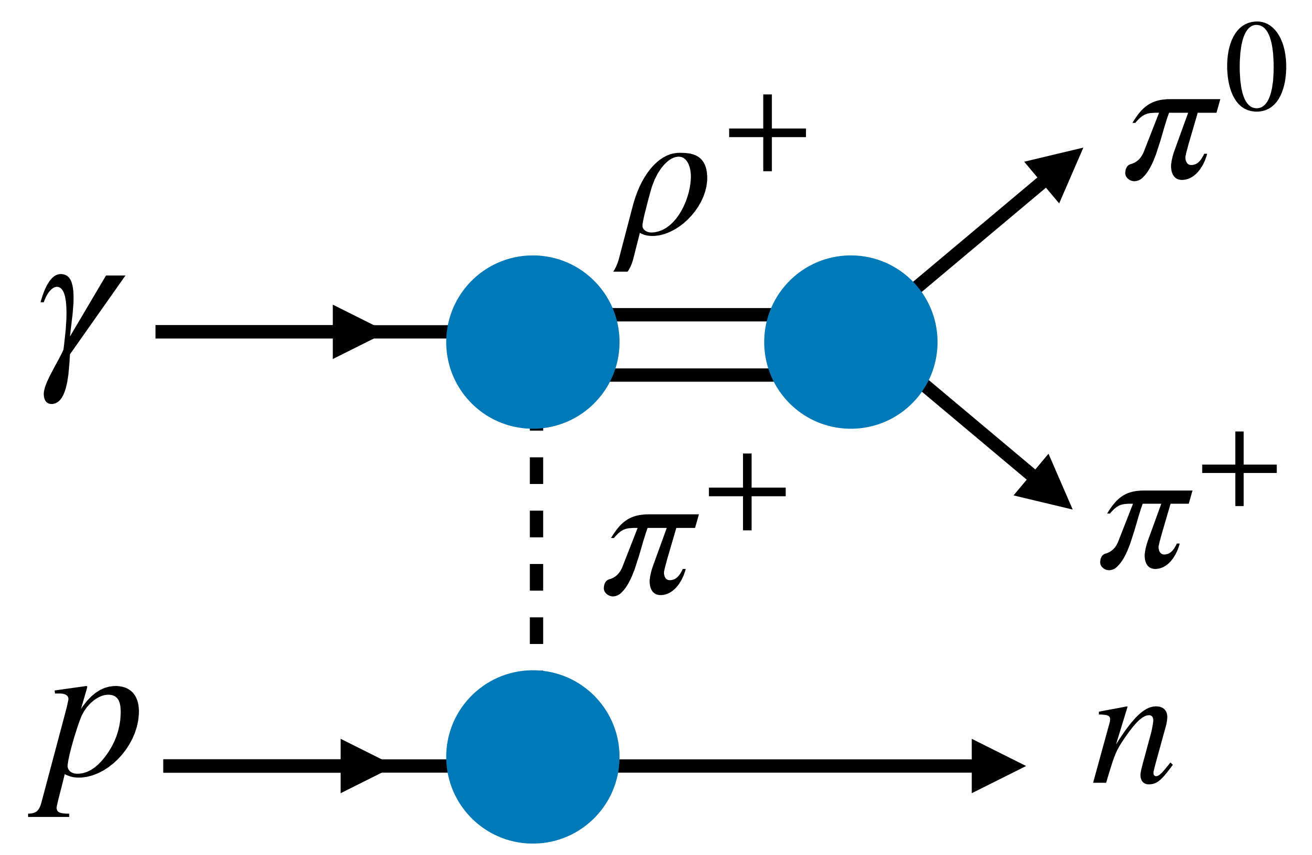
In the GJ frame,
the resonance has the same
helicity as in the t-channel



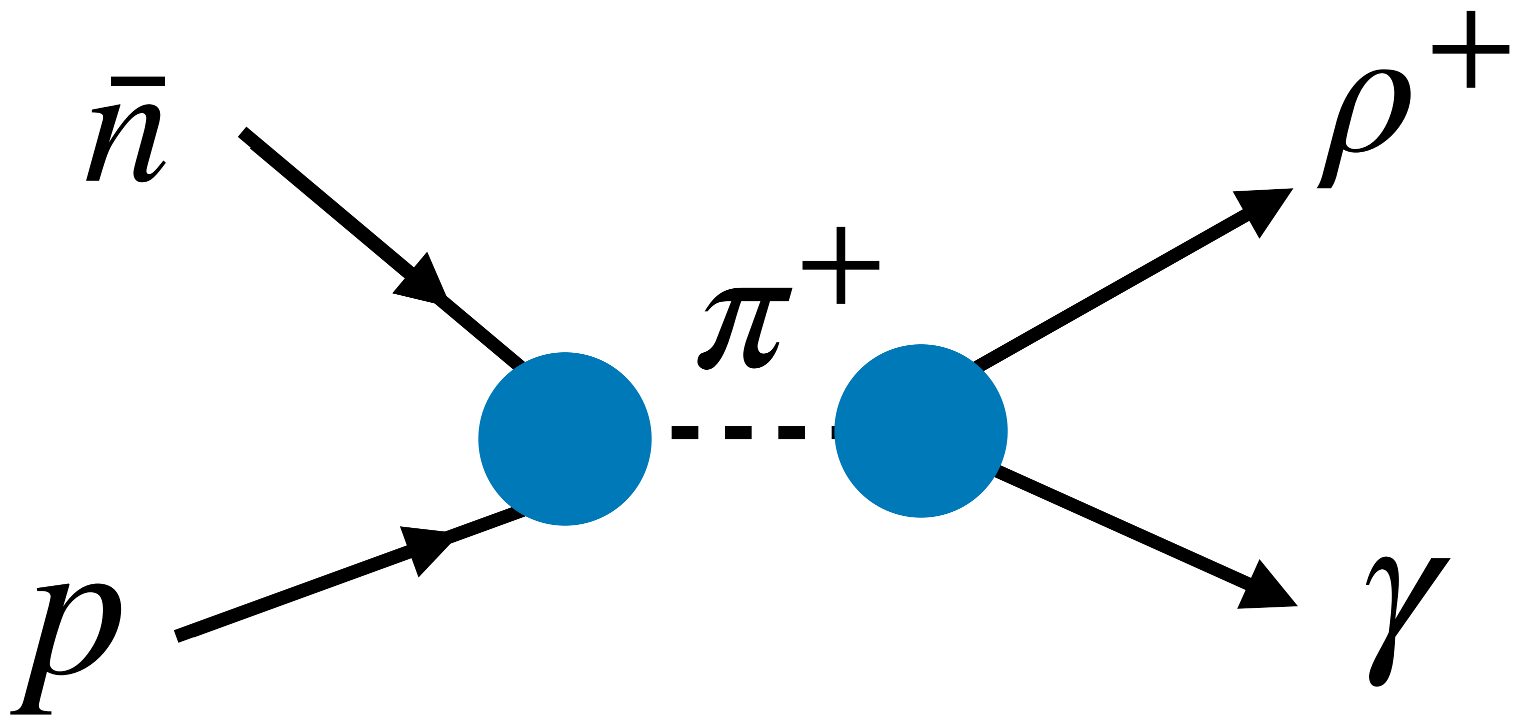



# T-channel frame

Consider the reaction




The crossed reaction is

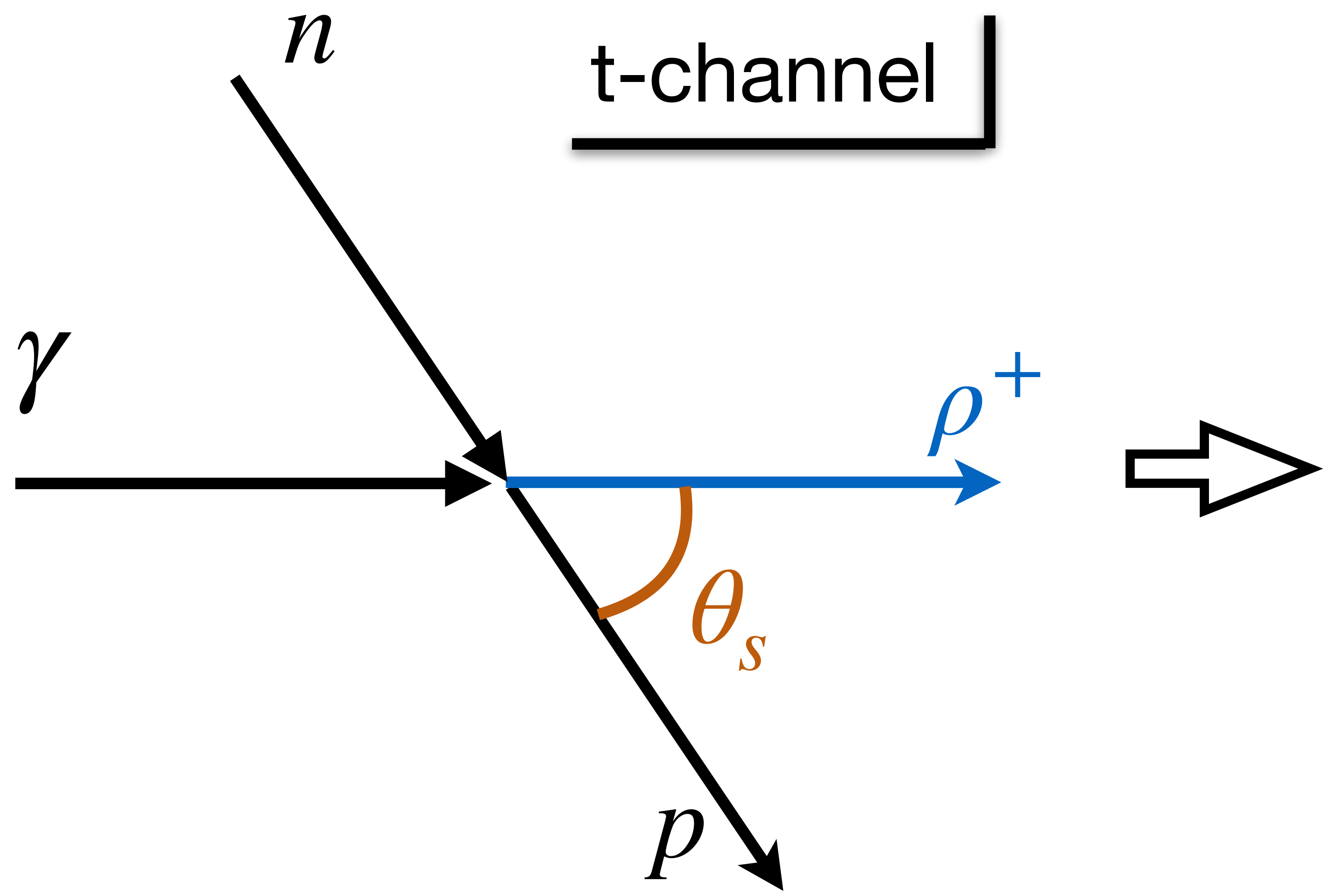


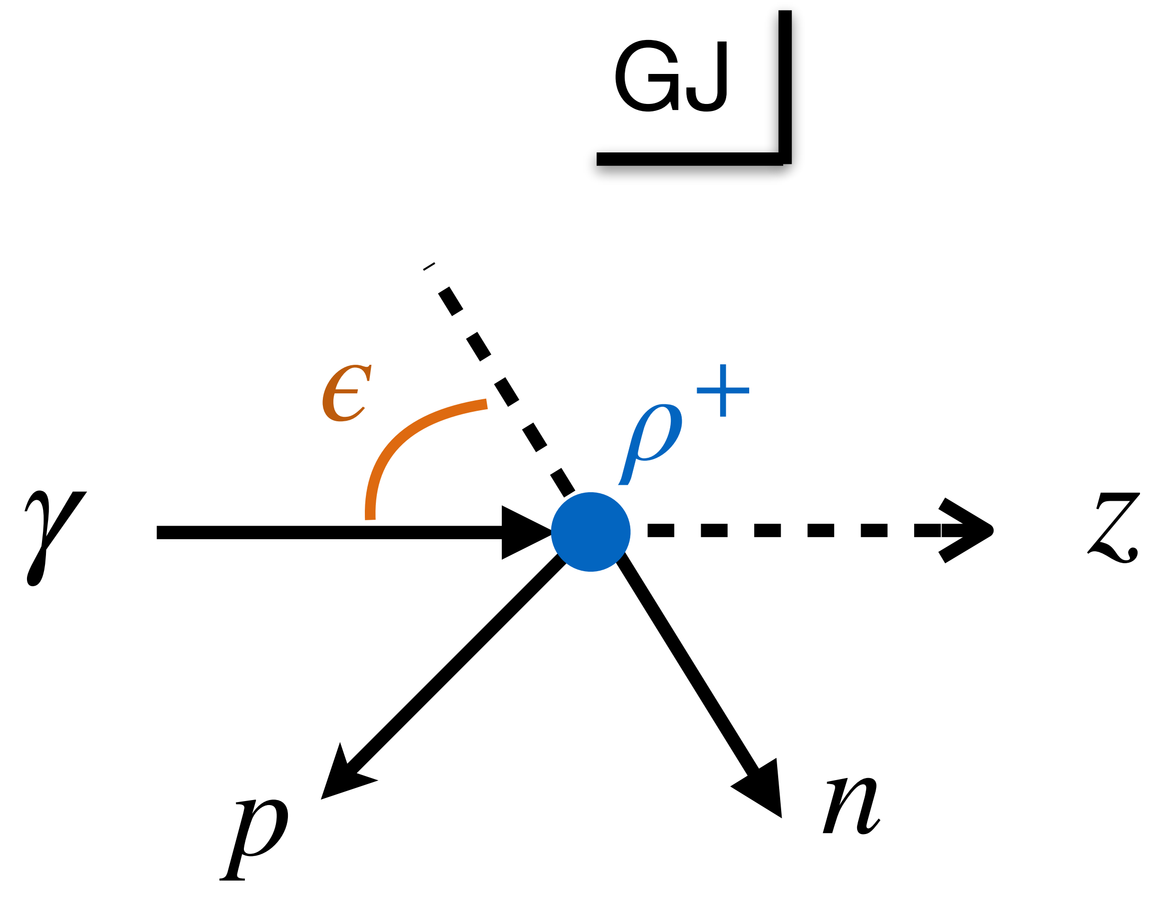

In the GJ frame,
the resonance has the same
helicity as in the t-channel

$$\rho^{GJ}_{m,m'} = \sum_{\lambda_\gamma,\lambda,\lambda} A^{GJ}_{\lambda_\gamma,\lambda,\lambda',m} (A^{GJ}_{\lambda_\gamma,\lambda,\lambda',m'})^*$$

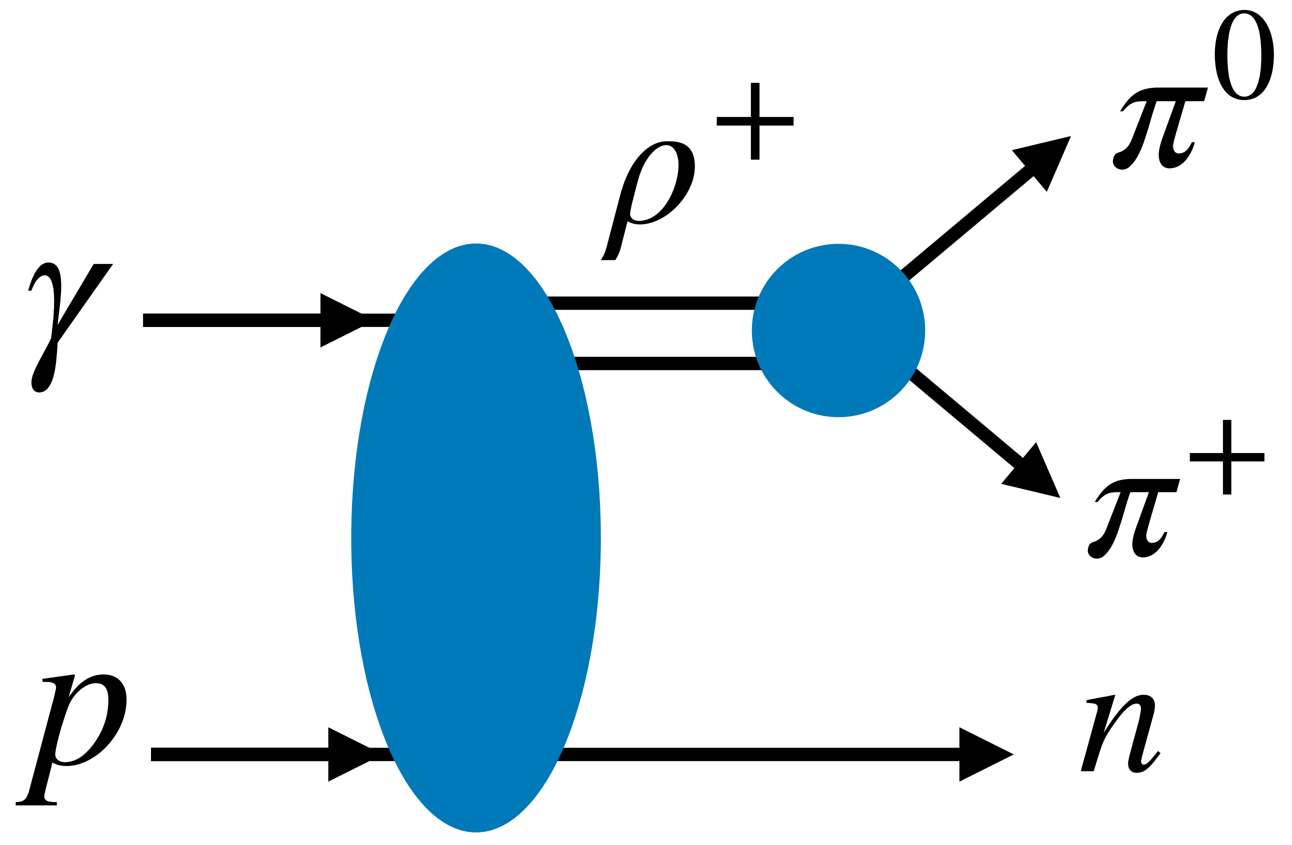
$$A^t_{\lambda_\gamma,\lambda,\lambda',m} \propto \delta_{\lambda,\lambda'} \delta_{\lambda_\gamma,m}$$

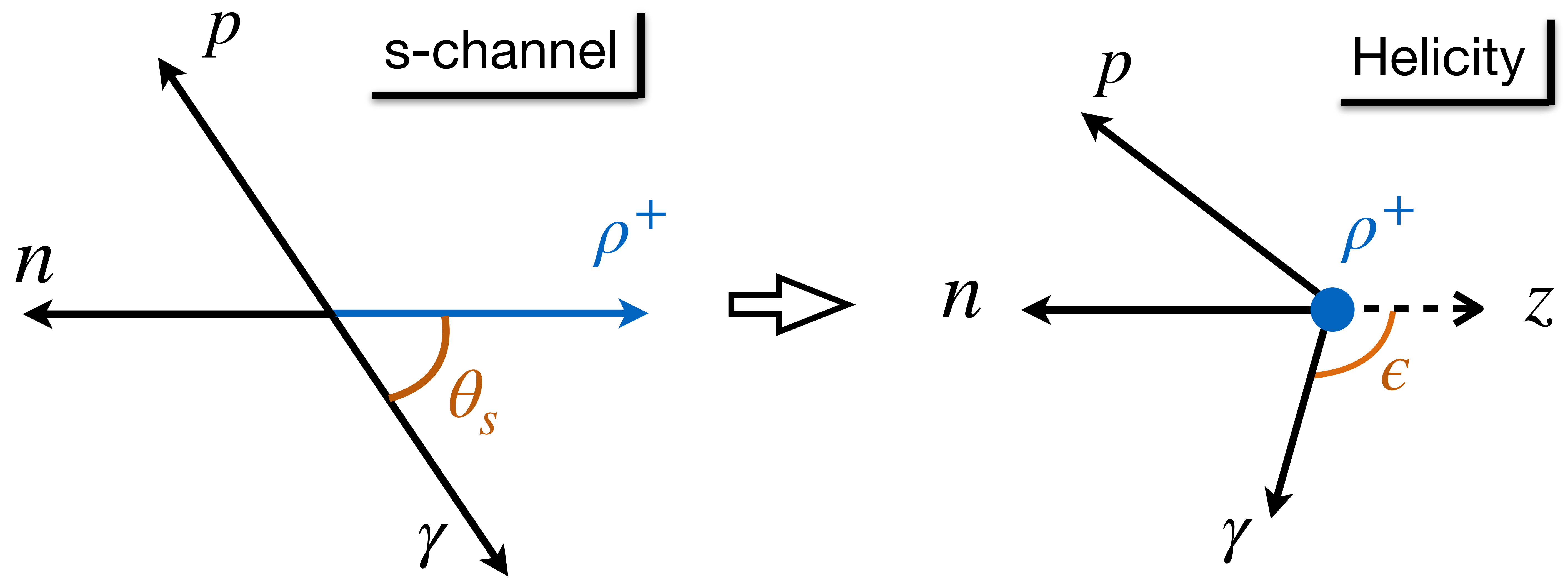
Only $\rho_{1,1} = \rho_{-1,-1}$ is non-zero!



# Frames in Meson Production

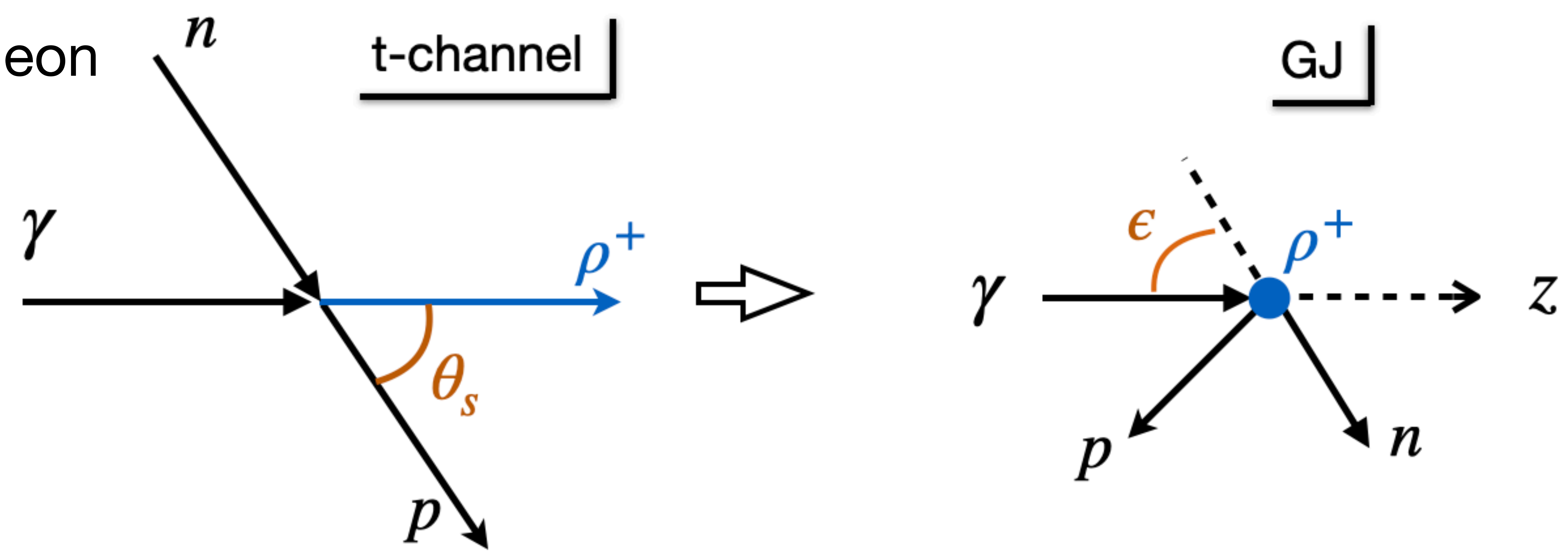



## In beam fragmentation:

Helicity frame: $z$ is opposite to recoil nucleon

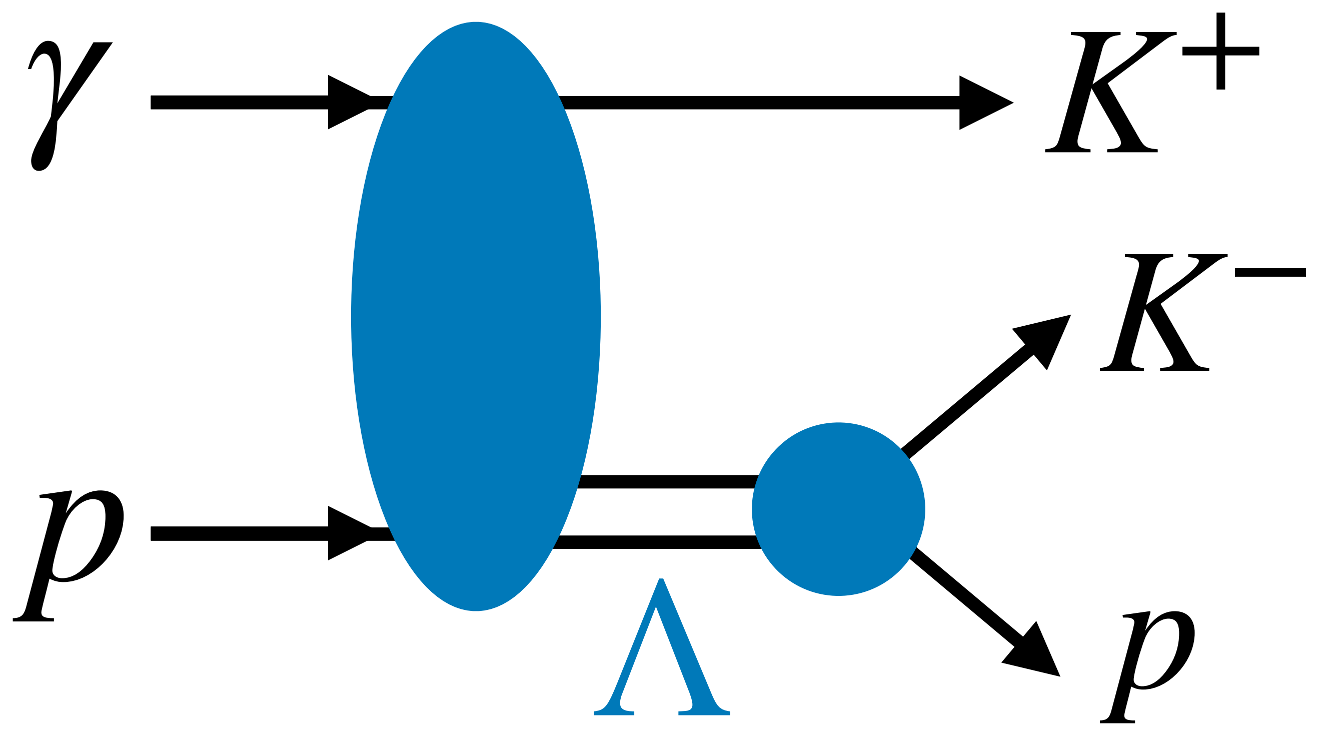
GJ frame: $z$ is parallel to beam

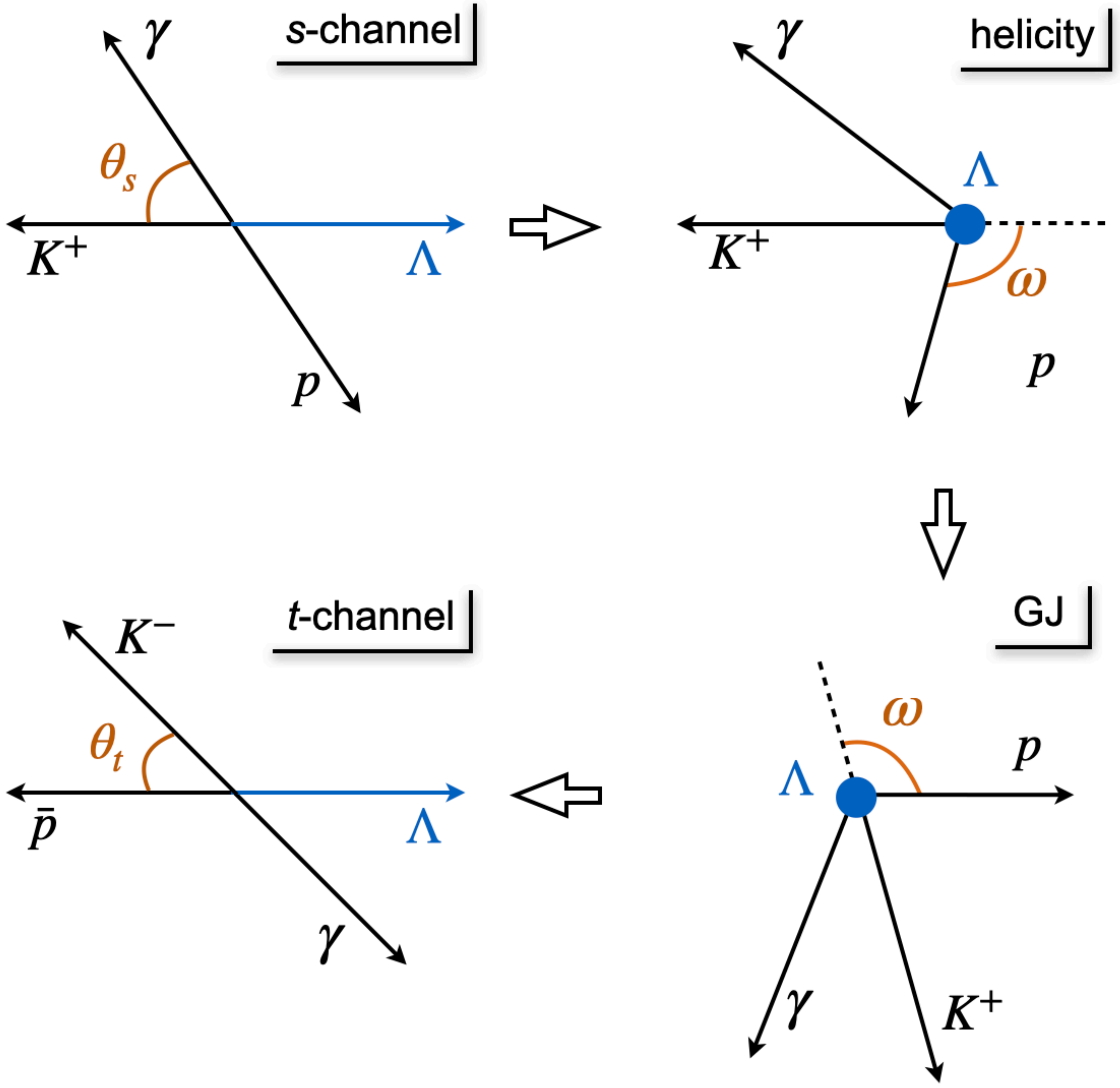



## Rotation of $\epsilon$ between frames

$$\rho^H_{m,m'} = \sum_{\lambda,\lambda'} d^J_{\lambda,m}(\epsilon)\ \rho^{GJ}_{m,m'}\ d^J_{\lambda',m'}(\epsilon)$$

# Baryon Production




## In target fragmentation:

Helicity frame: $z$ is opposite to recoil meson

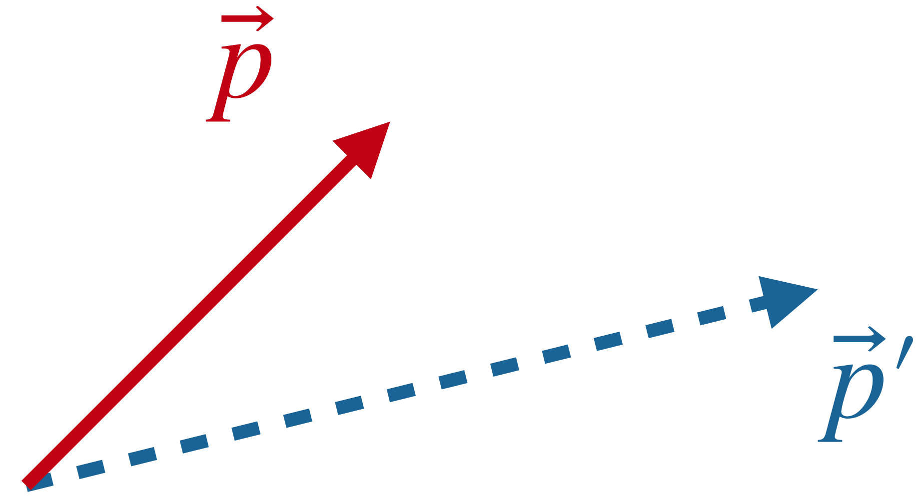
Gottfried-Jackson frame: $z$ is parallel to target

## Rotation of $\omega$ between frames

$$\rho^H_{m,m'} = \sum_{\lambda,\lambda'} d^J_{\lambda,m}(\omega)\ \rho^{GJ}_{m,m'}\ d^J_{\lambda',m'}(\omega)$$

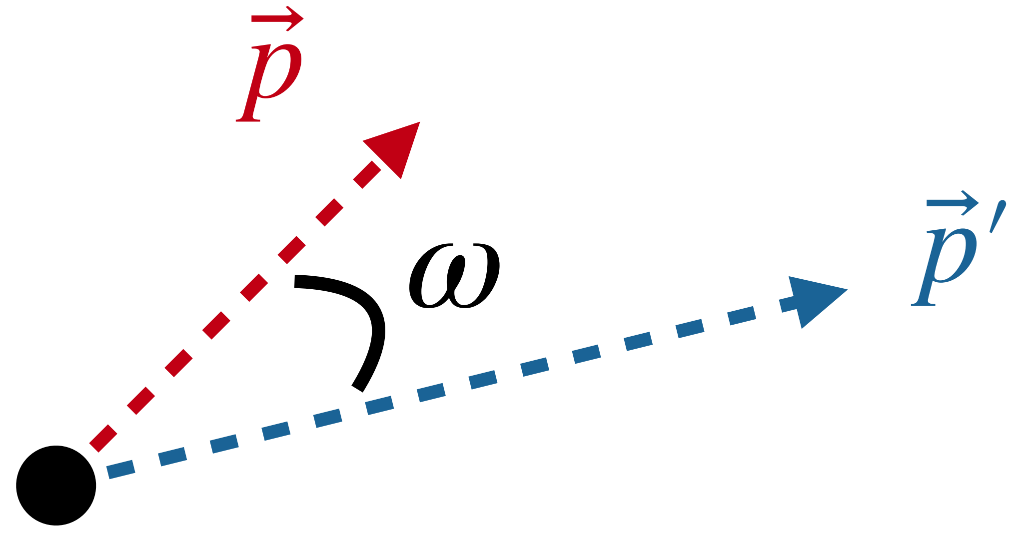
# Wigner Rotation for Helicity states

Helicity is the spin projection along the momentum




The plan including $\vec{p}$ and $\vec{p}'$ is the $x - z$ plane

$$\Lambda(\vec{p}' \leftarrow \vec{p})\,|\vec{p},\lambda\rangle = \sum_{\lambda'} d^{s}_{\lambda',\lambda}(\omega)\,|\vec{p}',\lambda'\rangle$$

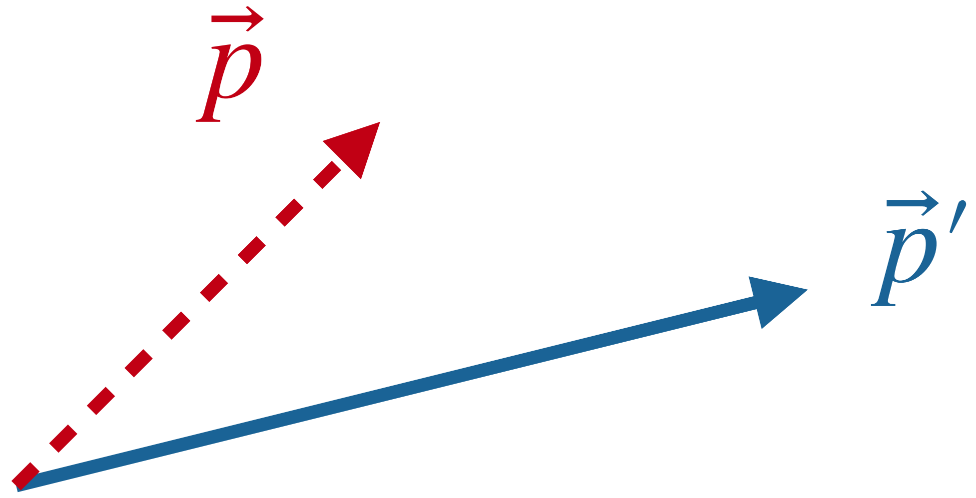



First boost to rest frame

Helicity becomes spin projection along the former momentum

Then rotate to align the $z$ to the future direction

The spin projection changes

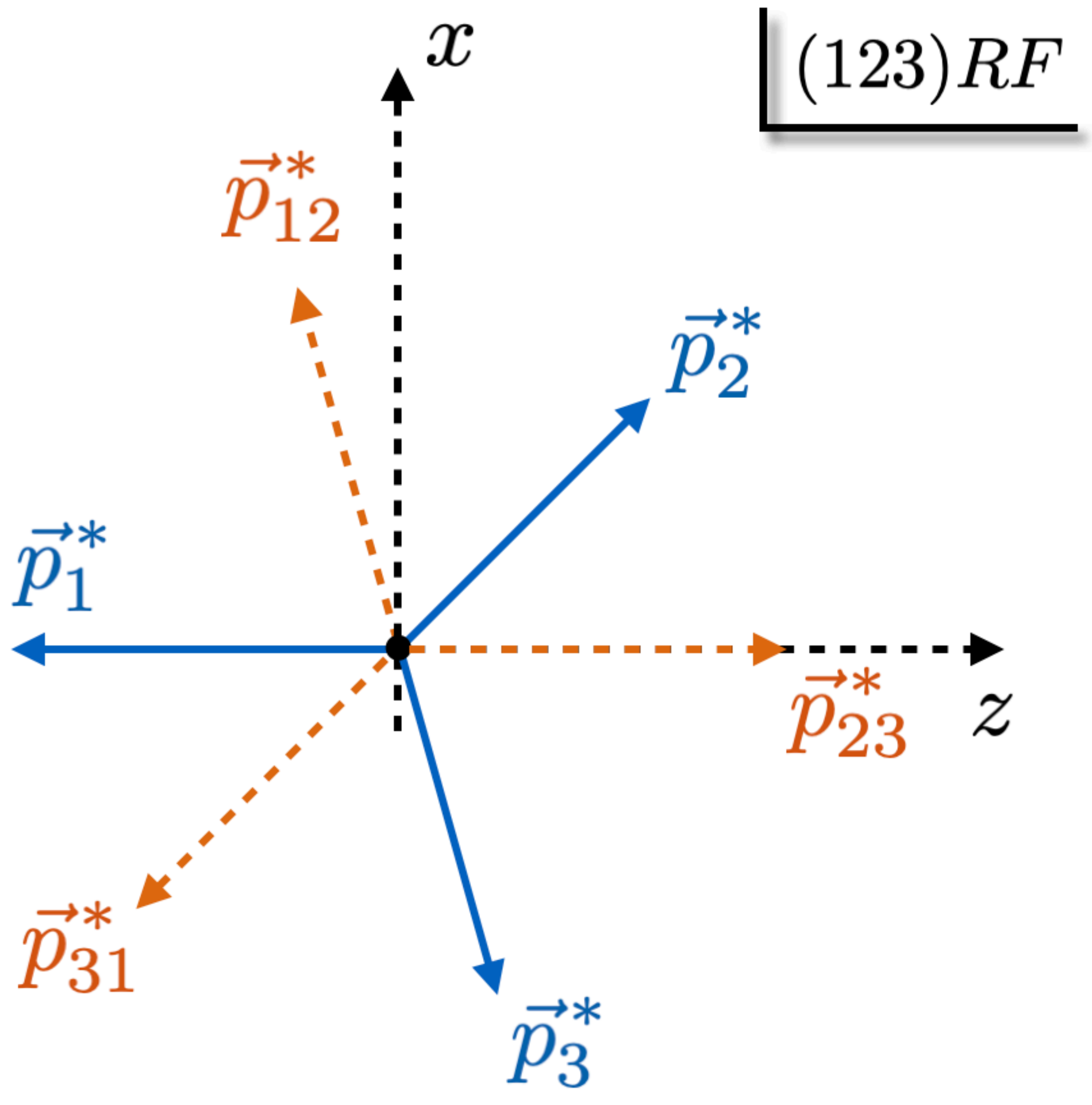


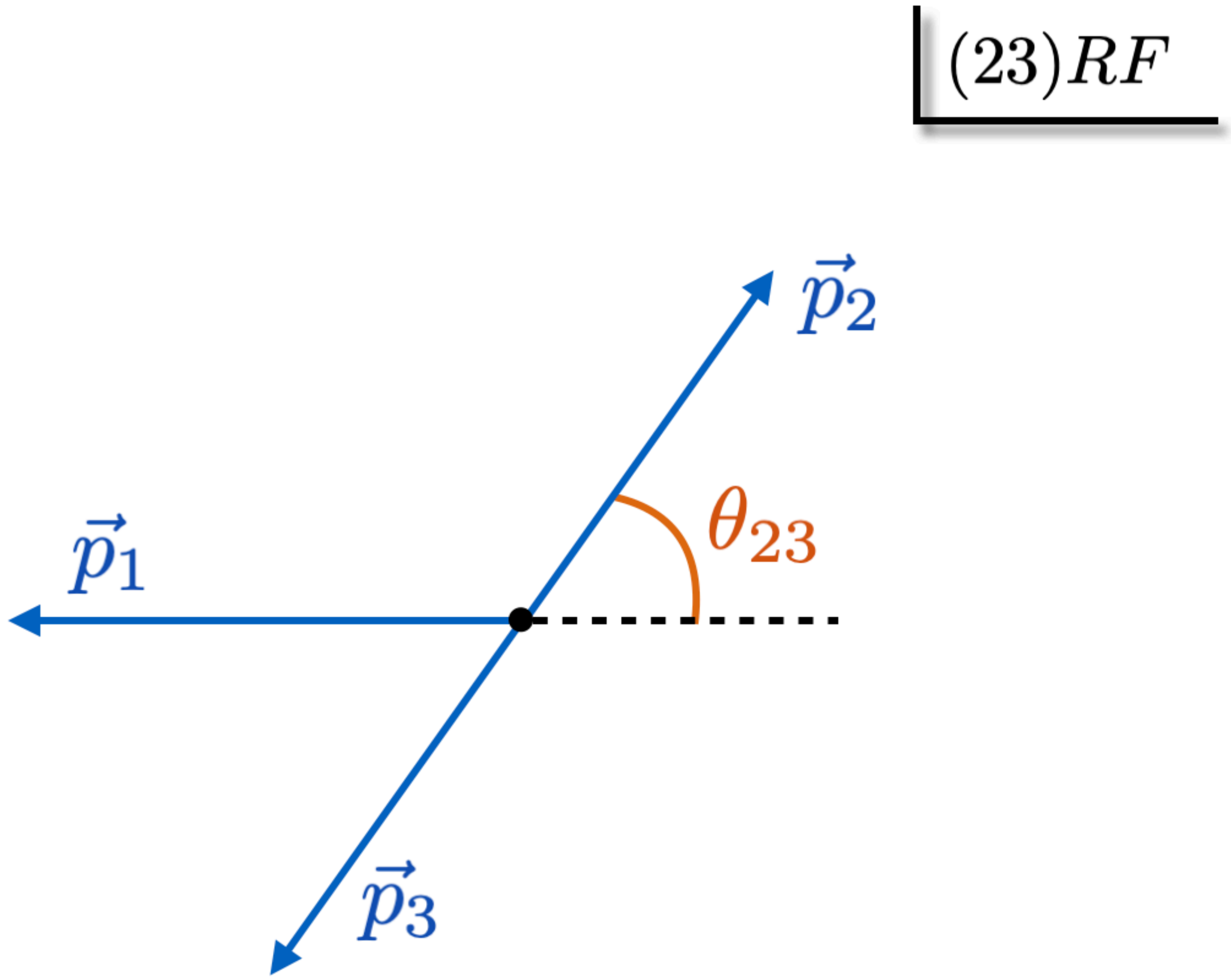

Finally boost to the final momentum

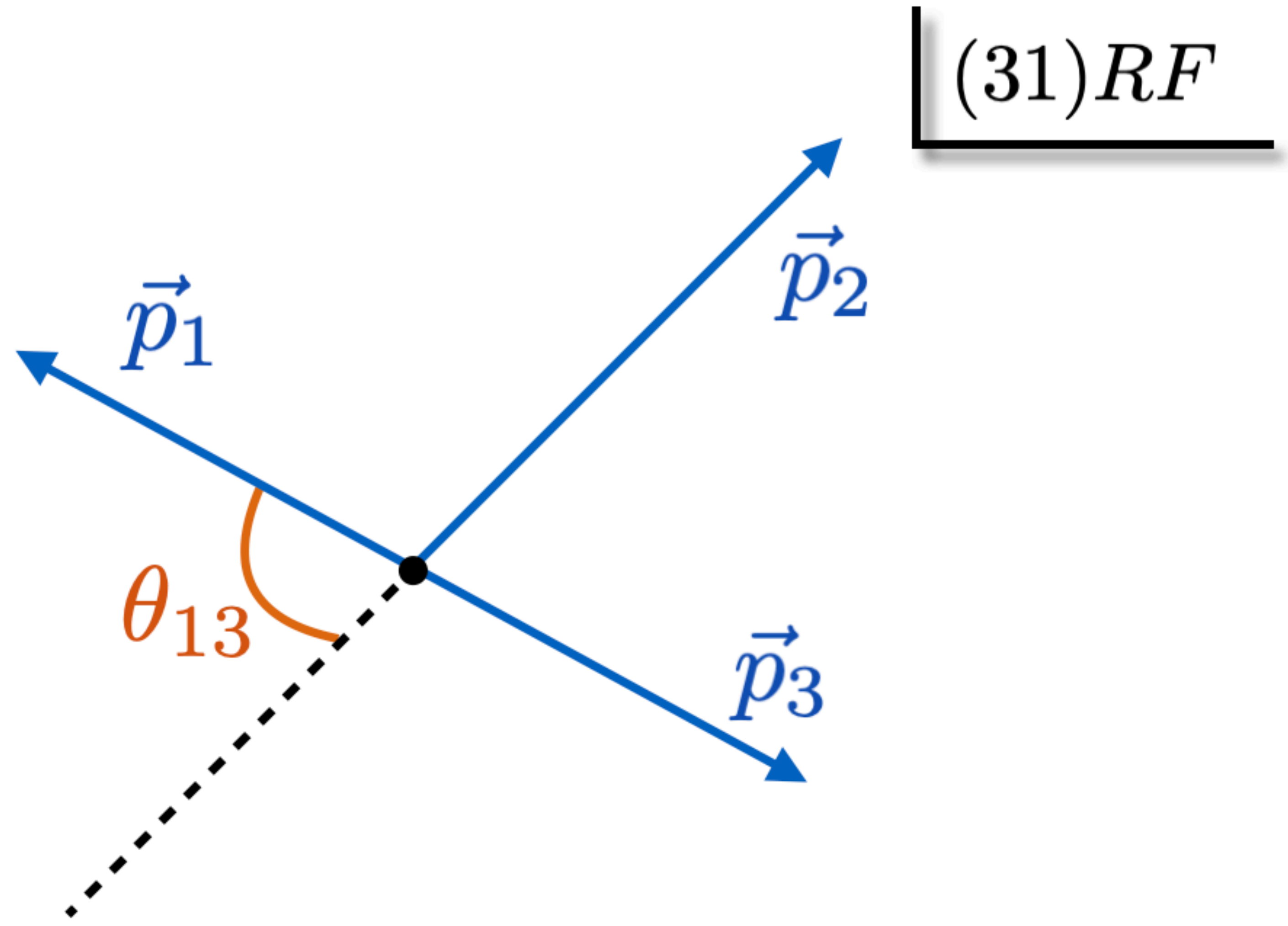
> The Wigner rotation is the angle between the two direction, as seen from the particle rest frame

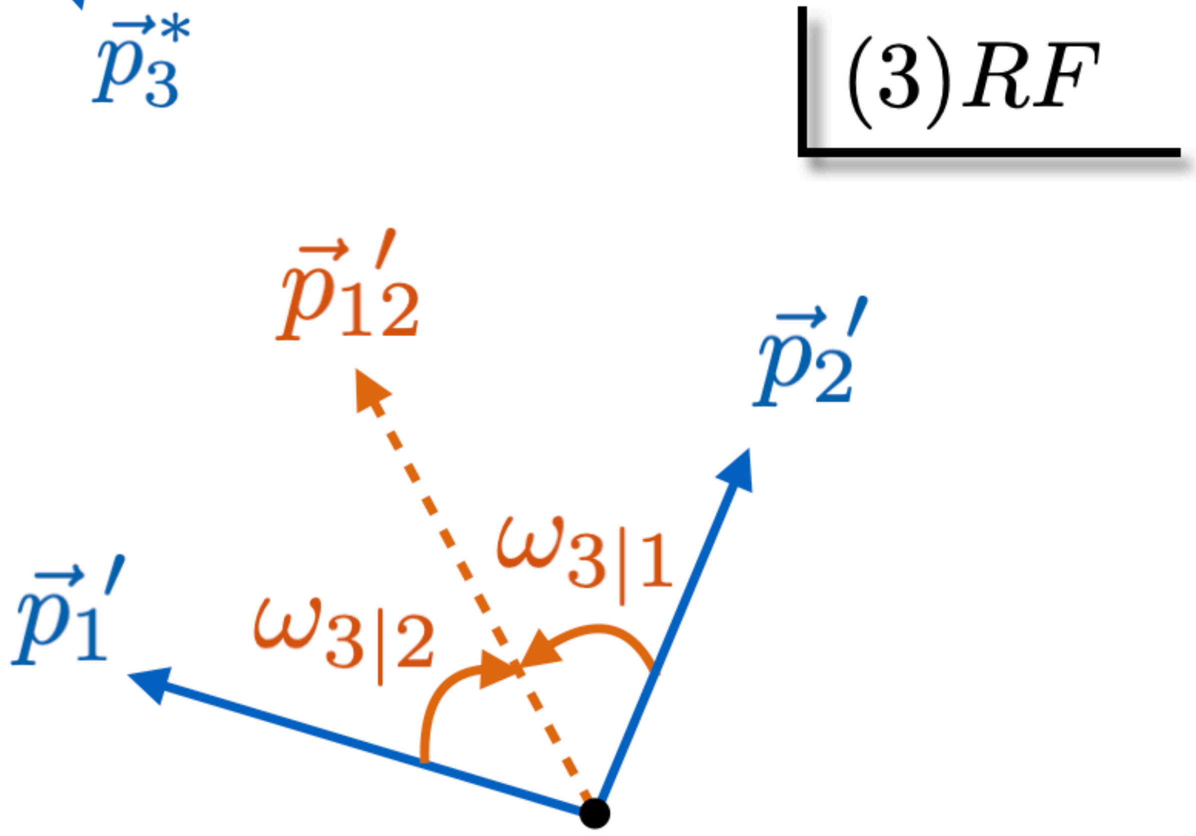
# Computing Wigner Rotation

Boosting from (23)RF to (123)RF will (Wigner) rotate the helicities of 2 and 3 (but not 1)



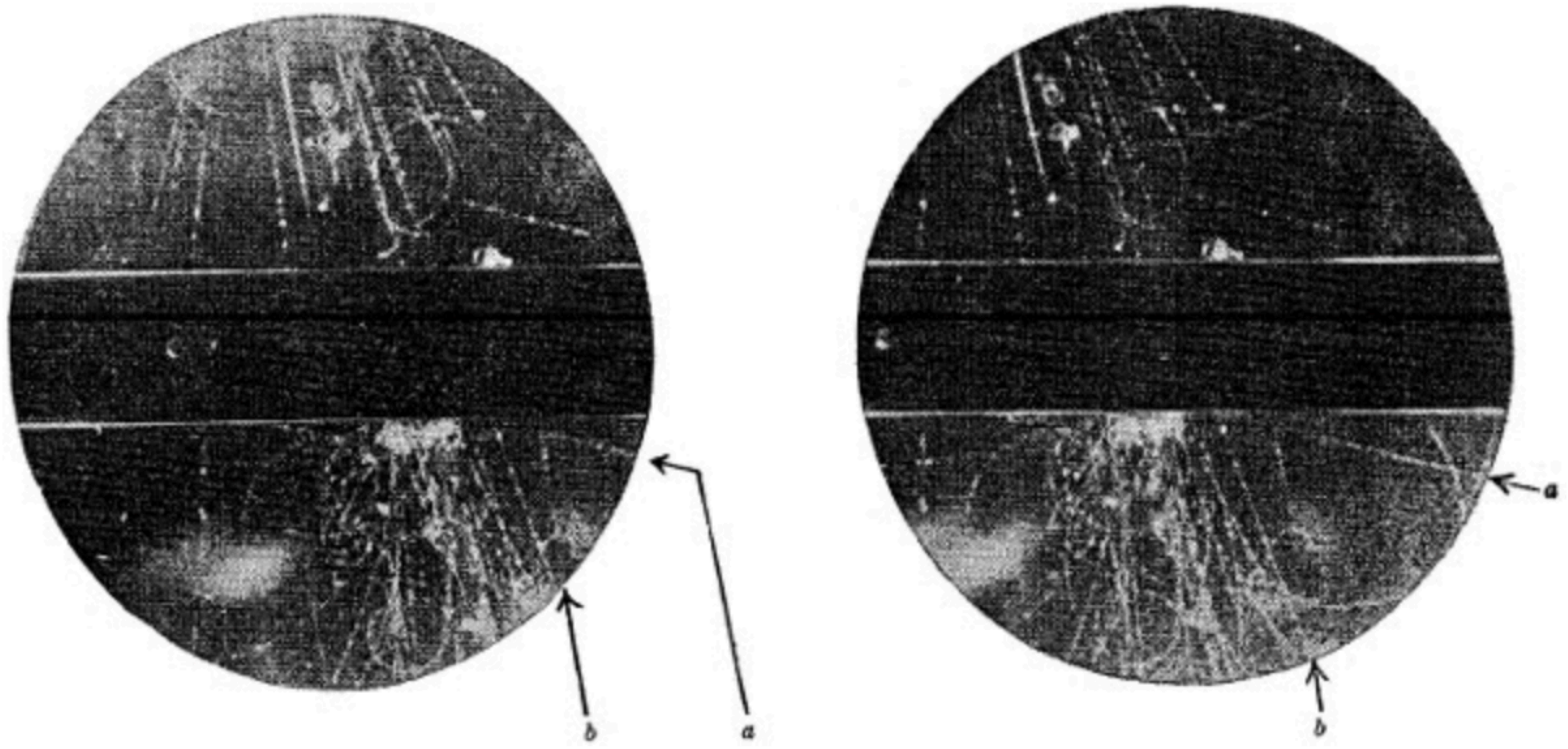

$\omega_{3|1}$ is the Wigner rotation of 3 from (23)RF to (123)RF

$\omega_{3|2}$ is the Wigner rotation of 3 from (31)RF to (123)RF

# Kaons

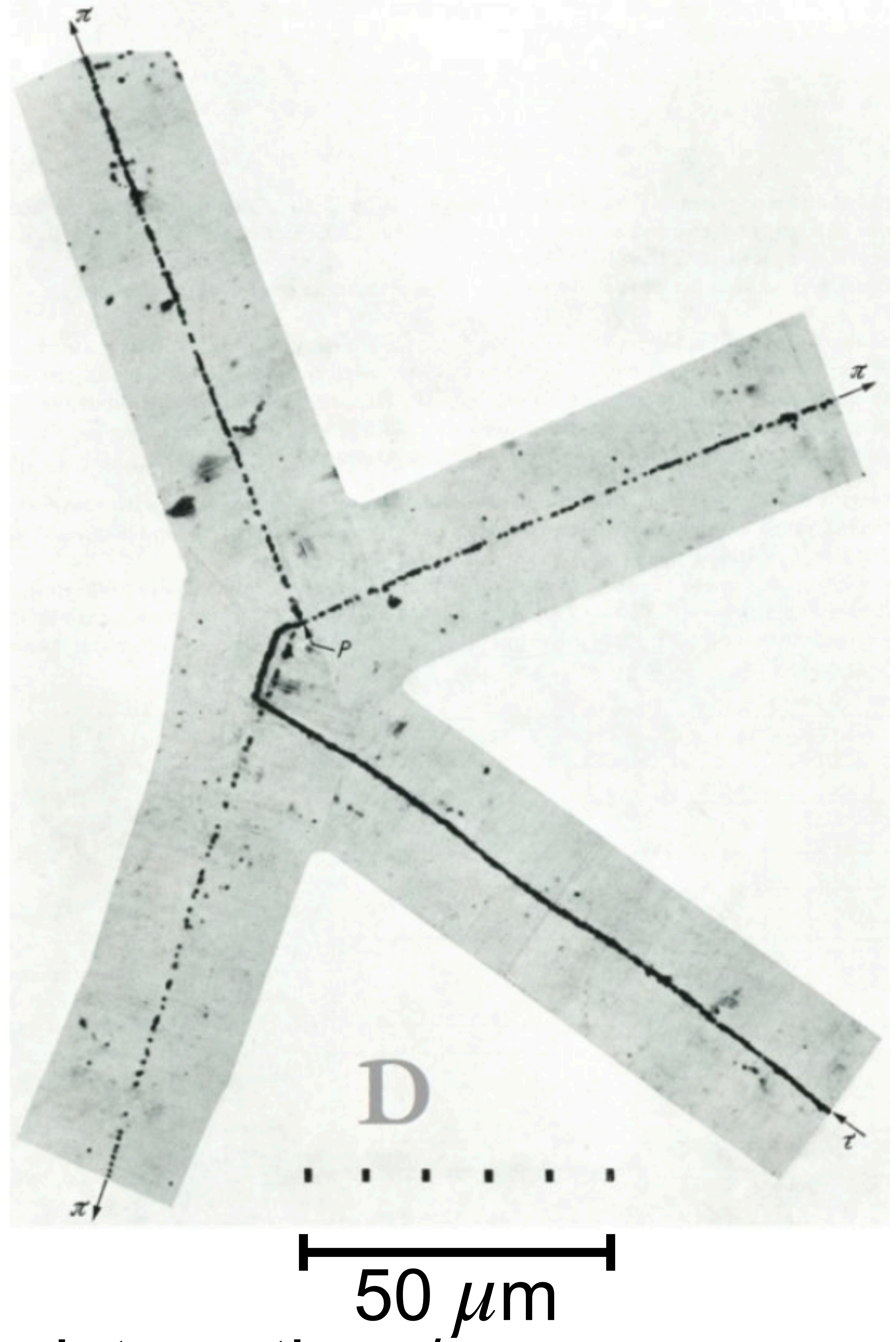
G.D. Rochester and C.C. Butler (1947)

discovered two “V” tracks in cloud chamber



R. Brown et al (1949)

discovered “K” tracks in cloud chamber




Source: https://www.cloudylabs.fr/wp/kaoninteractions/

# $\Lambda^0$ baryon

**Evidence Concerning the Existence of the New Unstable Elementary Neutral Particle**

V. D. Hopper and S. Biswas
*Department of Physics, University of Melbourne, Melbourne, Australia*
October 30, 1950

Observation from cosmic rays of the decay

$$\Lambda^0 \to p + \pi^-$$

Unexpected long life-time $\tau \sim 10^{-10}$ s

Resonance typical life-time $\tau \sim 10^{-23}$ s

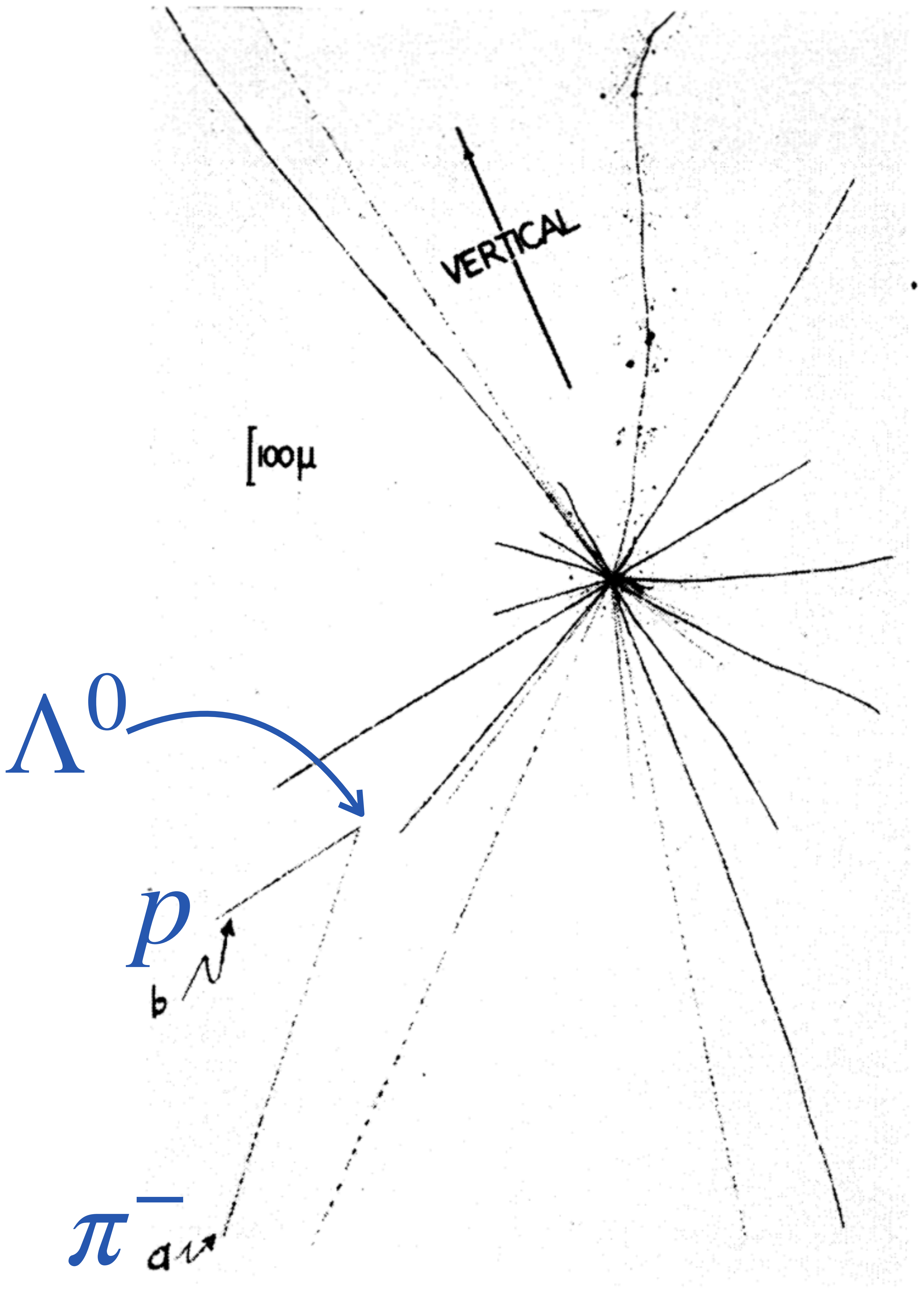



FIG. 1. Facsimile drawing which shows position of the two-pronged star relative to a large star. The plane of the two-pronged star does not coincide with the center of the large star. Track (a) corresponds to a meson and track (b) to a proton.

# $\Lambda^0$ baryon

LETTERS TO THE EDITOR

**Isotopic Spin and New Unstable Particles**

M. GELL-MANN
*Department of Physics and Institute for Nuclear Studies, University of Chicago, Chicago, Illinois*
(Received August 21, 1953)

Phys. Rev. 92 (1953) 833

M. Gell-Mann
1929 - 2019

At the end of this section it is worth to recall the reminiscences of Gell-Mann [50]. "Now let me return to the paper that I did sent off in August 1953 . . . . Isotopic Spin and the New Unstable Particles. That was not my title, which was: Isotopic Spin and Curious Particles. Physical Review rejected "Curious Particles". I tried "Strange Particles" and they rejected that too. They insisted on: "New Unstable Particles". That was the only phrase sufficiently pompous for the editors of the Physical Review. I should say that I have always hated the Physical Review Letters and almost twenty years ago I decided never again to publish in that journal, but in 1953 I was scarcely in the position to show around."

Source: https://www.fuw.edu.pl/~ajduk/hyperakw.pdf

Long list of new particles with "strange" properties

$$K^\pm, K^0, \bar{K}^0, \Lambda^0, \Sigma^\pm, \Sigma^0, \Xi^-, \Xi^0, \ldots$$

19 listed resonances in 1957, 26 in 1963

Long life time, appear in pair, etc

# $\phi$ meson

EXISTENCE AND PROPERTIES OF THE $\varphi$ MESON*

P. L. Connolly, E. L. Hart, K. W. Lai, G. London,† G. C. Moneti,‡ R. R. Rau, N. P. Samios, I. O. Skillicorn, and S. S. Yamamoto
Brookhaven National Laboratory, Upton, New York

and

M. Goldberg, M. Gundzik, J. Leitner, and S. Lichtman
Syracuse University, Syracuse, New York
(Received 27 March 1963)

Phys. Rev. Lett. 10 (1963) 371

- Exp. collaboration becomes larger
- Use of kaon beam
- ~ 40 events in a single experiment
- Don't seem to decay into pions

$$\phi \to K^+K^-$$

$$\phi \not\to \pi^+\pi^-$$

$$\phi \not\to \pi^+\pi^-\pi^0$$

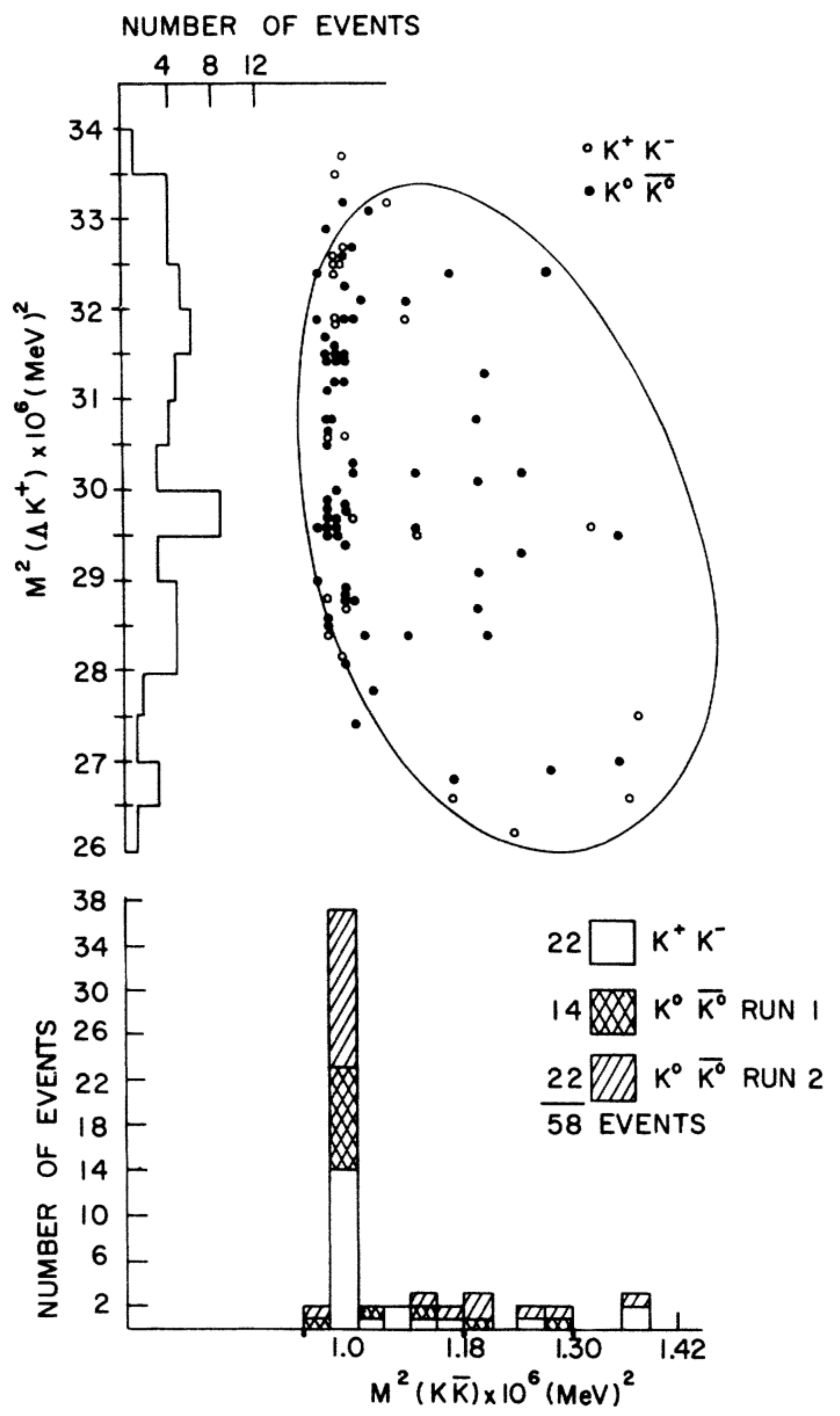



FIG. 1. Dalitz plot for the reaction $K^- + p \to \Lambda + K + \bar{K}$. The effective-mass distribution for $K\bar{K}$ and for $\Lambda K^+$ are projected on the abscissa and ordinate (see reference 7).

# The eightfold way

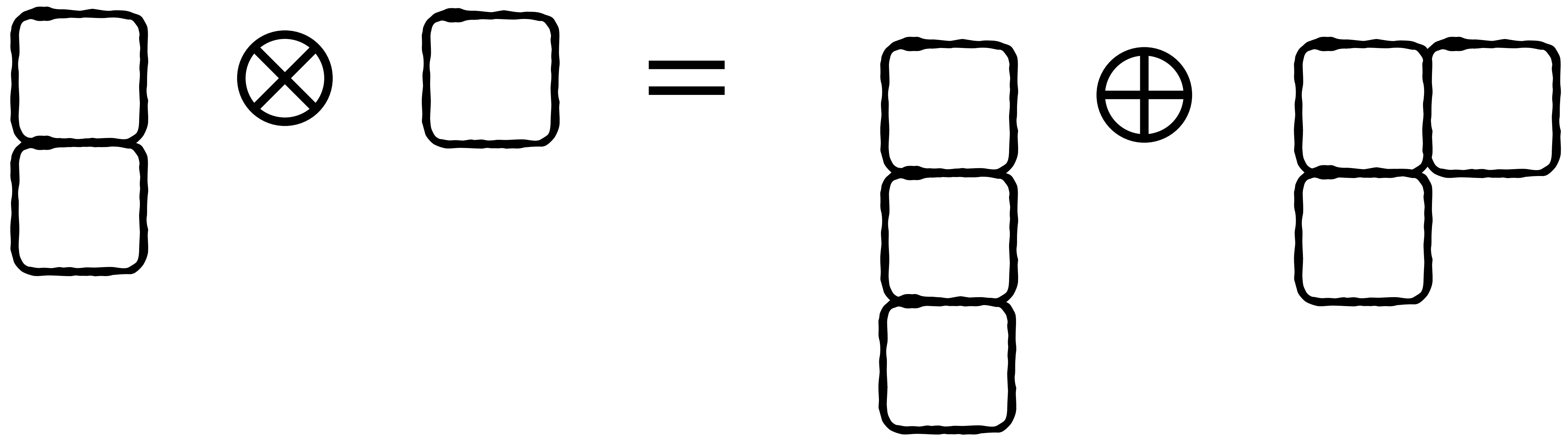
Rotation in 3 real dimensions: angular momentum

Rotation in 2 complex dimensions: isospin

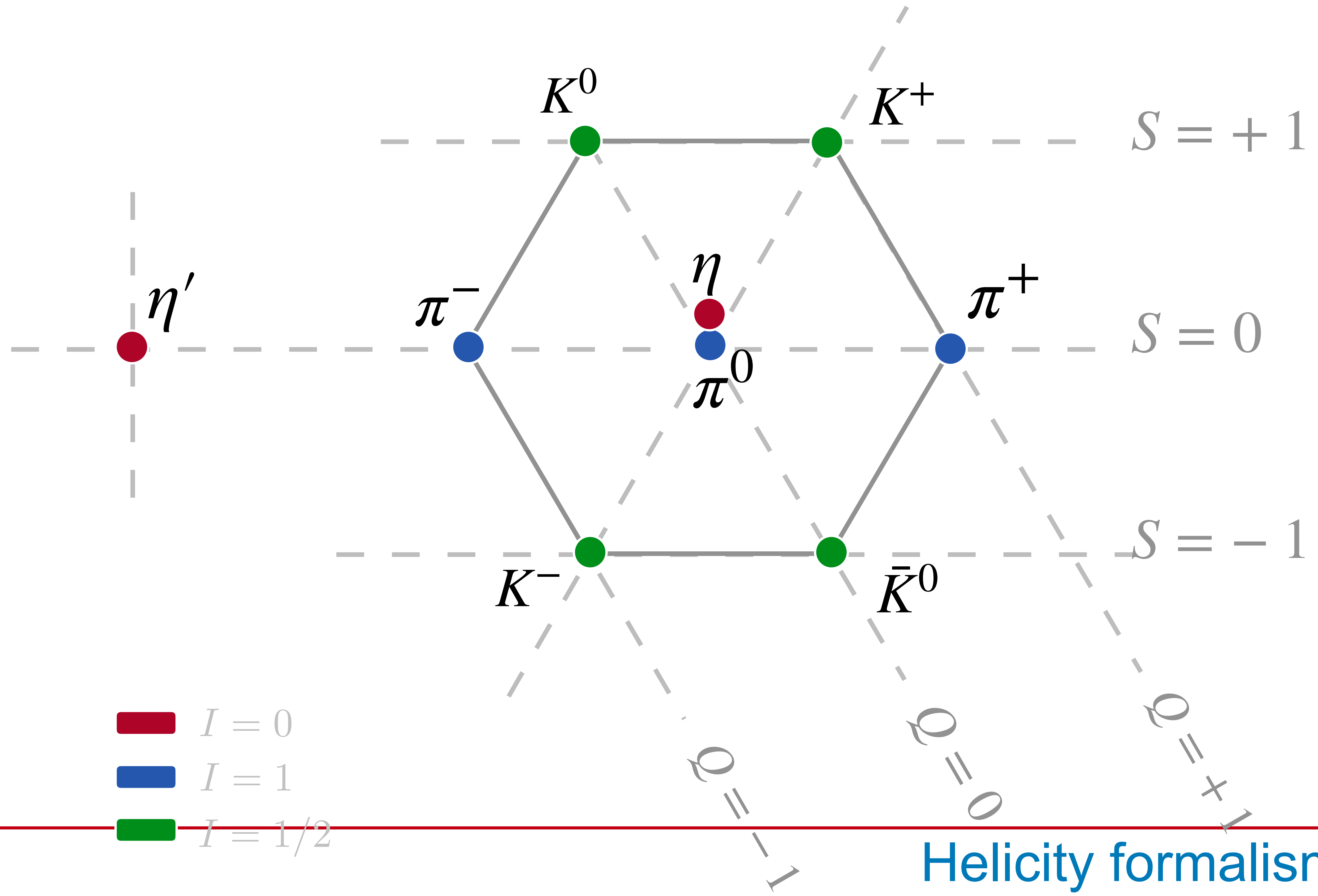
**Both are mathematically equivalent!**

What about rotation in 3 complex dimensions?

"Flavor" is the generalization of isospin

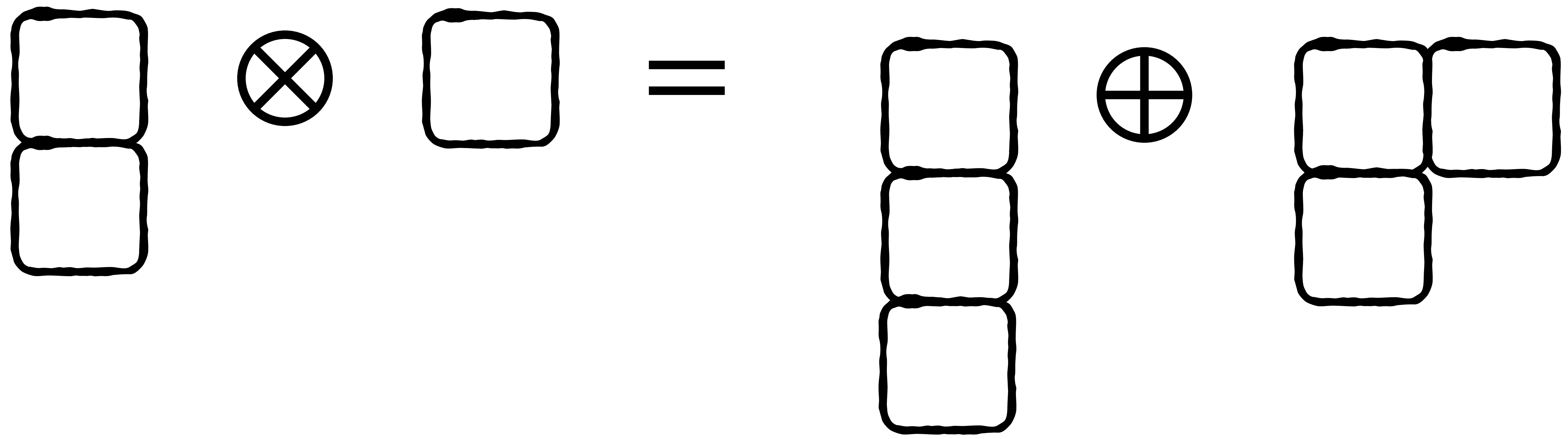


$$3^* \times 3 \qquad 1 + 8$$

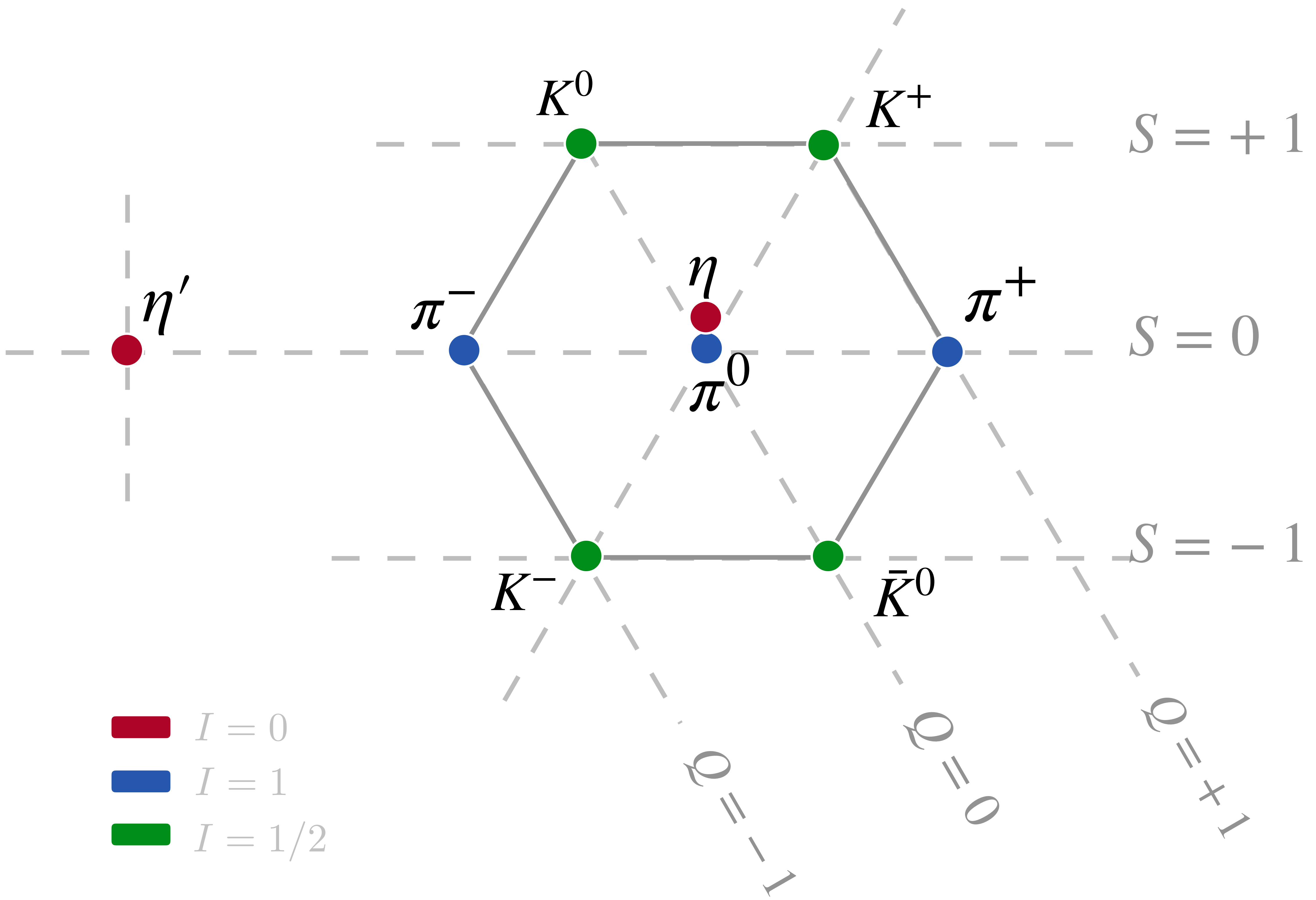


# The eightfold way

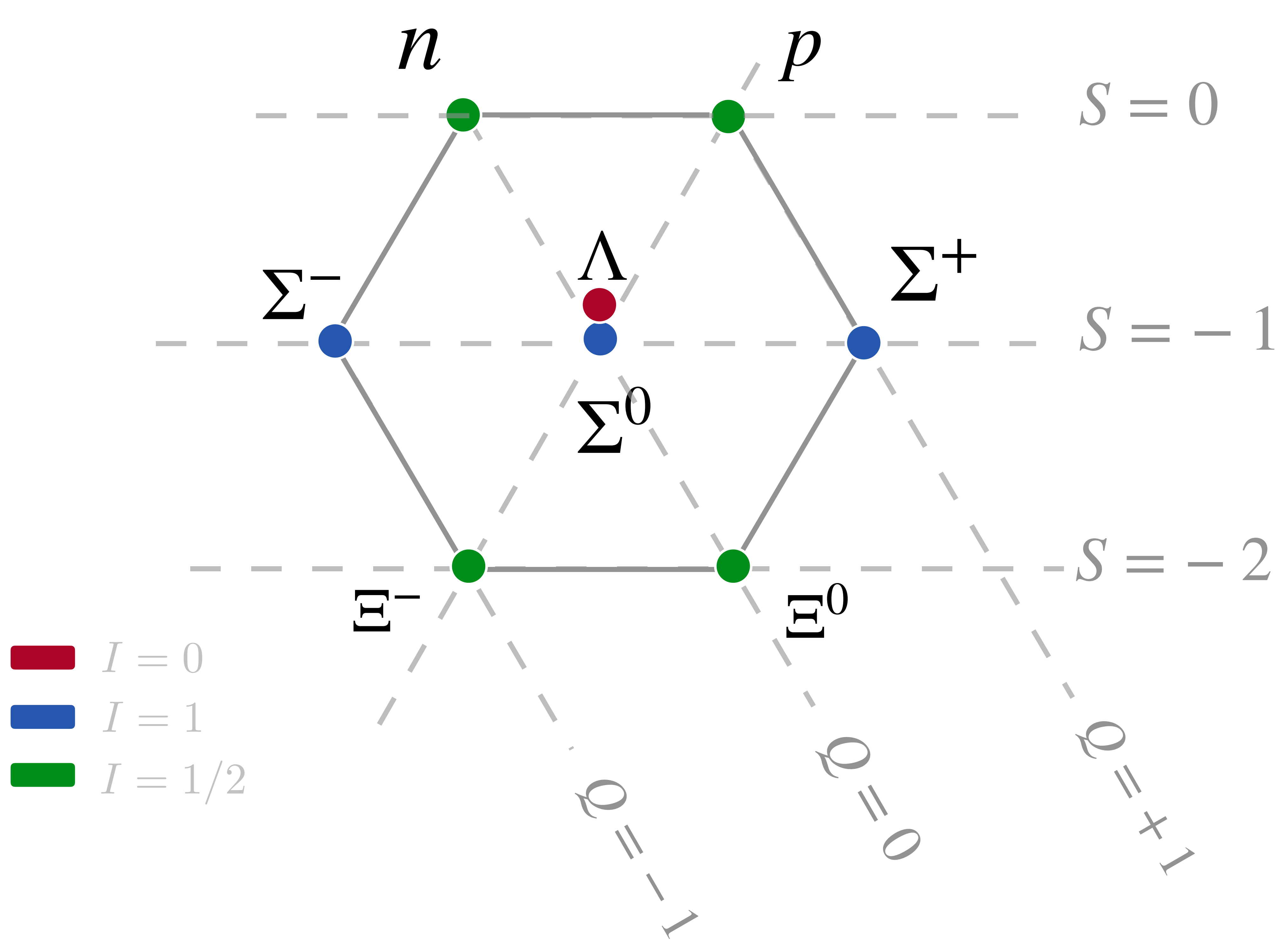
## Mesons split into a singlet and an octet



$$3^* \times 3 \qquad 1 + 8$$



# The baryon octet




$$m_N = m_0 + 3m_u$$

$$m_\Sigma = m_0 + 2m_u + m_s \qquad m_\Lambda = m_0 + 2m_u + m_s$$

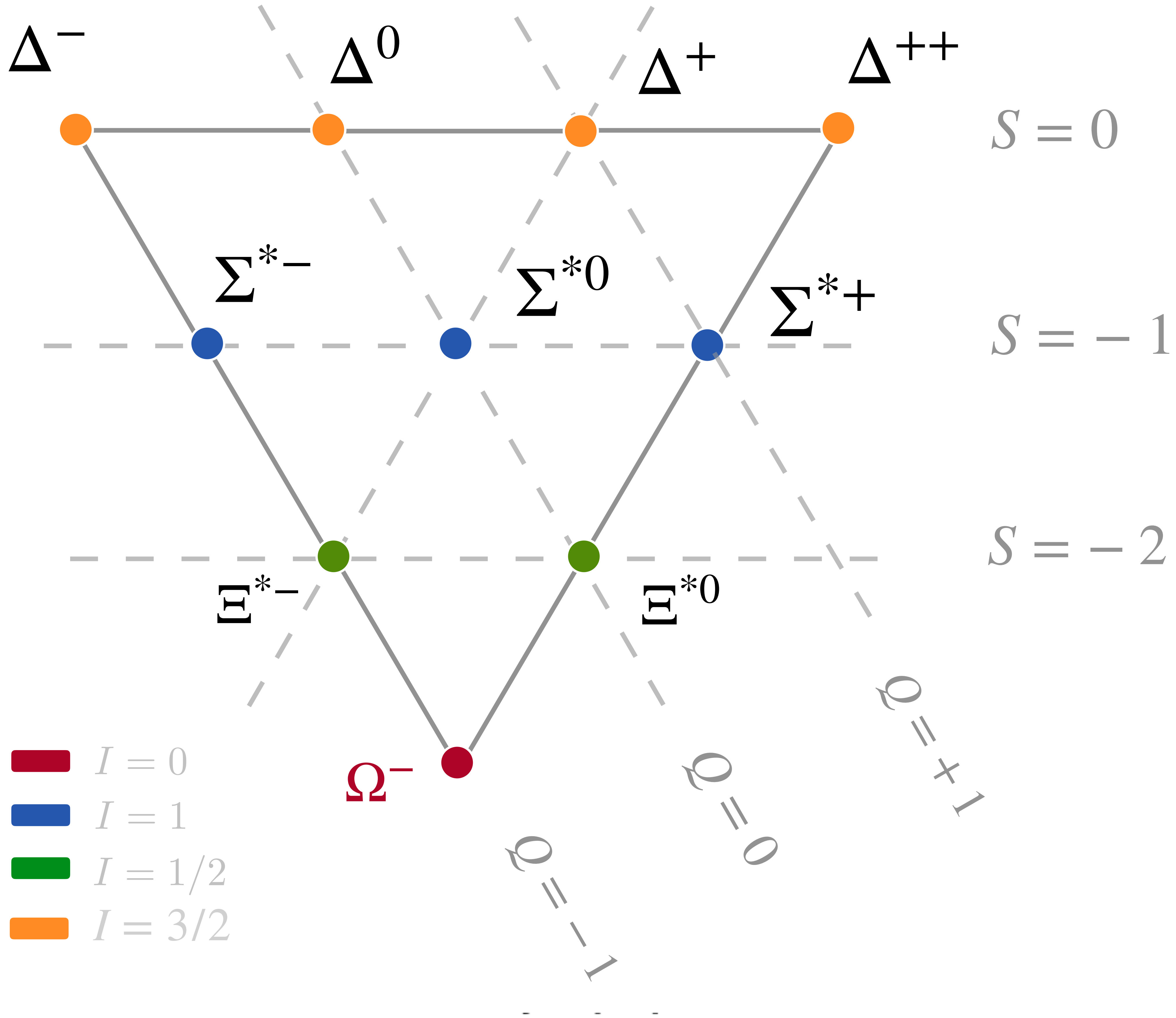
$$m_\Xi = m_0 + m_u + 2m_s$$

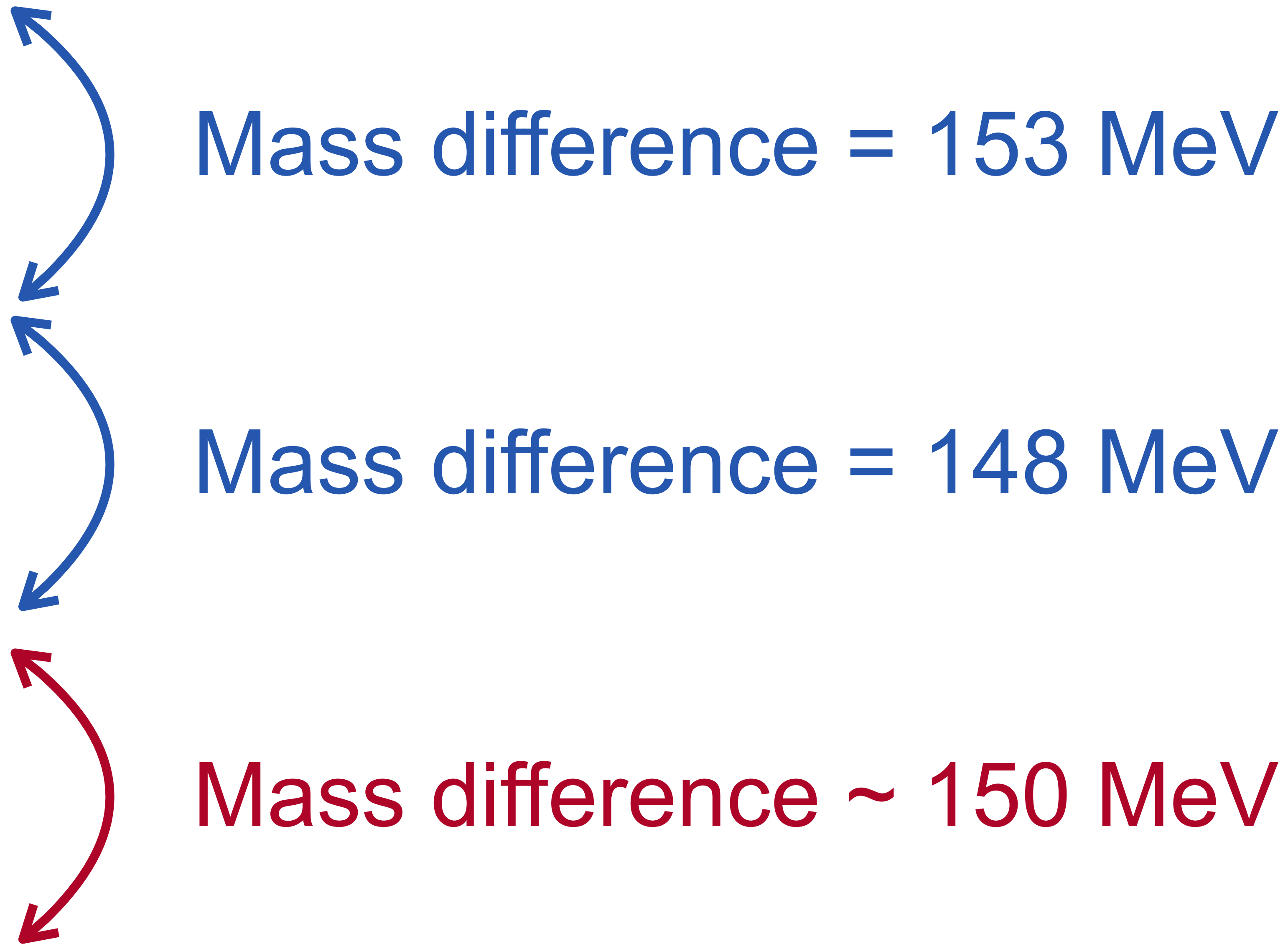
| | Name | Symbol | Isospin | Strangeness | Mass (MeV/$c^2$) |
|---|---|---|---|---|---|
| Octet | Nucleons | N | ½ | 0 | 939 |
| | Lambda baryons | Λ | 0 | −1 | 1116 |
| | Sigma baryons | Σ | 1 | −1 | 1193 |
| | Xi baryons | Ξ | ½ | −2 | 1318 |

Gell-Mann - Okubo mass relation

$$m_\Sigma + 3m_\Lambda = 2m_N + 2m_\Xi$$

# The baryon decuplet



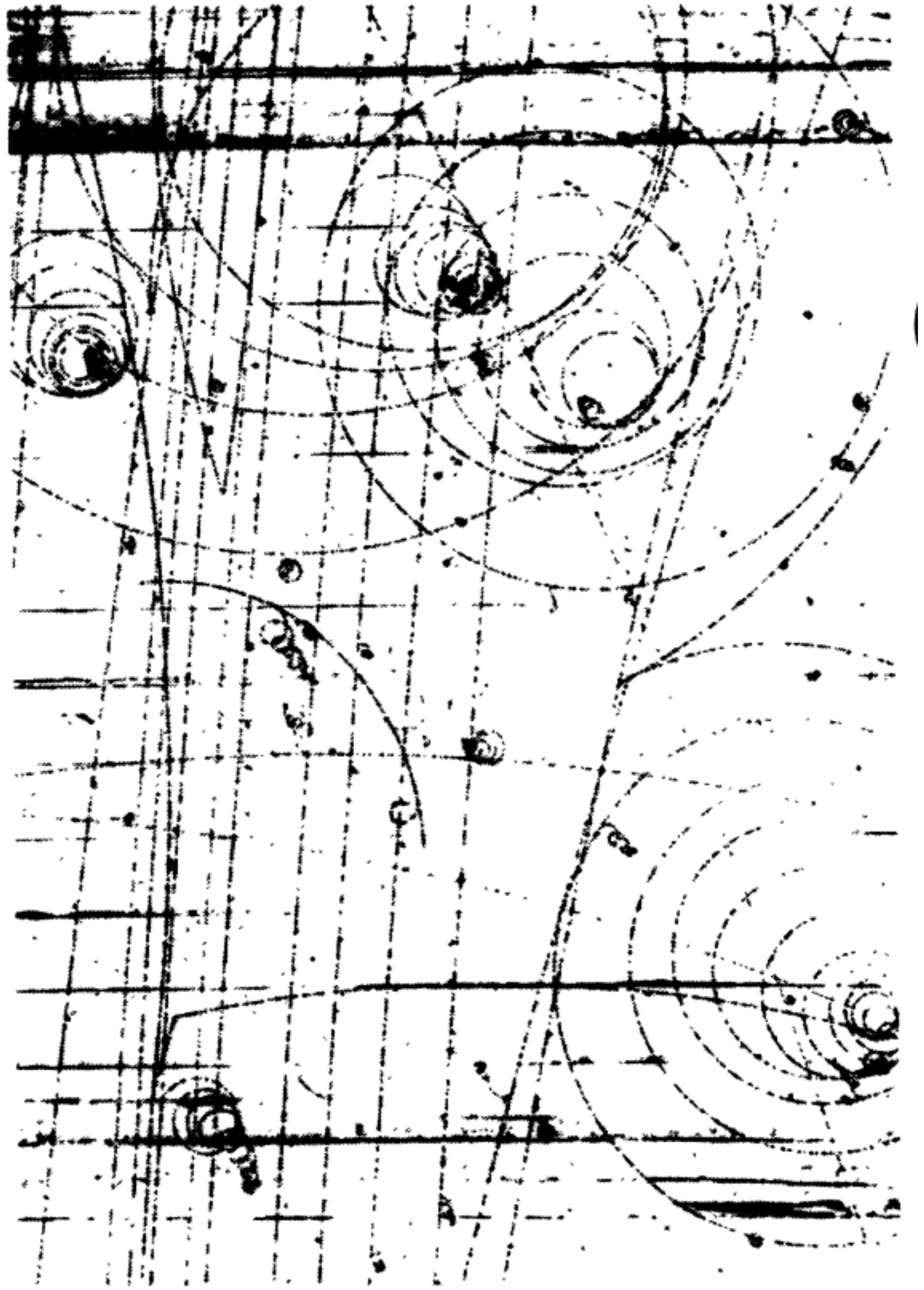

| | Name | Symbol | Isospin | Strangeness | Mass (MeV/$c^2$) |
|---|---|---|---|---|---|
| Decuplet | Delta baryons | Δ | 3⁄2 | 0 | 1232 |
| | Sigma baryons | Σ* | 1 | −1 | 1385 |
| | Xi baryons | Ξ* | ½ | −2 | 1533 |
| | Omega baryon | Ω | 0 | −3 | 1672 |

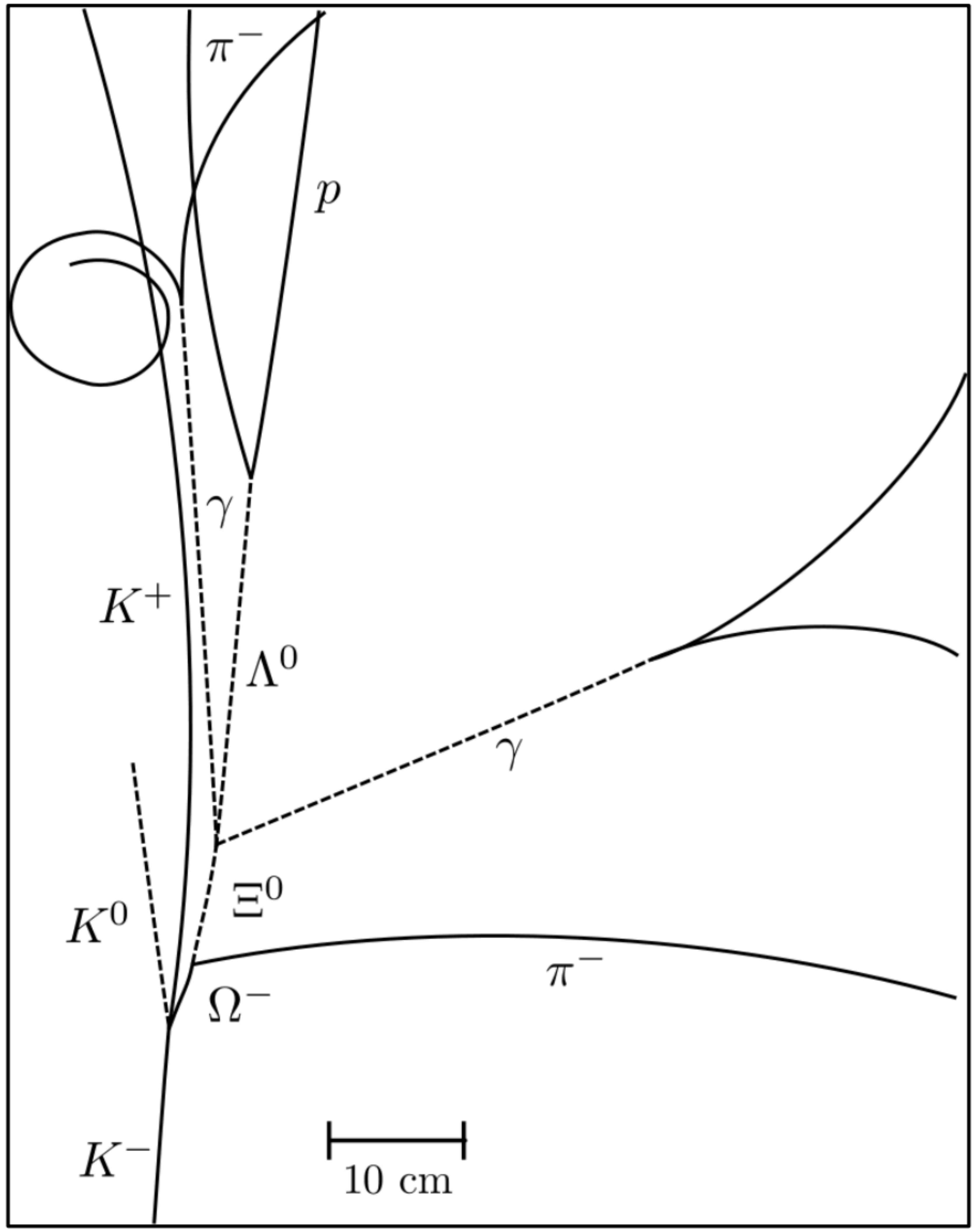
Prediction of a double strange baryon, with a negative electric charge and a mass around 1680 MeV by Gell-Mann in 1962

# Discovery of the $\Omega^-$ baryon

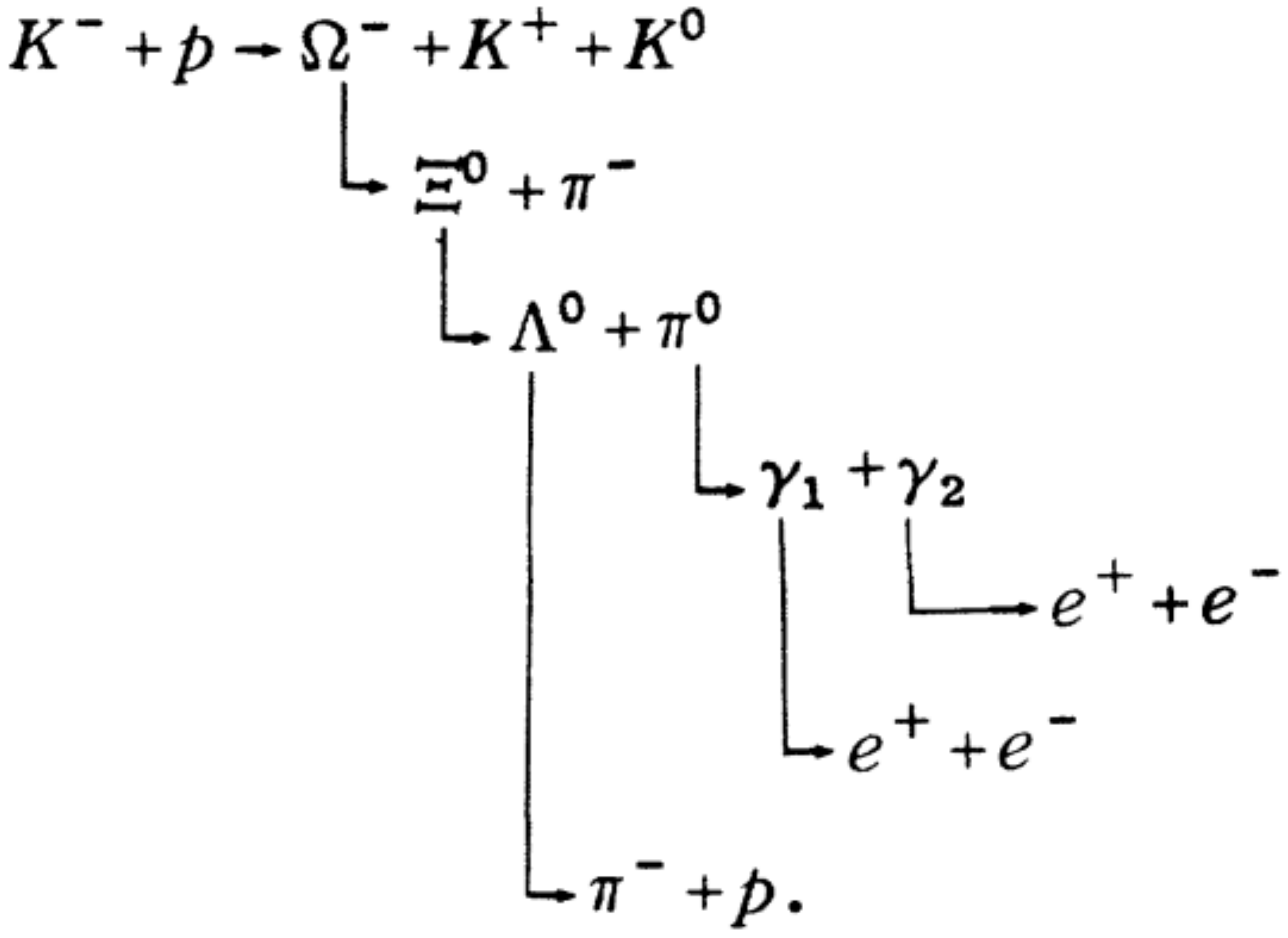
Barnes et al Phys. Rev. Lett. 12 (1964) 204

Can you spot the $\Omega^-$ baryon?



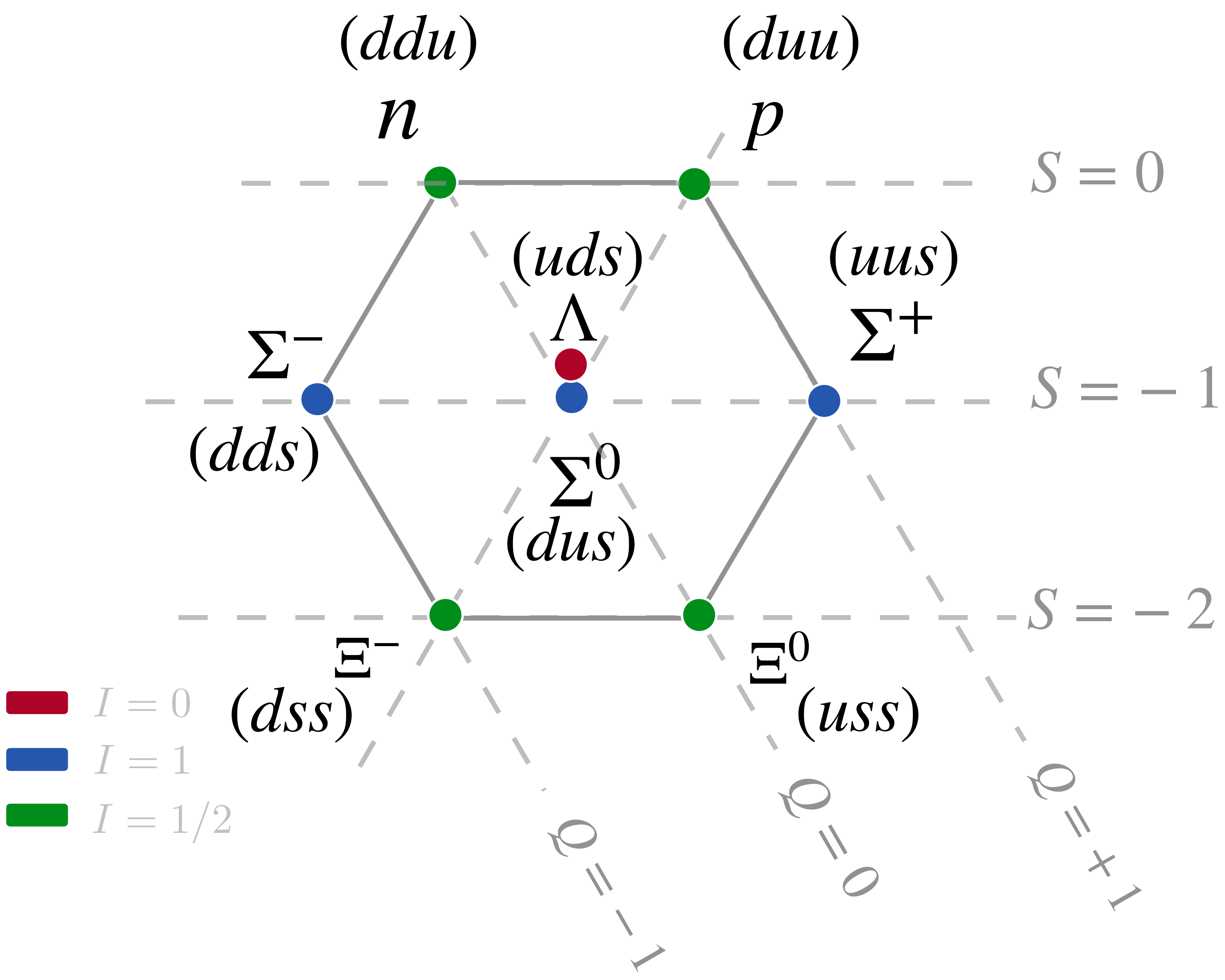


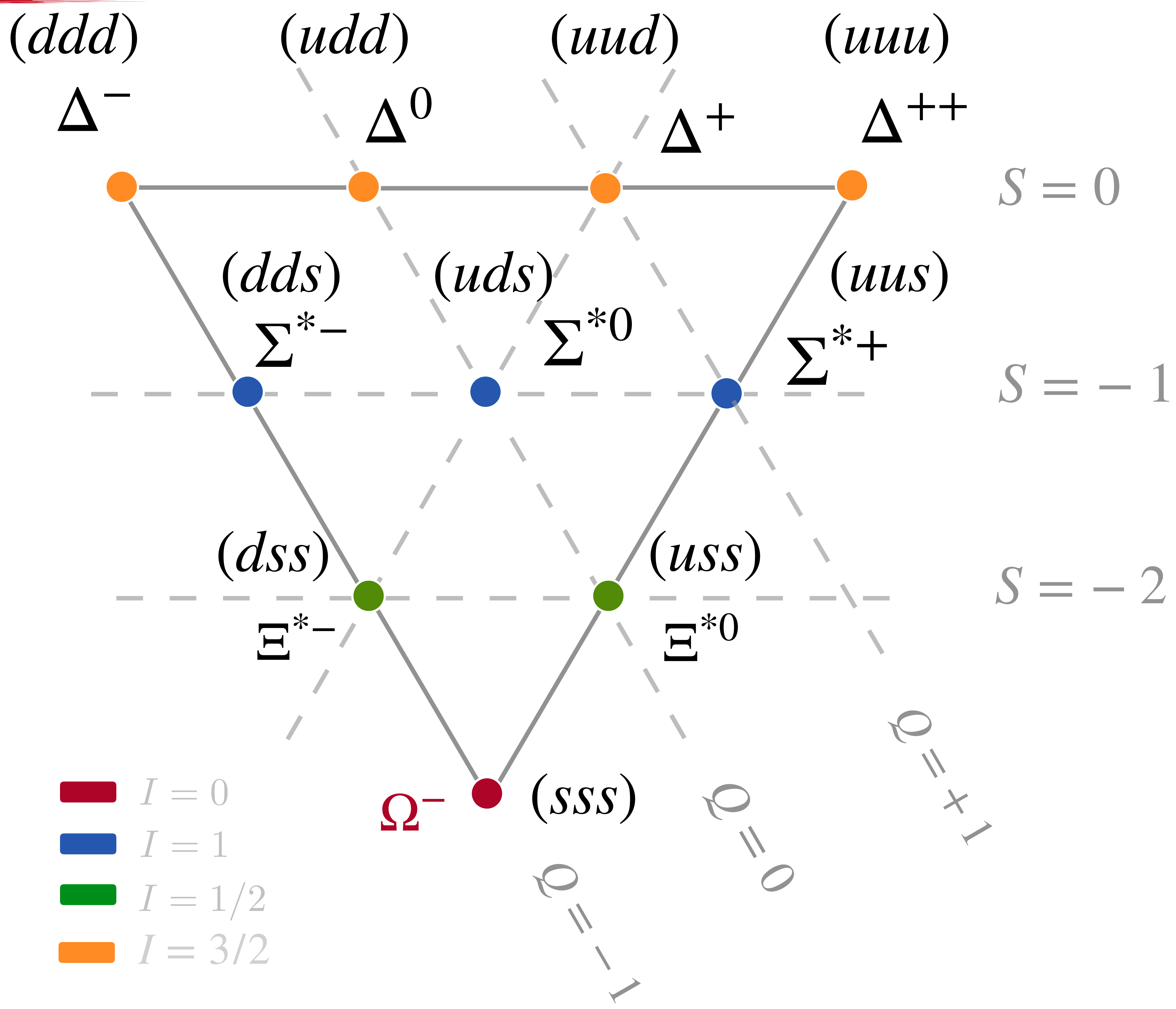

Strange cascade:




Neutral particles (dashed lines) don’t leave a track

# Three flavor of quarks




| | $I$ | $S$ | $\mathscr{B}$ | $e_c$ |
|---|---|---|---|---|
| $u$ | 1/2 | 0 | 1/3 | +2/3 |
| $d$ | 1/2 | 0 | 1/3 | −1/3 |
| $s$ | 0 | −1 | 1/3 | −1/3 |